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A posteriori error estimators for mixed finite element methods in linear elasticity

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Summary. Three a posteriori error estimators for $PEERS$ and $BDMS$ elements in linear elasticity are presented: one residual error estimator and two estimators based on the solution of auxiliary local problems with different boundary conditions. All of them are reliable and efficient with respect to the standard norm and furthermore robust for nearly incompressible materials.

1 Introduction

Adaptive finite element methods and a posteriori error estimators are indispensable tools in the numerical treatment of partial differential equations. The error estimators provide indicators for refining and coarsening the mesh and allow to control whether the error is below a given threshold.

There are many different error estimators for finite element methods for a variety of different equations and discretizations. An overview on the common methods in a posteriori error analysis is given in [14].

The estimators presented there deal with discretizations which are based on primal finite element methods, that is pure minimization problems and no saddle point problems. But, often a saddle point problem and the corresponding mixed finite element discretization are the more natural way to treat a partial differential equation numerically. In elasticity there are mainly two reasons to prefer the mixed method: first it is stable even for nearly incompressible materials and second it provides an acceptable precision simultaneously for displacements and stresses.

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A first error estimator for a mixed finite element discretization of the Poisson equation was developed by Braess and Verfürth [5]. This estimator is based on a mesh dependent norm which is not equivalent to the standard norm of the given problem. Alonso [1] and Carstensen [7] were the first who developed residual error estimators for this problem with respect to the standard norm. They circumvent the difficulties arising from the anisotropy of the norm by the use of the Helmholtz decomposition of square-integrable tensors. We also utilize the Helmholtz decomposition to derive a residual error estimator for the $PEERS$ element [2] and Stenberg's $BDMS$ family [13] of elements for the linear elasticity problem. This error estimator is similar to the one given by Carstensen in [8]. But, in contrast to Carstensen we do not use differently weighted norms for the upper and lower error bounds. Our estimator is reliable and efficient and furthermore robust for nearly incompressible materials.

In the last section we present two error estimators based on the solution of auxiliary local problems. These error estimators are similar to the ones developed by Bank and Weiser [4]. In order to deal with these problems and to obtain estimates with respect to the standard norms, we need the results on the stability of $BDMS$ elements which we presented in [12].

2 The linear elasticity problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal or polyhedral domain with boundary $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D, \Gamma_N \neq \emptyset$ and unit outer normal n. In the following $u : \Omega \to \mathbb{R}^d$ will denote the displacement, $\sigma : \Omega \to \mathbb{R}^{d\times d}$ the symmetric stress tensor, and $\varepsilon : \Omega \to \mathbb{R}^{d\times d}$ the strain tensor. We use the abbreviations $Du = \frac{1}{2}(\operatorname{grad} u + \operatorname{grad} u^T)$ for the symmetric gradient and $\operatorname{as}\sigma = \sigma - \sigma^T$ for the skew symmetric part of a tensor.

The linear elasticity problem is given by the boundary value problem

(2.1a) $$\varepsilon = Du \quad \text{in } \Omega$$

(2.1b) $$\varepsilon = C^{-1}\sigma \quad \text{in } \Omega$$

(2.1c) $$-\operatorname{div}\sigma = f \quad \text{in } \Omega$$

(2.1d) $$\operatorname{as}\sigma = 0 \quad \text{in } \Omega$$

(2.1e) $$u = 0 \quad \text{on } \Gamma_D$$

(2.1f) $$\sigma \cdot n = 0 \quad \text{on } \Gamma_N$$

where f is the given body load. C is the elasticity tensor

(2.2) $$\sigma = C\varepsilon = \lambda \operatorname{tr}(\varepsilon) I + 2\mu\varepsilon$$

where λ, $\mu > 0$ are the Lamé parameters, $\operatorname{tr}(\varepsilon)$ is the trace of ε, and I denotes the unit tensor. For simplicity we assume that the displacement on the Dirichlet boundary Γ_D vanishes and that there are no surface tractions on the Neumann boundary Γ_N.

The Hellinger-Reissner principle is a corresponding mixed variational formulation, in which the strain ε is eliminated:

Find $(\sigma, u, \gamma) \in H \times V \times W$ so that

$$a(\sigma, \tau) + b(\tau; u, \gamma) = 0 \tag{2.3a}$$

$$b(\sigma; v, \eta) = -(f, v) \tag{2.3b}$$

holds for all $(\tau, v, \eta) \in H \times V \times W$.

The bilinear forms a and b are given by

$$a(\sigma, \tau) = \int_\Omega C^{-1}\sigma : \tau \, dx, \tag{2.4a}$$

$$b(\sigma; v, \eta) = \int_\Omega \operatorname{div} \sigma \cdot v \, dx + \int_\Omega \sigma : \eta \, dx, \tag{2.4b}$$

where $\sigma : \tau = \sum_{1 \le i,j \le d} \sigma_{ij}\tau_{ij}$. The spaces H, V, W are defined by

$$\begin{aligned} H &= \{\sigma \in L^2(\Omega)^{d\times d} \mid \operatorname{div} \sigma \in L^2(\Omega)^d\}, & \|\sigma\|_H &= \sqrt{\|\operatorname{div} \sigma\|_{L^2}^2 + \|\sigma\|_{L^2}^2}, \\ V &= \{u \in L^2(\Omega)^d\}, & \|u\|_V &= \|u\|_{L^2}, \\ W &= \{\gamma \in L^2(\Omega)^{d\times d} \mid \gamma + \gamma^T = 0\}, & \|\gamma\|_W &= \|\gamma\|_{L^2}. \end{aligned}$$

The divergence of a tensor σ is taken row by row i.e. $(\operatorname{div} \sigma)_i = \sum_{1 \le j \le n} \partial_j \sigma_{ij}$. The additional Lagrangian parameter γ was introduced by Arnold, Brezzi, Douglas [2] in order to allow the construction of stable finite elements [6, §VII.2].

Arnold and Falk [3] have proven the unique solvability of the variational problem (2.3). In fact they have etablished the following stronger result:

Lemma 2.1 *Denote by Z the kernel of the bilinear form b. Then there are constants c_a and c_b which do not depend on λ such that*

$$a(\sigma, \tau) \le \frac{1}{\mu} \|\sigma\|_H \|\tau\|_H \qquad \forall \sigma, \tau \in H, \tag{2.6a}$$

$$a(\sigma, \sigma) \ge c_a \|\sigma\|_H^2 \qquad \forall \sigma \in H, \tag{2.6b}$$

$$b(\sigma; v, \eta) \le \|\sigma\|_H \|(v, \eta)\|_{V\times W} \qquad \forall \sigma \in H, (v, \eta) \in V \times W, \tag{2.6c}$$

$$\sup_{\substack{\sigma \in H \\ \sigma \ne 0}} \frac{b(\sigma; v, \eta)}{\|\sigma\|_H} \ge c_b \|(v, \eta)\|_{V\times W} \qquad \forall (v, \eta) \in V \times W. \tag{2.6d}$$

We can combine the two bilinear forms a and b to a single bilinear form

$$d[(\sigma, u, \gamma), (\tau, v, \eta)] = a(\sigma, \tau) + b(\tau; u, \gamma) + b(\sigma; v, \eta). \tag{2.7}$$

Standard arguments for saddle point problems [11, Lemma 2.9 and Lemma 2.10] and the previous lemma then imply the following stability estimate:

Lemma 2.2 *Set*

$$c_D = \sqrt{2}\max\{1, \frac{1}{\mu}\}, \quad c_d = \frac{1}{\sqrt{5}}\min\{c_a, \frac{\mu c_b^2}{\sqrt{2\mu^2 c_b^2 + 8}}\}.$$

Then d satisfies the estimates

$$\text{(2.8a)} \qquad \sup_{(\tau,v,\eta)\neq 0}\ \sup_{(\sigma,u,\gamma)\neq 0} \frac{d[(\sigma,u,\gamma),(\tau,v,\eta)]}{\|(\sigma,u,\gamma)\|_{H\times V\times W}\,\|(\tau,v,\eta)\|_{H\times V\times W}} \le c_D$$

$$\text{(2.8b)} \qquad \inf_{(\tau,v,\eta)\neq 0}\ \sup_{(\sigma,u,\gamma)\neq 0} \frac{d[(\sigma,u,\gamma),(\tau,v,\eta)]}{\|(\sigma,u,\gamma)\|_{H\times V\times W}\,\|(\tau,v,\eta)\|_{H\times V\times W}} \ge c_d$$

Thanks to this stability result, all forthcoming constants are independent of the Lamé parameter λ. Hence, the corresponding estimates are robust for nearly incompressible materials.

