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A finite element approximation of the Griffith's model in fracture mechanics

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Summary. The Griffith model for the mechanics of fractures in brittle materials is consider in the weak formulation of SBD spaces. We suggest an approximation, in the sense of $\Gamma -$ convergence, by a sequence of discrete functionals defined on finite elements spaces over structured and adaptive triangulations. The quasi-static evolution for boundary value problems is also taken into account and some numerical results are shown.

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1 Introduction

The evolution of irreversible cracks in brittle materials can be effectively described by Griffith's theory ([23], [19]). The basic idea is the following energy balance: let dl be the infinitesimal variation of crack length and let dE be the corresponding variation of linear elastic energy. Griffith suggested that the energy required to increase the crack should be γdl, for a parameter $\gamma > 0$ depending on material toughness. Then he postulated that the fracture evolves only when $-dE \geq \gamma dl$, which simply means that the release of elastic energy $-dE$ must be greater then the energy γdl spent to increase the fracture.

A rigorous mathematical formulation for these problems is given in the framework of the functions with bounded deformation (see [2], [17] and Section 2). Indeed, for these functions the symmetric part of the derivative (in the sense of distributions) is the measure

(1) $$Eu = Du^T + Du = \mathcal{E}u\, \mathcal{L}^n + (u^+ - u^-) \otimes \nu\, \mathcal{H}^{n-1} \llcorner J_u \,.$$

This derivative is the sum of two singular measures: the n-dimensional Lebesgue measure $\mathcal{L}^n$, with density $\mathcal{E}u$, and the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, concentrated on the discontinuity set J_u and which has density $(u^+ - u^-) \otimes \nu$. Thus this space seems to give the natural weak framework for the formulation of fracture problems, in particular, for a two dimensional isotropic material, the linearized elastic energy is given by the Hooke low

$$\int_\Omega W(\mathcal{E}u)\,dx = \int_\Omega \left(\mu|\mathcal{E}u|^2 + \frac{\lambda}{2}|tr(\mathcal{E}u)|^2\right)\,dx\,,$$

where $\mu > 0$ and $\lambda > 0$ are the Lamé coefficients, while the fracture energy of Griffith's type is given by

$$\gamma\mathcal{H}^1(J_u)\,,$$

where $\gamma > 0$ is called the material toughness.

In particular for the applications it is interesting to consider the evolution of the fracture when the displacement u_t (at time t) satisfies a Dirichlet boundary condition $u_t = g(t)$ on a set $\partial\Omega_D \subset \partial\Omega$ and when $\partial\Omega \setminus \partial\Omega_D$ is traction free (for the rigorous mathematical treatise of this problem see [16], [18] and recently [10]). In this situation, knowing the displacement u_t and the corresponding fracture J_{u_t} at time t, the behavior at time $t+dt$ is determined as

$$u_{t+dt} \in \operatorname{argmin}\left\{\int_\Omega W(\mathcal{E}v)\,dx + \gamma\mathcal{H}^1(J_v \setminus J_{u_t})\right\} \tag{2}$$

under the boundary condition $v = g(t+dt)$ in $\partial\Omega_D$ and the irreversibility condition $J_v \supset J_{u_t}$. Indeed, being u_{t+dt} a minimum point, for every function w such that $w = g(t+dt)$ in $\partial\Omega_D$ and $J_w = J_{u_t}$ we have

$$\int_\Omega W(\mathcal{E}u_{t+dt}) + \gamma\mathcal{H}^1(J_{u_{t+dt}} \setminus J_{u_t}) \le \int_\Omega W(\mathcal{E}w)\,dx$$

and then

$$\gamma\mathcal{H}^1(J_{u_{t+dt}} \setminus J_{u_t}) \le -\Big(\int_\Omega W(\mathcal{E}u_{t+dt}) - W(\mathcal{E}w)\,dx\Big)$$

which resembles the Griffith's criterion $-dE \ge \gamma dl$, and which suggests to introduction of the weak Griffith energy

$$\int_\Omega W(\mathcal{E}u)\,dx + \gamma\mathcal{H}^1(J_u)\,. \tag{3}$$

Clearly, in view of realistic simulations of the model, which are being developed (see [5]), we need first a discretization of the energy (3).

Following [11] and [20], we propose in this work an approximation of (3) by means of discrete functionals, defined on finite element spaces, and converging (in the sense of Γ-convergence) to energies of the form

$$\int_\Omega W(\mathcal{E}u)\,dx + \gamma \int_{J_u} \phi(\nu)\,d\mathcal{H}^1\,, \tag{4}$$

where the function $\phi(\nu)$ depends on the choice of the mesh. In particular the case $\phi(\nu) \equiv 1$, corresponding to isotropic fracture energies, is recovered using an adaptive triangulation.

