Numer. Math. (2004) 96: 525–581
Digital Object Identifier (DOI) 10.1007/s00211-003-0468-7

Numerische Mathematik

Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains

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Received August 28, 2001 / Revised version received January 9, 2003 /
Published online September 5, 2003 –

Summary. In this paper, we study the spectral properties of Dirichlet problems for second order elliptic equation with rapidly oscillating coefficients in a perforated domain. The asymptotic expansions of eigenvalues and eigenfunctions for this kind of problem are obtained, and the multiscale finite element algorithms and numerical results are proposed.

Mathematics Subject Classification (2000): 65F10, 35P15

1 Introduction

J.L. Lions (Cf. [21] pp121) proposed an open problem: **"Spectral problems connected with screens seem to be open. One can for instance consider a bounded open set Ω and take out from Ω a perforated plane screen F_ε; if we set $\Omega_\varepsilon = \Omega \backslash F_\varepsilon$, we consider $-\triangle$ in Ω_ε, subject to Dirichlet's boundary conditions on $\partial\Omega_\varepsilon$; which is the asymptotic expansion of the spectrum of this operator?"** In this paper, we wish to discuss this kind of problem.

This work is Supported by National Natural Science Foundation of China (grant # 19932030) and Special Funds for Major State Basic Research Projects (grant # TG2000067102)

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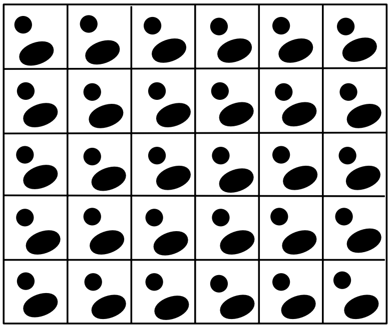

Fig. 1.1. Unbounded domain ω

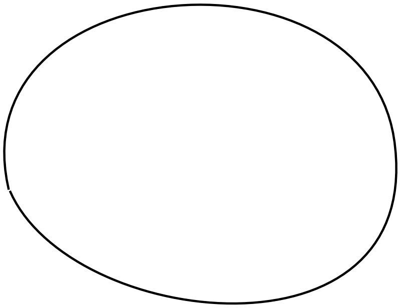

Fig. 1.2. Domain Ω

Let ω be an unbounded domain of R^n with a 1-periodic structure, i.e. ω is invariant under the shifts by any $z = (z_1, \cdots z_n) \in Z^n$.

Suppose that ω satisfies the following conditions

(B_1) ω is a smooth unbounded domain of R^n with a 1-periodic structure.

(B_2) The cell of periodicity $\omega \cap Q$ is a domain with a Lipschitz boundary.

(B_3) The set $Q \setminus \bar{\omega}$ and the intersection of $Q \setminus \bar{\omega}$ with the δ- neighborhood ($\delta < \frac{1}{4}$) of ∂Q consist of a finite number of Lipschitz domains separated from each other and from the edges of the cube Q by a positive distance.

A domain Ω^ε has the form: $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$, where Ω as shown in Figure 1.2 is a bounded Lipschitz convex domain of R^n without cavities, ω as shown in Figure 1.1 is a unbounded domain with 1-periodic structure satisfying the conditions $(B_1) - (B_3)$. Ω^ε is assumed to have a Lipschitz boundary, and the boundary of a domain Ω^ε can be rewritten as $\partial\Omega^\varepsilon = \Gamma_\varepsilon \cup S_\varepsilon$, where $\Gamma_\varepsilon = \partial\Omega \cap \varepsilon\omega, \quad S_\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$.

Throughout this paper the Einstein summation convention on repeated indices is adopted. C (with and without a subscript) denotes a generic positive constant, which is independent of ε unless otherwise stated.

Consider the Helmholtz problem of second order elliptic operator with highly oscillatory coefficients in a perforated domain Ω^ε as follows:

$$(1.1) \qquad \begin{cases} \mathcal{L}_\varepsilon U^\varepsilon(x) \equiv -\dfrac{\partial}{\partial x_i}(a_{ij}(\dfrac{x}{\varepsilon})\dfrac{\partial U^\varepsilon(x)}{\partial x_j}) + b(\dfrac{x}{\varepsilon})U^\varepsilon(x) \\ \qquad\qquad = \Lambda^\varepsilon \rho(\frac{x}{\varepsilon})U^\varepsilon(x) \quad \text{in} \quad \Omega^\varepsilon \\ U^\varepsilon(x) = 0, \quad \text{on} \quad \partial\Omega^\varepsilon \\ U^\varepsilon(x) \in H_0^1(\Omega^\varepsilon), \quad \displaystyle\int_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon})|U^\varepsilon(x)|^2 dx = 1. \end{cases}$$

We make the following assumptions:

(A_1). Let $\xi = \varepsilon^{-1}x$, the coefficients $a_{ij}(\xi),\ b(\xi) \quad \textit{and} \ \ \rho(\xi)$ be 1-periodic in ξ;

(A_2). $\gamma_0|\eta|^2 \le a_{ij}(\frac{x}{\varepsilon})\eta_i\eta_j \le \gamma_1|\eta|^2$, $\exists\gamma_0 > 0$, $\gamma_1 > 0$, $\forall(\eta_1, \cdots \eta_n) \in R^n$
(A_3). $a_{ij}(\frac{x}{\varepsilon}) = a_{ji}(\frac{x}{\varepsilon})$, $\quad \rho(\frac{x}{\varepsilon}) \ge \rho_0 = const > 0$, $\quad b(\frac{x}{\varepsilon}) \ge 0$;
(A_4). $a_{ij}(\frac{x}{\varepsilon})$, $\quad \rho(\frac{x}{\varepsilon})$, $\quad b(\frac{x}{\varepsilon}) \in L^\infty(\Omega^\varepsilon)$.

Lemma 1.1[24] *(the extension theorem) Let $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$ be a perforated domain. Then for any functions in $H^1(\Omega^\varepsilon)$ there exists a linear extension operator*
P_ε: $H^1(\Omega^\varepsilon) \to H^1(\Omega)$ such that

(1.2) $$\|P_\varepsilon v\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega^\varepsilon)}$$

(1.3) $$\|\nabla P_\varepsilon v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega^\varepsilon)}$$

By using Lemma 1.1, conditions $(A_2) - (A_4)$, and Fredholm's alternative theorem, one then has

Theorem 1.1 *Let $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$ be a perforated domain. If conditions $(A_2)-(A_4)$ are satisfied. Then the spectral problem (1.1) for $\mathcal{L}_\varepsilon$ in $H_0^1(\Omega^\varepsilon)$ has a countable set of solutions, $\lambda = \Lambda_k^\varepsilon$, $\quad u = U_k^\varepsilon(x)$, $\quad k = 1, 2, \cdots$. The eigenvalues Λ_k^ε, with the possible exception of the first few, are positive and $\Lambda_k^\varepsilon \to \infty$, as $k \to \infty$. The eigenfunctions $\{U_k^\varepsilon(x)\}$ form a basis in $L^2(\Omega^\varepsilon)$ and in $H_0^1(\Omega^\varepsilon)$; this basis can be orthonormalized in $L^2(\Omega^\varepsilon)$ and is orthogonal in the sense of the scalar product as follows*

(1.4) $$(u, v)_{H_\varepsilon} = \int\limits_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon})uv dx, \quad H_\varepsilon = L^2(\Omega^\varepsilon).$$

In some papers (see, e.g. [16,24]), it is proved that $\Lambda_k^\varepsilon = \varepsilon^{-2}\Lambda^0 + \lambda_k^\varepsilon$.

Let $\Theta(\xi)$, $\quad \xi = \frac{x}{\varepsilon}$, be the eigenfunction corresponding to the first eigenvalue Λ^0 of the following boundary value problem in unbounded domain ω with 1-periodic structure:

(1.5) $$\begin{cases} -\dfrac{\partial}{\partial\xi_i}(a_{ij}(\xi)\dfrac{\partial}{\partial\xi_j}\Theta(\xi)) = \Lambda^0\rho(\xi)\Theta(\xi), \quad \text{in} \quad \omega \\ \Theta(\xi) = 0, \quad \text{on} \quad \partial\omega, \quad \Theta(\xi) \text{ is 1-periodic in } \xi \\ \int\limits_{Q\cap\omega} \rho(\xi)\Theta^2(\xi)d\xi = 1 \end{cases}$$

Let us formally represent the k-th eigenfunction of (1.1) in the form:

(1.6) $$U_k^\varepsilon(x) = \Theta(\frac{x}{\varepsilon})u_k^\varepsilon(x)$$

It is easy to verify that $u_k^\varepsilon(x)$ must satisfy the following equation:

(1.7) $$\begin{cases} -\dfrac{\partial}{\partial x_i}(\Theta^2(\dfrac{x}{\varepsilon})a_{ij}(\dfrac{x}{\varepsilon})\dfrac{u_k^\varepsilon(x)}{\partial x_j}) + b(\dfrac{x}{\varepsilon})\Theta^2(\dfrac{x}{\varepsilon})u_k^\varepsilon(x) \\ \quad = \lambda_k^\varepsilon\rho(\dfrac{x}{\varepsilon})\Theta^2(\dfrac{x}{\varepsilon})u_k^\varepsilon(x) \qquad in \quad \Omega^\varepsilon \\ u_k^\varepsilon(x) = 0, \quad \text{on} \quad \Gamma_\varepsilon \end{cases}$$

where $\lambda_k^\varepsilon = \Lambda_k^\varepsilon - \varepsilon^{-2}\Lambda^0$, and $\int_{\Omega^\varepsilon} \Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})|u_k^\varepsilon(x)|^2 dx = 1$

O.A. Oleinik et al. (Ref. [24]), not only proved the convergence of λ_k^ε to $\lambda_k^{(0)}$, but also obtained the estimate $|\lambda_k^\varepsilon - \lambda_k^{(0)}| \le C_k\varepsilon, \quad C_k = const$, and studied the behavior of the eigenfunctions of problem (1.2) as $\varepsilon \to 0$.

The goals of this paper are to propose multiscale asymptotic expansions of eigenvalues and eigenfunctions for the Dirichlet problem of the second order elliptic equation in perforated domains, and to give rigorous error estimates in some cases. These expansions will play an essential role in numerical computation. We would like to apply them to construct multiscale finite element algorithms, and derive error estimates. Finally, numerical results are reported.

The remainder of this paper is organized as follows: some weight Sobolev spaces and auxiliary lemmas are introduced in the next section. The multiscale asymptotic expansions of eigenvalues and eigenfunctions degenerate on the boundary $\partial\Omega^\varepsilon$, and some error estimates are given in Section 3. §4 is devoted to the FE computations and error estimates of the related problems. In section 5, the multiscale FE computing formulation and the post-processing method are provided. Finally, some numerical results are reported.

2 Weight Sobolev spaces and auxiliary lemmas

To begin with, we wish to give some properties of weight function $\Theta(\xi)$.

Lemma 2.1[8] *If we assume that $a_{ij}(\xi)$, $\rho(\xi)$ are smooth functions of $\xi \in R^n$ and ω has smooth boundary, then $\Theta(\xi)$ is a smooth function in ω such that $\Theta(\xi) \neq 0$ in ω and $|\nabla_\xi\Theta(\xi)| \neq 0$ in a neighborhood of $\partial\omega$.*

Theorem 2.1 *Under the assumptions of Lemma 2.1, then the first eigenvalue Λ^0 of problem (1.5) is simple, and the corresponding eigenfunction $\Theta(\xi)$ has a constant sign in ω, and is unique up to a constant factor.*

Proof. Introduce the space

$$V_{per} = \{u \in H^1(Q\cap\omega, \partial\omega),\ u \text{ is 1-periodic in } \xi,\ \int_{Q\cap\omega} \rho(\xi)|u(\xi)|^2 d\xi = 1\},$$

and the functional

$$D(u) = \int_{Q\cap\omega} a_{ij}(\xi)\frac{\partial u(\xi)}{\partial\xi_i}\frac{\partial u(\xi)}{\partial\xi_j}d\xi.$$

Let $\Lambda^0 = \inf_{u\in V_{per}} D(u)$ and $\Theta(\xi)$ be a function giving the minimal value. If $\Theta(\xi) \in V_{per}$ then $|\Theta(\xi)| \in V_{per}$, and $D(|\Theta|) = D(\Theta)$. Hence the function

$|\Theta|$ is an eigenfunction corresponding to the eigenvalue $\lambda = \Lambda^0$. It follows from Lemma 2.1 that $|\Theta(\xi)|$ does not vanish anywhere in ω.

Now let $\Theta_0(\xi)$ be another eigenfunction corresponding to Λ^0. By using Schmidt's method, namely, substituting $\Theta_0(\xi)$ by the function $\Theta_1(\xi) = \Theta_0(\xi) + \tau\Theta(\xi)$, we can have the function Θ_1 is orthogonal to $\Theta(\xi)$, i.e. $\int\limits_{Q\cap\omega} \rho(\xi)\Theta_1(\xi) \cdot \Theta(\xi)d\xi = 0$. Since $\Theta(\xi) > 0$ in ω, the function $\Theta_1(\xi)$ changes its sign in ω. However, $|\Theta_1(\xi)|$ is an eigenfunction with respect to Λ^0, and $|\Theta_1|$ vanishes at an inner point of ω, Lemma 2.1 implies that $\Theta_1 \equiv 0$ □

Therefore we complete the proof of Theorem 2.1.

Before proceeding, we take time to introduce some notation and some conventions (see, e.g. [24]).

$\widehat{C}^\infty(\bar{\omega})$ is the space of infinitely differential functions in $\bar{\omega}$ which are 1-periodic in $x_1, \cdots x_n$.

$\widehat{C}_0^\infty(\omega)$ is the space of infinitely differential functions in ω that are 1-periodic in $x_1, \cdots x_n$, and vanish in a neighborhood of $\partial\omega$.

$W(\omega)$ is the completion of $\widehat{C}_0^\infty(\omega)$ with respect to the norm in $H^1(\omega\cap Q)$;
$\widehat{W}_2^1(\omega)$ is the completion of $\widehat{C}^\infty(\bar{\omega})$ with respect to the norm in $H^1(\omega\cap Q)$;
$\widehat{V}^1(\omega)$ is the completion of $\widehat{C}_0^\infty(\omega)$ in the norm

$$\|u\|^2_{\widehat{V}^1(\omega)} = \int\limits_{Q\cap\omega} |\Theta|^2(|u|^2 + |\nabla_\xi u|^2)d\xi \tag{2.1}$$

$\widehat{V}^0(\omega)$ is the completion of $\widehat{C}_0^\infty(\omega)$ in the norm

$$\|u\|^2_{\widehat{V}^0(\omega)} = \int\limits_{Q\cap\omega} |\Theta|^2|u|^2d\xi \tag{2.2}$$

$\widehat{V}(\omega)$ is the completion of $\widehat{C}_0^\infty(\omega)$ in the norm

$$\|u\|^2_{\widehat{V}(\omega)} = \int\limits_{Q\cap\omega} |\nabla_\xi\Theta(\xi)|^2|u|^2d\xi \tag{2.3}$$

Let $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$, and the spaces $V_0^{1,p}(\Omega^\varepsilon)$, $V^{0,p}(\Omega^\varepsilon)$, $V(\Omega^\varepsilon)$ be the completion of $C_0^\infty(\Omega^\varepsilon)$ in the respective norms

$$\|u\|_{V_0^{1,p}(\Omega^\varepsilon)} = \|\Theta(\frac{x}{\varepsilon})(|u| + |\nabla_x u|)\|_{L^p(\Omega^\varepsilon)} \tag{2.4}$$

$$\|u\|_{V_1^{2,p}(\Omega^\varepsilon)} = \|\Theta(\frac{x}{\varepsilon})(\sum_{|\alpha|\le 2} |D_x^\alpha u|)\|_{L^p(\Omega^\varepsilon)} \tag{2.5}$$

$$\|u\|_{V^{0,p}(\Omega^\varepsilon)} = \|\Theta(\frac{x}{\varepsilon})|u|\|_{L^p(\Omega^\varepsilon)} \tag{2.6}$$

$$\|u\|_{V(\Omega^\varepsilon)} = \|\nabla_\xi \Theta(\xi)|u|\|_{L^2(\Omega^\varepsilon)} \tag{2.7}$$

where $1 < p < +\infty$, in particular, we write $V_0^{1,2}(\Omega^\varepsilon) = V_0^1(\Omega^\varepsilon)$, *and* $V^{0,2}(\Omega^\varepsilon) = V^0(\Omega^\varepsilon)$, respectively. $\Theta(\frac{x}{\varepsilon}) = \Theta(\xi)$ is the eigenfunction corresponding to the first eigenvalue Λ^0 of (1.5), and $\Theta(\xi) > 0$, in $\xi \in \omega$.

Let $V^{1,p}(\Omega^\varepsilon)$, $V^{2,p}(\Omega^\varepsilon)$ be the completion of $C^\infty(\overline{\Omega}^\varepsilon)$ in the norm (2.4) and (2.5), respectively. In particular, $V^{1,2}(\Omega^\varepsilon) = V^1(\Omega^\varepsilon), V^2(\Omega^\varepsilon) = V^{2,2}(\Omega^\varepsilon)$

We introduce next some lemmas without any proofs (Ref. §3, Chap III of [24]).

Lemma 2.2 *The following imbeddings*

$$W(\omega) \subset \widehat{W}_2^1(\omega) \subset \widehat{V}^1(\omega) \tag{2.8}$$

$$\widehat{V}^1(\omega) \subset \widehat{V}^0(\omega) \tag{2.9}$$

$$\widehat{V}^1(\omega) \subset \widehat{V}(\omega) \tag{2.10}$$

are continuous. Moreover, the imbedding (2.9) is compact, and for any $v \in \widehat{V}^1(\omega)$ *we have* $\Theta(\xi)v(\xi) \in W(\omega)$

Lemma 2.3 *(the Poincaré inequality) For any* $u \in \widehat{V}^1(\omega)$ *such that*

$$\int_{Q\cap\omega} \Theta^2 u d\xi = 0 \tag{2.11}$$

the inequality

$$\|u\|^2_{\widehat{V}^1(\omega)} \le C \int_{Q\cap\omega} |\Theta|^2 |\nabla_\xi u|^2 d\xi \tag{2.12}$$

holds with a constant C independent of u.

Lemma 2.4 *For any* $u \in C_0^\infty(\Omega^\varepsilon)$ *the following inequalities are satisfied*

$$\int_{\Omega^\varepsilon} |\nabla_\xi \Theta(\frac{x}{\varepsilon})|^2 |u|^2 dx \le C_0 \int_{\Omega^\varepsilon} |\Theta(\frac{x}{\varepsilon})|^2 (|u|^2 + |\nabla_x u|^2) dx, \tag{2.13}$$

(2.14)

$$\int_{\Omega^\varepsilon} [|\nabla_x(\Theta(\frac{x}{\varepsilon}))u|^2 + |\Theta(\frac{x}{\varepsilon})u|^2] dx \le C_1 \int_{\Omega^\varepsilon} |\Theta(\frac{x}{\varepsilon})|^2 (|\nabla_x u|^2 + \frac{1}{\varepsilon^2}|u|^2) dx.$$

If $u \in V_0^1(\Omega^\varepsilon)$*, then it holds* $\Theta(\frac{x}{\varepsilon})u \in H_0^1(\Omega^\varepsilon)$*. The imbedding* $V_0^1(\Omega^\varepsilon) \subset V^0(\Omega^\varepsilon)$ *is compact and* $H^1(\Omega^\varepsilon, \Gamma_\varepsilon) \subset V_0^1(\Omega^\varepsilon)$*.*

Lemma 2.5 *Let the sequence $u^\varepsilon \in V_0^1(\Omega^\varepsilon)$ be such that*

$$\sup_\varepsilon \|u^\varepsilon\|_{V_0^1(\Omega^\varepsilon)} < \infty \tag{2.15}$$

Then there is a subsequence $\varepsilon' \to 0$ and a function $u_0 \in H_0^1(\Omega)$ such that $\|u_0 - u^{\varepsilon'}\|_{V^0(\Omega^{\varepsilon'})} \to 0$ as $\varepsilon' \to 0$

Lemma 2.6 *For any $u \in V_0^1(\Omega^\varepsilon)$ the following inequality of Friedrichs type*

$$\|u\|^2_{V_0^1(\Omega^\varepsilon)} \le C \int_{\Omega^\varepsilon} |\Theta(\frac{x}{\varepsilon})|^2 |\nabla_x u|^2 dx \tag{2.16}$$

holds with a constant C independent of ε.

3 Asymptotic expansions of eigenvalues and eigenfunctions degenerate on the boundary $\partial\Omega^\varepsilon$

In this section, let us consider the eigenvalue problem (1.7) degenerate on the surface S_ε of cavities.

Set formally

$$u^\varepsilon(x) \cong \sum_{l=0}^{+\infty} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u^0(x), \tag{3.1}$$

$$\lambda^\varepsilon \cong \sum_{i=0}^{+\infty} \varepsilon^i \lambda^{(i)}(\varepsilon), \tag{3.2}$$

In contrast to usual expression, now we use the following notation

$$D^\alpha v = \frac{\partial^l v}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}},\ \alpha = \{\alpha_1 \cdots \alpha_l\},\ \langle\alpha\rangle = l,\ \alpha_i = 1, 2, \cdots n \tag{3.3}$$

Inserting (3.1) and (3.2) into (1.7), and taking into account that $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi_i}$, one can formally obtain the following equality:

$$\begin{aligned}
0 &= \mathcal{A}_\varepsilon u^\varepsilon(x) - \lambda^\varepsilon \Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon}) u^\varepsilon(x) \\
&= -\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon}) a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u^\varepsilon(x)}{\partial x_j}) \\
&\quad + \Theta^2(\frac{x}{\varepsilon}) b(\frac{x}{\varepsilon}) u^\varepsilon(x) - \lambda^\varepsilon \Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon}) u^\varepsilon(x) \\
&= -\sum_{l=0}^{+\infty} \varepsilon^{l-2} \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} H_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u^0(x)
\end{aligned}$$

$$+\sum_{l=0}^{+\infty}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n} b(\xi)\Theta^2(\xi)N_{\alpha_1\cdots\alpha_l}(\xi)D^\alpha u^0(x)$$

$$-\sum_{s=0}^{+\infty}\varepsilon^s\sum_{i=0}^{s}\lambda^{(i)}\sum_{\alpha_1,\cdots,\alpha_{s-i}=1}^{n}\rho(\xi)\Theta^2(\xi)N_{\alpha_1\cdots\alpha_{s-i}}(\xi)D^\alpha u^0(x) \tag{3.4}$$

where

$$H_0(\xi)=\frac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\frac{\partial N_0(\xi)}{\partial\xi_j}) \tag{3.5}$$

$$\begin{aligned}H_{\alpha_1}(\xi)=&\frac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\frac{\partial N_{\alpha_1}(\xi)}{\partial\xi_j})+\frac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{i\alpha_1}(\xi)N_0(\xi))\\&+\Theta^2(\xi)a_{\alpha_1 j}(\xi)\frac{\partial N_0(\xi)}{\partial\xi_j},\end{aligned} \tag{3.6}$$

For $\langle\alpha\rangle=l\geq 2$

$$\begin{aligned}H_{\alpha_1\cdots\alpha_l}(\xi)=&\frac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial\xi_j})+\frac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi))\\&+\Theta^2(\xi)a_{\alpha_1 j}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_l}(\xi)}{\partial\xi_j}+\Theta^2(\xi)a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_l}(\xi).\end{aligned} \tag{3.7}$$

One of new ideas of this paper is to give the following relations, which are different from those of classical homogenization method (see, e.g. [2,18, 21,24]). Suppose that

$$\begin{cases}H_0(\xi)=0\\H_{\alpha_1}(\xi)=0\\H_{\alpha_1\alpha_2}(\xi)=\vartheta^{-1}\hat{a}_{\alpha_1\alpha_2}\\H_{\alpha_1\alpha_2\alpha_3}(\xi)=N_{\alpha_1}(\xi)\vartheta^{-1}\hat{a}_{\alpha_2\alpha_3}\\ \cdots\\H_{\alpha_1\alpha_2\cdots\alpha_l}(\xi)=N_{\alpha_1\cdots\alpha_{l-2}}(\xi)\vartheta^{-1}\hat{a}_{\alpha_{l-1}\alpha_l},\quad l\geq 4\end{cases} \tag{3.8}$$

where

$$\vartheta=\Big(\int_{Q\cap\omega}\Theta^2(\xi)d\xi\Big)^{-1}, \tag{3.9}$$

and $\hat{a}_{\alpha_1\alpha_2}$, $N_{\alpha_1\cdots\alpha_j}(\xi)$, $\alpha_j=1,2,\cdots n,\ j=0,1,2,\cdots$, will be defined later.