3 *PEERS* and *BDMS* elements

Let $\mathcal{T}_h$ be an admissible and shape-regular triangulation of Ω, with regularity parameter $\kappa = \sup_h \max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T}$. Here h_T denotes the diameter of T and ρ_T is the diameter of the largest ball inscribed into T. Given an element T of $\mathcal{T}_h$ we denote by $\lambda_0, \dots, \lambda_d$ its barycentric coordinates and introduce the bubble function $\psi_T = (d+1)^{d+1}\prod_{i=0}^{d}\lambda_i$.

As usual the operator curl is defined by

$$\begin{aligned}
\operatorname{curl} u &= \nabla \times u && \text{if } u:\Omega\to\mathbb{R}^3\\
\operatorname{curl} f &= (\partial_y f, -\partial_x f) && \text{if } f:\Omega\to\mathbb{R}\\
\operatorname{curl} u &= \partial_x u_2 - \partial_y u_1 && \text{if } u:\Omega\to\mathbb{R}^2.
\end{aligned}$$

Using these definitions we set

(3.1)

$$B_k(T) = \{(\sigma_{ij}) \in \mathbb{R}^{d\times d} \mid (\sigma_{i1}, \dots, \sigma_{id}) = \operatorname{curl}(\psi_T w_i),\ w_i \in \mathbb{P}_k(T)^l,\ i = 1, \dots, d,\ l = 2d-3\}.$$

Definition 3.1 *(Arnold, Brezzi, Douglas [2]) The plane elasticity element with reduced symmetry, called $PEERS$, is defined by*

$$\begin{aligned} RT_0(T) =&\{w \in L^2(T)^d \mid w = a + b\underline{x}, a \in \mathbb{R}^d,\ b \in \mathbb{R}, \\ &\underline{x} = (x_1, \dots, x_d)^T\}, \end{aligned} \tag{3.2a}$$

$$\begin{aligned} H_h =&\{\sigma_h \in H \mid \sigma_h|_T \in RT_0(T)^d \oplus B_0(T), T \in \mathcal{T}_h, \\ &\sigma_h \cdot n = 0 \ on\ \Gamma_N\}, \end{aligned} \tag{3.2b}$$

$$V_h =\{v_h \in V \mid v_h|_T \in \mathbb{P}_0(T)^d,\ T \in \mathcal{T}_h\}, \tag{3.2c}$$

$$W_h =\{\eta_h \in W \cap C(\Omega)^{d\times d} \mid \eta_h|_T \in \mathbb{P}_1(T)^{d\times d},\ T \in \mathcal{T}_h\}, \tag{3.2d}$$

$$PEERS =H_h \times V_h \times W_h. \tag{3.2e}$$

Definition 3.2 *(Stenberg [13]) For $k \geq 2$ the $BDMS$-element is defined by*

$$BDM_k(T) =\mathbb{P}_k(T)^d, \tag{3.3a}$$

$$\begin{aligned} H_h =&\{\sigma_h \in H \mid \sigma_h|_T \in BDM_k(T)^d \oplus B_{k-1}(T),\ T \in \mathcal{T}_h, \\ &\sigma_h \cdot n = 0 \ on\ \Gamma_N\}, \end{aligned} \tag{3.3b}$$

$$V_h =\{v_h \in V \mid v_h|_T \in \mathbb{P}_{k-1}(T)^d,\ T \in \mathcal{T}_h\}, \tag{3.3c}$$

$$W_h =\{\eta_h \in W \mid \eta_h|_T \in \mathbb{P}_k(T)^{d\times d},\ T \in \mathcal{T}_h\}, \tag{3.3d}$$

$$BDMS_k =H_h \times V_h \times W_h. \tag{3.3e}$$

Remark 3.3. For both elements, the space W_h is isomorphic to the space $\{\varphi \mid \varphi|_T \in \mathbb{P}_k(T)^l,\ T \in \mathcal{T}_h,\ l = 2d-3\}$. The corresponding isomorphism is given by

$$I_2 : \varphi \to \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \qquad \text{if } d = 2,$$

$$I_3 : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \to \begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ -\varphi_1 & 0 & \varphi_3 \\ -\varphi_2 & -\varphi_3 & 0 \end{pmatrix} \qquad \text{if } d = 3.$$

Denote by $Z_h = \{\sigma \in H_h \mid b(\sigma; v, \eta) = 0,\ \forall (v, \eta) \in V_h \times W_h\}$ the kernel of b in these finite element spaces. The following estimates for $PEERS$ are established in [2]

$$a(\sigma, \sigma) \geq c_a^{PEERS} \|\sigma\|_H^2 \qquad \forall \sigma \in Z_h, \tag{3.4a}$$

$$\sup_{\sigma \in H_h \setminus \{0\}} \frac{b(\sigma; v, \eta)}{\|\sigma\|_H} \geq c_b^{PEERS} \|(v, \eta)\|_{V\times W} \quad \forall (v, \eta) \in V_h \times W_h. \tag{3.4b}$$

Hence there exists a unique solution $(\sigma_h, u_h, \gamma_h) \in PEERS$ of the discrete problem (2.3).

In [12] we have established similar estimates for $BDMS_k$ with corresponding constants c_a^{BDMS} and c_b^{BDMS}. Again, this proves the unique sovability of the discrete problem.

As in the analytic case we obtain an inf-sup-condition for the bilinear form d with respect to the finite dimensional spaces.

4 A residual a posteriori error estimator

Following the techniques presented in [14] we use the bilinear form d defined in (2.7) to derive a reliable and efficient a posteriori error estimator, which is robust for nearly incompressible materials. In order to circumvent the difficulties that are due to the anisotropy of the $H(\mathrm{div})$-norm (cf. [5]), we adopt the suggestion of Alonso [1] and Carstensen [7] and use a Helmholtz decomposition for the stress tensors. In contrast to Carstensen's error estimator for the $BDMS$ element [8] we obtain estimates which use the same norm for the upper and lower bound of the estimator.

In the following (σ, u, γ) and $(\sigma_h, u_h, \gamma_h)$ are the solutions of the analytical (2.3) respectively the discretized problem using $PEERS$ or $BDMS$ elements. We denote by $e_\sigma = \sigma - \sigma_h$, $e_u = u - u_h$ and $e_\gamma = \gamma - \gamma_h$ the corresponding errors.

4.1 The residual

We define the residual $R(e_\sigma, e_u, e_\gamma)$ as a linear functional on $H \times V \times W$ by

$$\big\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\big\rangle = d[(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)]. \tag{4.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the corresponding spaces. Obviously the following Galerkin orthogonality holds

$$\big\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\big\rangle = 0 \quad \forall (\tau, v, \eta) \in H_h \times V_h \times W_h. \tag{4.2}$$

In order to derive an a posteriori error estimator, we need an L^2-representation of the residual. Denoting the inner product of L^2 by $(\cdot, \cdot)$, we derive from (2.3):

(4.3)

$$\begin{aligned}
&\big\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\big\rangle \\
&\quad = (C^{-1} e_\sigma, \tau) + (\mathrm{div}\,\tau, e_u) + (\tau, e_\gamma) + (\mathrm{div}\, e_\sigma, v) + (e_\sigma, \eta) \\
&\quad = -(C^{-1}\sigma_h, \tau) - (\mathrm{div}\,\tau, u_h) - (\tau, \gamma_h) + (\mathrm{div}\, e_\sigma, v) + (e_\sigma, \eta) \\
&\quad = -(C^{-1}\sigma_h, \tau) - (\mathrm{div}\,\tau, u_h) - (\tau, \gamma_h) \\
&\qquad - (f, v) - (\mathrm{div}\,\sigma_h, v) - (\mathrm{as}\,\sigma_h, \eta) \\
&\quad = -(C^{-1}\sigma_h + \gamma_h, \tau) - (\mathrm{div}\,\tau, u_h) - (f + \mathrm{div}\,\sigma_h, v) - (\mathrm{as}\,\sigma_h, \eta).
\end{aligned}$$