The discrete functionals are quite simple, indeed they are defined in the spaces of piecewise affine functions and they are of the form

$$G_\varepsilon(v_\varepsilon) = \sum_T \frac{1}{\varepsilon} \int_T f(\varepsilon, Dv_\varepsilon)\,dx\,,$$

where T denotes an element of the mesh and v_ε belongs to a finite element space of piecewise affine functions. The function f behaves as

$$f(\varepsilon, Dv_\varepsilon) \simeq \begin{cases} \varepsilon W(Ev_\varepsilon) & \text{when } \varepsilon|Dv_\varepsilon|^2 < \eta \\ f_\infty & \text{when } \varepsilon|Dv_\varepsilon|^2 \geq \eta\,, \end{cases}$$

for some constants $\eta > 0$. The physical interpretation is the following: in the elements T where $\varepsilon|Dv_\varepsilon|^2 < \eta$ the material remains elastic and does not present any fracture, indeed its energy is

$$\frac{1}{\varepsilon}\int_T f(\varepsilon, Dv_\varepsilon)\,dx \simeq \int_T W(Ev_\varepsilon)\,dx\,.$$

On the contrary if $\varepsilon|Dv_\varepsilon|^2 \geq \eta$ then, being $|T| = c\varepsilon^2$,

$$\frac{1}{\varepsilon}\int_T f(\varepsilon, Dv_\varepsilon)\,dx \simeq f_\infty \frac{|T|}{\varepsilon} = f_\infty\, c\, \varepsilon$$

which means that the energy is concentrated in a fracture because $f_\infty c\varepsilon$ is dimensionally equivalent to the Hausdorff measure $\mathcal{H}^1(J_u \cap T)$.

Unfortunately the proof of the convergence result relies on a density result which is known to be true only in SBV spaces. Overcoming this technical problem is possible but using another discretization [1], based on finite differences. Nevertheless this approximation seems to be sufficiently general to predict a realistic physical behavior, as shown by some numerical results for a model problem.

2 Preliminaries and notations

Let Ω be an open bounded polygonal set in $\mathbf{R}^2$. For a positive constant c, denote by $K(\Omega)$ the set of functions $u : \Omega \to \mathbf{R}^2$ such that $|u(x)| \leq c$ for a.e. $x \in \Omega$. The set Ω represents the initial configuration and K the constrain. Moreover let $\mathbf{M}^{2\times 2}$ be the space of 2×2 matrices with the norm $|M|^2 = \sum_{i,j} |m_{ij}|^2$ and let $\mathbf{M}^{2\times 2}_{sym}$ be the subspace of symmetric matrices.

The functional framework of the problem is given by the spaces $SBV^2(\Omega, \mathbf{R}^2)$ and $SBD^2(\Omega, \mathbf{R}^2)$. For a detailed theory of SBV and SBD functions we refer to [3] and [2], [17] respectively, here we will recall some definitions and basic properties.

For a function $u \in L^1(\Omega, \mathbf{R}^2)$ let S_u be the set of non-Lebesgue points of u. The function u is a function of bounded variation if its distributional derivative Du is a vector valued measure with finite total variation in Ω. For such functions it's possible to prove that the set S_u is countably rectifiable and that for $\mathcal{H}^1 - a.e.\, x \in S_u$ there exists (in a measure geometric sense) a normal ν_u and two one-sided traces u^+, u^-. The measure Du can be decomposed, by Radon-Nikodym theorem, as $Du = \nabla u\, \mathcal{L}^2 + D^s u$ where $\nabla u\, \mathcal{L}^2$ is absolutely continuous with respect to the Lebesgue measure and $D^s u$ is singular. If the singular part can be written as

$$D^s u = (u^+ - u^-) \otimes \nu\, \mathcal{H}^{n-1} \llcorner S_u \,,$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, then we say that u is a special function with bounded variation and we write $u \in SBV(\Omega, \mathbf{R}^2)$. If in addition $\nabla u \in L^2(\Omega, \mathbf{R}^2)$ and $\mathcal{H}^1(S_u) < \infty$ then we say that $u \in SBV^2(\Omega, \mathbf{R}^2)$.