Putting (3.8) into (3.4), one gets

(3.10)

$$0=\sum_{l=0}^{+\infty}\varepsilon^l\{\sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi)\frac{\partial^l}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}[-\sum_{\alpha_{l+1},\alpha_{l+2}=1}^{n}\hat{a}_{\alpha_{l+1}\alpha_{l+2}}\frac{\partial^2 u^0(x)}{\partial x_{\alpha_{l+1}}\partial x_{\alpha_{l+2}}}$$
$$+b(\xi)\Theta^2(\xi)u^0(x)]-\sum_{i=0}^{l}\lambda^{(i)}\sum_{\alpha_1,\cdots,\alpha_{l-i}=1}^{n}\rho(\xi)\Theta^2(\xi)N_{\alpha_1\cdots\alpha_{l-i}}(\xi)D^\alpha u^0(x)\}.$$

One concludes from the first equation of (3.8) that

$$\text{(3.11)}\qquad \begin{cases}\dfrac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\dfrac{\partial N_0(\xi)}{\partial\xi_j})=0, \quad \text{in} \quad \omega\\ N_0(\xi) \quad \text{is} \quad \text{1-periodic} \quad \text{in} \quad \xi\end{cases}$$

Consider the problem

(3.12a)

$$\begin{cases}\dfrac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\dfrac{\partial N(\xi)}{\partial\xi_j})=\Theta^2(\xi)F^0(\xi)+\dfrac{\partial}{\partial\xi_i}(\Theta^2(\xi)F^i(\xi)), \quad \text{in} \quad \omega\\ N(\xi) \text{ is 1-periodic in } \xi, \quad F^i(\xi)\in\widehat{V}^0(\omega), \quad i=0,1,\cdots n\end{cases}$$

A weak solution of this problem is defined as a function $N\in\widehat{V}^1(\omega)$ satisfying the following equality

$$\text{(3.12b)}\qquad \int_{Q\cap\omega}\Theta^2 a_{ij}\frac{\partial N}{\partial\xi_i}\frac{\partial\psi}{\partial\xi_j}d\xi=\int_{Q\cap\omega}[\Theta^2 F^i\frac{\partial\psi}{\partial\xi_i}-\Theta^2F^0\psi]d\xi$$

for any $\psi\in\widehat{V}^1(\omega)$

Let us remark the problem (3.12 a) admits a unique solution up to an additive constant iff

$$\text{(3.13)}\qquad \int_{Q\cap\omega}\Theta^2(\xi)F^0(\xi)d\xi=0$$

Combining (3.11) and (3.13) gives $N_0(\xi)=C$, for convenience, set $C\equiv 1$.

Applying the second equation of (3.8), we define

$$\text{(3.14)}\qquad \begin{cases}\dfrac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{ij}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial\xi_j})=-\dfrac{\partial}{\partial\xi_i}(\Theta^2(\xi)a_{i\alpha_1}(\xi)), \text{ in } \ Q\cap\omega\\ N_{\alpha_1}(\xi)=0 \quad \text{on} \quad \partial Q\end{cases}$$

Integrating on both sides of the third equation of (3.8) with respect to ξ over the unit cell $Q\cap\omega$, and bearing in mind that $N_{\alpha_1}(\xi)$, $N_{\alpha_1,\alpha_2}(\xi)$ are 1-periodic functions in ξ, we thus find

$$\text{(3.15)}\qquad \hat{a}_{ij}=\vartheta\int_{Q\cap\omega}\Theta^2(\xi)(a_{ij}(\xi)+a_{ik}(\xi)\frac{\partial N_j(\xi)}{\partial\xi_k})d\xi.$$

Using the third equation of (3.8) again, we define

(3.16)
$$\begin{cases} \dfrac{\partial}{\partial \xi_i}(\Theta^2(\xi)a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_i}(\Theta^2(\xi)a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi)) \\ \quad -\Theta^2(\xi)a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2}(\xi)}{\partial \xi_j} - \Theta^2(\xi)a_{\alpha_1\alpha_2}(\xi) + \vartheta^{-1}\hat{a}_{\alpha_1\alpha_2} \quad \text{in} \quad Q \cap \omega \\ N_{\alpha_1\alpha_2}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases}$$

Similarly, for $\langle \alpha \rangle = l \geq 3$, we define

(3.17)
$$\begin{cases} \dfrac{\partial}{\partial \xi_i}(\Theta^2(\xi)a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_i}(\Theta^2(\xi)a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi)) \\ \quad -\Theta^2(\xi)a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2\cdots\alpha_l}(\xi)}{\partial \xi_j} - \Theta^2(\xi)a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_l}(\xi) \\ \quad +N_{\alpha_1\cdots\alpha_{l-2}}(\xi)\vartheta^{-1}\hat{a}_{\alpha_{l-1}\alpha_l}, \quad \text{in} \quad Q \cap \omega \\ N_{\alpha_1\cdots\alpha_l}(\xi) = 0 \quad \text{on} \quad \partial Q \end{cases}$$

Remark 3.1 Existence and uniqueness of the solutions $N_{\alpha_1}(\xi), \cdots, N_{\alpha_1\cdots\alpha_l}(\xi)$ associated with respective problems (3.14), (3.16) and (3.17), can be easily proved by induction with respect to l on the basis of conditions (A_2) - (A_4), Lemma 2.6, and Lax-Milgram's lemma. Then they can be extended into the whole ω in 1-periodicity.

Using (3.10) and equating the power-like terms of ε, we have, for $l = 0$

$$\begin{aligned} &-\vartheta^{-1}\frac{\partial}{\partial x_{\alpha_1}}(\hat{a}_{\alpha_1\alpha_2}\frac{\partial u^0(x)}{\partial x_{\alpha_2}}) + \Theta^2(\xi)b(\xi)u^0(x) \\ &\quad = \lambda^{(0)}\Theta^2(\xi)\rho(\xi)u^0(x), \quad a.e \quad \xi \in \omega \end{aligned} \tag{3.18}$$

and for $l = 1$

$$\begin{aligned} &N_{\alpha_1}(\xi)\frac{\partial}{\partial x_{\alpha_1}}\big\{ -\vartheta^{-1}\frac{\partial}{\partial x_{\alpha_2}}(\hat{a}_{\alpha_2\alpha_3}\frac{\partial u^0(x)}{\partial x_{\alpha_3}}) + \Theta^2(\xi)b(\xi)u^0(x) \\ &\quad -\lambda^{(0)}\Theta^2(\xi)\rho(\xi)u^0(x)\big\} - \lambda^{(1)}(\varepsilon)\Theta(\xi)\rho(\xi)u^0(x) = 0 \end{aligned} \tag{3.19}$$

Equation (3.18) implies that

$$\lambda^{(1)}(\varepsilon)\hat{\rho}u^0(x) = 0 \tag{3.20}$$

Recalling $\hat{\rho} = \vartheta \int_{Q\cap\omega} \Theta^2(\xi)\rho(\xi)d\xi = \vartheta > 0, \quad \|u^0\|_{L^2(\Omega)} = 1$, it follows that $\lambda^{(1)}(\varepsilon) = 0$.

We have, following along the above lines, for $l = 2$

$$N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\{-\vartheta^{-1}\frac{\partial}{\partial x_{\alpha_3}}(\hat{a}_{\alpha_3\alpha_4}\frac{\partial u^0(x)}{\partial x_{\alpha_4}})$$
$$+\Theta^2(\xi)b(\xi)u^0(x)-\lambda^{(0)}\Theta^2(\xi)\rho(\xi)u^0(x)\}$$
$$-\lambda^{(1)}(\varepsilon)\Theta^2(\xi)\rho(\xi)N_{\alpha_1}(\xi)\frac{\partial u^0(x)}{\partial x_{\alpha_1}}-\lambda^{(2)}(\varepsilon)\Theta^2(\xi)\rho(\xi)u^0(x)=0 \tag{3.21}$$

Similarly, it follows from (3.18) and $\lambda^{(1)}(\varepsilon)=0$ that $\lambda^{(2)}(\varepsilon)=0$.

The remainder $l\geq 3$ can be determined successively.

On the other hand, from (3.18), an easy computation leads to the equation:

$$-\frac{\partial}{\partial x_{\alpha_1}}(\hat{a}_{\alpha_1\alpha_2}\frac{\partial u^0(x)}{\partial x_{\alpha_2}})=\vartheta(\lambda^{(0)}\rho(\xi)-b(\xi))\Theta^2(\xi)u^0(x)$$

Since $u^0(x)\not\equiv 0, x\in\Omega$, then there exist some points $x\in\Omega$ such that $u^0(x)\neq 0$, and

$$-\frac{1}{u^0(x)}\frac{\partial}{\partial x_{\alpha_1}}(\hat{a}_{\alpha_1\alpha_2}\frac{\partial u^0(x)}{\partial x_{\alpha_2}})=\vartheta(\lambda^{(0)}\rho(\xi)-b(\xi))\Theta^2(\xi)\equiv C \tag{3.22}$$

Integrating on the both sides of (3.22) with respect to ξ over $Q\cap\omega$, we thus find

$$C=\lambda^{(0)}\hat{\rho}-\hat{b} \tag{3.23}$$

where $\hat{f}=\vartheta\int_{Q\cap\omega}\Theta^2(\xi)\cdot f(\xi)d\xi$

As a matter of fact, we can deduce that equation (3.18) and $u^0(x)=0$ on $\partial\Omega$ are equivalent to the following homogenized Helmholtz equation associated with problem (1.7):

$$\begin{cases}\widehat{\mathcal{L}}u^0(x)\equiv-\dfrac{\partial}{\partial x_i}(\hat{a}_{ij}\dfrac{\partial u^0(x)}{\partial x_j})+\hat{b}u^0(x)=\lambda^{(0)}\hat{\rho}u^0(x), \quad \text{in} \quad \Omega\\ u^0(x)=0, \quad \text{on} \quad \partial\Omega\end{cases} \tag{3.24}$$

Recalling the definition of $N_{\alpha_1}(\xi)$, it is easy to verify that $\hat{a}_{lk}$ can be rewritten as follows:

$$\hat{a}_{lk}=\vartheta\int_{Q\cap\omega}\frac{\partial(N_l(\xi)+\xi_l)}{\partial\xi_i}\Theta^2(\xi)a_{ij}(\xi)\frac{\partial(N_k(\xi)+\xi_k)}{\partial\xi_j}d\xi \tag{3.25}$$

The fact that $a_{ij}=a_{ji}$ implies that $\hat{a}_{lk}=\hat{a}_{kl}$.

On the other hand, due to (3.25), setting $w_\eta=(N_l(\xi)+\xi_l)\eta_l$, then we derive

$$\hat{a}_{lk}\eta_l\eta_k=\vartheta\int_{Q\cap\omega}\frac{\partial w_\eta}{\partial\xi_i}\Theta^2 a_{ij}\frac{\partial w_\eta}{\partial\xi_j}d\xi\geq 0$$

Suppose that $\hat{a}_{lk}\eta_l\eta_k = 0$, for any $\eta \in R^n$, then $w_\eta = (N_l(\xi) + \xi_l)\eta_l = const$, for almost all $\xi \in Q \cap \omega$. Since $N_{\alpha_l}(\xi)$ are 1-periodic functions in ξ, $N_l(\xi) + \xi_l \neq const$, it follows that $\eta = 0$. Thus it is proved that $\widehat{\mathcal{L}}$ is a symmetric, positive-definite linear operator.

For an integer $M \geq 1$, we define

$$u_k^{\varepsilon,M}(x) = \sum_{l=0}^{M} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u_k^0(x), \quad x \in \Omega_0^\varepsilon \tag{3.26a}$$

$$\lambda_k^{\varepsilon,M} \equiv \lambda_k^{(0)}, \quad k = 1, 2, \cdots \tag{3.26b}$$

It is not difficult to see that $u_k^{\varepsilon,M} \in V^1(\Omega^\varepsilon)$. However, it should be mentioned that, generally speaking, we do not guarantee $u_k^{\varepsilon,M} \in V^2(\Omega^\varepsilon)$, due to $[\frac{\partial N_{\alpha_1\cdots\alpha_j}}{\partial n}]|_{\partial Q} \neq 0$, $j \geq 1, \alpha_j = 1, 2, \cdots, n$, which $[\frac{\partial N_{\alpha_1\cdots\alpha_j}}{\partial n}]|_{\partial Q}$ denotes the jump of normal derivative of $N_{\alpha_1\cdots\alpha_j}(\xi)$ on ∂Q. To this end, we need to make some assumptions on geometry and physical materials.

To begin with, we make the following assumption on the holes $T = Q \backslash \bar{\omega}$:

(H) T is symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2, \cdots, n$ of the unit cell (see Figure 3.1).

Next let us make the following assumptions on the physical materials:

(C₁) $a_{ii}^\varepsilon(x), \rho^\varepsilon(x)$ are symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2, \cdots, n$ of the unit cell;

(C₂) $a_{ij}^\varepsilon(x), i \neq j$ are anti-symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2, \cdots, n$ of the unit cell. In particular, $a_{ij} = 0, i \neq j$.

(C₃) $a_{ij}(\xi),\ \rho(\xi) \in C^{2,\gamma}(Q \cap \omega),\ 0 < \gamma \leq 1;\ b(\xi) \in L^\infty(Q \cap \omega)$.

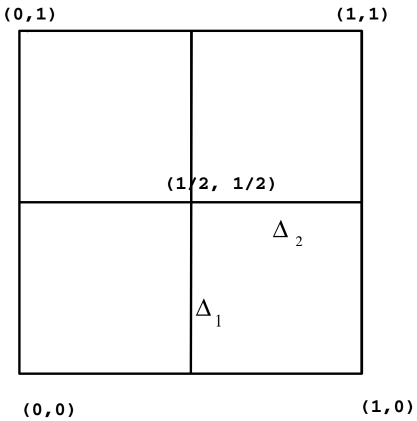

Fig. 3.1. The symmetry of $Q \cap \omega$

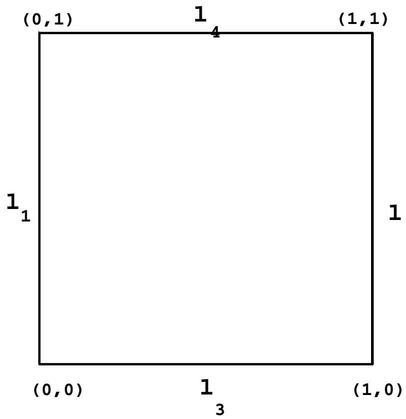

Fig. 3.2. The sides of ∂Q

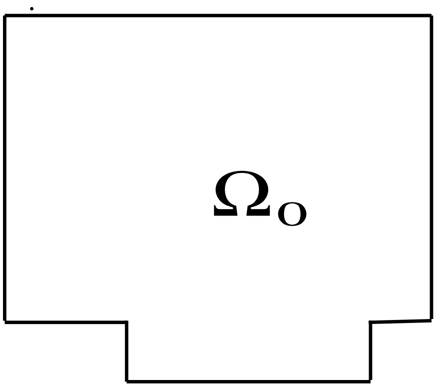

Fig. 3.3. Subdomain Ω_0

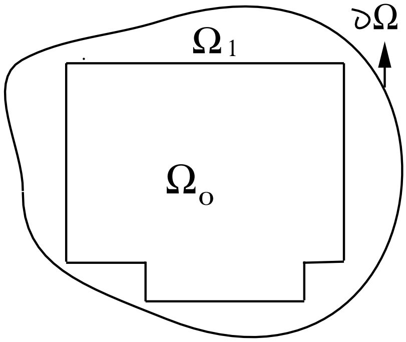

Fig. 3.4. Boundary layer

Let $\overline{\Omega}_0 = \bigcup_{z \in T_\varepsilon} \varepsilon(z + \overline{Q})$, $\overline{\Omega}_0^\varepsilon = \overline{\Omega}_0 \cap \varepsilon\omega$, where T_ε is the subset of Z^n consisting of all z, such that $\varepsilon(z + \bar{Q}) \subset \Omega, dist(\varepsilon(z + \bar{Q}), \partial\Omega) \geq C\varepsilon$, $\Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0, \Gamma^* = \partial\Omega_0$ as shown in Figs. 3.3, 3.4.

Proposition 3.1 *Let $u_k^{\varepsilon,M}$ be defined as in (3.26a). Under the assumptions $(A_1) - (A_3)$, $(B_1) - (B_3)$, $(C_1) - (C_3)$, and (H), then we can infer that $u_k^{\varepsilon,M} \in V^2(\Omega_0^\varepsilon), k = 1, 2, \cdots ; 1 \leq M \leq 3$.*

The proof of Proposition 3.1 can be found in the **Appendix A**.

Remark 3.2 Under the assumptions of Proposition 3.1, we can prove that homogenization method with zero boundary conditions on ∂Q presented in this paper is equivalent to the classical homogenization method with periodic boundary conditions on ∂Q(see, e.g. [2,6,16,18,21,24]). Furthermore, we will improve the theoretical results (Ref. Theorem 6.3 of [6] and Theorem 3.9, Chap. III in [24]), and obtain the higher-order asymptotic expansions of eigenfunctions and the better convergence results.

For $x \in \Omega_0^\varepsilon$, using (3.4)–(3.17), and taking into account Proposition 3.1, one derives

$$\begin{aligned} &\mathcal{A}_\varepsilon u_k^{\varepsilon,M}(x) - \lambda_k^{\varepsilon,M} \rho(\frac{x}{\varepsilon}) \Theta^2(\frac{x}{\varepsilon}) u_k^{\varepsilon,M}(x) \\ &\quad = \sum_{l=0}^{M} \varepsilon^{l-2} \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} H_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u_k^0(x) \\ &\qquad + \sum_{l=0}^{M-2} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} \Theta^2(\xi) b(\xi) N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u_k^0(x) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{l=0}^{M-2}\varepsilon^l \sum_{\alpha_1,\cdots\alpha_l=1}^{n} \lambda_k^{(0)}\Theta^2(\xi)\rho(\xi)N_{\alpha_1\cdots\alpha_l}(\xi)\\
&\times D^\alpha u_k^0(x)+\varepsilon^{M-1}F_0(x,\varepsilon)\\
&=\sum_{l=0}^{M-2}\varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi)\frac{\partial^l}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\\
&\times[\vartheta^{-1}\sum_{\alpha_{l+1},\alpha_{l+2}=1}^{n}\hat{a}_{\alpha_{l+1}\alpha_{l+2}}\frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_{l+1}}\partial x_{\alpha_{l+2}}}\\
&+\Theta^2(\xi)b(\xi)u_k^0(x)-\lambda_k^{(0)}\Theta^2(\xi)\rho(\xi)u_k^0(x)]+\varepsilon^{M-1}F_0(x,\varepsilon)\\
&=\varepsilon^{M-1}F_0(x,\varepsilon)
\end{aligned}
\tag{3.27}
$$

where $F_0(x,\varepsilon)$ is a sum of terms having the form $\varepsilon^i\psi(\xi)D^l u_k^0(x)$, $l \leq M$, $i \geq 0$, $\quad k = 1, 2, \cdots$, $\psi(\xi)$ is a bounded function, and $\| F_0(x,\varepsilon)\|_{L^2(\Omega^\varepsilon)} \leq C$, $\quad C$ is a constant independent of ε, x.

Just as J.L. Lions said that study of boundary layer in composite materials and perforated materials seem to be open (Cf. [21], pp. 121). In this paper, we give the boundary layer equation in such a way:

$$
\begin{cases}
-\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial w_k^\varepsilon(x)}{\partial x_j})\\
\quad +\Theta^2(\frac{x}{\varepsilon})\Big(b(\frac{x}{\varepsilon})-\lambda_k^{(0)}\rho(\frac{x}{\varepsilon})\Big)w_k^\varepsilon(x)=0, \quad x\in\Omega_1^\varepsilon\\
w_k^\varepsilon(x)=u_k^0(x), \quad x\in\partial\Omega_0\cap\partial\Omega_1\\
w_k^\varepsilon(x)=0, \quad x\in\Gamma_\varepsilon
\end{cases}
\tag{3.28}
$$

where $\lambda_k^{(0)}$, $u_k^0(x)$ are the k-th eigenvalue and eigenfunction associated with the homogenized Helmholtz equation(3.24), respectively, $k = 1, 2, \cdots$.

Define the operator $\mathcal{K}_\varepsilon : V^0(\Omega_1^\varepsilon) \to V^0(\Omega_1^\varepsilon)$, such that $\mathcal{K}_\varepsilon f^\varepsilon = \psi^\varepsilon$, where ψ^ε is the solution of the problem:

$$
\begin{cases}
\mathcal{Q}_\varepsilon(\psi^\varepsilon)\equiv-\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial\psi^\varepsilon}{\partial x_j})\\
\quad +\Theta^2(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon})\psi^\varepsilon=\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})f^\varepsilon, \quad \text{in} \quad \Omega_1^\varepsilon\\
\psi^\varepsilon=0, \quad \text{on} \quad \Gamma_\varepsilon\\
\psi^\varepsilon\in V_0^1(\Omega_1^\varepsilon), \quad f^\varepsilon\in V^0(\Omega_1^\varepsilon)
\end{cases}
\tag{3.29}
$$

Using conditions (A_2)–(A_3), one can verify that $\mathcal{Q}_\varepsilon : V_0^1(\Omega_1^\varepsilon) \to V^0(\Omega_1^\varepsilon)$ is a symmetric and positive-definite operator, therefore the inverse operator $\mathcal{K}_\varepsilon = \mathcal{Q}_\varepsilon^{-1} : V^0(\Omega_1^\varepsilon) \to V^0(\Omega_1^\varepsilon)$ is a bounded self-adjoint, compact operator due to the compact imbedding: $V_0^1(\Omega_1^\varepsilon) \to V^0(\Omega_1^\varepsilon)$ (see Lemma 2.4). Denotes by $\sigma_d(\mathcal{K}_\varepsilon)$ the set of the discrete spectra of $\mathcal{K}_\varepsilon$.

Suppose that

$$
(\lambda_k^{(0)}(\Omega))^{-1} \notin \sigma_d(\mathcal{K}_\varepsilon)
\tag{3.30}
$$

It follows from Fredholm's alternative theorem that the equation (3.28) has one and only one solution.

The following theorem shows that condition (3.30) is true in some cases.

Theorem 3.1 *Under the assumptions of Lemma 2.1, if assume that* $\lambda_1^{(0)}(\Omega)$ *is the first eigenvalue of (3.24), then we have, for a sufficiently small* $\varepsilon > 0$

$$(3.31) \qquad \left(\lambda_1^{(0)}(\Omega)\right)^{-1} \notin \sigma_d(\mathcal{K}_\varepsilon)$$

Proof. Given $\Omega_1^\varepsilon \subset \Omega^\varepsilon$, and $meas(\Omega^\varepsilon \setminus \overline{\Omega}_1) = meas(\Omega_0^\varepsilon) > 0$

Denote by $\lambda_1^\varepsilon(\Omega^\varepsilon)$, $\tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon)$ the first eigenvalues associated with equation (1.7) and the operator $\mathcal{Q}_\varepsilon$, respectively. The variational principle implies that $\lambda_1^\varepsilon(\Omega^\varepsilon) \le \tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon)$. Suppose that $\lambda_1^\varepsilon(\Omega^\varepsilon) = \tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon) = \mu$. Then the eigenfunction of the operator $\mathcal{Q}_\varepsilon$ with eigenvalue μ expanded by zero values on $\Omega^\varepsilon \setminus \overline{\Omega}_1$ is an eigenfunction in Ω^ε. However, it vanishes at some points of Ω^ε, contrary to the similar result of Lemma 2.1, therefore $\lambda_1^\varepsilon(\Omega^\varepsilon) < \tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon)$.

By virtue of Theorem 3.1 of Chap III of [24], we have $|\lambda_1^\varepsilon(\Omega^\varepsilon) - \lambda_1^{(0)}(\Omega)| < \varepsilon$. If $0 < \varepsilon < (\tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon) - \lambda_1^\varepsilon(\Omega^\varepsilon))/2$, then $\lambda_1^{(0)}(\Omega) < \tilde{\lambda}_1^\varepsilon(\Omega_1^\varepsilon)$, i.e. $(\lambda_1^{(0)}(\Omega))^{-1} \notin \sigma_d(\mathcal{K}_\varepsilon)$

The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let* $w_k^\varepsilon(x)$ *be the weak solution of boundary layer equation (3.28), and let* $a_{ij}(\frac{x}{\varepsilon})$, $b(\frac{x}{\varepsilon})$, $\rho(\frac{x}{\varepsilon})$ *satisfy conditions* $(A_2)-(A_4)$. *If* $(\lambda_k^{(0)})^{-1} \notin \sigma_d(\mathcal{K}_\varepsilon)$, *then it holds*

$$(3.32) \qquad \|w_k^\varepsilon\|_{V^1(\Omega_1^\varepsilon)} \le C\|u_k^0\|_{1,\Omega}, \quad k = 1, 2, \cdots$$

where C is independent of ε, u_k^0.