Since the stress tensors are in $L^2(\Omega)^{d\times d}$ they admit a Helmholtz decomposition rowwise. For its description we introduce the space

$$\Phi = \begin{cases} \{\psi \in H^1(\Omega) \,:\, \psi|_{\Gamma_D} = 0,\ \psi|_{\Gamma_N} = c\} & \text{if } d = 2, \\ \{\psi \in H^1(\Omega)^3 \,:\, \operatorname{div}\psi = 0,\ \operatorname{curl}\psi \cdot n|_\Gamma = 0\} & \text{if } d = 3. \end{cases}$$

Here, c is an arbitrary constant, if Γ_D and Γ_N consist of disjoint connected components, and equals 0 otherwise. Then the rowwise Helmholtz decomposition of a tensor τ is given by (cf. [10, §3])

$$\tau = G\tau + \operatorname{curl} R\tau \qquad \text{with } G\tau = \operatorname{grad}\tilde{G}\tau \tag{4.4}$$

where $\tilde{G}\tau \in (H^1(\Omega)/\mathbb{R})^d$ and $R\tau \in \Phi^d$. The proof of the upper bound on the error requires a certain regularity of this decomposition. More precisely, we have to assume that the estimates

$$|R\tau|_1 \le \|\tau\|_{L^2(\Omega)} \tag{4.5a}$$

$$\left|\tilde{G}\tau\right|_2 \le C\,\|\operatorname{div}\tau\|_{L^2(\Omega)} \tag{4.5b}$$

hold for all tensors $\tau \in H$, where $|\cdot|_1$ and $|\cdot|_2$ denote the H^1 respectively H^2-seminorm. This regularity assumption is satisfied if the boundary Γ is of class $C^{1,1}$ or if Ω is a convex polyhedron. (Note that the assumption $\tilde{G}\tau \in H^2(\Omega)$ is the critical one.)

We know from [12] that there exists an interpolation operator $\Pi_h : H \to H_h$, that fullfills a “commuting diagram property” (c.d.p.)

$$(\operatorname{div}\tau, u_h) = (\operatorname{div}\Pi_h\tau, u_h) \qquad \forall\tau \in H, u_h \in V_h, \tag{4.6}$$

and the error estimate

$$\|\tau - \Pi_h\tau\|_{L^2(T)} \le c_\Pi h_T\,|\tau|_{H^1(T)}\,. \tag{4.7}$$

Moreover, the relation

$$\int_E n(\tau - \Pi_h\tau)v_h = 0 \quad \forall\tau \in H, v_h \in V_h \tag{4.8}$$

holds for all edges respectively faces E. Using this operator we can write the first two terms of (4.3) as follows

(4.9)

$$
\begin{aligned}
&-(C^{-1}\sigma_h+\gamma_h,\tau)-(\operatorname{div}\tau,u_h)\\
&\quad=-(C^{-1}\sigma_h+\gamma_h,G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau)-(\operatorname{div}G\tau,u_h)\\
&\qquad-\underbrace{(\operatorname{div}\operatorname{curl}R\tau,u_h)}_{=0}\\
&\quad=-(C^{-1}\sigma_h+\gamma_h,G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau)-\underbrace{(\operatorname{div}\Pi_h G\tau,u_h)}_{\text{c.d.p.}}\\
&\quad=(C^{-1}\sigma_h+\gamma_h,(\Pi_h-Id)G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau)\\
&\quad=(C^{-1}\sigma_h+\gamma_h-\operatorname{grad}u_h,(\Pi_h-Id)G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau)\\
&\qquad+\underbrace{(\operatorname{grad}u_h,(\Pi_h-Id)G\tau)}_{=0}\\
&\quad=(C^{-1}\sigma_h+\gamma_h-\operatorname{grad}u_h,(\Pi_h-Id)G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau).
\end{aligned}
$$

Equations (4.3) and (4.9) and integration by parts elementwise yield the following L^2-representation of the residual

(4.10)

$$
\begin{aligned}
&\left\langle R(e_\sigma,e_u,e_\gamma),(\tau,v,\eta)\right\rangle\\
&\quad=(C^{-1}\sigma_h+\gamma_h-\operatorname{grad}u_h,(\Pi_h-Id)G\tau)-(C^{-1}\sigma_h+\gamma_h,\operatorname{curl}R\tau)\\
&\qquad-(f+\operatorname{div}\sigma_h,v)-(\sigma_h,\eta)\\
&\quad=(C^{-1}\sigma_h+\gamma_h-\operatorname{grad}u_h,(\Pi_h-Id)G\tau)\\
&\qquad-(f+\operatorname{div}\sigma_h,v)-(\operatorname{as}\sigma_h,\eta)\\
&\qquad-\sum_{T\in\mathcal{T}_h}(\operatorname{curl}(C^{-1}\sigma_h+\gamma_h),R\tau)_T-\sum_{E\in\mathcal{E}_h}([\gamma_t(C^{-1}\sigma_h+\gamma_h)]_E,R\tau)_E.
\end{aligned}
$$

Here, γ_t is the trace operator in tangential direction, $[\cdot]_E$ denotes the jump across an edge respectively face E and $\mathcal{E}_h$ is the set of all interior edges respectively faces of the triangulation.

4.2 The case $\mu = 1$

In a first step we give an error estimate for the case that the Lamé parameter μ equals 1. A scaling argument will then yield a similar estimate for arbitrary μ.

4.2.1 Upper bound We denote with I_h Clément's interpolation operator [9] and set $\eta_h = 0$, $v_h = 0$ and $\tau_h = \operatorname{curl} I_h R\tau$. Invoking the Galerkin orthogonality we obtain

(4.11)

$$\begin{aligned}
&\big\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\big\rangle \\
&\quad = \big\langle R(e_\sigma, e_u, e_\gamma), (\tau - \tau_h, v - v_h, \eta - \eta_h)\big\rangle \\
&\quad = -(C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h, (Id - \Pi_h)G\tau) - (f + \operatorname{div} \sigma_h, v) - (\sigma_h, \eta) \\
&\qquad - \sum_{T \in \mathcal{T}_h} (\operatorname{curl}(C^{-1}\sigma_h + \gamma_h), (Id - I_h)R\tau)_T \\
&\qquad - \sum_{E \in \mathcal{E}_h} ([\gamma_t(C^{-1}\sigma_h + \gamma_h)]_E, (Id - I_h)R\tau)_E.
\end{aligned}$$

Due to (4.7) and (4.5) we have

$$\|(Id - \Pi_h)G\tau\|_{0,T} \le c_\Pi h_T \, |G\tau|_{1,T} = c_\Pi h_T \left|\tilde{G}\tau\right|_{2,T},$$

where $\|.\|_{0;T}$ and $|.|_{k,T}$ denote the $L^2(T)$-norm and the $H^k(T)$-seminorm respectively.

The error estimates for Clément's interpolation operator and (4.5) yield

$$\begin{aligned}
\|(Id - I_h)R\tau\|_{0,T} &\le c_I h_T \, |R\tau|_{1,\widetilde{\omega}_T} \\
\|(Id - I_h)R\tau\|_{0,E} &\le c_I h_T^{\frac{1}{2}} \, |R\tau|_{1,\widetilde{\omega}_E}.
\end{aligned}$$

Here, the domains $\widetilde{\omega}_T$ and $\widetilde{\omega}_E$ consist of the union of all elements that share at least a vertex with T respectively E and $\|.\|_{0,E}$ denotes the $L^2(E)$-norm.

These estimates and the Cauchy-Schwarz inequalities for integrals and sums imply that

$$\begin{aligned}
&\left|\big\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\big\rangle\right| \\
&\quad \le \Bigg\{ \sum_{T \in \mathcal{T}_h} \left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T} c_\Pi h_T \left|\tilde{G}\tau\right|_{2,T} \\
&\qquad + \sum_{T \in \mathcal{T}_h} \|f + \operatorname{div} \sigma_h\|_{0,T} \, \|v\|_{0,T} + \sum_{T \in \mathcal{T}_h} \|\operatorname{as} \sigma_h\|_{0,T} \, \|\eta\|_{0,T} \\
&\qquad + \sum_{T \in \mathcal{T}_h} \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T} c_I h_T \, |R\tau|_{1,\widetilde{\omega}_T} \\
&\qquad + \sum_{E \in \mathcal{E}_h} \left\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]\right\|_{0,E} c_I h_T^{\frac{1}{2}} \, |R\tau|_{1,\widetilde{\omega}_E} \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\le c\Big\{\sum_{T\in\mathcal{T}_h}\Big(\|\operatorname{div}\sigma_h + f\|_{0,T}^2 + \|\operatorname{as}\sigma_h\|_{0,T}^2 \\
&\quad + h_T^2 \left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}^2 + h_T^2 \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 \\
&\quad + \sum_{E\in\partial T} h_E \left\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]\right\|_{0,E}^2\Big)^2\Big\}^{\frac12} \\
&\quad \Big\{\left|\tilde{G}\tau\right|_2^2 + |R\tau|_1^2 + \|v\|_0^2 + \|\eta\|_0^2\Big\}^{\frac12} \\
&\le c'\Big\{\sum_{T\in\mathcal{T}_h}\Big(\|\operatorname{div}\sigma_h + f\|_{0,T}^2 + \|\operatorname{as}\sigma_h\|_{0,T}^2 \\
&\quad + h_T^2 \left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}^2 + h_T^2 \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 \\
&\quad + \sum_{E\in\partial T} h_E \left\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]\right\|_{0,E}^2\Big)^2\Big\}^{\frac12} \|(\tau, v, \eta)\|_{H\times V\times W} .
\end{aligned}
$$

In the second step we have used the fact that the domains $\widetilde{\omega}_T$ and $\widetilde{\omega}_E$ only consist of a finite number of elements, this number being bounded by the regularity parameter κ.

These estimates and the inf-sup-condition for the bilinear form d finally yield

$$
\begin{aligned}
(4.12)\qquad & c_d \left\|(e_\sigma, e_u, e_\gamma)\right\|_{H\times V\times W} \\
&\le \sup_{(\tau,v,\eta)\ne 0} \frac{\left|\left\langle R(e_\sigma, e_u, e_\gamma), (\tau, v, \eta)\right\rangle\right|}{\|(\tau, v, \eta)\|_{H\times V\times W}} \\
&\le c\Big\{\sum_{T\in\mathcal{T}_h}\Big(\|\operatorname{div}\sigma_h + f_h\|_{0,T}^2 + \|\operatorname{as}\sigma_h\|_{0,T}^2 \\
&\quad + h_T^2 \left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}^2 \\
&\quad + h_T^2 \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 \\
&\quad + \sum_{E\in\partial T} h_E \left\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]\right\|_{0,E}^2\Big)^2\Big\}^{\frac12} \\
&\quad + c\,\|f - f_h\|_V .
\end{aligned}
$$

As usual in a posteriori error analysis, we have replaced the function f by a suitable finite element approximation f_h, e.g., the L^2-projection of f into V_h.

4.2.2 Lower bound To prove the converse of inequality (4.12) we estimate each residual term separately by inserting suitable local test-functions in (2.3). To this end we denote by ψ_E the product of the affine nodal shape functions corresponding to the vertices of a given edge respectively face E multiplied by d^d. The support ω_E of ψ_E then consists of the two elements that share E. The following lemma is the special case $p = 2$ of [14, Lemma 3.3].

Lemma 4.1 *The following estimates hold for all elements T, all edges respectively faces E of T, and all polynomials v and ρ defined on T respectively E:*

(4.13a) $$\|v\|_{0,T}^2 \le c_1\,(v, \psi_T v)_T\,,$$

(4.13b) $$\|\rho\|_{0,E}^2 \le c_2\,(\rho, \psi_E \rho)_E\,,$$

(4.13c) $$\|\operatorname{grad}(\psi_T v)\|_{0,T} \le c_3 h_T^{-1}\,\|v\|_{0,T}\,,$$

(4.13d) $$\|\operatorname{grad}(\psi_E \rho)\|_{0,T} \le c_4 h_E^{-1}\,\|\psi_E \rho\|_{0,T}\,,$$

(4.13e) $$\|\psi_E \rho\|_{0,T} \le c_5 h_E^{\frac{1}{2}}\,\|\rho\|_{0,E}\,.$$

Here, the polynomial ρ is continued to a polynomial on the whole space in the canonical way. The constants $c_1, \ldots, c_5$ depend on the polynomial degree of v and ρ and on the shape parameter κ.

The weak formulation (2.3) immediately implies

(4.14)
$$\begin{aligned}
&\|\operatorname{div}\sigma_h + f_h\|_{0,T} + \|\operatorname{as}\sigma_h\|_{0,T} \\
&\quad\le \|\operatorname{div}\sigma_h + f\|_{0,T} + \|\operatorname{as}\sigma_h\|_{0,T} + \|f - f_h\|_{0,T} \\
&\quad= \|\operatorname{div} e_\sigma\|_{0,T} + \|\operatorname{as} e_\sigma\|_{0,T} + \|f - f_h\|_{0,T} \\
&\quad= \|\operatorname{div} e_\sigma\|_{0,T} + 2\,\|e_\sigma\|_{0,T} + \|f - f_h\|_{0,T} \\
&\quad\le 2\sqrt{2}\,\|e_\sigma\|_{H,T} + \|f - f_h\|_{0,T}\,.
\end{aligned}$$

Set $\rho_T = \psi_T(C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h)$. Since $\mu = 1$ we conclude from (2.6) and lemma 4.1 that

$$\begin{aligned}
&c_1^{-1}\,\left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}^2 \\
&\quad\le \left(C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h, \rho_T\right) \\
&\quad= \left(C^{-1}\sigma_h + \gamma_h, \rho_T\right) + (u_h, \operatorname{div}\rho_T) \\
&\quad= a(\sigma_h, \rho_T) + b(\rho_T; u_h, \gamma_h) \\
&\quad= -a(e_\sigma, \rho_T) - b(\rho_T; e_u, e_\gamma) \\
&\quad\le \|e_\sigma\|_{0,T}\,\|\rho_T\|_{0,T} + \|e_u\|_{0,T}\,\|\operatorname{div}\rho_T\|_{0,T} + \left\|e_\gamma\right\|_{0,T}\,\|\rho_T\|_{0,T} \\
&\quad\le \|e_\sigma\|_{0,T}\,\|\rho_T\|_{0,T} + c_3 h_T^{-1}\,\|e_u\|_{0,T}\,\|\rho_T\|_{0,T} + \left\|e_\gamma\right\|_{0,T}\,\|\rho_T\|_{0,T} \\
&\quad\le \sqrt{2 + c_3^2 h_T^{-2}}\,\left\|(e_\sigma, e_u, e_\gamma)\right\|_{0,T}\,\left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}\,.
\end{aligned}$$

Dividing by $h_T^{-1}\,\left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T}$ this yields

(4.15)
$$\begin{aligned}
&h_T\,\left\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\right\|_{0,T} \\
&\quad\le c_1\sqrt{2 + c_3^2 h_T^{-2}}\,h_T\,\left\|(e_\sigma, e_u, e_\gamma)\right\|_{0,T}\,.
\end{aligned}$$

Next we set $\rho_T = \psi_T \operatorname{curl}(C^{-1}\sigma_h + \gamma_h)$ and obtain

$$\begin{aligned}
c_1^{-1} & \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T}^2 \\
& \leq \left(\operatorname{curl}(C^{-1}\sigma_h + \gamma_h), \rho_T \right)_{0,\Omega} \\
& = \left(C^{-1}\sigma_h + \gamma_h, \operatorname{curl} \rho_T \right)_{0,\Omega} \\
& = a(\sigma_h, \operatorname{curl} \rho_T) + b(\operatorname{curl} \rho_T; u_h, \gamma_h) \\
& = -a(e_\sigma, \operatorname{curl} \rho_T) - b(\operatorname{curl} \rho_T; e_u, e_\gamma) \\
& \leq \|e_\sigma\|_{0,T} \|\operatorname{curl} \rho_T\|_{0,T} + \|\operatorname{curl} \rho_T\|_{0,T} \left\| e_\gamma \right\|_{0,T} \\
& \leq c_4 h_T^{-1} \sqrt{2} \left\{ \|e_\sigma\|_{0,T}^2 + \left\| e_\gamma \right\|_{0,T}^2 \right\}^{\frac{1}{2}} \|\rho_T\|_{0,T} \\
& \leq c_4 h_T^{-1} \sqrt{2} \left\| (e_\sigma, e_u, e_\gamma) \right\|_{0,T} \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T} .
\end{aligned}$$