Theorem 3.2 is an immediate consequence of Theorem 3.6, Chap.III of [24].

Theorem 3.3 *Let* $w^\varepsilon(x)$ *be defined as in (3.28), and* $\Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0 \subset R^2$ *be shown in Figure 3.4. For convenience, we do omit the subscript k under no confusion. Under the hypotheses of Theorem 3.2, if* $a_{ij}^\varepsilon \in C(\overline{\Omega}^\varepsilon)$, $\nabla_\xi a_{ij}(\xi) \in L^\infty(\Omega^\varepsilon)$, *then there exists* $1 < p_0 < +\infty$, *such that*

$$(3.33) \qquad w^\varepsilon(x) \in V^{2,p}(\Omega_1^\varepsilon), \quad 1 < p \le p_0$$

$$(3.34) \qquad \|w^\varepsilon\|_{V^{2,p}(\Omega_1^\varepsilon)} \le C\varepsilon^{-2}\|u^0\|_{2,p,\Omega}$$

We refer the reader to **Appendix B** *for details.*

For $M \geq 1$, define

(3.35)
$$\tilde{u}_k^{\varepsilon,M}(x) = \begin{cases} u_k^{\varepsilon,M}(x) = u_k^0(x) + \sum_{l=1}^{M} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u_k^0(x) & x \in \overline{\Omega}_0^\varepsilon \\ w_k^\varepsilon(x) \quad x \in \Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0, \quad k = 1, 2, \cdots \end{cases}$$

where $N_{\alpha_1,\cdots\alpha_l}(\xi)$, $1 \leq l \leq M$ are defined as in (3.14), (3.16) and (3.17), respectively.

The fact that $u_k^{\varepsilon,M}(x)|_{\partial\Omega_0\cap\partial\Omega_1} = u_k^0(x)|_{\partial\Omega_0\cap\partial\Omega_1} = w_k^\varepsilon(x)|_{\partial\Omega_0\cap\partial\Omega_1}$ implies that $\tilde{u}_k^{\varepsilon,M}(x) \in V^1(\Omega^\varepsilon)$. However, generally speaking, there exists the jump of normal derivative of $\tilde{u}_k^{k,M}$ on the interface $\partial\Omega_0 \cap \partial\Omega_1$, i.e. $[\frac{\partial \tilde{u}_k^{\varepsilon,M}}{\partial n}]|_{\partial\Omega_0\cap\partial\Omega_1} \neq 0$. To this end, we need to treat it with some regularization operators.

Set

$$\mathcal{V}_1 = \{x \in \Omega_0 : \quad dist(x, \partial\Omega_0) > \frac{\delta}{2}\}$$
$$\mathcal{V}_2 = \{x \in (R^n \setminus \overline{\Omega}_0) : dist(x, \partial\Omega_0) > \frac{\delta}{2}, \; dist(x, \partial\Omega) < \delta\} \tag{3.36}$$
$$\mathcal{V}_3 = \{x \in \Omega : \quad dist(x, \partial\Omega_0) < \delta\}$$

It is obvious to see that $\overline{\Omega} \subset \cup_{l=1}^3 \mathcal{V}_l$

It follows from the resolution of unity theorem that there exist a set of functions $\{\psi_l(x)\}_{l=1}^3$ such that: (1) $\psi_l(x) \in C_0^\infty(\mathcal{V}_l)$; (2) $\sum_{l=1}^3 \psi_l(x) \equiv 1$, $\forall x \in \Omega$

Set $\Omega_0'' = \Omega_0 \setminus \overline{\mathcal{V}}_3$, $\Omega_1'' = \Omega_1 \setminus \overline{\mathcal{V}}_3$, and choose a sufficiently small $\delta > 0$ such that $\delta \leq C \cdot \varepsilon^M$, $\quad M \geq 2$

Define

(3.37)
$$\hat{u}_k^{\varepsilon,M}(x) = \psi_1(x) \cdot \tilde{u}_k^{\varepsilon,M}(x) + \psi_2(x) \cdot \tilde{u}_k^{\varepsilon,M}(x) + J_\delta * (\psi_3(x) \cdot \tilde{u}_k^{\varepsilon,M}(x))$$

where J_δ is a regularization operator (Ref. Section 2.17 of [1]).

Therefore we have $\hat{u}_k^{\varepsilon,M}(x) \in V^1(\Omega^\varepsilon)$ and $[\frac{\partial \hat{u}_k^{\varepsilon,M}}{\partial n}]|_{\partial\Omega_0\cap\partial\Omega_1} = 0$.

Recalling (1.6), we define

$$U_k^{\varepsilon,M}(x) = \Theta(\frac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M}(x) \tag{3.38}$$

Lemma 3.1[24] *Let A: $H \to H$ be a linear self-adjoint compact operator in a Hilbert space H. Let $\mu \in R^1$, and let $u \in H$, be such that $\|u\|_H = 1$, and*

$$\|Au - \mu u\|_H \leq \beta, \qquad \beta = const > 0, \tag{3.39}$$

Then there exists an eigenvalue μ_i of operator A such that

$$|\mu_i - \mu| \leq \beta \tag{3.40}$$

Moreover, for any $d > \beta$ there exists a vector $\bar{u} \in H$, such that

$$\|u - \bar{u}\|_H \leq 2\beta d^{-1}, \qquad \|\bar{u}\|_H = 1 \tag{3.41}$$

and $\bar{u}(x)$ is a linear combination of the eigenvectors of operator A corresponding to the eigenvalues within the interval $[\mu - d, \mu + d]$

Lemma 3.2[24] *Let λ_k^ε and $\lambda_k^{(0)}$ be the k-th eigenvalues of problems (1.7) and (3.24), respectively. Then holds*

$$|\lambda_k^\varepsilon - \lambda_k^{(0)}| \leq C_k\varepsilon, \quad k = 1, 2, \cdots \tag{3.42}$$

where C_k is a constant independent of ε.

Suppose that the multiplicity of $\lambda^{(0)} = \lambda_k^{(0)}$ is equal to t, i.e. $\lambda_{k-1}^{(0)} < \lambda_k^{(0)} = \cdots = \lambda_{k+t-1}^{(0)} < \lambda_{k+t}^{(0)}, \quad \lambda_0^{(0)} = 0$, and $u_k^0(x)$ is an eigenfunction of problem (3.24) corresponding to $\lambda_k^{(0)}$, $\|u_k^0\|_{L^2(\Omega)} = 1$. Then for every $\varepsilon \in (0, 1)$ there is a function $\bar{u}_k^\varepsilon$ such that

$$\|\bar{u}_k^\varepsilon - u_k^0\|_{V^0(\Omega^\varepsilon)} \leq M_k\varepsilon \tag{3.43}$$

where M_k is a constant independent of ε, $u_k^0(x)$; $\bar{u}_k^\varepsilon(x)$ is a linear combination of eigenfunctions of problem (1.7) corresponding to the eigenvalue $\lambda_k^\varepsilon, \cdots, \lambda_{k+t-1}^\varepsilon$.

Now we would like to give the following error estimate, which is an important theoretical result in this paper.

Theorem 3.4 *Let $(\lambda_k^\varepsilon, u_k^\varepsilon)$ be the eigenpairs of problem (1.7), $k = 1, 2, \cdots$, $\lambda_0^\varepsilon = 0$, and let $\tilde{u}_k^{\varepsilon,M}(x)$, $\lambda_k^{\varepsilon,M}$ be defined as in (3.35) and (3.26b), respectively. If condition (3.30) is satisfied, under the assumptions of Proposition 3.1, then*

$$|\lambda_k^\varepsilon - \lambda_k^{\varepsilon,M}| \leq \begin{cases} C_1(k) \cdot \varepsilon, & if \quad M = 0, 1 \\ C_1(k) \cdot min(\varepsilon, \gamma_M), & if \quad 2 \leq M \leq 4, \end{cases} \tag{3.44}$$

where $\gamma_M = \varepsilon^{M-1} + \left(\int\limits_{\partial\Omega_0 \cap \partial\Omega_1} \left([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})] \right)^2 d\Gamma \right)^{1/2}$, *and* $[\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]$ *denotes the jump of the normal derivative of* $\tilde{u}_k^{\varepsilon,M}$ *on the interface of* $\partial\Omega_0 \cap \partial\Omega_1$.

Moreover, if the multiplicity of the eigenvalues $\lambda_k^{(0)}$ are equal to t, then

$$\|\tilde{u}_k^{\varepsilon,M} - \bar{u}_k^{\varepsilon}\|_{V^0(\Omega^\varepsilon)} \leq \begin{cases} C_2(k)\cdot\varepsilon, & \text{if} \quad M=0,1 \\ C_2(k)\cdot min(\varepsilon,\gamma_M), & \text{if} \quad 2\leq M\leq 4, \end{cases} \tag{3.45}$$

where $\bar{u}_k^{\varepsilon}$ is a linear combination of eigenfunctions of problem (1.10) corresponding to $\lambda_k^\varepsilon, \cdots \lambda_{k+t-1}^\varepsilon$

In particular, if the eigenvalue $\lambda_k^{(0)}$ is simple, then

$$\|\tilde{u}_k^{\varepsilon,M} - u_k^{\varepsilon}\|_{V^0(\Omega^\varepsilon)} \leq \begin{cases} C_2(k)\cdot\varepsilon, & \text{if} \quad M=0,1 \\ C_2(k)\cdot min(\varepsilon,\gamma_M), & \text{if} \quad 2\leq M\leq 4, \end{cases} \tag{3.46}$$

Proof. Denote by $\mathcal{H}_\varepsilon$ the space $V^0(\Omega^\varepsilon)$ equipped with the scalar product

$$(u,v)_{\mathcal{H}_\varepsilon} = \int\limits_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon})\Theta^2(\frac{x}{\varepsilon})uv dx$$

Consider the following auxiliary problem:

$$\begin{cases} \mathcal{A}_\varepsilon\varphi^\varepsilon \equiv -\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial\varphi^\varepsilon(x)}{\partial x_j}) + \Theta^2(\frac{x}{\varepsilon})b(\frac{x}{\varepsilon})\varphi^\varepsilon(x) \\ \qquad = \Theta^2(\frac{x}{\varepsilon})f^\varepsilon(x), \quad \text{in} \quad \Omega^\varepsilon \\ \varphi^\varepsilon(x) = 0, \quad \text{on} \quad \Gamma_\varepsilon \\ \varphi^\varepsilon(x)\in V_0^1(\Omega^\varepsilon), \quad f^\varepsilon(x)\in V^{-1}(\Omega^\varepsilon) \end{cases} \tag{3.47}$$

By using conditions $(A_2)-(A_4)$, Lemma 2.6, and Lax-Milgram's lemma, one can prove that there is a unique weak solution $\varphi^\varepsilon \in V_0^1(\Omega^\varepsilon)$, for any $f^\varepsilon \in V^{-1}(\Omega^\varepsilon)$, where $V^{-1}(\Omega^\varepsilon)$ denotes the dual space of $V_0^1(\Omega^\varepsilon)$. In other words, $\mathcal{A}_\varepsilon : V_0^1(\Omega^\varepsilon) \to V^{-1}(\Omega^\varepsilon)$ is a symmetric, positive homeomorphism mapping. Let $\mathcal{N}_\varepsilon = \mathcal{A}_\varepsilon^{-1}$, then $\mathcal{N}_\varepsilon$ is a compact, self-adjoint, and positive-definite operator in $\mathcal{H}_\varepsilon$ due to the imbedding $V_0^1(\Omega^\varepsilon) \to V^0(\Omega^\varepsilon)$ is compact (see Lemma 2.4).

Assuming that Ω^ε is a bounded convex Lipschitz's domain, under the assumptions of this theorem, by virtue of *a-priori* estimates of PDEs, we can prove that $u_k^\varepsilon(x)\in V^2(\Omega^\varepsilon)$, and $u_k^{\varepsilon,M}(x)\in V^2(\Omega_0^\varepsilon)$(Ref. **Appendix A**), and $w_k^\varepsilon(x)\in V^{2,p}(\Omega_1^\varepsilon)$, $1<p\leq p_0<+\infty$, (Ref. **Appendix B**).

If $x\in\Omega_0^\varepsilon$, combining (3.35) and (3.27), and recalling Proposition 3.1, one gets

$$\mathcal{A}_\varepsilon\tilde{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M}(x) = \varepsilon^{M-1}F_0(x,\varepsilon) \tag{3.48}$$

If $x\in\Omega_1^\varepsilon$, combining (3.35) and (3.28) yields

(3.49)

$$\mathcal{A}_\varepsilon\tilde{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M}(x) = \mathcal{A}_\varepsilon w_k^\varepsilon(x) - \lambda_k^{(0)}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})w_k^\varepsilon(x) = 0$$

From (3.37), (3.48) and (3.49), it thus follows that

$$(3.50)\quad \begin{cases} \mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M}(x) - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}(x) = \hat{F}_0(x,\varepsilon), \quad x \in \Omega^\varepsilon \\ \hat{u}_k^{\varepsilon,M}(x) = 0, \quad x \in \Gamma_\varepsilon \end{cases}$$

For any $v \in V^{-1}(\Omega^\varepsilon)$, observe that

$$\begin{aligned} (\hat{F}_0, v)_{\Omega^\varepsilon} &= (\mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}, v)_{\Omega^\varepsilon} \\ &= (\mathcal{A}_\varepsilon u_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})u_k^{\varepsilon,M}, v)_{\Omega_0''\cap\varepsilon\omega} \\ &\quad + (\mathcal{A}_\varepsilon w_k^\varepsilon - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})w_k^\varepsilon, v)_{\Omega_1''\cap\varepsilon\omega} \\ &\quad + (\mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}, v)_{\mathcal{V}_3\cap\Omega_0^\varepsilon} \\ &\quad + (\mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\tfrac{x}{\varepsilon})\rho(\tfrac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}, v)_{\mathcal{V}_3\cap\Omega_1^\varepsilon} \end{aligned} \tag{3.51}$$

On the other hand, one has

$$\begin{aligned} &(\mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}, v)_{\mathcal{V}_3\cap\Omega_0^\varepsilon} \\ &= (\mathcal{A}_\varepsilon u_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})u_k^{\varepsilon,M}, v)_{\mathcal{V}_3\cap\Omega_0^\varepsilon} \\ &\quad + (\mathcal{A}_\varepsilon \Lambda_0(x) - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\Lambda_0(x), v)_{\mathcal{V}_3\cap\Omega_0^\varepsilon} \end{aligned} \tag{3.52a}$$

where $\Lambda_0(x) = \psi_3(x)u_k^{\varepsilon,M}(x) - J_\delta * (\psi_3(x)\tilde{u}_k^{\varepsilon,M}(x))$

Similarly

$$\begin{aligned} &(\mathcal{A}_\varepsilon \hat{u}_k^{\varepsilon,M} - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\hat{u}_k^{\varepsilon,M}, v)_{\mathcal{V}_3\cap\Omega_1^\varepsilon} \\ &= (\mathcal{A}_\varepsilon w_k^\varepsilon - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})w_k^\varepsilon, v)_{\mathcal{V}_3\cap\Omega_1^\varepsilon} \\ &\quad + (\mathcal{A}_\varepsilon \Lambda_1(x) - \lambda_k^{\varepsilon,M}\Theta^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})\Lambda_1(x), v)_{\mathcal{V}_3\cap\Omega_1^\varepsilon} \end{aligned} \tag{3.52b}$$

where $\Lambda_1(x) = \psi_3(x)w_k^\varepsilon(x) - J_\delta * (\psi_3(x)\tilde{u}_k^{\varepsilon,M}(x))$

Following along the lines of the proof of Theorem 3.16 of [1], one gets

$$\|\Lambda_0\|_{V^1(\mathcal{V}_3\cap\Omega_0^\varepsilon)} \le \delta, \quad \|\Lambda_1\|_{V^1(\mathcal{V}_3\cap\Omega_1^\varepsilon)} \le \delta \tag{3.53}$$

Denote by $\|f^0\|_*$ the norm in $V^{-1}(\Omega^\varepsilon)$:

$$\|f^0\|_* = \sup_v \left\{ |(f^0, v)_{L^2(\Omega^\varepsilon)}|, v \in V_0^1(\Omega^\varepsilon), \|v\|_{V_0^1(\Omega^\varepsilon)} = 1 \right\} \tag{3.54}$$

From (3.50), (3.52 a), (3.52 b), (3.53), (3.54), (3.48) and (3.49), one can obtain

$$\begin{aligned}\|\hat{F}_0\|_* &\leq C\Big\{\varepsilon^{M-1} + \varepsilon^{-1}\cdot\|\Lambda_0\|_{V^1(\mathcal{V}_3\cap\Omega_0^\varepsilon)} + \varepsilon^{-1}\cdot\|\Lambda_1\|_{V^1(\mathcal{V}_3\cap\Omega_1^\varepsilon)} \\ &\quad + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\} \\ &\leq C\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\} \qquad (3.55)\end{aligned}$$

The fact $\mathcal{N}_\varepsilon = \mathcal{A}_\varepsilon^{-1}$ implies that the above equation (3.50) is equivalent to the following equation

$$(3.56)\qquad \begin{cases} \hat{u}_k^{\varepsilon,M}(x) - \lambda_k^{\varepsilon,M}\mathcal{N}_\varepsilon(\Theta^2\rho\hat{u}_k^{\varepsilon,M}) = \mathcal{N}_\varepsilon(\hat{F}_0), & \text{in} \quad \Omega^\varepsilon \\ \hat{u}_k^{\varepsilon,M}(x) = 0, & \text{on} \quad \Gamma_\varepsilon \end{cases}$$

Let us apply Lemma 3.1 to (3.56), setting

$$\begin{aligned} &u(x) = \Big(\|\hat{u}_k^{\varepsilon,M}\|_{\mathcal{H}_\varepsilon}\Big)^{-1}\hat{u}_k^{\varepsilon,M}, \qquad \mathcal{A} = \mathcal{N}_\varepsilon(\Theta^2\rho\hat{u}_k^{\varepsilon,M}) \\ &\lambda = \lambda_k^{\varepsilon,M} = \lambda_k^{(0)}, \qquad H = \mathcal{H}_\varepsilon \quad \beta = \|\mathcal{N}_\varepsilon(\hat{F}_0)\|_{\mathcal{H}_\varepsilon}\cdot\Big(\|\hat{u}_k^{\varepsilon,M}\|_{\mathcal{H}_\varepsilon}\Big)^{-1}\end{aligned}$$

Lemma 3.1 ensures that

$$\begin{aligned} &\Big|\Big(\lambda_k^{\varepsilon,M}\Big)^{-1} - \Big(\lambda_{n(k)}^\varepsilon\Big)^{-1}\Big| \\ &= \Big|\Big(\lambda_k^{(0)}\Big)^{-1} - \Big(\lambda_{n(k)}^\varepsilon\Big)^{-1}\Big| \\ &\leq C\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}, \quad 2\leq M\leq 4 \qquad (3.57)\end{aligned}$$

It follows from Lemma 3.2 that $\lambda_k^\varepsilon \to \lambda_k^{(0)}$, as $\varepsilon \to 0, k = 1, 2, \cdots$. So, for a fixed k, there is a small neighborhood of point $\lambda_k^{(0)}$ which contains a eigenvalue λ_k^ε such that $\lambda_{n(k)}^\varepsilon = \lambda_k^\varepsilon$, Therefore

$$|\lambda_k^{(0)} - \lambda_k^\varepsilon| \leq C_1\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}$$

By using Lemma 3.1 again, we can conclude that

$$\|\hat{u}_k^{\varepsilon,M} - \bar{u}_k^\varepsilon\|_{V^0(\Omega^\varepsilon)} \leq C_2\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}$$

In particular, if the eigenvalue $\lambda_k^{(0)}$ of (3.24) is simple, then one can choose $\bar{u}_k^\varepsilon = c_0 u_k^\varepsilon$, $c_0 = const$, such that

$$\|\hat{u}_k^{\varepsilon,M} - u_k^{\varepsilon,M}\|_{V^0(\Omega^\varepsilon)} \leq C_2\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}$$

Following along the lines of the proof of Theorem 3.16 of [1], one derives

$$\|\hat{u}_k^{\varepsilon,M} - \tilde{u}_k^{\varepsilon,M}\|_{V^1(\Omega^\varepsilon)} \le \delta \le C\varepsilon^M$$

Therefore

$$\|u^\varepsilon - \tilde{u}_k^{\varepsilon,M}\|_{V^0(\Omega^\varepsilon)} \le C\Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1} \Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\} \tag{3.58}$$

On the other hand, using Lemma 3.2 and the trace theorem, one gets

$$\|u^\varepsilon - \tilde{u}_k^{\varepsilon,M}\|_{V^0(\Omega^\varepsilon)} \le C\cdot\varepsilon \tag{3.59}$$

Combining (3.58) and (3.59) yields

$$\begin{aligned}&\|u^\varepsilon - \tilde{u}_k^{\varepsilon,M}\|_{V^0(\Omega^\varepsilon)} \le C\cdot min(\varepsilon,\gamma_M),\\ &\gamma_M = \varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\},\quad 2\le M\le 4\end{aligned} \tag{3.60}$$

For $M = 0, 1$, following along the lines of proof of Lemma 3.2, and using the trace theorem, one can derive the error estimates.

Therefore we complete the proof of Theorem 3.4.

Corollary 3.1 *Let $\Lambda_k^\varepsilon, \lambda_k^\varepsilon, \lambda_k^{(0)}$ be the eigenvalues of (1.1), (1.7) and (3.24), respectively. Under the assumptions of Theorem 3.4, then*

$$\begin{aligned}&\Lambda_k^\varepsilon = \varepsilon^{-2}\Lambda^0 + \lambda_k^\varepsilon\\ &|\lambda_k^\varepsilon - \lambda_k^{\varepsilon,M}| \le \begin{cases} C_1(k)\cdot\varepsilon, & if \quad M = 0,1\\ C_1(k)\cdot min(\varepsilon,\gamma_M), & if \quad 2\le M\le 4,\end{cases}\end{aligned} \tag{3.61}$$

Moreover, if the multiplicity of the eigenvalues $\lambda_k^{(0)}$ are equal to t, then

$$\|\Theta(\frac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M} - \overline{U}_k^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le \begin{cases} C_2(k)\cdot\varepsilon, & if \quad M = 0,1\\ C_2(k)\cdot min(\varepsilon,\gamma_M), & if \quad 2\le M\le 4,\end{cases} \tag{3.62}$$

where $\overline{U}_k^\varepsilon$ is a linear combination of eigenfunctions of problem (1.1) corresponding to $\Lambda_k^\varepsilon, \cdots \Lambda_{k+t-1}^\varepsilon$

In particular, if the eigenvalue $\lambda_k^{(0)}$ is simple, then

$$\|\Theta(\frac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M} - U_k^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le \begin{cases} C_2(k)\cdot\varepsilon, & if \quad M = 0,1\\ C_2(k)\cdot min(\varepsilon,\gamma_M), & if \quad 2\le M\le 4,\end{cases} \tag{3.63}$$

where $\gamma_M = \varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}, \quad 2\le M\le 4.$

Corollary 3.2 *Under the assumptions of Theorem 3.4, then the following error estimates hold:*

(3.64)
$$\|\tilde{u}_k^{\varepsilon,M} - \bar{u}_k^{\varepsilon}\|_{V^1(\Omega^\varepsilon)} \leq \begin{cases} C_2(k) \cdot \varepsilon^{1/2}, & if \quad M = 0, 1 \\ C_2(k) \cdot min(\varepsilon^{1/2}, \gamma_M^{1/2}), & if \quad 2 \leq M \leq 4, \end{cases}$$

(3.65)
$$\|\Theta(\frac{x}{\varepsilon})\tilde{u}_k^{\varepsilon,M} - \overline{U}_k^{\varepsilon}\|_{H^1(\Omega^\varepsilon)} \begin{cases} C_2(k) \cdot \varepsilon^{1/2}, & if \quad M = 0, 1 \\ C_2(k) \cdot min(\varepsilon^{1/2}, \gamma_M^{1/2}), & if \quad 2 \leq M \leq 4, \end{cases}$$

where $\gamma_M = \varepsilon^{M-1} + \left(\int\limits_{\partial\Omega_0 \cap \partial\Omega_1} \left([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\right)^2 d\Gamma\right)^{1/2}\Big\}, \quad 2 \leq M \leq 4.$

Remark 3.3 It should be pointed out that, if Ω^ε is the union of entire cells, under the assumptions of Theorem 3.4, and $u_k^0 \in H^{M+2}(\Omega)$, then we have $|\lambda_k^\varepsilon - \lambda_k^{\varepsilon,M}| \leq C(k)\varepsilon^{M-1}$, and $\|u_k^{\varepsilon,M} - \bar{u}_k^\varepsilon\|_{V^0(\Omega^\varepsilon)} \leq C(k)\varepsilon^{M-1}, \quad 2 \leq M \leq 4.$

4 Finite element methods for the related problems

For simplicity, we here discuss only 2-D problems without loss of generality.