Division by $h_T^{-1} \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T}$ yields

$$h_T \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T} \leq c_1 c_3 \sqrt{2} \left\| (e_\sigma, e_u, e_\gamma) \right\|_{0,T} . \tag{4.16}$$

Now we insert $\rho_E = \psi_E [\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E$ in (2.3) and apply lemma 4.1 with $\rho = [\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E$. This yields

$$\begin{aligned}
c_2^{-1} & \left\| [\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E \right\|_{0,E}^2 \\
& \leq \left([\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E, \rho_E \right)_{0,E} \\
& = \sum_{T \in \omega_E} \left\{ - \left(\operatorname{curl}(C^{-1}\sigma_h + \gamma_h), \rho_E \right)_{0,T} + \left(C^{-1}\sigma_h + \gamma_h, \operatorname{curl} \rho_E \right)_{0,T} \right\} \\
& = \sum_{T \in \omega_E} - \left(\operatorname{curl}(C^{-1}\sigma_h + \gamma_h), \rho_E \right)_{0,T} \\
& \quad - a(e_\sigma, \operatorname{curl} \rho_E) - b(\operatorname{curl} \rho_E; e_u, e_\gamma) \\
& \leq \sum_{T \in \omega_E} \left\{ \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T} \|\rho_E\|_{0,T} \right. \\
& \quad \left. + \|e_\sigma\|_{0,T} \|\operatorname{curl} \rho_E\|_{0,T} + \left\| e_\gamma \right\|_{0,T} \|\operatorname{curl} \rho_E\|_{0,T} \right\} \\
& \leq \sum_{T \in \omega_E} \left\{ c_1 c_3 \sqrt{2} h_T^{-1} \left\| (e_\sigma, e_u, e_\gamma) \right\|_{0,T} \|\rho_E\|_{0,T} \right. \\
& \quad \left. + c_4 h_T^{-1} \|e_\sigma\|_{0,T} \|\rho_E\|_{0,T} + c_4 h_T^{-1} \left\| e_\gamma \right\|_{0,T} \|\rho_E\|_{0,T} \right\} \\
& \leq c_6 h_T^{-\frac{1}{2}} \sum_{T \in \omega_E} \left\{ c_1 c_3 \sqrt{2} \left\| (e_\sigma, e_u, e_\gamma) \right\|_{0,T} \|\rho_E\|_{0,E} \right. \\
& \quad \left. + c_4 c_5 \|e_\sigma\|_{0,T} \|\rho_E\|_{0,E} + c_4 c_5 \left\| e_\gamma \right\|_{0,T} \|\rho_E\|_{0,E} \right\}
\end{aligned}$$

$$\le c_6(c_1c_3\sqrt{2}+2c_4c_5)h_T^{-\frac{1}{2}}\,\|\rho_E\|_{0,E}\sum_{T\in\omega_E}\left\|(e_\sigma,e_u,e_\gamma)\right\|_{0,T}$$
$$\le c_6(c_1c_3\sqrt{2}+2c_4c_5)\sqrt{2}\left\|(e_\sigma,e_u,e_\gamma)\right\|_{0,\omega_E}$$
$$\times\, h_T^{-\frac{1}{2}}\left\|[\gamma_t(C^{-1}\sigma_h+\gamma_h)]_E\right\|_{0,E}.$$

Dividing by $h_T^{-\frac{1}{2}}\left\|[\gamma_t(C^{-1}\sigma_h+\gamma_h)]_E\right\|_{0,E}$ this gives

$$\begin{aligned}(4.17)\qquad & h_T^{\frac{1}{2}}\left\|[\gamma_t(C^{-1}\sigma_h+\gamma_h)]_E\right\|_{0,E}\\ &\le c_2c_6(c_1c_3\sqrt{2}+2c_4c_5)\sqrt{2}\left\|(e_\sigma,e_u,e_\gamma)\right\|_{0,\omega_E}.\end{aligned}$$

Combining the previous estimates we finally arrive at the lower bound

$$\begin{aligned}(4.18)\qquad & \Big\{\sum_{T\in\mathcal{T}_h}\Big(\|\operatorname{div}\sigma_h+f_h\|_{0,T}^2+\|\operatorname{as}\sigma_h\|_{0,T}^2\\ &\quad +h_T^2\left\|C^{-1}\sigma_h+\gamma_h-\operatorname{grad}u_h\right\|_{0,T}^2\\ &\quad +h_T^2\left\|\operatorname{curl}(C^{-1}\sigma_h+\gamma_h)\right\|_{0,T}^2\\ &\quad +\sum_{E\in\partial T}h_E\left\|[\gamma_t(C^{-1}\sigma_h+\gamma_h)]\right\|_{0,E}^2\Big)^2\Big\}^{\frac{1}{2}}\\ &\le C\left\|(e_\sigma,e_u,e_\gamma)\right\|_{H\times V\times W}+\|f-f_h\|_V\end{aligned}$$

with a constant C independent of h and λ.

4.3 Arbitrary μ

In order to extend our results to arbitrary μ we consider the scaling

$$\begin{aligned}\bar\sigma&=\tfrac{1}{\mu}\sigma, & \bar\sigma_h&=\tfrac{1}{\mu}\sigma_h,\\ \bar u&=u, & \bar u_h&=u_h,\\ \bar\gamma&=\gamma, & \bar\gamma_h&=\gamma_h,\\ \bar f&=\tfrac{1}{\mu}f, & \bar f_h&=\tfrac{1}{\mu}f_h.\end{aligned}$$

and define a modified bilinear form $\bar a$ by

$$\bar a(\bar\sigma,\tau)=\int_\Omega \bar C^{-1}\bar\sigma:\tau\,dx \qquad\qquad \bar C^{-1}=\mu C^{-1}.$$

Then $(\bar\sigma,\bar u,\bar\gamma)$ and $(\bar\sigma_h,\bar u_h,\bar\gamma_h)$ are the solutions of the saddle point problems

$$\begin{aligned}\bar a(\bar\sigma,\tau)+b(\tau;\bar u,\bar\gamma)&=0\\ b(\bar\sigma;v,\eta)&=-(\bar f,v)\end{aligned}$$

in $H \times V \times W$ respectively $H_h \times V_h \times W_h$. This is the linear elasticity problem and its discretization with Lamé parameter $\bar{\lambda} = \frac{\lambda}{\mu}$ und $\bar{\mu} = 1$. This leads to the following definition.

Definition 4.2 *The residual a posteriori error estimator η_R is defined by:*

$$
\begin{aligned}
\eta_{R,T}^2 = & \frac{1}{\mu^2} \|\operatorname{div} \sigma_h + f_h\|_{0,T}^2 + \frac{1}{\mu^2} \|\operatorname{as} \sigma_h\|_{0,T}^2 \\
& + h_T^2 \left\| C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h \right\|_{0,T}^2 \\
& + h_T^2 \left\| \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) \right\|_{0,T}^2 \\
& + \sum_{E \in \partial T} h_E \left\| [\gamma_t(C^{-1}\sigma_h + \gamma_h)] \right\|_{0,E}^2
\end{aligned} \tag{4.19}
$$

and

$$
\eta_R = \Big\{ \sum_{T \in \mathcal{T}_h} \eta_{R,T}^2 \Big\}^{\frac{1}{2}}. \tag{4.20}
$$

The previous transformation and inequalities (4.12) and (4.18) prove the next theorem.

Theorem 4.3 *If Ω is a convex polyhedron and (σ, u, γ), $(\sigma_h, u_h, \gamma_h)$ are the solutions of* (2.3) *and of the associated discrete problem, then η_R is a reliable and efficient a posteriori error estimator, which is robust for nearly incompressible materials. There are constants c and C independent of h, λ and μ so that the following estimates hold*

(4.21a)

$$
c \left\| (\tfrac{1}{\mu} e_\sigma, e) \right\|_{H \times V \times W} \le \eta_R + \tfrac{1}{\mu} \|f - f_h\|_V ,
$$

$$
\eta_{R,T} \le C \Big(\left\| (\tfrac{1}{\mu} e_\sigma, e) \right\|_{H \times V \times W(\omega_T)} + \tfrac{1}{\mu} \|f - f_h\|_{0,T} \Big). \tag{4.21b}
$$

Here $\|.\|_{H \times V \times W(\omega_T)}$ denotes the restriction of $\|.\|_{H \times V \times W}$ to the domain $\omega_T \subset \Omega$ which consists of all elements that share an edge respectively face with T.

5 Estimators based on auxiliary local problems

In this section we want to treat error estimators based on the solution of auxiliary problems. These are saddle point problems similar to the original one, but based on small patches of elements. In order to ensure the reliability and efficiency of the error estimators, we must assure that appropriate norms of the solutions of the auxiliary problems do not depend on the diameter of the patches. This is achieved by the following lemma.

Lemma 5.1 *The constant c_d in the inf-sup-condition* (2.8b) *with respect to a single element $T \in \mathcal{T}_h$ does not depend on the element's diameter h_T. It only depends on the condition number of the transformation of T onto the reference simplex $\hat{T}$, i.e. the shape parameter h_T/ρ_T.*

Proof. The lemma is proven by transforming to the reference element, invoking stability of the linear elasticity problem on the reference element and transforming back to the element T (cf. [11] for details). □

5.1 Local problems with neumann boundary conditions

We want to treat local auxiliary problems, which are based on a single element $T \in \mathcal{T}_h$. Furthermore we want to impose pure Neumann boundary conditions. Since the displacement of a linear elasticity problem with pure Neumann boundary conditions is unique only up to rigid body motions, we must factor out the rigid body motions of the element T. These are given by

(5.1)
$$R_T = \begin{cases} \{v \in (L^2(T))^2 \mid v = (a,b) + c(-x_2, x_1),\ a,b,c \in \mathbb{R}\} & \text{if } d = 2, \\ \{v \in (L^2(T))^3 \mid v = a + b \times x,\ a,b \in \mathbb{R}^3\} & \text{if } d = 3. \end{cases}$$

The bilinear form a is still coercive and we obtain the following inf-sup-condition for b

$$\inf_{\substack{u \in V/R_T \\ \gamma \in W}} \sup_{\sigma \in H} \frac{b(\sigma; u, \gamma)}{\|\sigma\|_H \, \|(u,\gamma)\|_{V/R_T \times W}} \geq c_b.$$

We split $V_h(T)$ into the rigid body motions R_T and the remaining part $\tilde{V}_h(T)$

$$V_h(T) = R_T \oplus \tilde{V}_h(T), \qquad \tilde{V}_h(T) = V_h(T)/R_T.$$

Using this decomposition we define

$$\overline{BDMS}_l(T) = H_h(T) \times \tilde{V}_h \times W_h(T),$$

where $l \geq 2$ is the polynomial degree in the finite element spaces.

Since the error $(e_\sigma, e_u, e_\gamma) \in H \times V \times W$ is a solution of

$$a(e_\sigma, \tau) + b(\tau; e_u, e_\gamma) = -a(\sigma_h, \tau) - b(\tau; u_h, \gamma_h) \tag{5.2a}$$

$$b(e_\sigma; v, \eta) = -(f, v) - b(\sigma_h; v, \eta) \tag{5.2b}$$

for all $(\tau, v, \eta) \in H \times V \times W$, we consider the following auxiliary local problem: Find $(\sigma_T, u_T, \gamma_T) \in \overline{BDMS}_l(T)$ such that

$$a(\sigma_T, \tau) + b(\tau; u_T, \gamma_T) = -a(\sigma_h, \tau) - b(\tau; u_h, \gamma_h), \tag{5.3a}$$

$$b(\sigma_T; v, \eta) = -(f_h, v) - b(\sigma_h; v, \eta) \tag{5.3b}$$

holds for all $(\tau, v, \eta) \in \overline{BDMS}_l(T)$.

The coercivity of a and the inf-sup-condition for b immediately imply:

Proposition 5.2 *Problem* (5.3) *admits a unique solution.*

Definition 5.3 *We define the a posteriori error estimator* η_N^l *by:*

$$\eta_{N,T}^l = \left\| \left(\tfrac{1}{\mu}\sigma_T, u_T, \gamma_T\right) \right\|_{H\times V\times W(T)}, \tag{5.4a}$$

$$\eta_N^l = \Big(\sum_{T\in\mathcal{T}_h} {\eta_{N,T}^l}^2 \Big)^{\frac{1}{2}}. \tag{5.4b}$$

Here $\|.\|_{H\times V\times W(T)}$ *denotes the restriction of* $\|.\|_{H\times V\times W}$ *to* T.

Remark 5.4. In order to implement the error estimator η_N^l one has to construct a basis for the space $\tilde{V}_h(T)$. This can be achieved by taking the standard basis of $V_h(T)$ and dropping those degrees of freedom that belong to the rigid body motions. Afterwards one has to compute the stiffness matrix for each element $T \in \mathcal{T}_h$ and solve the associated local auxiliary problems.

Theorem 5.5 *Assume that* Ω *is a convex polyhedron, that the discretization consists of* $BDMS$ *elements with polynomial degree* $k \geq 2$*, and that the polynomial degree* l *of the auxiliary problem* (5.3) *satisfies* $l \geq k + 2d$*. Then* η_N^l *is a robust, reliable and efficient a posteriori error estimator. There are constants* c *and* C*, independent of* h*,* λ *and* μ *such that:*

(5.5a)
$$c \left\| \left(\tfrac{1}{\mu}e_\sigma, e\right) \right\|_{H\times V\times W} \leq \eta_N^l + \tfrac{1}{\mu}\|f - f_h\|_V,$$

$$\eta_{N,T}^l \leq C\Big(\left\| \left(\tfrac{1}{\mu}e_\sigma, e\right) \right\|_{H\times V\times W(T)} + \tfrac{1}{\mu}\|f - f_h\|_{0,T} \Big). \tag{5.5b}$$

It is sufficient to prove this theorem for the case $\mu = 1$. The same scaling as for the residual error estimator yields the assertion for arbitrary μ.

Lemma 5.6 *If* $\mu = 1$ *and* $l \geq k + 2d$ *then* $\eta_{N,T}^l$ *and* η_N^l *yield local and global lower bounds for the error* $\|(e_\sigma, e_u, e_\gamma)\|_{H\times V\times W(T)}$*. We obtain the estimates*

$$\eta_{N,T}^l \leq C\Big(\|(e_\sigma, e_u, e_\gamma)\|_{H\times V\times W(T)} + \|f - f_h\|_{0,T} \Big), \tag{5.6a}$$

$$\eta_N^l \leq \sqrt{2}C\Big(\|(e_\sigma, e_u, e_\gamma)\|_{H\times V\times W} + \|f - f_h\|_V \Big). \tag{5.6b}$$

The constant C *depends on the regularity parameter* κ *but does not depend on the diameter* h_T *nor on the Lamé parameter* λ.

Proof. This lemma follows directly from the inf-sup condition (2.8b)

$$\begin{aligned}
c_d\eta^l_{N,T} &= c_d\,\|(\sigma_T, u_T, \gamma_T)\|_{H\times V\times W(T)}\\
&\le \sup_{\substack{(\tau,v,\eta)\in BDMS_l(T)\\ \|\tau,v,\eta\|_{H\times V\times W}=1}} \big(a(\sigma_T,\tau)+b(\tau;u_T,\gamma_T)+b(\sigma_T;v,\eta)\big)\\
&= \sup_{\substack{(\tau,v,\eta)\in BDMS_l(T)\\ \|\tau,v,\eta\|_{H\times V\times W}=1}} \big(a(\sigma_h,\tau)+b(\tau;u_h,\gamma_h)-(f_h,v)-b(\sigma_h;v,\eta)\big)\\
&= \sup_{\substack{(\tau,v,\eta)\in BDMS_l(T)\\ \|\tau,v,\eta\|_{H\times V\times W}=1}} \big(a(\sigma_h,\tau)+b(\tau;u_h,\gamma_h)-(f,v)+(f-f_h,v)\\
&\qquad\qquad -b(\sigma_h;v,\eta)\big)\\
&= \sup_{\substack{(\tau,v,\eta)\in BDMS_l(T)\\ \|\tau,v,\eta\|_{H\times V\times W}=1}} \big(a(e_\sigma,\tau)\\
&\quad + b(\tau;e_u,e_\gamma)+b(e_\sigma;v,\eta)+(f-f_h,v)\big).
\end{aligned}$$

Invoking the continuity of a and b and Hölder's inequality this yields

$$\eta^l_{N,T} \le c_d^{-1}\sqrt{2}\big(\big\|(e_\sigma,e_u,e_\gamma)\big\|_{H\times V\times W(T)} + \|f-f_h\|_{0,T}\big).$$

The global bound is an immediate consequence of the local one. □

In order to establish the reliability of the estimator, we prove that $\eta^l_{N,T}$ is an upper bound for the terms in $\eta_{R,T}$. Invoking (4.12) this will complete the proof of theorem 5.5.