4.1 FEM for Computing the First Eigenvalue Λ^0 and Eigenfunction $\Theta(\xi)$ of Problem (1.5)

To begin with, one can see that the variational formulation of the first eigenvalue problem (1.5) is the following:

(4.1)
$$\begin{cases} \int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial\Theta(\xi)}{\partial\xi_j}\frac{\partial v(\xi)}{\partial\xi_i}d\xi = \Lambda^0 \int\limits_{Q\cap\omega} \rho(\xi)\Theta(\xi)v(\xi)d\xi, \quad \forall v \in H^1(Q\cap\omega, \partial\omega) \\ \Theta(\xi) \in H^1(Q\cap\omega, \partial\omega), \quad \Theta(\xi) \text{ is 1-periodic in } \xi \\ \int\limits_{Q\cap\omega} \rho(\xi)|\Theta(\xi)|^2 d\xi = 1 \end{cases}$$

Let $\mathcal{J}^{h_0} = \{K\}$ be a regular family of triangulations of the unit cell $Q\cap\omega$, $h_0 = \max_K\{h_K\}$. Define a linear finite element space

(4.2)
$$W_{h_0} = \left\{v \in C(\overline{Q\cap\omega}) : \quad v|_K \in P_1(K), \quad v|_{\partial\omega} = 0\right\} \subset H^1(Q\cap\omega, \partial\omega)$$

Define finite element solution $(\Lambda^0_{h_0}, \Theta_{h_0})$ corresponding to (Λ^0, Θ) satisfies the following integral equality

(4.3)
$$\begin{cases} \int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial\Theta_{h_0}}{\partial\xi_j}\frac{\partial v_{h_0}}{\partial\xi_i}d\xi = \Lambda^0_{h_0}\int\limits_{Q\cap\omega}\rho(\xi)\Theta_{h_0}(\xi)v_{h_0}(\xi)d\xi, \quad \forall v_{h_0}\in W_{h_0} \\ \Theta_{h_0}(\xi) \quad \text{is 1-periodic in } \xi, \quad \int\limits_{Q\cap\omega}\rho(\xi)|\Theta_{h_0}(\xi)|^2 d\xi = 1 \end{cases}$$

and

$$\Lambda^0_{h_0} = \inf_{v\in W_{h_0}} D(v) = \inf_{v\in W_{h_0}} \int\limits_{Q\cap\omega} a_{ij}(\xi)\frac{\partial v}{\partial\xi_j}\frac{\partial v}{\partial\xi_i}d\xi$$

We can easily obtain the following results:

Proposition 4.1 *Assume that $a_{ij}(\xi)$, $\rho(\xi)$ satisfy conditions $(A_2)-(A_3)$, and $a_{ij}(\xi), \rho(\xi)\in C(\overline{\Omega}^{\varepsilon})$, $\nabla_\xi a_{ij}(\xi)\in L^\infty(\Omega^\varepsilon)$, then holds*

$$\Lambda^0 \le \Lambda^0_{h_0} \le \Lambda^0 + Ch_0^2 \tag{4.4}$$

$$\|\Theta(\xi)-\Theta_{h_0}(\xi)\|_{0,Q\cap\omega} \le Ch_0^2\|\Theta\|_{2,Q\cap\omega} \tag{4.5}$$

4.2 FEM for calculating periodic functions $N_{\alpha_1\cdots\alpha_j}(\xi)$

Suppose that $\mathcal{J}^{h_0}=\{K\}$ is the same partition of $Q\cap\omega$ as in §4.1. Define a linear finite element space

(4.6)
$$V_{h_0} = \{v\in C(\overline{Q\cap\omega}) : v|_K\in P_1(K), \quad v|_{\partial Q}=0\}\subset H^1(Q\cap\omega,\partial Q)$$

Proposition 4.2 *Let $N_{\alpha_1\cdots\alpha_j}(\xi)$, $\alpha_j=1,2,\cdots n$, $j=1,\cdots,l$ be the weak solutions associated with problems (3.14), (3.16) and (3.17), respectively, and let $N^{h_0}_{\alpha_1\cdots\alpha_j}(\xi)$ be the corresponding FE solutions of $N_{\alpha_1\cdots\alpha_l}(\xi)$ in V_{h_0}. If $a_{ij}(\xi), \rho(\xi)\in C(\overline{\Omega}^{\varepsilon})$, $\nabla_\xi a_{ij}(\xi)\in L^\infty(\Omega^\varepsilon)$, then it holds*

$$\|N_{\alpha_1\cdots\alpha_l}-N^{h_0}_{\alpha_1\cdots\alpha_l}\|_{\widehat{V}^1(\omega)} \le Ch_0\Big(\sum_{j=1}^{l}\|N_{\alpha_1\cdots\alpha_j}\|_{2,Q\cap\omega}\Big) \tag{4.7}$$

where $C>0$ is independent of $h_0, \varepsilon, N_{\alpha_1\cdots\alpha_j}$, $j=1,\cdots,l$.

4.3 FEM for computing eigenvalues and eigenfunctions of the Homogenized helmholtz equation

In practice, we need to solve the modified homogenized Helmholtz equation as follows

(4.8)
$$\begin{cases} \widehat{\mathcal{L}}_{h_0}\tilde{u}^0(x) \equiv -\frac{\partial}{\partial x_i}(\hat{a}_{ij}^{h_0}\frac{\partial \tilde{u}^0(x)}{\partial x_j}) + \hat{b}^{h_0}\tilde{u}^0(x) = \tilde{\lambda}^0\hat{\rho}^{h_0}\tilde{u}^0(x), \quad \text{in } \Omega \\ \tilde{u}^0(x) = 0 \quad \text{on } \partial\Omega \end{cases}$$

where

$$\hat{a}_{ij}^{h_0} = \vartheta_{h_0}\int\limits_{Q\cap\omega} \Theta_{h_0}^2(\xi)(a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N_j^{h_0}(\xi)}{\partial \xi_k})d\xi \tag{4.9}$$

and

$$\hat{b}^{h_0} = \vartheta_{h_0}\int\limits_{Q\cap\omega} \Theta_{h_0}^2(\xi)b(\xi)d\xi, \quad \hat{\rho}^{h_0} = \vartheta_{h_0}\int\limits_{Q\cap\omega} \Theta_{h_0}^2(\xi)\rho(\xi)d\xi \tag{4.10}$$

Note that $\Theta_{h_0}(\xi)$, $N_j^{h_0}(\xi)$ are defined as in (4.3) and (4.7), respectively.

Proposition 4.3 *Suppose that mesh parameter h_0 is sufficiently small, then the coefficients $(\hat{a}_{ij}^{h_0})$ satisfy the following properties:*

$$\hat{a}_{ij}^{h_0} = \hat{a}_{ji}^{h_0} \tag{4.11}$$

$$\bar{\mu}_1\sum_{i=1}^{2}\eta_i^2 \le \sum_{i,j=1}^{2}\hat{a}_{ij}^{h_0}\eta_i\eta_j \le \bar{\mu}_2\sum_{i=1}^{2}\eta_i^2, \quad \forall(\eta_1,\eta_2)\in R^2 \tag{4.12}$$

where $\bar{\mu}_1, \bar{\mu}_2 > 0$ are constants independent of h_0.

Proof. One can directly verify that

$$\hat{a}_{ij}^{h_0} = \vartheta_{h_0}\int\limits_{Q\cap\omega} \Theta_{h_0}^2(\xi)\frac{\partial}{\partial \xi_l}(N_i^{h_0}(\xi) + \xi_i)a_{lm}(\xi)\frac{\partial}{\partial \xi_m}(N_j^{h_0}(\xi) + \xi_j)d\xi$$

The condition $a_{lm}(\xi) = a_{ml}(\xi)$ yields $\quad \hat{a}_{ij}^{h_0} = \hat{a}_{ji}^{h_0}$

Let $\quad \hat{a}_{ij}^{h_0} - \hat{a}_{ij} = \hat{r}_{ij} = \hat{r}_{ij}^{(1)} + \hat{r}_{ij}^{(2)}$, where

$$\hat{r}_{ij}^{(1)} = \vartheta\int\limits_{Q\cap\omega} \Theta^2(\xi)\frac{\partial}{\partial \xi_l}(N_i^{h_0}(\xi) - N_i(\xi))a_{lm}(\xi)\frac{\partial}{\partial \xi_m}(N_j^{h_0}(\xi) - N_j(\xi))d\xi$$

and

$$\hat{r}_{ij}^{(2)} = (\vartheta_{h_0} - \vartheta) \int\limits_{Q\cap\omega} \Theta_{h_0}^2(\xi)(a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N_j^{h_0}(\xi)}{\partial \xi_k})d\xi$$
$$+\vartheta \int\limits_{Q\cap\omega} (\Theta_{h_0}^2(\xi) - \Theta^2(\xi))(a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N_j^{h_0}(\xi)}{\partial \xi_k})d\xi$$

It follows from Proposition 4.1, 4.2, and Cauchy-Schwarz inequality that

$$\|\hat{r}_{ij}^{(1)}\|_F + \|\hat{r}_{ij}^{(2)}\|_F \le Ch_0^2\|N_i\|_{2,Q\cap\omega}\|N_j\|_{2,Q\cap\omega}$$

where $\|A\|_F$ denotes the Frobenius norm of a matrix A, and C is a constant independent of h_0.

Choosing a sufficiently small $h_0 > 0$ such that

$$Ch_0^2\|N_i\|_{2,Q\cap\omega}\|N_j\|_{2,Q\cap\omega} \le \frac{\hat{\mu}_1}{32},$$

then it holds

$$\frac{\hat{\mu}_1}{2}\sum_{i=1}^{2}\eta_i^2 \le \sum_{i,j=1}^{2}\hat{a}_{ij}^{h_0}\eta_i\eta_j = \sum_{i,j=1}^{2}\hat{a}_{ij}\eta_i\eta_j + \sum_{i,j=1}^{2}\hat{r}_{ij}\eta_i\eta_j \le (\hat{\mu}_2 + \frac{\hat{\mu}_1}{2})\sum_{i=1}^{2}\eta_i^2$$

where $\bar{\mu}_1 = \frac{\hat{\mu}_1}{2} > 0$, $\bar{\mu}_2 = \hat{\mu}_2 + \frac{\hat{\mu}_1}{2} > 0$ are independent of h_0. □

Next we will analyze the perturbation of the eigenvalues and eigenfunctions as computing numerically the coefficients of the homogenized differential operator.

Theorem 4.1 *Assume that* $(\lambda_k^{(0)}, u_k^0)$, *and* $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0)$, $\quad k = 1, 2\cdots$ *are the eigenvalues and eigenfunctions of the eigenvalue problems (3.24) and (4.8), respectively. Then it holds*

$$|\tilde{\lambda}_k^{(0)} - \lambda_k^{(0)}| \le C_k h_0^2 \|N_i\|_{2,Q\cap\omega}^2 \tag{4.13}$$

Moreover, if the multiplicity of the eigenvalue $\lambda_k^{(0)}$ *is equal to* t, *i.e.*

$$\lambda_{k-1}^{(0)} < \lambda_k^{(0)} = \cdots = \lambda_{k+t-1}^{(0)} < \lambda_{k+t}^{(0)}, \quad \lambda_0^{(0)} = 0$$

then

$$\|u_k^0 - \bar{u}_k^0\|_{L^2(\Omega)} \le C_k h_0^2 \|N_i\|_{2,Q\cap\omega}^2 \tag{4.14}$$

where $\bar{u}_k^0$ *is a linear combination of eigenfunctions of problem (4.8) corresponding to the eigenvalues* $\tilde{\lambda}_k^{(0)}, \cdots \tilde{\lambda}_{k+t-1}^{(0)}$.

The proof of Theorem 4.1 can be found in **Appendix C**.

By making use of the interior regularity estimates for PDEs, we can prove the following theorem without any difficulty (Cf. [13, 17]).

Theorem 4.2 *Under the assumptions of Theorem 4.1, assume that* $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$ *as shown in Figure 3.3, and* $u_k^0(x) \in H^{M+2}(\Omega')$, *then the following estimate holds:*

$$\|u_k^0(x) - \tilde{u}_k^0(x)\|_{s,\Omega_0} \leq Ch_0^2 \|N_i\|_{2,Q}^2 \|u_k^0\|_{M+2,\Omega'} \tag{4.15}$$

where $s = 0, 1, \cdots, M, k = 1, 2, \cdots, 1 \leq M \leq 4$

For simplicity, suppose $\Omega \subset R^2$ is a bounded smooth domain, $J^h = \{e\}$ is a regular family of subdivisions of Ω, and satisfies the following properties:

(F_1). The elements are uniform rectangles in the interior domain $\Omega_0 \subset\subset \Omega$;
(F_2). The elements are regular triangles in region $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, and the elements are (curved) triangles near the boundary $\partial\Omega$;
(F_3). Any face of any element e_1 is either a subset of the boundary $\partial\Omega$, or a face of another element e_2 in the subdivision.

Define a finite element space: $r \geq 1$

$$S_0^h(\Omega) = \{v \in C(\overline{\Omega}) : \quad v|_e \in \overline{P}_r(e), \quad v|_{\partial\Omega} = 0\} \subset H_0^1(\Omega) \tag{4.16}$$

where $\overline{P}_r = \begin{cases} Q_r, & e \quad \text{is a rectangle} \\ P_r, & e \quad \text{is a triangle} \end{cases}$, here we follow P.Ciarlet's notations of finite element spaces (see [5]).

The discrete variational formulation of the modified homogenized Helmholtz equation (4.8) is the following:

$$A(\tilde{u}_{k,h}^0, v_h) = \tilde{\lambda}_{k,h}^0 \hat{\rho}^{h_0}(\tilde{u}_{k,h}^0, v_h), \quad \forall v_h \in S_0^h(\Omega), \quad k = 1, 2, \cdots \tag{4.17}$$

where

$$A(u, v) = \int_\Omega \left(\hat{a}_{ij}^{h_0} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \hat{b}^{h_0} uv \right) dx \tag{4.18}$$

Next we will concentrate on discussing FE computations and the post-processing technique of the first eigenvalue and eigenfunction associated with problem (4.8). In practice, the proposed method in this paper is suitable for the computations of other eigenvalues and eigenfunctions.

To begin with, let us introduce some notation. Set $\|w\|_A^2 = A(w, w)$, where $A(\cdot, \cdot)$ as shown in (4.18). Define a Ritz-Galerkin projection operator $R_h : H_0^1(\Omega) \to S_0^h(\Omega)$ such that

$$A(u - R_h u, v_h) = 0, \quad u \in H_0^1(\Omega), \quad \forall v_h \in S_0^h(\Omega) \tag{4.19}$$

Proposition 4.4 *Assume that* $(\tilde{\lambda}_1^{(0)}, \tilde{u}_1^0)$ *and* $(\tilde{\lambda}_{1,h}^{(0)}, \tilde{u}_{1,h}^0)$ *are the first eigenvalues and eigenfunctions of problems (4.8) and (4.17), respectively. Then the following relations hold*

$$0 \le \frac{A(w,w)}{(w,w)} - \tilde{\lambda}_1^{(0)} \le \frac{\|w - \tilde{u}_1^0\|_A^2}{(w,w)}, \quad \forall w \in H_0^1(\Omega) \tag{4.20}$$

$$0 < \tilde{\lambda}_1^{(0)} \le \frac{a(\tilde{u}_{1,h}^0, \tilde{u}_{1,h}^0)}{(\tilde{u}_{1,h}^0, \tilde{u}_{1,h}^0)} = \tilde{\lambda}_{1,h}^{(0)} \le \frac{a(v,v)}{(v,v)}, \quad \forall v \in S_0^h(\Omega) \tag{4.21}$$

and

$$0 \le \tilde{\lambda}_{1,h}^{(0)} - \tilde{\lambda}_1^{(0)} \le \frac{a(R_h\tilde{u}_1^0, R_h\tilde{u}_1^0)}{(R_h\tilde{u}_1^0, R_h\tilde{u}_1^0)} - \tilde{\lambda}_1^{(0)} \le \frac{\|R_h\tilde{u}_1^0 - \tilde{u}_1^0\|_b^2}{(R_h\tilde{u}_1^0, R_h\tilde{u}_1^0)} \tag{4.22}$$

Proof. (4.21) is a straightforward consequence of this relation $S_0^h(\Omega) \subset H_0^1(\Omega)$. It remains to give the proofs of (4.20) and (4.22).

Remark that $\|w\|_A^2 = \tilde{\lambda}_1^{(0)}(w, \tilde{u}_1^0)^2 + \|w - (w, \tilde{u}_1^0)\tilde{u}_1^0\|_A^2$, hence we have

$$\tilde{\lambda}_1^{(0)} = \min_{w \in H_0^1(\Omega),\ w \ne 0} \frac{\|w\|_A^2}{\|w\|_0^2} \le \frac{\|w\|_A^2}{\|w\|_0^2} \le \tilde{\lambda}_1^{(0)} + \frac{\|w - (w, \tilde{u}_1^0)\tilde{u}_1^0\|_A^2}{\|w\|_0^2} \tag{4.23}$$

On the other hand, one can directly verify that

$$A(w - (w, \tilde{u}_1^0)\tilde{u}_1^0, \alpha\tilde{u}_1^0) = 0, \quad \forall \alpha \in R$$

Consequently

$$\|w - \tilde{u}_1^0\|_A^2 = \|w - (w, \tilde{u}_1^0)\tilde{u}_1^0\|_A^2 + \|(w, \tilde{u}_1^0)\tilde{u}_1^0 - \tilde{u}_1^0\|_A^2 \ge \|w - (w, \tilde{u}_1^0)\tilde{u}_1^0\|_A^2$$

From (4.23), one gets

$$0 \le \frac{A(w,w)}{(w,w)} - \tilde{\lambda}_1^{(0)} \le \frac{\|w - (w, \tilde{u}_1^0)\tilde{u}_1^0\|_A^2}{\|w\|_0^2} \le \frac{\|w - \tilde{u}_1^0\|_A^2}{\|w\|_0^2}$$

Putting $w = R_h\tilde{u}_1^0$ into (4.20), and recalling (4.21), one derives

$$0 \le \tilde{\lambda}_{1,h}^{(0)} - \tilde{\lambda}_1^{(0)} \le \frac{\|R_h\tilde{u}_1^0 - \tilde{u}_1^0\|_A^2}{(R_h\tilde{u}_1^0, R_h\tilde{u}_1^0)}$$

Theorem 4.3 *Under the assumptions of Theorem 4.2. Suppose* $\tilde{u}_1^0 \in H^{r+1}(\Omega)$, *then the following estimate holds*

$$0 \le \tilde{\lambda}_{1,h}^{(0)} - \tilde{\lambda}_1^{(0)} \le Ch^{2r} \tag{4.24}$$

Proof. It is well known that

$$\|R_h\tilde{u}_1^0 - \tilde{u}_1^0\|_0 \le Ch^2\|\tilde{u}_1^0\|_2$$
$$\|R_h\tilde{u}_1^0 - \tilde{u}_1^0\|_A^2 \le Ch^{2r}\|\tilde{u}_1^0\|_{r+1}^2$$

Hence we can choose a sufficiently small $h > 0$ such that

$$\|R_h\tilde{u}_1^0\|_0 \ge \|\tilde{u}_1^0\|_0 - Ch^2\|\tilde{u}_1^0\|_2 \ge \frac{1}{2}$$

The use of inequality of (4.22) gives

$$0 \le \tilde{\lambda}_{1,h}^{(0)} - \tilde{\lambda}_1^{(0)} \le Ch^{2r}\|\tilde{u}_1^0\|_{r+1}^2$$

To implementing post-processing technique, let us recall some superconvergence results for computing the first eigenfunction of (4.17).

For convenience, set $\lambda = \tilde{\lambda}_1^{(0)}, \quad \lambda_h = \tilde{\lambda}_{1,h}^{(0)}, \quad u_h = \tilde{u}_{1,h}^0$, and assume that H_λ is the eigenspace of the operator $\widehat{\mathcal{L}}_{h_0}$ with respect to eigenvalue $\lambda = \tilde{\lambda}_1^{(0)}$. Define a projection operator $P : L^2 \to H_\lambda$, such that

$$Pu = \sum_{i=1}^{l}(u, u_i)u_i \tag{4.25}$$

where $u_i, i = 1, \cdots l$, form a set of orthonormal basis of H_λ.

Let K be the inverse operator of $\widehat{\mathcal{L}}_{h_0}$, then K is a bounded self-adjoint compact operator due to Proposition 4.3.

Lemma 4.1[20] *Let $R_h : H_0^1(\Omega) \to S_0^h(\Omega)$ be the Ritz-Galerkin projection operator. Then we have, for $q_0 > 2,\ 1 < q < q_0$*

$$\|\lambda_h R_h K - \lambda K\|_\infty \to 0, \quad \text{as } h \to 0 \tag{4.26}$$

$$\|R_h K\|_\infty \le C \tag{4.27}$$

Lemma 4.2[20] *Let $u = Pu_h \in H_\lambda \subset W^{r+1,q}(\Omega), \quad q > 2$. Then the following estimates hold*

$$\|u\|_{r+1,q} \le C \tag{4.28}$$

$$\|R_h K(I - R_h)u\|_\infty \le Ch^{r+2}, \quad (r \ge 2) \tag{4.29}$$

Proposition 4.5 *Let (λ, H_λ) be the solution of problem (4.8) and (λ_h, V_λ) be the solution of problem (4.17), and $V_\lambda \subset S_0^h(\Omega)$. If $H_\lambda \subset W^{r+1,q} \cap H_0^1(\Omega)$, $q > 2$, then there exists $u \in H_\lambda$ such that*

$$\|R_h u - u_h\|_{0,\infty} \le Ch^{r+2}, \quad (r \ge 2) \tag{4.30}$$

where $u_h \in V_\lambda$.

Proof. Set $u = Pu_h = \sum_{j=1}^{l}(u_h, u_i)u_i$, and $\bar{u} = R_h u - u_h - P(R_h u - u_h)$

Observe that $(\bar{u}, u) = 0$, for any $u \in H_\lambda$, and hence $\bar{u} \in H_\lambda^\perp$.

It follows from Fredholm's alternative theorem that the operator $(I - \lambda K)$ has a bounded inverse operator. Consequently, there exists a constant $\delta > 0$ such that

(4.31)
$$\begin{aligned}\delta\|\bar{u}\|_{0,\infty} &\le \|(I-\lambda K)\bar{u}\|_{0,\infty}\\ &= \|(I-\lambda K)(R_h u - u_h)\|_{0,\infty}\\ &= \|\lambda R_h K(I-R_h)u + (\lambda_h R_h K - \lambda K)(R_h u - u_h)\\ &\quad +(\lambda-\lambda_h)R_h K R_h u\|_{0,\infty}, \quad (\text{since } u_h = \lambda_h R_h K u_h,\ u = \lambda K u)\\ &\le \lambda\|R_h K(I-R_h)u\|_{0,\infty} + \|\lambda_h R_h K - \lambda K\|_\infty \|R_h u - u_h\|_{0,\infty} + Ch^{2r}\end{aligned}$$

Hence due to $Pu - Pu_h = 0$, one derives

$$\begin{aligned}\|P(R_h u - u_h)\|_{0,\infty} &= \|P(R_h u - u)\|_{0,\infty}\\ &= \|\sum_{j=1}^{l}(R_h u - u, u_j)u_j\|_{0,\infty}\\ &\le \sum_{j=1}^{l}|(R_h u - u, u_j)|\|u_j\|_{0,\infty}\\ &= \sum_{j=1}^{l}\tfrac{1}{\lambda}|a(R_h u - u, u_j - u_j^I)|\|u_j\|_{0,\infty}\\ &\le Ch^{2r}\sum_{j=1}^{l}\|u\|_{r+1}\|u_j\|_{r+1}\|u_j\|_{0,\infty} \le Ch^{2r}\|u\|_{r+1}\end{aligned}$$

The use of the triangle inequality gives

(4.32)
$$\|R_h u - u_h\|_{0,\infty} \le \|\bar{u}\|_{0,\infty} + \|P(R_h u - u_h)\|_{0,\infty} \le \|\bar{u}\|_{0,\infty} + Ch^{2r}$$

Putting (4.31) into (4.32), one obtains

$$\begin{aligned}&(1 - \tfrac{1}{\delta}\|\lambda_h R_h K - \lambda K\|_\infty)\|R_h u - u_h\|_{0,\infty}\\ &\quad\le \tfrac{1}{\delta}\|R_h K(I-R_h)u\|_{0,\infty} + Ch^{2r}\|u\|_{r+1}\end{aligned}$$

Using Lemma 4.1 and Lemma 4.2, and choosing a sufficiently small $h > 0$, one gets

$$\frac{1}{2}\|R_h u - u_h\|_{0,\infty} \le Ch^{r+2}\|u\|_{r+1,q} \tag{4.33}$$

Let now show how the above superconvergence results can be applied to implement post-processing technique of $D^\alpha \tilde{u}_1^0(x)$, where $\tilde{u}_1^0$ is the first eigenfunction associated with problem (4.8).