Lemma 5.7 *If $\mu = 1$ and $l \ge k + 2d$, the following estimates holds*

(5.7a) $\|\operatorname{div}\sigma_h + f_h\|^2_{0,T} + \|\operatorname{as}\sigma_h\|^2_{0,T} \le 2c_1\,\eta^l_{N,T}$,

(5.7b) $h_T\,\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|_{0,T} \le c_1\sqrt{2h_T^2 + c_3^2}\,\eta^l_{N,T}$,

(5.7c) $h_T\,\big\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\big\|_{0,T} \le c_1c_3\sqrt{2}\,\eta^l_{N,T}$,

(5.7d) $h_E^{\frac12}\,\big\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]\big\|_{0,E} \le 2c_2(c_1c_3 + c_4)c_5\Big\{\sum_{T\subset\omega_E} {\eta^l_{N,T}}^2\Big\}^{\frac12}$.

Proof. We consider the local test-function

$$(0, v_T, \eta_T) = (0, \psi_T(\operatorname{div}\sigma_h + f_h), 2\psi_T\operatorname{as}\sigma_h).$$

Since $(\operatorname{div}\sigma_h + f_h) \in V_h$ we have $v_T \in \mathbb{P}_{k+d}$ and $\operatorname{as}\sigma_h \in \mathbb{P}_{k+d-1}$, hence $\eta_T \in \mathbb{P}_{k+2d}$. The condition $l \ge k + 2d$ therefore implies $(0, v_T, \eta_T) \in BDMS_l(T)$. Since v_T vanishes on the boundary of T, we have $v_T \in \widetilde{V}_h$ and thus $(0, v_T, \eta_T) \in \overline{BDMS_l}(T)$. Hence we can insert $(0, v_T, \eta_T)$ as a test function in the auxiliary problem. Invoking lemma 4.1 we obtain

$$\begin{aligned}\|(\operatorname{div}\sigma_h + f_h, \operatorname{as}\sigma_h)\|^2_{V\times W(T)} &= \|\operatorname{div}\sigma_h + f_h\|^2_{0,T} + \|\operatorname{as}\sigma_h\|^2_{0,T} \\ &\le c_1\Big(\int_T (\operatorname{div}\sigma_h + f_h)\cdot v_T\,dx \\ &\quad + \int_T \sigma_h : \eta_T\,dx\Big) \\ &= c_1\big((f_h, v_T) + b(\sigma_h; v_T, \eta_T)\big) \\ &= c_1\,|b(\sigma_T; v_T, \eta_T)|\,.\end{aligned}$$

The continuity of b and $\psi_T \le 1$ yield

$$\begin{aligned}|b(\sigma_T; v_T, \eta_T)| &\le \|\sigma_T\|_{0,T}\,\|(v_T, \eta_T)\|_{V\times W(T)} \\ &\le 2\eta^l_{N,T}\,\|(\operatorname{div}\sigma_h + f_h, \operatorname{as}\sigma_h)\|_{V\times W(T)}\,.\end{aligned}$$

Division by $\|(\operatorname{div}\sigma_h + f_h, \operatorname{as}\sigma_h)\|_{V\times W(T)}$ on both sides of the inequality proves the first estimate.

For the second estimate we consider the local test-function

$$(\tau_T, 0, 0) = (\psi_T(C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h), 0, 0).$$

We have $\tau_T \in \mathbb{P}_{k+2d}$ and therefore $(\tau_T, 0, 0) \in \overline{BDMS_l}(T)$. With lemma 4.1 and integration by parts in the second term we get

$$\begin{aligned}&\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|^2_{0,T} \\ &\quad\le c_1\Big(\int_T C^{-1}\sigma_h : \tau_T\,dx - \int_T \operatorname{grad} u_h : \tau_T\,dx + \int_T \gamma_h : \tau_T\,dx\Big), \\ &\quad= c_1\big(a(\sigma_h, \tau_T) + b(\tau_T; u_h, \gamma_h)\big).\end{aligned}$$

The definition of the auxiliary problem (5.3) and the continuity of a and b imply

$$\begin{aligned}&\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|^2_{0,T} \\ &\quad\le c_1\big(a(\sigma_T, \tau_T) + b(\tau_T; u_T, \gamma_T)\big) \\ &\quad\le c_1\big(\|\sigma_T\|_{0,T}\,\|\tau_T\|_{0,T} + \|\operatorname{div}\tau_T\|_{0,T}\,\|u_T\|_{0,T} + \|\tau_T\|_{0,T}\,\|\gamma_T\|_{0,T}\big) \\ &\quad\le c_1\big(\|\sigma_T\|_{0,T}\,\|\tau_T\|_{0,T} + c_3 h_T^{-1}\,\|\tau_T\|_{0,T}\,\|u_T\|_{0,T} + \|\tau_T\|_{0,T}\,\|\gamma_T\|_{0,T}\big) \\ &\quad\le c_1\sqrt{2 + c_3^2 h_T^{-2}}\,\|(\sigma_T, u_T, \gamma_T)\|_{H\times V\times W(T)}\,\|\tau_T\|_{0,T}\,.\end{aligned}$$

Using the definition of τ_T and $\eta^l_{N,T}$ and the fact that $|\psi_T| \le 1$ we obtain

$$\begin{aligned}&\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|^2_{0,T} \\ &\quad\le c_1\sqrt{2 + c_3^2 h_T^{-2}}\,\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|_{0,T}\,\eta^l_{N,T}.\end{aligned}$$

Division by $h_T^{-1}\,\big\|C^{-1}\sigma_h + \gamma_h - \operatorname{grad} u_h\big\|$ proves the second estimate.

Next we set $\tau_T = \psi_T \operatorname{curl}(C^{-1}\sigma_h + \gamma_h)$ and have $(\tau_T, 0, 0) \in \overline{BDMS_l}(T)$. Lemma 4.1, integration by parts and the definition of the auxiliary problem (5.3) yield

$$\begin{aligned}
\left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 &\le c_1 \int_T \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) : \tau_T \, dx \\
&= c_1 \int_T (C^{-1}\sigma_h + \gamma_h) : \operatorname{curl} \tau_T \, dx \\
&= c_1 \big(a(\sigma_h, \operatorname{curl} \tau_T) + b(\operatorname{curl} \tau_T; u_h, \gamma_h)\big) \\
&= c_1 \big(a(\sigma_T, \operatorname{curl} \tau_T) + b(\operatorname{curl} \tau_T; u_T, \gamma_T)\big).
\end{aligned}$$

The continuity of a and b and $|\psi_T| \le 1$ imply

$$\begin{aligned}
\left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 &\le c_1 \big(\|\sigma_T\|_{0,T} \|\operatorname{curl} \tau_T\|_{0,T} + \|\operatorname{curl} \tau_T\|_{0,T} \|\gamma_T\|_{0,T}\big) \\
&\le c_1 \sqrt{2} \, \|(\sigma_T, 0, \gamma_T)\|_{H \times V \times W(T)} \|\operatorname{curl} \tau_T\|_{0,T} \\
&\le c_1 \sqrt{2} c_3 h_T^{-1} \|\tau_T\|_{0,T} \, \eta_{N,T}^l \\
&\le c_1 c_3 \sqrt{2} h_T^{-1} \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T} \eta_{N,T}^l .
\end{aligned}$$

Division by $h_T^{-1} \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}$ establishes the third estimate.