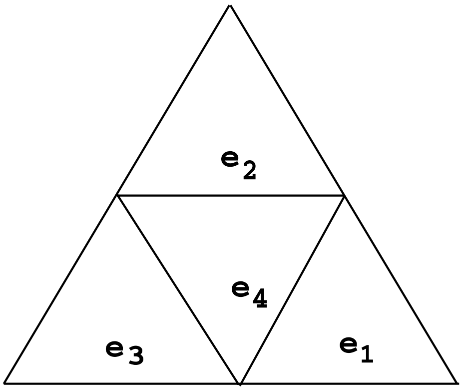

Fig. 4.1. Triangular mesh

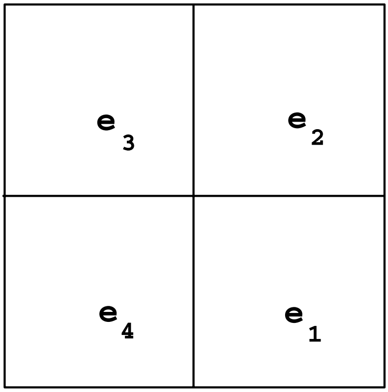

Fig. 4.2. Rectangular mesh

By using the nodal values of the bi-r-th (r-th)finite element solution, we can construct a bi-2r-th (2r-th) interpolation function at a new larger element with respect to a coarse mesh as shown in Figure 4.1 and 4.2, which is called as the interpolated FEM by Lin Qun et al., see [20]. Denote by $\mathcal{I}_{2h}^{(2r)}$ the bi-2r-th (2r-th) order interpolation operator.

Lemma 4.3[20] *Let* $\mathcal{I}_h : H^1(\Omega) \to S_0^h(\Omega)$ *be a usual Lagrange's interpolation operator. Then the interpolation operators* $\mathcal{I}_h$ *and* $\mathcal{I}_{2h}^{(2r)}$ *satisfy the following properties:*

$$\|\mathcal{I}_{2h}^{(2r)} u\|_{m,p} \leq C\|u\|_{m,p} \quad 1 \leq p \leq \infty,\ m = 0, 1,\ \forall u \in S^h(\Omega_0) \tag{4.34}$$

where $C > 0$ *does depend on* r, p *but is independent of* u, h.

$$(\mathcal{I}_{2h}^{(2r)})^2 = \mathcal{I}_{2h}^{(2r)}, \quad \mathcal{I}_{2h}^{(2r)}\mathcal{I}_h = \mathcal{I}_{2h}^{(2r)}, \quad \mathcal{I}_h\mathcal{I}_{2h}^{(2r)} = \mathcal{I}_h$$

$$\forall P_i \in T_0^h,\ \mathcal{I}_{2h}^{(2r)}u(P_i) = \mathcal{I}_h u(P_i) = u(P_i), \quad u \in C(\bar{\Omega}_0)$$

where T_0^h *is the set of nodal points of* J^h *restricted to* $\overline{\Omega}_0$

$$\|u - \mathcal{I}_{2h}^{(2r)}u\|_{m,p,E} \leq Ch^{2r+1-m}\|u\|_{2r+1,p,E} \tag{4.35}$$

$$\forall u \in W^{2r+1,p}(E), \quad m = 0, 1,\ 1 \leq p \leq +\infty, \forall E \in J^{2h}|_{\Omega_0},$$

Theorem 4.4 *Assume that* $(\tilde{\lambda}_1^{(0)}, \tilde{u}_1^0(x))$ *are the first eigenvalue and eigenfunction of problem (4.8), respectively, and* $(\tilde{\lambda}_{1,h}^{(0)}, \tilde{u}_{1,h}^0(x))$ *are the corresponding FE solutions of* $(\tilde{\lambda}_1^{(0)}, \tilde{u}_1^0(x))$ *in* $S_0^h(\Omega)$. *Let the partition* J^h *satisfy the above conditions* $(F_1) - (F_2)$*, and,* $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$. *Then the following error estimate holds:*

$$\|\tilde{u}_1^0(x) - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{1,h}^0(x)\|_{0,\Omega_0} + h\|\tilde{u}_1^0(x) - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{1,h}^0(x)\|_{1,\Omega_0} \leq Ch^{r+2} \tag{4.36}$$

where $C > 0$ *is independent of* $h, h_0, r \geq 2$

Proof. By using Lemma 4.3, Proposition 4.5, and recalling the inverse inequality in a finite element space(Cf.[5]), one derives

$$\begin{aligned}
&\|\mathcal{I}_{2h}^{(2r)}\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{1,h}^0\|_{1,\Omega_0} \\
&\quad = \|\mathcal{I}_{2h}^{(2r)}(\mathcal{I}_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0)\|_{1,\Omega_0} \\
&\quad \leq C\|\mathcal{I}_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{1,\Omega_0} \\
&\quad \leq C\|I_h\tilde{u}_1^0 - R_h\tilde{u}_1^0\|_{1,\Omega_0} + \|R_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{1,\Omega_0} \\
&\quad \leq Ch^{r+1}\|\tilde{u}^0\|_{r+2,\Omega_0} + C\|\tilde{u}^0 - \tilde{u}_h^0\|_{-s,\Omega_1} + Ch^{-1}\|R_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{0,\Omega_0} \\
&\quad \leq Ch^{r+1}\|\tilde{u}_1^0\|_{r+2,\Omega'}, \qquad s \geq 0
\end{aligned}$$

and consequently

$$\begin{aligned}
\|\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{1,h}^0\|_{1,\Omega_0} &\leq \|\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2r)}\tilde{u}_1^0\|_{1,\Omega_0} + \|\mathcal{I}_{2h}^{(2r)}\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{1,h}^0\|_{1,\Omega_0} \\
&\leq Ch^{r+1}\|\tilde{u}_1^0\|_{r+2,\Omega'}
\end{aligned}$$

The remainder can be completed similarly. □

4.4 FEM for solving boundary layer equation

In practice, we need to solve the modified boundary value problem as follows,

(4.37)

$$\begin{cases}
-\frac{\partial}{\partial x_i}(\Theta_{h_0}^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial \tilde{w}_k^\varepsilon(x)}{\partial x_j}) + \Theta_{h_0}^2(\frac{x}{\varepsilon})(b(\frac{x}{\varepsilon}) - \tilde{\lambda}_{k,h}^{(0)}\rho(\frac{x}{\varepsilon}))\tilde{w}_k^\varepsilon(x) = 0, \quad x \in \Omega_1^\varepsilon \\
\tilde{w}_k^\varepsilon(x) = 0, \quad x \in \Gamma_\varepsilon \\
\tilde{w}_k^\varepsilon(x) = \tilde{u}_{k,h}^0(x) \quad x \in \partial\Omega_0 \cap \partial\Omega_1, \quad k = 1, 2, \cdots
\end{cases}$$

where $\Theta_{h_0}(\xi)$ is FE solution of $\Theta(\xi)$ in W_{h_0}, $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0(x)$ is FE solution of $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ in $S_0^h(\Omega)$.

Let $\mathcal{F}^{h_1} = \{e\}$ be a regular family of triangulations of subdomain $\Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0$ as shown in Figure 3.4, where $h_1 = \max_{e\in\mathcal{F}^{h_1}}\{h_e\}, \quad 0 < \frac{h_1}{\varepsilon^2} << 1.$

Define a linear finite element space:

$$S_{h_1}^\varepsilon(\Omega_1^\varepsilon) = \{v \in C(\overline{\Omega}_1^\varepsilon): \quad v|_e \in P_1(e), \quad v|_{\Gamma_\varepsilon \cup \partial\Omega_0} = 0\} \tag{4.38}$$

From (3.28), respectively (4.37), one can prove that

(4.39)

$$\begin{cases}
-\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial(w_k^\varepsilon - \tilde{w}_k^\varepsilon)}{\partial x_j}) + \Theta^2(\frac{x}{\varepsilon})(b(\frac{x}{\varepsilon}) - \lambda_k^{(0)}\rho(\frac{x}{\varepsilon}))(w_k^\varepsilon - \tilde{w}_k^\varepsilon)(x) \\
\qquad = \frac{\partial}{\partial x_i}(\Theta^2 - \Theta_{h_0}^2)\frac{\partial \tilde{w}_k^\varepsilon}{\partial x_j} - (\Theta^2 - \Theta_{h_0}^2)(b(\frac{x}{\varepsilon}) - \tilde{\lambda}_k^{(0)}\rho(\frac{x}{\varepsilon}))\tilde{w}_k^\varepsilon \\
\qquad\quad +(\Theta^2 - \Theta_{h_0}^2)\lambda_k^{(0)}\rho(\frac{x}{\varepsilon})\tilde{w}_k^\varepsilon(x) + \Theta_{h_0}^2(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})(\lambda_k^{(0)} - \tilde{\lambda}_{k,h}^{(0)})\tilde{w}_k^\varepsilon, \quad x \in \Omega_1^\varepsilon \\
w_k^\varepsilon - \tilde{w}_k^\varepsilon(x) = 0, \quad x \in \Gamma_\varepsilon \\
w_k^\varepsilon - \tilde{w}_k^\varepsilon(x) = u_k^0(x) - \tilde{u}_{k,h}^0(x) \quad x \in \partial\Omega_0 \cap \partial\Omega_1
\end{cases}$$

Theorem 4.5 *Suppose that $w_k^\varepsilon \in H^1(\Omega_1^\varepsilon, \Gamma_\varepsilon)$ solves the variational equation (3.28) with $(\lambda_k^{(0)}, u_k^0(x)$ defined as (3.24), and $\tilde{w}_k^\varepsilon \in H^1(\Omega_1^\varepsilon, \Gamma_\varepsilon)$ solves the variational equation (4.37) with $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0)$ defined as in (4.17). Let $\tilde{w}_{k,h_1}^\varepsilon(x)$ be the FE solution of $\tilde{w}_k^\varepsilon(x)$ in $S_{h_1}^\varepsilon(\Omega_1)$. Under the assumptions of Theorem 3.3, then we have the following estimate:*

(4.40)

$$\|w_k^\varepsilon(x) - \tilde{w}_{k,h_1}^\varepsilon(x)\|_{V^{1,p}(\Omega_1^\varepsilon)} \le C\Big\{(\frac{h_1}{\varepsilon^2}) + h_0 + h^r\Big\}, \quad 1 < p \le p_0 < +\infty$$

where C is constant independent of ε, h_0, h, h_1; and h_0, h, h_1 are the mesh sizes associated with $Q \cap \omega, \Omega, \Omega_1^\varepsilon$, respectively. $r \ge 2$ is the degree of piecewise polynomials of $S_0^h(\Omega)$ defined as in (4.16).

Proof. By Theorem 3.3, we have

$$\begin{aligned} \|\tilde{w}_k^\varepsilon(x) - \tilde{w}_{k,h_1}^\varepsilon(x)\|_{V^{1,p}(\Omega_1^\varepsilon)} &\le Ch_1\|\tilde{w}_k^\varepsilon\|_{V^{2,p}(\Omega_1^\varepsilon)} \\ &\le C(\frac{h_1}{\varepsilon^2})\|\tilde{u}_{k,h}^0\|_{2,p,\Omega} \le C(\frac{h_1}{\varepsilon^2})\|\tilde{u}_k^0\|_{2,p,\Omega} \end{aligned} \tag{4.41}$$

On the other hand, by using (4.37), Proposition 4.1, Theorem 4.1 and Theorem 4.3, one gets

$$\|w_k^\varepsilon(x) - \tilde{w}_k^\varepsilon(x)\|_{V_0^1(\Omega_1^\varepsilon)} \le C\{h^r + h_0\} \tag{4.42}$$

Combining (4.41) with (4.42), and using the triangle inequality, we can complete the proof of (4.40). □

5 Multiscale finite element method

To begin with, let us introduce the first-order difference quotient as follows (Cf. [3]):

$$\delta_{x_i}\tilde{u}_{k,h}^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} [\frac{\partial \tilde{u}_{k,h}^0}{\partial x_i}]_e(N_p) \tag{5.1}$$

where $\sigma(N_p)$ denotes the set of elements with node N_p; $\tau(N_p)$ is the number of elements of $\sigma(N_p)$; $\tilde{u}_{k,h}^0(x)$ denotes the FE solution of $\tilde{u}_k^0(x)$ in $S_0^h(\Omega)$; $[\frac{\partial \tilde{u}_{k,h}^0}{\partial x_i}]_e(N_p)$ is the value of the derivative $\frac{\partial \tilde{u}_{k,h}^0}{\partial x_i}$ at node N_p relative to element e.

Similarly, define any higher-order difference quotients as follows:

(5.2)

$$\delta_{x_{\alpha_1},\cdots x_{\alpha_l}}^l \tilde{u}_{k,h}^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} [\sum_{j=1}^d \delta_{x_{\alpha_1},\cdots x_{\alpha_{l-1}}}^{l-1} \tilde{u}_{k,h}^0(P_j)\frac{\partial \psi_j}{\partial x_{\alpha_l}}]_e(N_p)$$

where d is the number of nodes in e, P_j are the nodes of e, $\psi_j(x)$ are Lagrange's shape functions, $j = 1, 2 \cdots d$

Therefore, the multi-scale finite element computing formulation can be written as follows:

(5.3)
$$\tilde{u}^{\varepsilon,M,h_0}_{k,h,h_1}(N_p) = \begin{cases} \tilde{u}^0_{k,h}(N_p) + \sum_{l=1}^{M} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=l}^{n} N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi(N_p))\delta^l_{x_{\alpha_1}\cdots x_{\alpha_l}}\tilde{u}^0_{k,h}(N_p), \ N_p \in \overline{\Omega}^\varepsilon_0 \\ \tilde{w}^\varepsilon_{k,h_1}(N_p), \quad N_p \in \Omega^\varepsilon_1 \end{cases}$$

(5.4)
$$U^{\varepsilon,M,h_0}_{k,h,h_1}(N_p) = \Theta_{h_0}(\xi(N_p))\tilde{u}^{\varepsilon,M,h_0}_{k,h,h_1}(N_p)$$

where the integer $2 \leq M \leq 4$, $h_0,\ h,\ h_1$ are the mesh parameters of $Q \cap \omega, \Omega, \Omega^\varepsilon_1$, respectively.

To improve computing accuracy for $\tilde{u}^{\varepsilon,M,h_0}_{k,h,h_1}$, we will make use of the post-processing technique (see Theorem 4.4).

(5.5)
$$\mathcal{P}\tilde{u}^{\varepsilon,M,h_0}_{k,h,h_1}(x) = \begin{cases} \mathcal{I}^{(2r)}_{2h}\tilde{u}^0_{k,h}(x) + \sum_{l=1}^{M} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi)\delta^l_{x_{\alpha_1}\cdots x_{\alpha_l}}\mathcal{I}^{(2r)}_{2h}\tilde{u}^0_{k,h}(x), \ x \in \overline{\Omega}^\varepsilon_0 \\ \tilde{w}^\varepsilon_{k,h_1}(x), \quad x \in \Omega^\varepsilon_1 \end{cases}$$

Let us recall that $I^{(2r)}_{2h}$ denotes the bi-2r-th (2r-th) order interpolation operator.

(5.6)
$$\mathcal{P}U^{\varepsilon,M,h_0}_{k,h,h_1}(x) = \Theta_{h_0}(\xi) \cdot \mathcal{P}\tilde{u}^{\varepsilon,M,h_0}_{k,h,h_1}(x)$$

Finally, let us give some convergence results for the eigenvalues and eigenfunctions.

Theorem 5.1 *Let* $(\Lambda^\varepsilon_k, U^\varepsilon_k(x))$, $(\lambda^\varepsilon_k, u^\varepsilon_k(x))$, $(\lambda^{(0)}_k, u^0_k(x))$, $(\tilde{\lambda}^{(0)}_k, \tilde{u}^0_k(x))$ *be the eigenpairs of problems (1.1), (1.7), (3.24) and (4.8), respectively; and let* $(\Lambda^0, \Theta(\xi))$ *be the first eigenpair of problem (1.5), and* $w^\varepsilon_k(x)$ *and* $\tilde{w}^\varepsilon_k(x)$ *be solutions of boundary layer equations (3.28) and (4.37) corresponding to* $(\lambda^{(0)}_k, u^0_k(x))$ *and* $(\tilde{\lambda}^{(0)}_{k,h}, \tilde{u}^0_{k,h}(x))$*, respectively. Let* $N_{\alpha_1\cdots\alpha_l}(\xi)$, $\alpha_j = 1, 2, \cdots,\ 1 \leq j \leq l,\ l \geq 1$ *be periodic functions defined in unit cell* $Q \cap \omega$*. Let* $(\tilde{\lambda}^{(0)}_{k,h}, \tilde{u}^0_{k,h})$, $(\Lambda^0_{h_0}, \Theta^0_{h_0}(\xi))$, $N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi)$, *and* $\tilde{w}^\varepsilon_{k,h_1}(x)$ *be the FE solutions associated with* $(\tilde{\lambda}^{(0)}_k, \tilde{u}^0_k)$, $(\Lambda^0, \Theta^0(\xi))$, $N_{\alpha_1\cdots\alpha_l}(\xi)$, *and* $\tilde{w}^\varepsilon_k(x)$*, respectively. Under the assumptions of Theorem 3.4, then the following error estimates hold:*

$$\Lambda^\varepsilon_k = \varepsilon^{-2}\Lambda^0 + \lambda^\varepsilon_k$$

(5.7)
$$|\lambda^\varepsilon_k - \tilde{\lambda}^{(0)}_{k,h}| \leq \begin{cases} C \cdot \left(\varepsilon + h_0 + h^{2r}\right), & \text{if} \quad M = 0, 1 \\ C \cdot \left\{min(\varepsilon, \gamma_M) + h_0 + h^{2r}\right\}, & \text{if} \quad 2 \leq M \leq 4, \end{cases}$$

$$(5.8)\quad \|u_k^\varepsilon(x) - \mathcal{P}\tilde{u}_{k,h,h_1}^{\varepsilon,M,h_0}(x)\|_{V^1(\Omega_0^\varepsilon)} \leq \begin{cases} C \cdot \Big\{\varepsilon^{1/2} + h_0 + h^{2r-M}\Big\}, & \text{if} \quad M = 0,1 \\ C \cdot \Big\{min(\varepsilon^{1/2}, \gamma_M^{1/2}) + h_0 + h^{2r-M}\Big\}, & \text{if} \quad 2 \leq M \leq 4, \end{cases}$$

$$(5.9)\quad \|U_k^\varepsilon(x) - \mathcal{P}U_{k,h,h_1}^{\varepsilon,M,h_0}(x)\|_{H^1(\Omega_0^\varepsilon)} \leq \begin{cases} C \cdot \Big\{\varepsilon^{1/2} + h_0 + h^{2r-M}\Big\}, & \text{if} \quad M = 0,1 \\ C \cdot \Big\{min(\varepsilon^{1/2}, \gamma_M^{1/2}) + h_0 + h^{2r-M}\Big\}, & \text{if} \quad 2 \leq M \leq 4, \end{cases}$$

$$(5.10)\quad \|U_k^\varepsilon(x) - \Theta_{h_0}(\xi)\tilde{w}_{k,h_1}^\varepsilon(x)\|_{1,p,\Omega_1^\varepsilon} \leq \begin{cases} C \cdot \Big\{\varepsilon^{1/2} + h_0 + h^r + \Big(\frac{h_1}{\varepsilon^2}\Big)\Big\}, & \text{if} \quad M = 0,1 \\ C \cdot \Big\{min(\varepsilon^{1/2}, \gamma_M^{1/2}) + h_0 + h^r + \frac{h_1}{\varepsilon^2}\Big)\Big\}, & \text{if} \quad 2 \leq M \leq 4, \end{cases}$$

where $\gamma_M = \Big\{\varepsilon^{M-1} + \Big(\int_{\partial\Omega_0\cap\partial\Omega_1}\Big([\sigma_\varepsilon(\tilde{u}_k^{\varepsilon,M})]\Big)^2 d\Gamma\Big)^{1/2}\Big\}, 2 \leq M \leq 4$*, and* $1 < p \leq p_0 < +\infty, C > 0$ *is a constant independent of* ε, h_0, h, h_1; *and* $\Omega_0 \subset\subset \Omega$ *is the union of periodic cells,* $\Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0$*,* $r \geq 2$ *denotes the degree of piecewise polynomials in* $S_0^h(\Omega)$ *defined as in (4.16), and* $h_0, h,$ h_1 *are the mesh parameters of* $Q \cap \omega$*,* $\Omega, \Omega_1^\varepsilon$*, respectively,* $2r \geq M + 1$*,* $0 < h_1 << \varepsilon^2, \quad k = 1, 2, \cdots$

Proof. For $x \in \overline{\Omega}_0^\varepsilon$, from (3.61), (4.14), (4.36), (4.7), (4.15) and (4.36), we obtain

$$(5.11)\quad \begin{aligned} u_k^\varepsilon(x) - \mathcal{P}\tilde{u}_{k,h,h_1}^{\varepsilon,M,h_0}(x) &= u_k^\varepsilon(x) - u_k^{\varepsilon,M}(x) + u_k^{\varepsilon,M}(x) - \mathcal{P}\tilde{u}_{k,h,h_1}^{\varepsilon,M,h_0}(x) \\ &= u_k^\varepsilon(x) - u_k^{\varepsilon,M}(x) + u_k^0(x) - \tilde{u}_k^0(x) + \tilde{u}_k^0(x) - \mathcal{I}_{2h}^{(2r)}\tilde{u}_{k,h}^0(x) \\ &\quad + \sum_{l=1}^M \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^n (N_{\alpha_1\cdots\alpha_l}(\xi) - N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi))D^\alpha u_k^0(x) \\ &\quad + \sum_{l=1}^M \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^n N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)D^\alpha(u_k^0(x) - \tilde{u}_k^0(x)) \\ &\quad + \sum_{l=1}^M \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^n N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)(D^\alpha\tilde{u}_k^0(x) - \delta_{x_{\alpha_1}\cdots x_{\alpha_l}}^l \mathcal{I}_{2h}^{(2r)}\tilde{u}_{k,h}^0(x)) \end{aligned}$$

and consequently

$$\|u_k^\varepsilon(x) - \mathcal{P}\tilde{u}_{k,h,h_1}^{\varepsilon,M,h_0}(x)\|_{V^1(\Omega_0^\varepsilon)} \leq \begin{cases} C \cdot \{\varepsilon^{1/2} + h_0 + h^{2r-M}\}, & \text{if } M = 0,1 \\ C \cdot \{min(\varepsilon^{1/2}, \gamma_M^{1/2}) + h_0 + h^{2r-M}, & \text{if } 2 \leq M \leq 4 \end{cases}$$

On the other hand, for $x \in \Omega_1^\varepsilon$, we have

$$u_k^\varepsilon(x) - \tilde{w}_{k,h_1}^\varepsilon(x) = u_k^\varepsilon(x) - w_k^\varepsilon(x) + w_k^\varepsilon(x) - \tilde{w}_{k,h_1}^\varepsilon(x) \tag{5.12}$$

It follows from (3.45) and (4.40) that

$$\|u_k^\varepsilon(x) - \tilde{w}_{k,h_1}^\varepsilon(x)\|_{V^{1,p}(\Omega_1^\varepsilon)} \leq \begin{cases} C \cdot \left\{\varepsilon^{1/2} + h_0 + h^r + \left(\frac{h_1}{\varepsilon^2}\right)\right\}, & \text{if} \quad M = 0, 1 \\ C \cdot \left\{min(\varepsilon^{1/2}, \gamma_M^{1/2}) + h_0 + h^r + \frac{h_1}{\varepsilon^2}\right)\right\}, & \text{if} \quad 2 \leq M \leq 4, \end{cases}$$

where $\Omega_1^\varepsilon = \Omega^\varepsilon \setminus \overline{\Omega}_0$

To summarize the above results, we can complete the proof of Theorem 5.1. □

6 Numerical results

In this section, let us show some numerical results of the first eigenvalue and eigenfunction.