Finally we consider an element T and an edge respectively face E thereof and set $\tau_E = \psi_E \gamma_t [C^{-1}\sigma_h + \gamma_h]_E$. Here the polynomial $[C^{-1}\sigma_h + \gamma_h]_E$ is continued to the whole space in the canonical way. Due to the assumption on the polynomial degree l we have $(\operatorname{curl} \tau_E, 0, 0) \in \overline{BDMS_l}(T)$. Integration by parts and lemma 4.1 yield

$$\begin{aligned}
&\left\|[\gamma_t(C^{-1}\sigma_h + \gamma_h)]_E\right\|_{0,E}^2 \\
&\quad \le c_2 \int_E [\gamma_t(C^{-1}\sigma_h + \gamma_h)]_E : \tau_E \, ds \\
&\quad = c_2 \Big(-\int_{\omega_E} \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) : \tau_E \, dx + \int_{\omega_E} (C^{-1}\sigma_h + \gamma_h) : \operatorname{curl} \tau_E \, dx\Big).
\end{aligned}$$

The first term can be estimated by invoking the Cauchy-Schwarz inequality and the fourth estimate of lemma 4.1

$$\begin{aligned}
\int_{\omega_E} \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) : \tau_E \, dx &\le \Big\{ \sum_{T \subset \omega_E} \left\|\operatorname{curl}(C^{-1}\sigma_h + \gamma_h)\right\|_{0,T}^2 \Big\}^{\frac{1}{2}} \|\tau_E\|_{0,\omega_E} \\
&\le \Big\{ \sum_{T \subset \omega_E} (c_1 c_3 \sqrt{2} \, h_T^{-1} \eta_{N,T}^l)^2 \Big\}^{\frac{1}{2}} \|\tau_E\|_{0,\omega_E} \\
&\le c_1 c_3 \sqrt{2} \, h_T^{-1} \Big\{ \sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big\}^{\frac{1}{2}} \|\tau_E\|_{0,\omega_E} .
\end{aligned}$$

Using once again lemma 4.1 we obtain

$$\begin{aligned}
&\int_{\omega_E} \operatorname{curl}(C^{-1}\sigma_h + \gamma_h) : \tau_E \, dx \\
&\quad \le c_1 c_3 \sqrt{2}\, h_T^{-1} \Big\{ \sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big\}^{\frac{1}{2}} c_5 \sqrt{2h_T}\, \big\| [\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E \big\|_{0,E} \\
&\quad \le 2 c_1 c_3 c_5\, h_T^{-\frac{1}{2}} \Big\{ \sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big\}^{\frac{1}{2}} \big\| [\gamma_t (C^{-1}\sigma_h + \gamma_h)]_E \big\|_{0,E} .
\end{aligned}$$

Invoking the continuity of a and b and lemma 4.1 the second term can be estimated by

$$\begin{aligned}
&\int_{\omega_E} (C^{-1}\sigma_h + \gamma_h) : \operatorname{curl} \tau_E \, dx \\
&\quad = a(\sigma_h, \operatorname{curl} \tau_E) + b(\operatorname{curl} \tau_E; u_h, \gamma_h) \\
&\quad = \sum_{T \subset \omega_E} a(\sigma_T, \operatorname{curl} \tau_E) + b(\operatorname{curl} \tau_E; u_T, \gamma_T) \\
&\quad \le \sum_{T \subset \omega_E} \|\sigma_T\|_{0,T} \, \|\operatorname{curl} \tau_E\|_{0,T} + \|\operatorname{curl} \tau_E\|_{0,T} \, \|\gamma_T\|_{0,T} \\
&\quad \le \sqrt{2}\, \|\operatorname{curl} \tau_E\|_{0,\omega_E} \Big(\sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big)^{\frac{1}{2}} \\
&\quad \le 2 c_4 c_5 h_T^{-\frac{1}{2}} \Big\{ \sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big\}^{\frac{1}{2}} \big\| \gamma_t [C^{-1}\sigma_h + \gamma_h]_E \big\|_{0,E} .
\end{aligned}$$

Combining these estimates and dividing by $h_T^{-\frac{1}{2}} \big\| \gamma_t [C^{-1}\sigma_h + \gamma_h]_E \big\|_{0,E}$ we obtain

$$h_T^{\frac{1}{2}} \big\| \gamma_t [C^{-1}\sigma_h + \gamma_h]_E \big\|_{0,E} \le 2 c_2 (c_1 c_3 + c_4) c_5 \Big\{ \sum_{T \subset \omega_E} {\eta_{N,T}^l}^2 \Big\}^{\frac{1}{2}}. \tag{5.8}$$

This proves the last estimate of the lemma. □

Lemma 5.7 proves the second estimate of theorem 5.5. Hence $\eta_{N,T}$ is a reliable and efficient error estimator. Since our estimates do not depend on the Lamé-parameter λ, this estimator is also robust for nearly incompressible materials.

5.2 Local problems with dirichlet boundary conditions

It is also possible to impose Dirichlet boundary conditions on the auxiliary problems. But due to the nature of these boundary conditions the local problem must be posed on a larger patch of elements. We will derive an error estimator which is posed on the patches ω_T that consist of the union of all elements sharing an edge respectively face with a given element T.

Proposition 5.8 *For every $T \in \mathcal{T}_h$ and every $l \geq k$ there is a unique $(\sigma_{\omega_T}, u_{\omega_T}, \gamma_{\omega_T}) \in BDMS_l(\omega_T)$ such that*

$$a(\sigma_{\omega_T}, \tau) + b(\tau; u_{\omega_T}, \gamma_{\omega_T}) = -a(\sigma_h, \tau) - b(\tau; u_h, \gamma_h), \tag{5.9a}$$

$$b(\sigma_{\omega_T}; v, \eta) = -(f_h, v) - b(\sigma_h; v, \eta) \tag{5.9b}$$

holds for all $(\tau, v, \eta) \in BDMS_l(\omega_T)$.

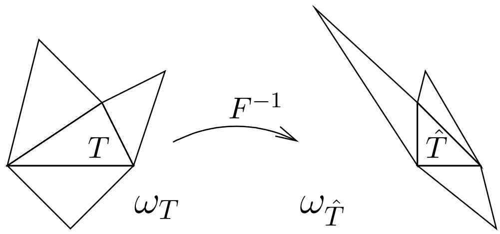

Fig. 1. The mapping of ω_T onto $\omega_{\hat{T}}$

Remark 5.9. Lemma 5.1 deals only with the scaling of c_d under affine transformations of a single element. In order to get a reliable and efficient a posteriori error estimator based on the auxiliary problem (5.9) we need an analogous lemma with respect to domains of the type ω_T. Applying an affine transformation to ω_T we can map T onto the reference element $\hat{T}$. The remaining three respectively four elements are mapped onto arbitrary elements adjacent to $\hat{T}$. Hence we get an arbitrary patch $\omega_{\hat{T}}$. Due to the regularity of the triangulation $\mathcal{T}_h$ the set of all possible patches $\omega_{\hat{T}}$ is a compact set. Since c_d depends continuously on the transformation it attains its minimum $\bar{c}_d$ on this compact set. Obviously, this minimum is still independent of the Lamé parameter λ.

Definition 5.10 *We define the a posteriori error estimator η_D^l by*

$$\eta_{D,T}^l = \left\| (\tfrac{1}{\mu}\sigma_{\omega_T}, u_{\omega_T}, \gamma_{\omega_T}) \right\|_{H \times V \times W(T)}, \tag{5.10a}$$

$$\eta_D^l = \Big(\sum_{T \in \mathcal{T}_h} {\eta_{D,T}^l}^2 \Big)^{\frac{1}{2}}. \tag{5.10b}$$

With the same arguments as in the previous section one can prove that $\eta_{D,T}$ provides a reliable and efficient a posteriori error estimator, robust with respect to nearly incompressible materials.

Theorem 5.11 *If Ω is a convex polyhedron and (σ, u, γ), $(\sigma_h, u_h, \gamma_h)$ are the solution of* (2.3) *respectively its discretization in $BDMS_k$ and if the* *polynomial degree l of the auxiliary local problem* (5.9) *fullfills* $l \geq k + 2d$*, then there are constants c and C, independent of h, λ, and μ, such that the estimates*

(5.11a)
$$c \left\| \left(\tfrac{1}{\mu} e_\sigma, e\right) \right\|_{H \times V \times W} \leq \eta^l_D + \tfrac{1}{\mu} \| f - f_h \|_V \, ,$$

(5.11b)
$$\eta^l_{D,T} \leq C\Big(\left\| \left(\tfrac{1}{\mu} e_\sigma, e\right) \right\|_{H \times V \times W(\omega_T)} + \tfrac{1}{\mu} \| f - f_h \|_{0,T} \Big).$$

hold.

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