Example 6.1 Consider the Helmholtz equation as follows:

(6.1)
$$\begin{cases} \mathcal{L}_\varepsilon U^\varepsilon(x) = -\frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon})\frac{\partial U^\varepsilon(x)}{\partial x_j}) + b(\frac{x}{\varepsilon})U^\varepsilon(x) = \Lambda^\varepsilon \rho(\frac{x}{\varepsilon})U^\varepsilon(x), & \text{in} \quad \Omega^\varepsilon \\ U^\varepsilon(x) = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \end{cases}$$

where $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$ as shown in Figure 6.1, the periodicity cell $Q \cap \omega$ as shown in Figure 6.2, $\partial\omega \cap Q$ denotes the surface of cavities restricted to $Q \cap \omega$, $\varepsilon = \mathbf{\frac{1}{12}}$

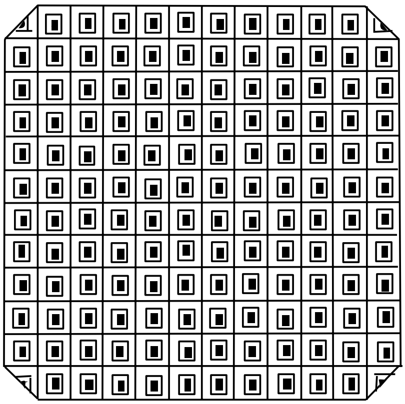

Fig. 6.1. Domain Ω^ε

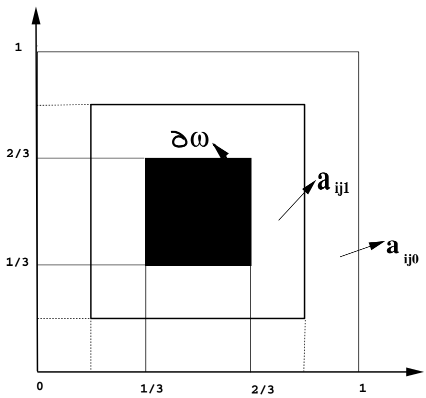

Fig. 6.2. Unit cell $Q \cap \omega$

Since it is impossible to find the first eigenvalue and eigenfunction by using analytic method, we have to replace $(\Lambda^\varepsilon, U^\varepsilon(x))$ with their FE solutions in a very refined mesh. For solving directly the original problem, we need to implement the triangular subdivision of Ω^ε, which is such that the discontinuities of the coefficients a_{ij} coincide with sides of the triangulations (which needed 37696 triangles and consequently was longer and more expensive)

Here we use the subspace iterative method for computing numerically the first eigenvalue and eigenfunction.

In (6.1), assume that $b(\frac{x}{\varepsilon}) = 0, \quad \rho(\frac{x}{\varepsilon}) = 1$

Case 1: $a_{ij0} = \delta_{ij}, \quad a_{ij1} = 20.0\delta_{ij};$
Case 2: $a_{ij0} = \delta_{ij}, \quad a_{ij1} = 0.002\delta_{ij};$
Case 3: $a_{ij0} = \delta_{ij}, \quad a_{ij1} = 0.004\delta_{ij};$
Case 4: $a_{ij0} = a_{ij1} = \delta_{ij};$
Case 5: $a_{ij0} = \delta_{ij}, \quad a_{ij1} = 114.0\delta_{ij};$
Case 6: $a_{11} = a_{22} = \dfrac{1}{4 + (sin(2\pi x/\varepsilon) + sin(2\pi y/\varepsilon))}, a_{12} = a_{21} = 0,$
$\varepsilon = \dfrac{1}{12}$

Note that $\delta_{ij} = 1$, **if** $i = j$; $\delta_{ij} = 0$, **if** $i \neq j$.

It is interesting to compare with computational amount shown in Table 1.

Notice that symbols I, II, III, IV denote different partitions over the domain Ω for solving numerically the homogenized Helmholtz equation, respectively.

Remark 6.1 Let us have a glance at Table 3. By comparing with the above numerical results, we can deduce that the first eigenvalues between the

Table 1. The comparison with computational amount

	original equation	cell problem	homogenized equation				boundary layer
			I	II	III	IV	
elements	37696	1296	10224	2556	1136	284	3024
nodes	19813	1369	5245	1345	613	165	1764

Table 2. The computational results of relative eigenvalues

	Λ^0	homogenized problems λ^0			
		I	II	III	IV
case 1	46.6267	15.6964	15.7280	15.7805	16.0637
case 2	0.02879	10.2488	10.2694	10.3037	10.4886
case 3	0.05754	10.2684	10.2890	10.3234	10.5086
case 4	11.4159	15.4760	15.5072	15.5589	15.8382
case 5	56.7032	14.1307	14.1592	14.2064	14.4614
case 6	3.05553	4.14434	4.06663	4.0802	4.15343

Table 3. The comparison of computational results: A. eigenvalues

	original problem Λ^ε	approximate solutions $\varepsilon^{-2}\Lambda^0+\lambda^{(0)}$			
		I	II	III	IV
case 1	7893.03423	7161.94335	7161.97494	7162.02744	7162.31064
case 2	14.81526	14.39436	14.41498	14.44926	14.63417
case 3	19.31837	18.55469	18.57536	18.60970	18.79497
case 4	1748.16067	1659.3776	1659.40873	1659.4605	1659.73971
case 5	9215.64367	8179.3855	8179.41392	8179.46118	8179.71614
case 6	452.12864	444.13998	444.06227	444.0758	444.1491

Table 4. The comparison of computational results: B. eigenfunctions

	$\frac{\|e_0\|_{L^2}}{\|\Theta u^0\|_{L^2}}$	$\frac{\|e_1\|_{L^2}}{\|U_1^\varepsilon\|_{L^2}}$	$\frac{\|e_2\|_{L^2}}{\|U_2^\varepsilon\|_{L^2}}$	$\frac{\|e_0\|_{H^1}}{\|\Theta u^0\|_{H^1}}$	$\frac{\|e_1\|_{H^1}}{\|U_1^\varepsilon\|_{H^1}}$	$\frac{\|e_2\|_{H^1}}{\|U_2^\varepsilon\|_{H^1}}$
Case 1	0.040199	0.039678	0.039765	0.061445	0.061531	0.061464
Case 2	2.051521	0.024299	0.024305	1.480773	0.030492	0.030489
Case 3	2.048649	0.005567	0.005577	1.478613	0.015898	0.015882
Case 4	1.246650	0.021465	0.021211	0.915519	0.036834	0.036413
Case 5	0.155731	0.054466	0.054471	0.142533	0.078790	0.078787
Case 6	1.177821	0.101529	0.101532	0.911825	0.243411	0.234077

original problem and the corresponding homogenized equation are very close, and this is a very important conclusion in engineering applications.

Now let us introduce some notations. $U^\varepsilon(x)$ denotes the FE approximate solution of the first eigenfunction for the original problem (6.1) by direct numerical computation in a very refined mesh. $u^0(x)$ denotes the FE approximate solution of the first eigenfunction for the homogenized Helmholtz equation (3.24) in a coarse mesh I. $U_1^\varepsilon(x)$, $U_2^\varepsilon(x)$ are respectively the first-order and the second-order multi-scale FE solutions calculated by multi-scale FE scheme (5.4), set $e_0(x) = U^\varepsilon(x) - \Theta(\xi)u^0(x)$, $e_1(x) = U^\varepsilon(x) - U_1^\varepsilon(x)$, $e_2(x) = U^\varepsilon(x) - U_2^\varepsilon(x)$.

The comparison of some computational results for the first eigenfunctions will be shown in Table 4 (also see Figures 6.4–6.7). It should be stated that the proposed method in this paper is a robust one for calculating the eigenvalues and eigenfunctions of second order Helmholtz equation in periodically domains (with stiff inclusions).

Concluding Remarks It should be mentioned that the proposed method in this paper can be applied to solve elastic systems of second order elliptic equations without any difficulty. From the numerical point of view, this method is also suitable for calculating the eigenvalues and eigenfunctions of second order Helmholtz equation in a perforated domain with a quasi-periodic structure.

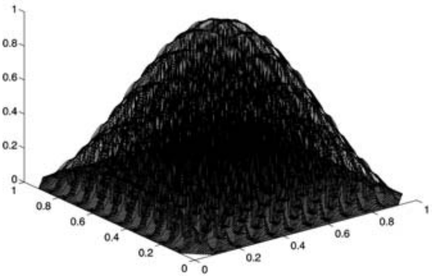

Fig. 6.4a. Case2:solution U^{ε}

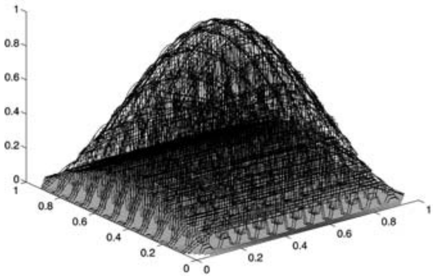

Fig. 6.4b. Case2: MFEM U_2^{ε}

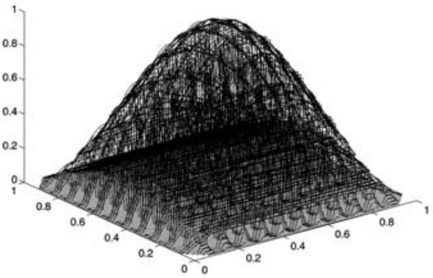

Fig. 6.4c. Case2: MFEM U_1^{ε}

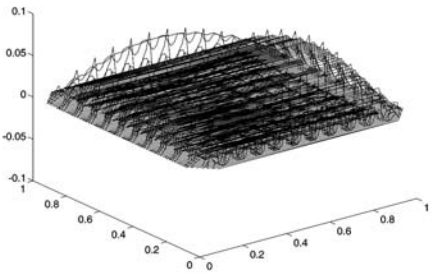

Fig. 6.4d. Case2: $e_2(x) = U^{\varepsilon} - U_2^{\varepsilon}$

Finally, we would like to state that the theoretical results and numerical algorithm presented in this paper have wide applications in physics, mechanics and industry engineering. If one wishes to calculate the eigenvalues (such as the natural frequencies, energy levels) of composite materials in a perforated domain, then only one needs to solve the first eigenvalue problem over unit cell(see (1.5)) and the homogenized Helmholtz equation in whole domain Ω (see (3.24)), due to Theorem 3.4. According to the superposition principle, the natural vibration modes of original problem consist of two parts, the first one is the natural vibration modes of the homogenized problem,which reflect the macroscopic behaviors of structure; and the second one are the periodic solutions $N_{\alpha_1\cdots\alpha_l}(\xi), l \geq 1, \alpha_j = 1, \cdots, n$, which depict the local fluctuations of solutions considered.

Appendix A: The proof of proposition 3.1

For simplicity, we here consider only 2-D problem without loss of generality.

We can directly prove the following lemmas under some assumptions on the geometry of the unit cell and physical materials.

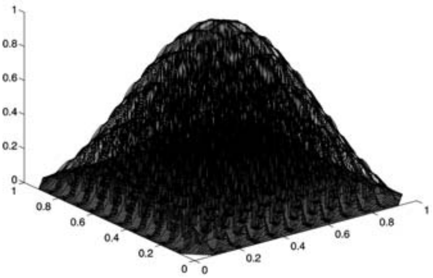

Fig. 6.5a. Case3:solution U^{ε}

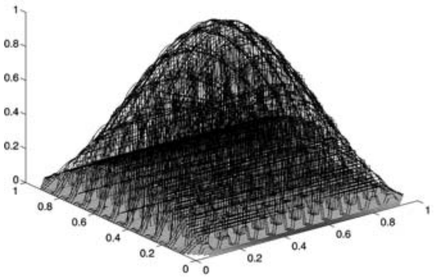

Fig. 6.5b. Case3: MFEM U_2^{ε}

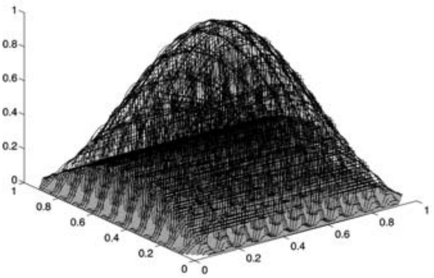

Fig. 6.5c. Case3: MFEM U_1^{ε}

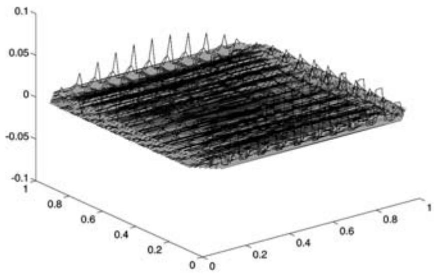

Fig. 6.5d. Case3: $e_2(x) = U^{\varepsilon} - U_2^{\varepsilon}$

Lemma A.1 *Let $\Theta(\xi)$ be the first eigenfunction of problem (1.5). Under the assumptions (H), $(A_1)-(A_3)$, $(B_1)-(B_3)$, $(C_1)-(C_2)$, and $a_{ij}(\xi) \in C^0(\overline{Q \cap \omega})$, $\rho(\xi) \in L^{\infty}(Q \cap \omega)$, then we can show that $\Theta(\xi)$ is symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2, \cdots, n$.*

Lemma A.2 *Let $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi), \cdots, N_{\alpha_1\cdots\alpha_l}(\xi), \alpha_j = 1, 2, \cdots, n, j = 1, 2, \cdots, l$ be the solutions of problems (3.14), (3.16) and (3.17), respectively. Under the assumptions of Lemma A.1, then we can show that $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi), \cdots, N_{\alpha_1\cdots\alpha_l}(\xi), \alpha_j = 1, 2, \cdots, n, j = 1, 2, \cdots, l$ are symmetric or anti-symmetric with respect to the middle hyperplanes $\Delta_i, i = 1, 2, \cdots, n$. For example,*

$$N_1(\xi_1, \xi_2) = -N_1(1-\xi_1, \xi_2), \qquad N_1(\xi_1, \xi_2) = N_1(\xi_1, 1-\xi_2);$$
$$N_2(\xi_1, \xi_2) = N_2(1-\xi_1, \xi_2), \qquad N_2(\xi_1, \xi_2) = -N_2(\xi_1, 1-\xi_2);$$
$$N_{11}(\xi_1, \xi_2) = N_{11}(1-\xi_1, \xi_2), \qquad N_{11}(\xi_1, \xi_2) = N_{11}(\xi_1, 1-\xi_2);$$
$$N_{22}(\xi_1, \xi_2) = N_{22}(1-\xi_1, \xi_2), \qquad N_{22}(\xi_1, \xi_2) = N_{22}(\xi_1, 1-\xi_2);$$
$$N_{12}(\xi_1, \xi_2) = -N_{12}(1-\xi_1, \xi_2), \qquad N_{12}(\xi_1, \xi_2) = -N_{12}(\xi_1, 1-\xi_2);$$
$$N_{21}(\xi_1, \xi_2) = -N_{21}(1-\xi_1, \xi_2), \qquad N_{21}(\xi_1, \xi_2) = -N_{21}(\xi_1, 1-\xi_2).$$

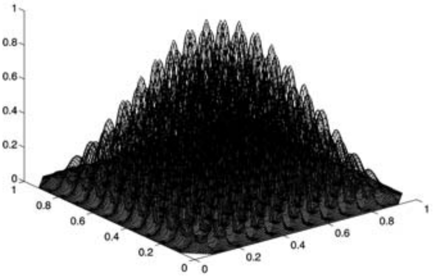

Fig. 6.6a. Case4:solution U^{ε}

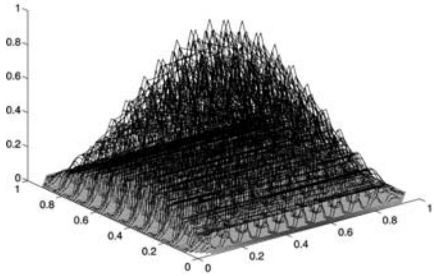

Fig. 6.6b. Case4: MFEM U_2^{ε}

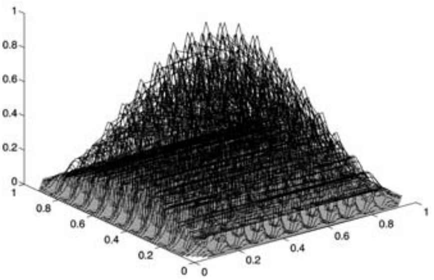

Fig. 6.6c. Case4: MFEM U_1^{ε}

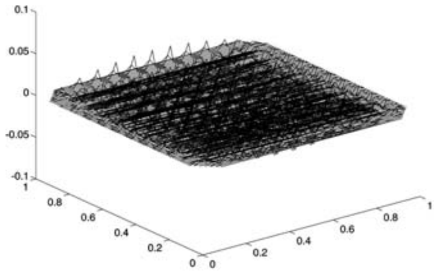

Fig. 6.6d. Case4: $e_1(x) = U^{\varepsilon} - U_1^{\varepsilon}$

Recalling (3.14), under the assumption (C_1) *(see §3), we have*

$$\text{(A.1)} \qquad \begin{cases} \dfrac{\partial}{\partial \xi_k}\Big[\Theta^2(\xi)\Big(a_{kj}(\xi)\dfrac{\partial N_1(\xi)}{\partial \xi_j} + a_{k1}(\xi)\Big)\Big] = 0, \textit{in} \quad Q \cap \omega \\ N_1(\xi) = 0 \quad \textit{on} \quad \partial Q \end{cases}$$

Set $\Lambda_{11} = \Theta^2(\xi)\Big(a_{11}(\xi) + \sum_{j=1}^{2} a_{1j}(\xi)\dfrac{\partial N_1(\xi)}{\partial \xi_j}\Big)$, $\Lambda_{21} = \Theta^2(\xi)\Big(a_{21}(\xi) + \sum_{j=1}^{2} a_{2j}\dfrac{\partial N_1}{\partial \xi_j}\Big)$. *Hence (A.1) can be rewritten as the following form:*

$$\text{(A.2)} \qquad \frac{\partial}{\partial \xi_1}(\Lambda_{11}) + \frac{\partial}{\partial \xi_2}(\Lambda_{21}) = 0$$

Set $v(\xi) = (e^{i2\pi m_1 \xi_1} - 1)(e^{i2\pi m_2 \xi_2} - 1)$, $m_1 \neq 0, m_2 \neq 0$. *It is obvious that* $v(\xi) \in H^1(Q \cap \omega, \partial Q)$.

The variational formulation of problem (A.2) is the following:

$$i2\pi \sum_{k=1}^{2} m_k \int_{Q\cap\omega} \Lambda_{k1}(\xi)(e^{i2\pi m\cdot\xi} - e^{i2\pi m_k \xi_k})d\xi = 0$$

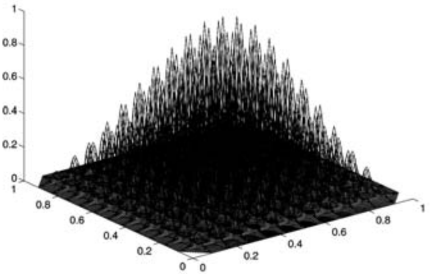

Fig. 6.7a. Case5:solution U^{ε}

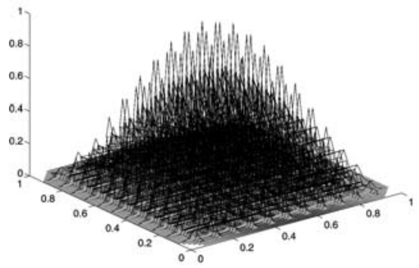

Fig. 6.7b. Case5: MFEM U_2^{ε}

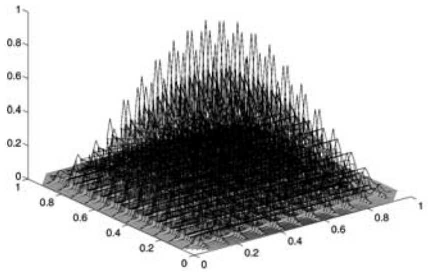

Fig. 6.7c. Case5: MFEM U_1^{ε}

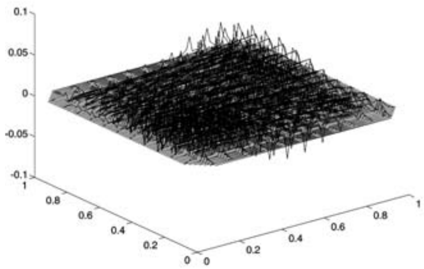

Fig. 6.7d. Case5: $e_1(x) = U^{\varepsilon} - U_1^{\varepsilon}$

and consequently

$$m_1(c^{(1)}_{m_1m_2} - c^{(1)}_{m_10}) + m_2(d^{(1)}_{m_1m_2} - d^{(1)}_{0m_2}) = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.3}$$

where $c^{(1)}_{m_1m_2} = \int\limits_{Q\cap\omega} \Lambda_{11} e^{i2\pi m\cdot\xi} d\xi, \qquad c^{(1)}_{m_10} = \int\limits_{Q\cap\omega} \Lambda_{11} e^{i2\pi m_1\xi_1} d\xi,$

$$d^{(1)}_{m_1m_2} = \int\limits_{Q\cap\omega} \Lambda_{21} e^{i2\pi m\cdot\xi} d\xi, \qquad d^{(1)}_{0m_2} = \int\limits_{Q\cap\omega} \Lambda_{21} e^{i2\pi m_2\xi_2} d\xi.$$

Lemma A.3 *Under the assumptions of* (H), $(A_1) - (A_3)$, $(B_1) - (B_3)$, *and* $(C_1) - (C_3)$, *then we can prove that the corresponding Fourier series of the function* $F(\xi_1, \xi_2) = \Lambda_{11}(\xi_1, \xi_2)$ *is absolutely uniform convergence on* $[0, 1] \times [0, 1]$.

Proof. Set $F(\xi_1, \xi_2) = \Lambda_{11}(\xi_1, \xi_2) = \Theta^2(\xi)\Big(a_{11}(\xi) + \sum\limits_{j=1}^{2} a_{1j}(\xi)\dfrac{\partial N_1(\xi)}{\partial \xi_j}\Big)$.

Under the assumption of (C_4), using Theorem 6.17, Theorem 6.18 of [13], and Theorem 7.2 of [14], one can show that the following process is valid.

For $m_1 \neq 0$, and $m_2 \neq 0$, let $\tilde{c}^{(1)}_{m_1 m_2}$ denote the m_1, m_2 th Fourier coefficient of $F_{\xi_2\xi_1}$; thus

$$\tilde{c}^{(1)}_{m_1 m_2} = \int_0^1 \int_0^1 F_{\xi_2\xi_1} e^{i2\pi m\cdot\xi} d\xi$$

If we integrate by parts with respect to ξ_2, holding ξ_1 fixed, then by periodicity we obtain

$$\tilde{c}^{(1)}_{m_1 m_2} = -i2\pi m_2 \int_0^1 \int_0^1 F_{\xi_1} e^{i2\pi m\cdot\xi} d\xi$$

Integrating by parts again yields

$$\tilde{c}^{(1)}_{m_1 m_2} = -4\pi^2 m_1 m_2 c^{(1)}_{m_1 m_2} \tag{A.4}$$

If we let $\tilde{c}^{(1)}_{0m_2}$, for $m_2 \neq 0$, denote the m_2-th Fourier coefficient of F_{ξ_2}, then $\tilde{c}^{(1)}_{0m_2} = -i2\pi m_2 c^{(1)}_{0m_2}$, and similarly, for $m_1 \neq 0$, we have $\tilde{c}^{(1)}_{m_1 0} = -i2\pi m_1 c^{(1)}_{m_1 0}$. Finally, we define $\tilde{c}^{(1)}_{00}$ to 0. Bessel's inequality applied to F_{ξ_1}, F_{ξ_2}, and $F_{\xi_1\xi_2}$ implies that

$$\sum_{m_1 m_2 = -\infty}^{+\infty} |\tilde{c}^{(1)}_{m_1 m_2}|^2 \leq \int_0^1 \int_0^1 [F^2_{\xi_1} + F^2_{\xi_2} + F^2_{\xi_2\xi_1}] d\xi_1 d\xi_2 \tag{A.5}$$

and by using Cauchy inequality, we obtain:

(A.6)

$$\begin{aligned} \sum |c^{(1)}_{m_1 m_2}| &= |c^{(1)}_{00}| + \sum{}' |\tilde{c}^{(1)}_{m_1 m_2}| \frac{1}{m_1 m_2} + \sum{}' |\tilde{c}^{(1)}_{m_1 0}| \frac{1}{m_1} + \sum{}' |\tilde{c}^{(1)}_{0 m_2}| \frac{1}{m_2} \\ &\leq |c^{(1)}_{00}| + \Big(\sum{}' |\tilde{c}^{(1)}_{m_1 m_2}|^2\Big)^{1/2} \Big(\sum{}' \frac{1}{m_1^2 m_2^2}\Big)^{1/2} \\ &\quad + \Big(\sum{}' |\tilde{c}^{(1)}_{m_1 0}|^2\Big)^{1/2} \Big(\sum{}' \frac{1}{m_1^2}\Big)^{1/2} + \Big(\sum{}' |\tilde{c}^{(1)}_{0 m_2}|^2\Big)^{1/2} \Big(\sum{}' \frac{1}{m_2^2}\Big)^{1/2} \end{aligned}$$

Because of (A.5), the last sum above is finite. Consequently, we complete the proof of Lemma (A.4). □

For the reader's convenience we give here the following lemmas, which are some well-known results.

Lemma A.4 *Assume that $f(x, y)$ is continuous on a bounded domain $R(a \leq x \leq b, c \leq y \leq d)$, and there exists a continuous partial derivative $f'_y(x, y)$. Then the following equality holds, for $c < y < d$*

$$\frac{\partial}{\partial y}\int_a^b f(x, y)dx = \int_a^b f'_y(x, y)dx \tag{A.7}$$

Lemma A.5 *Suppose that $f(x, y)$ is a even function on domain $[-a, a] \times [-b, b]$, i.e. $f(-x, y) = -f(x, y)$, $f(x, -y) = -f(x, y)$. Then it holds*

$$f(x, y) \sim \sum_{m,n=1}^{\infty} D_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \tag{A.8}$$

where $D_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x, y) sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}\, dxdy$

Lemma A.6 *Suppose $u_n(x) \in [a, b], n = 1, 2, \cdots$, and $\{u_n(x)\}$ is uniform convergence on $[a, b]$, i.e. $\sum_{n=1}^{+\infty} u_n(x) = S(x)$, then $\sum_{n=1}^{+\infty}\int_a^b u_n(x)dx = \int_a^b S(x)dx$.*

Theorem A.1 *Under the assumptions of Lemma A.3, we have*

$$m_1 c^{(1)}_{m_1m_2} + m_2 d^{(1)}_{m_1m_2} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.9}$$

Furthermore

$$m_1 c^{(1)}_{m_10} + m_2 d^{(1)}_{0m_2} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.10}$$

Proof. Integrating on both sides of (A.2) with respect to ξ_1, one gets

$$\int_{1/2}^{\xi_1} \frac{\partial}{\partial \xi_1}(\Lambda_{11})d\xi_1 + \int_{1/2}^{\xi_1} \frac{\partial}{\partial \xi_2}(\Lambda_{21})\, d\xi_1 = 0$$

Lemma A.4 ensures that

$$\Lambda_{11} + \frac{\partial}{\partial \xi_2}\left(\int_{1/2}^{\xi_1} \Lambda_{21} d\xi_1\right) = \phi_2(\xi_2)$$

Set $\tilde{\Lambda}_{21} = \int_{1/2}^{\xi_1} \Lambda_{21}\, d\xi_1$. One can directly verify that $\tilde{\Lambda}_{21}$ is symmetric and anti-symmetric with respect to the middle hyperplane Δ_1 and Δ_2, respectively, due to Λ_{21} are all symmetric with respect to the middle hyperplanes Δ_1, Δ_2.

From Lemma A.5, we have

$$\tilde{\Lambda}_{21}(\xi_1,\xi_2) \sim \sum (-1)^{m_1+m_2+1}\alpha_{m_1m_2} cos 2\pi m_1\xi_1 \cdot sin 2\pi m_2\xi_2 \tag{A.11}$$

It follows from Lemma A.3, lemma A.5 and equation (A.8) that

$$\frac{\partial}{\partial \xi_2}(\tilde{\Lambda}_{21}) = -\sum_{m_1,m_2}(-1)^{m_1+m_2}A_{m_1m_2} cos 2\pi m_1\xi_1 \cdot cos 2\pi m_2\xi_2 + \phi_2(\xi_2)$$

Using Lemma A.6, and integrating on both sides of the above equation with respect to ξ_2, one gets

$$\begin{aligned}\tilde{\Lambda}_{21} = &-\sum_{m_1,m_2}\frac{(-1)^{m_1+m_2}}{2\pi m_2}A_{m_1m_2} cos 2\pi m_1\xi_1 \cdot sin 2\pi m_2\xi_2\\ &+\int_{1/2}^{\xi_2}\phi_2(\xi_2)d\xi_2 + \phi_1(\xi_1)\end{aligned} \tag{A.12}$$

Comparing (A.11) and (A.12) gives

$$2\pi m_2\alpha_{m_1m_2} = A_{m_1m_2}, \quad m_1 \neq 0, m_2 \neq 0 \tag{A.13}$$

On the other hand, recalling the definition of the coefficients of Fourier series and integrating by parts, we have

$$\begin{aligned}\alpha_{m_1m_2} &= \int_0^1\int_0^1 \tilde{\Lambda}_{21} cos 2\pi m_1\xi_1 \cdot sin 2\pi m_2\xi_2 d\xi_1 d\xi_2\\ &= \int_0^1 sin 2\pi m_2\xi_2 d\xi_2 \int_0^1 (\int_{1/2}^{\xi_1} \Lambda_{21}(t,\xi_2)dt) cos 2\pi m_1\xi_1 d\xi_1\\ &= -\frac{1}{2\pi m_1}\int_0^1\int_0^1 \Lambda_{21}(\xi) sin 2\pi m_1\xi_1 \cdot sin 2\pi m_2\xi_2 d\xi_1 d\xi_2\end{aligned} \tag{A.14}$$

Let us recall that Λ_{11}, Λ_{21} are symmetric and anti-symmetric with respect to the middle hyperplanes Δ_1, Δ_2, respectively. Lemma A.5 implies that

$$\Lambda_{11} \sim \sum (-1)^{m_1+m_2}A_{m_1m_2} cos 2\pi m_1\xi_1 \cdot cos 2\pi m_2\xi_2 \tag{A.15}$$

$$\Lambda_{21} \sim \sum (-1)^{m_1+m_2}B_{m_1m_2} sin 2\pi m_1\xi_1 \cdot sin 2\pi m_2\xi_2 \tag{A.16}$$

From (A.13) and (A.14), we get

$$m_1A_{m_1m_2} + m_2B_{m_1m_2} = 0,\ m_1 \neq 0, m_2 \neq 0$$

Comparing with the relation of the coefficients between real Fourier's series and complex Fourier's series(see [28], pp190), one can get

$$m_1 c^{(1)}_{m_1 m_2} + m_2 d^{(1)}_{m_1 m_2} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.17}$$

Combining (A.3) and (A.17) yields

$$m_1 c^{(1)}_{m_1 0} + m_2 d^{(1)}_{0 m_2} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.18}$$

Therefore we complete the proof of Theorem A.1. □

Theorem A.2 *Under the assumptions of Theorem A.1, we can infer that* $\sigma_\xi(N_1), \sigma_\xi(N_2)$ *are continuous on the boundary* ∂Q*, where* $\sigma_\xi(N_1), \sigma_\xi(N_2)$ *denote the normal derivatives of* $N_1(\xi)$ *and* $N_2(\xi)$*, respectively.*

Proof. Recalling the definition of $N_1(\xi)$, we have

$$\begin{cases} \frac{\partial}{\partial \xi_k}\Big[\Theta^2(\xi)\Big(a_{kj}(\xi)\frac{\partial N_1(\xi)}{\partial \xi_j} + a_{k1}(\xi)\Big)\Big] = 0,\ \xi \in Q \cap \omega \\ N_1(\xi) = 0,\ \xi \in \partial Q \end{cases} \tag{A.19}$$

Set $v_1(\xi) = e^{i2\pi m_2 \xi_2} - 1$, the variational formulation of (A.19) is the following:

$$\int_{l_1} [\Lambda_{11}] v_1(\xi) d\Gamma + i2\pi m_2 d^{(1)}_{0,m_2} = 0 \tag{A.20}$$

where side l_1 as shown in Figure 3.2, and $[A_{11}]$ denotes the jump on the side l_1 of ∂Q.

Because of $\Lambda_{11}|_{l_1} = \Lambda_{11}|_{l_2}$, *i.e.* $[\Lambda_{11}]|_{l_1} = 0$, one gets $d^{(1)}_{0,m_2} = 0$.

Putting it into (A.18), one has $c^{(1)}_{m_1 0} = 0$.

Similarly set $v_2(\xi) = e^{i2\pi m_1 \xi_1} - 1$, the variational formulation of (A.19) is the following:

$$\int_{l_3} [\Lambda_{21}] v_2(\xi) d\Gamma + i2\pi m_1 c^{(1)}_{m_1 0} = 0 \tag{A.21}$$

where side l_3 as shown in Figure 3.2, and $[A_{21}]$ denotes the jump on the side l_3 of ∂Q.

Consequently

$$\int_{l_3} [\Lambda_{21}] v_2(\xi) d\Gamma = 0 \tag{A.22}$$

Integrating directly on both sides of (A.19), we have

$$\int_{\partial Q} \sigma_\xi(N_1)d\Gamma = 0 \tag{A.23}$$

Observing that $\int_{l_1}[\Lambda_{11}]d\Gamma = 0$, thanks to Lemma A.2, and using (A.23), one derives $\int_{l_3}[\Lambda_{21}]d\Gamma = 0$.

To summarized the above results, we can deduce that

$$\int_{l_3} [\Lambda_{21}]e^{i2\pi m_1\xi_1}d\xi_1 = 0, \forall m_1 \in Z$$

The completeness of the function family $\{e^{i2\pi m_1\xi_1}\}_{m_1=-\infty}^{+\infty}$ implies that $[\Lambda_{21}]|_{l_3} = 0, a.e.$. Hence we can include that $\sigma_\xi(N_1)$ is continuous on ∂Q.

The remainder can be completed similarly.

Remark A.1 From Lemma A.1 and Lemma A.2, we know that $N_1(\xi)(N_2(\xi))$ is anti-symmetric(symmetric) and symmetric(anti-symmetric) with respect to the middle hyperplanes Δ_1, Δ_2, respectively, and hence we have $\int_{Q\cap\omega}\Theta^2(\xi)N_k(\xi)d\xi = 0, k = 1, 2$. Therefore the proposed method in this paper is equivalent to the classical homogenization method (see,e.g.[2, 6, 16, 18]).

It remains to prove that $\sigma_\xi(N_{\alpha_1\alpha_2})$, $\sigma_\xi(N_{\alpha_1\alpha_2\alpha_3})$ are continuous on the boundary $\partial Q, \alpha_i = 1, 2$.

From Lemma A.2, we can directly verify that $\sigma_\xi(N_{12}), \sigma_\xi(N_{21})$ are continuous on the boundary ∂Q. Hence next we only consider $\sigma_\xi(N_{11}), \sigma_\xi(N_{22})$.

From (3.16), we have:

$$\begin{cases} \dfrac{\partial}{\partial \xi_k}\Big[\Theta^2(\xi)(a_{kk}(\xi)\dfrac{\partial N_{11}}{\partial \xi_k} + a_{k1}(\xi)N_1(\xi))\Big] = -\Theta^2(\xi)\Big[a_{11}\dfrac{\partial N_1}{\partial \xi_1} + a_{11}(\xi)\Big] \\ \qquad\qquad +\theta^{-1}\hat{a}_{11}, \xi \in Q\cap\omega \\ N_{11}(\xi) = 0, \quad \xi \in \partial Q \end{cases} \tag{A.24}$$

Set $\Lambda_{1,11} = \Theta^2(\xi)\Big(a_{11}(\xi)N_1(\xi) + \sum_{j=1}^{2} a_{1j}(\xi)\dfrac{\partial N_{11}}{\partial \xi_j}\Big)$,

$$\Lambda_{2,11} = \Theta^2(\xi)\Big(a_{21}(\xi)N_1(\xi) + \sum_{j=1}^{2} a_{2j}(\xi)\frac{\partial N_{11}}{\partial \xi_j}\Big).$$

Similarly to (A.3), one gets

$$\begin{aligned} &-i2\pi m_1(c^{(11)}_{m_1m_2} - c^{(11)}_{m_10}) - i2\pi m_2(d^{(11)}_{m_1m_2} - d^{(11)}_{0m_2}) \\ &= -c^{(1)}_{m_1m_2} + c^{(1)}_{m_10} + c^{(1)}_{0m_2}, m_1 \neq 0, m_2 \neq 0 \end{aligned} \tag{A.25}$$

where $c^{(11)}_{m_1m_2} = \int\limits_{Q\cap\omega} \Lambda_{1,11}(\xi)e^{i2\pi m\cdot\xi}d\xi, \quad c^{(11)}_{m_10} = \int\limits_{Q\cap\omega} \Lambda_{1,11}(\xi)e^{i2\pi m_1\xi_1}d\xi;$

$$d^{(11)}_{m_1m_2} = \int\limits_{Q\cap\omega} \Lambda_{2,11}(\xi)e^{i2\pi m\cdots\xi}d\xi, \qquad d^{(11)}_{0m_2} = \int\limits_{Q\cap\omega} \Lambda_{2,11}(\xi)e^{i2\pi m_2\xi_2}d\xi,$$

and $c^{(1)}_{m_1m_2}, c^{(1)}_{m_10}, c^{(1)}_{0m_2}$ are stated as in (A.3).

Set $G(\xi_1, \xi_2) = \frac{\partial}{\partial\xi_2}(\Lambda_{2,11})$

$$= \frac{\partial}{\partial\xi_2}\Big(\Theta^2(\xi)(a_{21}(\xi)N_1(\xi) + \sum_{j=1}^{2} a_{2j}(\xi)\frac{\partial N_{11}}{\partial\xi_j})\Big).$$

Following the lines of the proof of Theorem A.1, we can obtain the following theorem:

Theorem A.3 *Under the assumptions of Theorem A.1, then the following equalities hold:*

$$-i2\pi m_1 c^{(11)}_{m_1m_2} - i2\pi m_2 d^{(11)}_{m_1m_2} = -c^{(1)}_{m_1m_2},\ m_1 \neq 0, m_2 \neq 0 \tag{A.26}$$

Furthermore

$$i2\pi m_1 c^{(11)}_{m_10} + i2\pi m_2 d^{(11)}_{0m_2} = c^{(1)}_{m_10} + c^{(1)}_{0m_2},\ m_1 \neq 0, m_2 \neq 0 \tag{A.27}$$

Theorem A.4 *Under the assumptions of Theorem A.3, we can prove that* $\sigma_\xi(N_{\alpha_1\alpha_2}), \alpha_j = 1, 2,$ *are continuous on the boundary* ∂Q*, where* $\sigma_\xi(N_{\alpha_1\alpha_2})$ *denote the normal derivatives of* $N_{\alpha_1\alpha_2}(\xi)$*, respectively.*

Proof. Set $v_1(\xi) = (e^{i2\pi m_2\xi_2} - 1)$, $v_2(\xi) = (e^{i2\pi m_1\xi_1} - 1)$, then the variational formulations of equation (A.24) are the following:

$$\int\limits_{l_1} [\Lambda_{1,11}(\xi)]v_1(\xi)d\Gamma = i2\pi m_2 d^{(11)}_{0m_2} - c^{(1)}_{0m_2};$$

$$\int\limits_{l_3} [\Lambda_{2,11}(\xi)]v_2(\xi)d\gamma = i2\pi m_1 c^{(11)}_{m_10} - c^{(1)}_{m_10}$$

Theorem A.3 ensures that

$$\begin{aligned} &\int\limits_{l_1} [\Lambda_{1,11}(\xi)]v_1(\xi)d\Gamma + \int\limits_{l_3} [\Lambda_{2,11}(\xi)]v_2(\xi)d\Gamma \\ &= i2\pi m_2 d^{(11)}_{0m_2} + i2\pi m_1 c^{(11)}_{m_10} - c^{(1)}_{0m_2} - c^{(1)}_{m_10} = 0 \end{aligned} \tag{A.28}$$

Set $\lambda_{m_2} = \int\limits_{l_1}[\Lambda_{1,11}]e^{i2\pi m_2\xi_2}d\xi_2, \quad \mu_{m_1} = \int\limits_{l_3}[\Lambda_{2,11}]e^{i2\pi m_1\xi_1}d\xi_1, \quad m_1 \neq 0$ $m_2 \neq 0; \quad \lambda_0 = \int\limits_{l_1}[\Lambda_{1,11}]d\xi_2, \quad \mu_0 = \int\limits_{l_3}[\Lambda_{2,11}]d\xi_1$

(A.28) implies that

$$\lambda_{m_2} - \lambda_0 + \mu_{m_1} - \mu_0 = 0 \tag{A.29}$$

Integrating directly on both sides of (A.24), one gets $\lambda_0 + \mu_0 = 0$, and consequently

$$\lambda_{m_2} + \mu_{m_1} = 0,\ m_1 \neq 0, m_2 \neq 0 \tag{A.30}$$

Let $m_2 \to +\infty$, for any fixed m_1, and using Riemann-Lebesgue lemma (see [28]), one derives $\mu_{m_1} = 0$, for $\forall m_1 \neq 0$. Similarly, one has $\lambda_{m_2} = 0$, for $\forall m_2 \neq 0$. Hence one can deduce that $[\Lambda_{1,11}] = const$, $[\Lambda_{21}] = const$. Therefore it implies that $\sigma_\xi(N_{11})$ has a constant sign in the neighborhood of ∂Q.

The fact $\int\limits_{\partial Q} \sigma_\xi(N_{11}) d\Gamma = 0$ gives $\sigma_\xi(N_{11})|_{\partial Q} = 0, a.e.$. Similarly, one can prove that $\sigma_\xi(N_{22})|_{\partial Q} = 0, a.e.$. Furthermore, we have $c^{(11)}_{m_1 0} = 0$ thanks to $c^{(1)}_{m_1 0} = 0$.

The proof of Theorem A.4 is complete.

Following the lines of proofs of Theorem A.2 and Theorem A.4, one can prove the following theorem:

Theorem A.5 *Under the assumptions of Theorem A.3, we have* $\sigma_\xi(N_{\alpha_1\alpha_2\alpha_3})$, $\alpha_j = 1, 2$, *are continuous on the boundary* ∂Q, *where* $\sigma_\xi(N_{\alpha_1\alpha_2\alpha_3})$ *denote the normal derivatives of* $N_{\alpha_1\alpha_2\alpha_3}(\xi)$, *respectively.*

Remark A.2 It should be mentioned that, generally speaking, $\sigma_\xi(N_{1111})$ is not continuous on the boundary ∂Q, due to $\int\limits_{\partial Q} \Theta^2(\xi)\sigma_\xi(N_{1111}) d\Gamma = \int\limits_{Q\cap\omega} \Theta^2(\xi)(\sum\limits_{j=1}^{2} a_{1j} \frac{\partial N_{111}}{\partial \xi_j} + a_{11}(\xi)N_{11}(\xi) - \vartheta^{-1}\hat{a}_{11}N_{11}(\xi))d\xi = \int\limits_{Q\cap\omega} \Theta^2(\xi) (\sum\limits_{j=1}^{2} a_{1j} \frac{\partial N_{111}}{\partial \xi_j})d\xi \neq 0$. Therefore, under the assumptions of Theorem A.2, we can at most obtain the fourth asymptotic expansion.

To summarized the above results, we have the following result:

Theorem A.6 *Under the assumptions of Theorem A.3, then the following regularity estimates hold :* $u_k^{\varepsilon,M} \in V^2(\Omega_0^\varepsilon)$, $1 \leq M \leq 3$, *in other words,* $U_k^{\varepsilon,M} = \Theta \cdot u_k^{\varepsilon,M} \in H^2(\Omega_0^\varepsilon)$, *where* $\Omega_0^\varepsilon \subset\subset \Omega^\varepsilon$.

Appendix B: Regularity estimates of the solution for boundary layer equation (see (3.28))

For simplicity, here let us consider only 2-D problems, and it can be treated similarly in other cases (Cf. [15]).

We first discuss the boundary value problem over concave domain $\Omega_1 \subset R^2$ as shown in Figure 3.4 as follows:

$$\text{(B.1a)} \qquad \begin{cases} -\triangle u = f(x) & \text{in} \quad \Omega_1 \\ u(x) = 0 & \text{on} \quad \partial\Omega_1 \end{cases}$$

Let $\{\sigma_j\}_{j=1}^N$ denote the angular points of Ω_1, respectively, and $\beta_j\pi, \quad j = 1, \cdots N$ are the corresponding internal angles, namely, $\beta_1 \le \beta_2 \le \cdots \le \beta_N, \quad \gamma_j = \frac{1}{\beta_j}$

It is obvious that $1 < \beta_N \le 2, \quad \frac{1}{2} \le \gamma_N < 1$. Suppose that

$$\text{(B.1b)} \qquad V_j = \{x \in \Omega_1 : |x - \sigma_j| < r_j\}, \quad j = 1, \cdots, N$$

satisfy

$$\text{(B.1c)} \qquad V_i \cap V_j = \emptyset, \quad V_0 = \Omega_1 \setminus \cup_{j=1}^N \overline{V}_j$$

Lemma B.1[15] Let u be the unique solution of problem (B.1a). If $f \in L^2(\Omega)$, then it holds

$$\text{(B.2)} \qquad u(x) = \sum_{j=1}^N c_j(f)u_j + U(x)$$

where $U(x) \in H^2(\Omega_1) \cap H_0^1(\Omega_1), \quad \|U\|_2 \le C\|f\|_0$, and the constants $c_j(f)$ satisfy
$|c_j(f)| \le C\|f\|_0$

Note that u_j are some functions independent of f, u, and satisfy the following conditions

(Γ_1) If $\gamma_j > 1$, then $u_j(x) \equiv 0$;

(Γ_2) $u_j(x) \equiv 0$ outside of V_j;

(Γ_3) If $\frac{1}{2} < \gamma_j < 1$, then there exists the following formula in a neighborhood of σ_j;

$$\text{(B.3)} \qquad u_j = \rho^{\gamma_j}\sin\gamma_j\theta, \quad \text{if} \quad (\rho, \theta) \in V_j$$

where $V_j = \{(\rho, \theta) : \quad 0 < \rho < r_j, \quad 0 < \theta < \beta_j\pi\}$

Remark B.1 By using (B.3), one can show that

$$\text{(B.4)} \qquad |D^k u| \le C\rho^{\gamma_j - |k|}$$

in a neighborhood of σ_j

It remains to complete the proof of Theorem 3.3.

It follows from the finite covering theorem that there exist the finite points $P_1, \cdots P_s$, and the corresponding neighborhoods $\mathcal{O}_l, \quad l = 1, \cdots s$, such that

(i) $\cup_{l=1}^{s}\mathcal{O}_l \supset \overline{\Omega}_1$;
(ii) $diam(\mathcal{O}_l) \le \varepsilon R_0, \quad R_0$ will be chosen later.
(iii) $\mathcal{I}_i = \{j : \mathcal{O}_j \cap \mathcal{O}_i \neq \emptyset, \}, \quad \sigma(\mathcal{I}_i) \le s_0$, where $\sigma(\mathcal{I}_i)$ denotes the number of elements in $\mathcal{I}_i$, respectively, $i = 1, \cdots s$ and s_0 is a constant.

By means of the resolution of unity theorem, there exist a set of functions $\phi_l(x) \in C_0^\infty(R^n), \quad l = 1, \cdots s$, such that $0 \le \phi_l(x) \le 1$, supp$\phi_l(x) \subset \mathcal{O}_l$, and $\sum_{l=1}^{s} \phi_l(x) \equiv 1, \quad$ in $\quad \Omega_1$

Let $\mathcal{D}_\varepsilon \cdot = -\frac{\partial}{\partial x_i}(\Theta^2(\frac{x}{\varepsilon})a_{ij}(\frac{x}{\varepsilon})\frac{\partial}{\partial x_j})\cdot, \quad w^\varepsilon = \sum_{l=1}^{s} w_l^\varepsilon, \quad w_l^\varepsilon = \phi_l \cdot w^\varepsilon$, and consequently

$$\mathcal{D}_\varepsilon w_l^\varepsilon = \phi_l \cdot \mathcal{D}_\varepsilon w^\varepsilon + \eta_l \tag{B.5}$$

where

(B.6)
$$\begin{aligned}\eta_l = &-\frac{\partial \phi_l}{\partial x_i}[\Theta^2(\frac{x}{\varepsilon})\frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon}))w^\varepsilon + 2a_{ij}(\frac{x}{\varepsilon})\Theta(\frac{x}{\varepsilon})\nabla_x(\Theta(\frac{x}{\varepsilon}))w^\varepsilon(x)\\ &+\Theta^2(\tfrac{x}{\varepsilon})a_{ij}(\tfrac{x}{\varepsilon})\frac{\partial w^\varepsilon}{\partial x_j}] - \Theta^2(\tfrac{x}{\varepsilon})a_{ij}(\tfrac{x}{\varepsilon})[\frac{\partial \phi_l}{\partial x_i}\frac{\partial w^\varepsilon}{\partial x_j} + w^\varepsilon \frac{\partial^2 \phi_l}{\partial x_i \partial x_j}]\end{aligned}$$

Recalling the definition of $\Theta(\xi)$ (see (1.5)) and Lemma 2.4, and taking into account condition $\nabla_\xi a_{ij}(\xi) \in L^\infty(\Omega^\varepsilon)$, one derives

(B.7)
$$\|\Theta^{-1}(\frac{x}{\varepsilon})\eta_l\|_{0,p,\mathcal{O}_l\cap\Omega_1^\varepsilon} = \|\Theta^{-1}(\frac{x}{\varepsilon})\eta_l\|_{0,p,\mathcal{O}_l\cap\Omega_1} \le C\frac{1}{\varepsilon^2}\|w^\varepsilon\|_{V^{1,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)}$$

$\forall R > 0$, let

$$\omega_\varepsilon(R) = \max_{i,j} \max_{|x-x'|<\varepsilon R} |a_{ij}(\frac{x}{\varepsilon}) - a_{ij}(\frac{x'}{\varepsilon})|, \quad x, x' \in \Omega_1^\varepsilon$$

For any fixed $x_0 \in \mathcal{O}_l \cap \Omega_1^\varepsilon$, set $A^\varepsilon = (a_{ij}(\frac{x_0}{\varepsilon}))$, it follows from condition (A_3) that there exists a orthogonal matrix T such that

$$TA^\varepsilon T' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$$

where T' denotes the transpose of a matrix T.

We have $\lambda_i \ge \sigma > 0, \quad i = 1, 2$, due to condition (A_2). If let $B = D^{-1/2}T$, then $BA^\varepsilon B' = I$, and $\|B\| \le \sqrt{\|D^{-1}\|} \le \sqrt{\frac{1}{\sigma}}$. Hence we have $\|B^{-1}\| = \|D^{1/2}\| \le \sum_{i=1}^{2} \lambda_i = \sum_i a_{ii}(\frac{x_0}{\varepsilon}) \le M_0$, where M_0 is a positive constant independent of ε.

If let $\hat{\mathcal{O}}_l = B(\mathcal{O}_l \cap \Omega_1)$, $\quad \hat{\mathcal{O}}_1^\varepsilon = B(\mathcal{O}_1 \cap \Omega_1^\varepsilon)$, $\quad$ then $\hat{v}(y) = v(B^{-1}y) \in V^{2,p}(\hat{\mathcal{O}}_l^\varepsilon)$, for any $v \in V^{2,p}(\mathcal{O}_l \cap \Omega_1^\varepsilon)$, where p will be stated below.

(B.8) $$C\|v\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} \le \|\hat{v}\|_{V^{2,p}(\hat{\mathcal{O}}_l^\varepsilon)} \le C'\|v\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)}$$

Let

$$\begin{aligned} g(x) &= -\Theta^2(\tfrac{x}{\varepsilon})a_{ij}(\tfrac{x_0}{\varepsilon})\frac{\partial^2 w_l^\varepsilon}{\partial x_i \partial x_j} \\ &= -\Theta^2(\tfrac{x}{\varepsilon})(a_{ij}(\tfrac{x_0}{\varepsilon}) - a_{ij}(\tfrac{x}{\varepsilon}))\frac{\partial^2 w_l^\varepsilon}{\partial x_i \partial x_j} - \Theta^2(\tfrac{x}{\varepsilon})a_{ij}(\tfrac{x}{\varepsilon})\frac{\partial^2 w_l^\varepsilon}{\partial x_i \partial x_j} \\ &= -(a_{ij}(\tfrac{x_0}{\varepsilon}) - a_{ij}(\tfrac{x}{\varepsilon}))\Theta^2(\tfrac{x}{\varepsilon})\frac{\partial^2 w_l^\varepsilon}{\partial x_i \partial x_j} + (\lambda^{(0)}\rho(\tfrac{x}{\varepsilon}) - b(\tfrac{x}{\varepsilon}))\Theta^2(\tfrac{x}{\varepsilon})w_l^\varepsilon(x) \\ &\quad + \eta_l(x) + \frac{\partial}{\partial x_i}(\Theta^2(\tfrac{x}{\varepsilon})a_{ij}(\tfrac{x}{\varepsilon}))\frac{\partial w_l^\varepsilon}{\partial x_j} \end{aligned}$$

From (1.8), respectively (2.14), we have

(B.9a) $$\int\limits_{\mathcal{O}_l\cap\Omega_1^\varepsilon} |\nabla_x(\Theta(\frac{x}{\varepsilon}))|^2 dx \le C_1\varepsilon^{-2} \int\limits_{\mathcal{O}_l\cap\Omega_1^\varepsilon} |\Theta(\frac{x}{\varepsilon})|^2 dx$$

If assume that $\Theta(\frac{x}{\varepsilon})$ is a smooth function, and $diam(\mathcal{O}_l)$ is sufficiently small, using proof by contradiction, then one can prove that

(B.9b) $$|\nabla_x(\Theta(\frac{x}{\varepsilon})| \le C_0\varepsilon^{-1}|\Theta(\frac{x}{\varepsilon})|, \quad a.e. x \in \mathcal{O}_l \cap \Omega_1^\varepsilon$$

From (B.7), (B.8), (B.9a), (B.9b) and condition $\nabla_\xi a_{ij}(\xi) \in L^\infty(\Omega^\varepsilon)$, we obtain

(B.10)
$$\begin{aligned} \|\Theta^{-1}\cdot g\|_{0,p,\mathcal{O}_l\cap\Omega_1^\varepsilon} &= \|\Theta^{-1}\cdot g\|_{0,p,\mathcal{O}_1\cap\Omega_1} \\ &\le \omega_\varepsilon(R)\|w_l^\varepsilon\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} + C\tfrac{1}{\varepsilon^2}\|w^\varepsilon\|_{V^{1,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} \end{aligned}$$

On the other hand, if let $\hat{w}_l^\varepsilon(y) = w_l^\varepsilon(B^{-1}y)$, $\quad \hat{g}(y) = g(B^{-1}y)$, $\quad \hat{\Theta}(y) = \Theta(\varepsilon^{-1}B^{-1}(y))$, then one gets

$$a_{ij}(\frac{x_0}{\varepsilon})\frac{\partial^2 w_l^\varepsilon}{\partial x_i \partial x_j} = \triangle\hat{w}_l^\varepsilon(y), \quad y = Bx$$

Moreover

$$-\hat{\Theta}(y)\triangle\hat{w}_l^\varepsilon(y) = \hat{\Theta}^{-1}(y)\cdot\hat{g}(y)$$

Using Lemma B.1, and replacing $L^p(\Omega_1)$ with $L^p_\Theta(\Omega_1)$, $\quad 1 < p < +\infty$, we can deduce that there exists $\hat{w}_l^\varepsilon(x) \in V^{2,p}(\hat{\mathcal{O}}_l)$ such that

(B.11)
$$\|\hat{\Theta}(y)\frac{\partial^2\hat{w}_l^\varepsilon}{\partial y_i \partial y_j}\|_{0,p,\hat{\mathcal{O}}_l} \le C(p)\|\Theta(y)\triangle\hat{w}_l^\varepsilon\|_{0,p,\hat{\mathcal{O}}_l} = C(p)\|\hat{\Theta}^{-1}\cdot\hat{g}\|_{0,p,\hat{\mathcal{O}}_l}$$

where $L^p_\Theta(\Omega_1) = \left\{ f(x) : \int_{\Omega_1} \Theta^p(\frac{x}{\varepsilon})|f|^p dx < +\infty \right\}$, and $1 < p \leq p_0 = \frac{2\beta_N}{2\beta_N - 1} < +\infty$, and β_N is the maximum internal angle of $B\Omega_1$.

From (B.8), (B.11) and (B.10), one obtains

$$\|w_l^\varepsilon\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} \leq C(p)\{\omega_\varepsilon(R)\|w_l^\varepsilon\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} + \frac{1}{\varepsilon^2}\|w_l^\varepsilon\|_{V^{1,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)}\}$$

Since $a_{ij}(\frac{x}{\varepsilon}) \in C(\overline{\Omega}^\varepsilon)$, then there exists a constant $R_0 > 0$ such that

$$\omega_\varepsilon(R) < \frac{1}{3C(p)} \quad \text{for} \quad 0 < R < R_0$$

and consequently

$$\begin{aligned}\|w_l^\varepsilon\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} &\leq C(p)\varepsilon^{-2}\{\|w_l^\varepsilon\|_{V^{1,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} + \|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\} \\ &\leq C(p)\varepsilon^{-2}\|u^0\|_{2,p,\mathcal{O}_l\cap\Omega_1}\end{aligned}$$

Therefore

$$\|w^\varepsilon\|_{V^{2,p}(\Omega_1^\varepsilon)} = \|\sum_{l=1}^{s} w_l^\varepsilon\|_{V^{2,p}(\Omega_1^\varepsilon)} \leq \sum_{l=1}^{s} \|w_l^\varepsilon\|_{V^{2,p}(\mathcal{O}_l\cap\Omega_1^\varepsilon)} \leq C(p)\varepsilon^{-2}\|u^0\|_{2,p,\Omega}$$

Appendix C: The difference between the eigenvalues and eigenfunctions of the homogenized Helmholtz equation (3.24) and those of the modified homogenized Helmholtz equation (4.8)

Here we formulate some results in the spectral theory of linear abstract operator, which are useful for applications considered below.

Let $\mathcal{H}_\tau, 0 < \tau \leq 1$, be a family of Hilbert spaces with scalar products $(u, v)_{\mathcal{H}_\tau}$, and let $\mathcal{H}_0$ be a Hilbert space with a scalar product $(u, v)_{\mathcal{H}_0}$. Consider bounded linear operators $\mathcal{B}_\tau : \mathcal{H}_\tau \to \mathcal{H}_\tau, \mathcal{B}_0 : \mathcal{H}_0 \to \mathcal{H}_0$. We assume that spaces $\mathcal{H}_\tau, \mathcal{H}_0$ and operators $\mathcal{B}_\tau, \mathcal{B}_0$ are subject to the following conditions.

(I). There exist continuous linear operators $\mathcal{R}_\tau : \mathcal{H}_0 \to \mathcal{H}_\tau$ such that

$$\|\mathcal{R}_\tau u\|_{\mathcal{H}_\tau} \leq c_0\|u\|_{\mathcal{H}_0}, \quad \forall u \in \mathcal{H}_0 \tag{C.1}$$

where the constant c_0 is independent of τ; moreover,

$$\lim_{\tau\to 0}(u^\tau, v^\tau)_{\mathcal{H}_\tau} = (u^0, v^0)_{\mathcal{H}_0}, \tag{C.2}$$

provided that

$$\lim_{\tau\to 0}\|u^\tau - \mathcal{R}_\tau u^0\|_{\mathcal{H}_\tau} = 0, \quad \lim_{\tau\to 0}\|v^\tau - \mathcal{R}_\tau v^0\|_{\mathcal{H}_\tau} = 0,$$
$$u^\tau, \quad v^\tau \in \mathcal{H}_\tau, \quad u^0, \quad v^0 \in \mathcal{H}_0$$

(II). The operators $\mathcal{B}_\tau$, $\mathcal{B}_0$ are positive, compact and self-adjoint, and the norms $\|\mathcal{B}_\tau\| = \|\mathcal{B}_\tau\|_{\mathcal{L}(\mathcal{H}_\tau)}$ are bounded by a constant independent of τ.

(III). If $f^\tau \in \mathcal{H}_\tau$, $f^0 \in \mathcal{H}_0$ and

$$\lim_{\tau\to 0} \|f^\tau - \mathcal{R}_\tau f^0\|_{\mathcal{H}_\tau} = 0 \tag{C.3}$$

then

$$\lim_{\tau\to 0} \|\mathcal{B}_\tau f^\varepsilon - \mathcal{R}_\tau \mathcal{B}_0 f^0\|_{\mathcal{H}_\tau} = 0 \tag{C.4}$$

(IV). For any sequence $f^\tau \in \mathcal{H}_\tau$ such that $\sup_\tau \|f^\tau\|_{\mathcal{H}_\tau} < \infty$, there exists a subsequence $f^{\tau'}$ and a vector $w^0 \in \mathcal{H}_0$ such that

$$\|\mathcal{B}_{\tau'} f^{\tau'} - \mathcal{R}_{\tau'} w^0\|_{\mathcal{H}_{\tau'}} \to 0, \quad \text{as} \quad \tau' \to 0 \tag{C.5}$$

Consider the spectral problems for the operators $\mathcal{B}_\tau$:

$$\begin{aligned} & u_\tau^k \in \mathcal{H}_\tau, \quad \mathcal{B}_\tau u_\tau^k = \mu_\tau^k u_\tau^k, \quad k = 1, 2, \cdots \\ & \mu_\tau^1 \geq \mu_\tau^2 \geq \cdots \geq \mu_\tau^k, \quad \mu_\tau^k > 0 \\ & (u_\tau^l, u_\tau^m) = \delta_{lm}, \end{aligned} \tag{C.6}$$

and the spectral problem for $\mathcal{B}_0$:

$$\begin{aligned} & u_0^k \in \mathcal{H}_0, \quad \mathcal{B}_0 u_0^k = \mu_0^k u_0^k, \quad k = 1, 2, \cdots \\ & \mu_0^1 \geq \mu_0^2 \geq \cdots \geq \mu_0^k, \quad \mu_0^k > 0 \\ & (u_0^l, u_0^m) = \delta_{lm}, \end{aligned} \tag{C.7}$$

where δ_{lm} is the Kronecker symbol.

Lemma C.1[16, 24] *Let the space $\mathcal{H}_\tau$, $\mathcal{H}_0$ and operators $\mathcal{B}_\tau$, $\mathcal{B}_0$ satisfy conditions I–IV, then for sufficiently small τ*

$$|\mu_\tau^k - \mu_0^k| \leq 2 \sup_{u \in N(\mu_0^k, \mathcal{B}_0),\ \|u\|_{\mathcal{H}_0} = 1} \|\mathcal{B}_\tau \mathcal{R}_\tau u - \mathcal{R}_\tau \mathcal{B}_0 u\|_{\mathcal{H}_\tau}, \quad k = 1, 2, \cdots \tag{C.8}$$

where μ_τ^k, μ_0^k are eigenvalues of problems (C.6) and (C.7), respectively. $N(\mu_0^k, \mathcal{B}_0) = \{u \in \mathcal{H}_0, \mathcal{B}_0 u = \mu_0^k u\}$ is the eigenspace of operator $\mathcal{B}_0$ corresponding to the eigenvalue μ_0^k.

Lemma C.2[16, 24] *Assume that $k \geq 1$, $t \geq 1$ are integers, and*

$$\mu_0^{k-1} > \mu_0^k = \cdots = \mu_0^{k+t-1} > \mu_0^{k+t} \tag{C.9}$$

i.e. the multiplicity of the eigenvalue μ_0^k is equal to t (here $\mu_0^0 = \infty$). Then for any $w \in N(\mu_0^k, \mathcal{B}_0)$, $\|w\|_{\mathcal{H}_0} = 1$, there exists a linear combination $\bar{u}_\tau$ of eigenvectors $u_\tau^k, \cdots u_\tau^{k+t-1}$ of problem (C.6), such that

$$\text{(C.10)} \qquad \|\bar{u}_\tau - \mathcal{R}_\tau w\|_{\mathcal{H}_\tau} \le M_k \|\mathcal{B}_\tau \mathcal{R}_\tau w - \mathcal{R}_\tau \mathcal{B}_0 w\|_{\mathcal{H}_\tau}$$

where the constant M_k does not depend on τ.

It remains to complete the proof of Theorem 4.1.

Proof. In Lemma C.1, Lemma C.2, choose $0 < \tau = h_0 << 1$, $\mathcal{H}_{h_0} = \mathcal{H}_0 = L^2(\Omega)$, $\mathcal{R}_{h_0} \equiv I$ is an identity operator.

Define the operators $\mathcal{B}_{h_0} : \mathcal{H}_{h_0} \to \mathcal{H}_{h_0}$ setting $\mathcal{B}_{h_0} f^{h_0} = w^{h_0}$, where w^{h_0} is the weak solution of the following problem:

$$\text{(C.11)} \qquad \begin{cases} \widehat{\mathcal{L}}_{h_0} w^{h_0} = f^{h_0} & \text{in} \quad \Omega \\ w^{h_0} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $w^{h_0} \in H^1(\Omega)$, $f^{h_0} \in L^2(\Omega)$.

It follows from Proposition 4.3 that the norm $\|\mathcal{B}_{h_0}\|$ is bounded. The compactness of the operator $\mathcal{B}_{h_0} : \mathcal{H}_{h_0} \to \mathcal{H}_{h_0}$ is due to the compact imbedding $H_0^1(\Omega) \subset L^2(\Omega)$. The fact that $\widehat{\mathcal{L}}_{h_0}$ is symmetric guarantees that $\mathcal{B}_{h_0}$ is a self-adjoint operator in $\mathcal{H}_{h_0}$, since

$$\text{(C.12)} \qquad \begin{aligned} &(\mathcal{B}_{h_0} f^{h_0}, g^{h_0})_{\mathcal{H}_{h_0}} = (\mathcal{B}_{h_0} f^{h_0}, g^{h_0})_{L^2(\Omega)} = (w^{h_0}, g^{h_0})_{L^2(\Omega)} \\ &= (w^{h_0}, \widehat{\mathcal{L}}_{h_0} v^{h_0})_{L^2(\Omega)} = (\widehat{\mathcal{L}}_{h_0} w^{h_0}, v^{h_0})_{L^2(\Omega)} = (f^{h_0}, \mathcal{B}_{h_o} g^{h_0})_{L^2(\Omega)} \end{aligned}$$

where $w^{h_0} = \mathcal{B}_{h_0} f^{h_0}$, $v^{h_0} = \mathcal{B}_{h_0} g^{h_0}$.

Below we need to verify that conditions (I)–(IV) are valid on purpose to use Lemma C.1 and Lemma C.2.

It is easy to see that Condition I is valid due to $\mathcal{R}_{h_0} \equiv I$.

In a similar way, we define the operator $\mathcal{B}_0 : \mathcal{H}_0 \to \mathcal{H}_0$ by $\mathcal{B}_0 f = w$, where w is the solution of the following Dirichlet problem:

$$\text{(C.13)} \qquad \begin{cases} \widehat{\mathcal{L}} w = f & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $w \in H_0^1(\Omega)$, $f \in L^2(\Omega)$

Thus condition II has also been verified.

From (C.11) and (C.13), one obtains

(C.14)

$$-\frac{\partial}{\partial x_i}(\hat{a}_{ij}^{h_0} \frac{\partial (w^{h_0} - w)}{\partial x_j}) + \langle \hat{b} \rangle (w^{h_0} - w) = -\frac{\partial}{\partial x_i}(\hat{r}_{ij} \frac{\partial w}{\partial x_j}) + f^{h_0}(x) - f(x)$$

where $\hat{r}_{ij} = \hat{a}_{ij}^{h_0} - \hat{a}_{ij}$.

Since $w^{h_0}(x) - w(x) \in H_0^1(\Omega)$, it follows from Proposition 4.2 and Poincaré-Friedrichs' inequality that

(C.15)
$$\begin{aligned}\|w^{h_0} - w\|^2_{1,\Omega} &\leq CB(w^{h_0} - w, w^{h_0} - w)\\ &\leq C\Big\{\int_\Omega \hat{r}_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial(w^{h_0} - w)}{\partial x_i}dx + \int_\Omega (f^{h_0} - f)\cdot(w^{h_0} - w)dx\Big\}\\ &\leq C\Big\{h_0^2\|N_i\|_{2,Q}\|N_j\|_{2,Q}\|w\|_{1,\Omega}\|w^{h_0} - w\|_{1,\Omega} + C\|f^{h_0}\\ &\quad - f\|_{0,\Omega}\|w^{h_0} - w\|_{0,\Omega}\Big\}\end{aligned}$$

where $B(u,v) = \int_\Omega \Big(\hat{a}^{h_0}_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \langle\hat{b}\rangle uv\Big)dx.$

Consequently

$$\|w^{h_0} - w\|_{1,\Omega} \leq Ch_0^2\|N_i\|^2_{2,Q}\|w\|_{1,\Omega} + C\|f^{h_0} - f\|_{0,\Omega} \tag{C.16}$$

If $f^{h_0} \to f$ in $L^2(\Omega)$ as $h_0 \to 0$, by using (C.16), we have

$$w^{h_0} \to w \quad \text{in} \quad H_0^1(\Omega) \quad \text{as} \quad h_0 \to 0 \tag{C.17}$$

Therefore condition III holds, too.

Returning to Condition IV, for any sequence $f^{h_0} \in L^2(\Omega)$ such that

$$\sup_{h_0}\|f^{h_0}\|_{L^2(\Omega)} < \infty$$

Since $L^2(\Omega)$ is a reflexive Hilbert space, it follows from Eberlein's Theorem that there exists a subsequence $f^{h_0'} \in L^2(\Omega)$ such that $f^{h_0'} \overset{w}{\rightharpoonup} f \in L^2(\Omega)$. Similarly to (C.15), one can obtain

$$\begin{aligned}&\|w^{h_0'} - w\|^2_{1,\Omega}\\ &\quad\leq CB(w^{h_0'} - w, w^{h_0'} - w)\\ &\quad\leq C\int_\Omega \hat{r}_{ij}\frac{\partial w}{\partial x_j}\frac{\partial(w^{h_0'} - w)}{\partial x_j}dx + C\int_\Omega (f^{h_0'} - f)\cdot(w^{h_0'} - w)dx\\ &\quad= J_1 + J_2\end{aligned} \tag{C.18}$$

$J_1 \to 0$ as $h_0 \to 0$ is due to the fact $\|\hat{r}_{ij}\|_F \leq Ch_0^2\|N_i\|_{2,Q}\|N_j\|_{2,Q}$. Since $f^{h_0'} \overset{w}{\rightharpoonup} f \in L^2(\Omega)$. as $h_0' \to 0$, then $J_2 \to 0$, as $h_0' \to 0$. Thus $w^{h_0'} \to w$ in $H^1(\Omega)$, as $h_0' \to 0$. Therefore condition IV is verified.

Setting $\mu^k_{h_0} = (\tilde{\lambda}^{(0)}_k)^{-1}$, $\quad \mu^k_0 = (\lambda^{(0)}_k)^{-1}$, it follows from Lemma C.2 that

(C.19)
$$|(\tilde{\lambda}^{(0)}_k)^{-1} - (\lambda^{(0)}_k)^{-1}| \leq 2 \sup_{w\in N((\lambda^{(0)}_k)^{-1},\mathcal{B}_0),\quad \|w\|_{\mathcal{H}_0}=1} \|\mathcal{B}_{h_0}w - \mathcal{B}_0 w\|_{\mathcal{H}_{h_0}}$$

where $N((\lambda^{(0)}_k)^{-1}, \mathcal{B}_0)$ defined as in Lemma C.2.

For any $w \in N((\lambda_k^{(0)})^{-1}, \mathcal{B}_0)$, $\quad \|w\|_{\mathcal{H}_0} = 1$, define

$$\begin{cases} \widehat{\mathcal{L}}_{h_0} v^{h_0} = w & \text{in} \quad \Omega \\ v^{h_0} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{C.20}$$

$$\begin{cases} \widehat{\mathcal{L}} v = w & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{C.21}$$

The same computation as in the proof of Proposition 4.2 shows that

$$\|v^{h_0} - v\|_{1,\Omega} \le C h_0^2 \|N_i\|_{2,Q}^2 \tag{C.22}$$

and moreover

$$|(\tilde{\lambda}_k^{(0)})^{-1} - (\lambda_k^{(0)})^{-1}| \le 2 \sup_{w \in N((\lambda_k^{(0)})^{-1}, \mathcal{B}_0),\ \|w\|_{\mathcal{H}_0}=1} \|\mathcal{B}_{h_0} w - \mathcal{B}_0 w\|_{\mathcal{H}_{h_0}}$$
$$\le C\|v^{h_0} - v\|_{1,\Omega} \le C h_0^2 \|N_i\|_{2,Q}^2$$

i.e.

$$|\tilde{\lambda}_k^{(0)} - \lambda_k^{(0)}| \le C_k h_0^2 \|N_i\|_{2,Q}^2$$

Lemma C.2 ensures that

$$\|u_k^0 - \bar{u}_k^0\|_{0,\Omega} \le C_k h_0^2 \|N_i\|_{2,Q}^2 \qquad \square$$

Acknowledgement. The authors thank to The State Key Laboratory of Science-Engineering Computing (China) for kind assistance. For various improvement, such as style and presentation, I am especially indebted to Prof. Dr. Zenger and referees.

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