The case of SBD function is slightly more complicated due to the presence of symmetric gradients. For SBD function it is still not known if for $\mathcal{H}^1 - a.e.\; x \in S_u$ there exists a normal ν_u and the traces u^+, u^-. Nevertheless we can define the set $J_u \subset S_u$ to be the collection of points such that there exist the one-sided approximate limits u^+ and u^- with respect to a suitable direction ν_u. This is enough to develop our theory, i.e. to prove compactness and lower semicontinuity theorems. In particular we say that a function $u \in L^1(\Omega, \mathbf{R}^2)$ is a function with bounded deformation if the symmetric distributional gradient $Eu = \nabla u + \nabla u^T$ is a vector valued measure with finite total variation. Moreover, as in the case of SBV functions, if Eu can be written as

$$Eu = \mathcal{E}u\, \mathcal{L}^n + (u^+ - u^-) \otimes \nu\, \mathcal{H}^{n-1} \llcorner J_u$$

then we say that u is a special function with bounded deformation and we write $u \in SBD(\Omega, \mathbf{R}^2)$. Finally, if $\mathcal{E}u \in L^2(\Omega, \mathbf{R}^2)$ and if $\mathcal{H}^1(S_u) < \infty$ we say that $u \in SBD^2(\Omega, \mathbf{R}^2)$.

Even if the Γ-convergence holds in $SBV^2(\Omega, \mathbf{R}^2)$ the proof of Γ-liminf requires the following compactness and lower semicontinuity result in $SBD^2(\Omega, \mathbf{R}^2)$ which slightly generalizes Theorem 1.1 and Corollary 1.2 in [4] to the case of anisotropic energies.

Proposition 1 *Let Ω be an open bounded set in $\mathbf{R}^2$, let $\phi : \mathbf{R}^2 \to \mathbf{R}$ be convex, positively 1-homogeneous and pair, let $f : \mathbf{M}^{2\times 2}_{sym} \to \mathbf{R}$ be convex and lower semicontinuous, let $\{u_j\}$ be a sequence in $SBD^2(\Omega, \mathbf{R}^2)$ such that*

$$\int_\Omega |\mathcal{E}u_j|^2 dx + \mathcal{H}^1(J_{u_j}) + \|u_j\|_\infty \le c < +\infty \tag{5}$$

then there exists a function $u \in SBD(\Omega, \mathbf{R}^2)$ and subsequence (not relabeled) such that

$$u_j \to u \quad in\ L^1_{loc}(\Omega, \mathbf{R}^2)\ , \tag{6}$$

$$\int_\Omega f(\mathcal{E}u)dx \le \liminf_{j\to+\infty} \int_\Omega f(\mathcal{E}u_j)dx\ , \tag{7}$$

$$\mathcal{H}^1(J_u) \le \liminf_{j\to+\infty} \mathcal{H}^1(J_{u_j})\ , \tag{8}$$

$$\int_{J_u} \phi(\nu)d\mathcal{H}^1 \le \liminf_{j\to+\infty} \int_{J_{u_j}} \phi(\nu)d\mathcal{H}^1\ . \tag{9}$$

Proof. From the compactness and lower semicontinuity result contained in [4] follows the existence of a subsequence, denoted as $\{u_j\}$ such that $u_j \to u$ in $L^1_{loc}(\Omega, \mathbf{R}^2)$ and such that inequalities (7) and (8) holds.

To prove (9) let us first consider a function $\phi(v)$ defined as $\phi(v) = |\langle v, \zeta\rangle|$ for $\zeta \in S^1$. For $\xi \in S^1$ let $\Pi_\xi = \{x \in \mathbf{R}^2 : \langle x, \xi\rangle = 0\}$, Ω_ξ be the projection of Ω on the line Π_ξ and $\Omega_{y,\xi} = \{t \in \mathbf{R} : \text{for } y \in \Pi_\xi \quad y + t\xi \in \Omega\}$. Moreover for $y \in \Pi_\xi$ and $\xi \in S^1$, given $v \in SBD(\Omega, \mathbf{R}^2)$ let $J_{v,\xi} = \{x \in J_v : \langle v^+(x) - v^-(x), \xi\rangle \ne 0\}$, $v_{y,\xi} : \Omega_{y,\xi} \to \mathbf{R}$ be defined as $v_{y,\xi} = \langle v(y+t\xi), \xi\rangle$ and finally let $A_{y,\xi}(v)$ and $B_{y,\xi}(v)$ be respectively the total variation of the absolutely continuous part and the counting measure of singular part of $v'_{y,\xi}$ in $\Omega_{y,\xi}$, namely

$$A_{y,\xi}(v) = |v'_{y,\xi}|(\Omega_{y,\xi}) \qquad B_{y,\xi}(v) = \mathcal{H}^0(J_{v_{y,\xi}}) = \#(J_{v_{y,\xi}})\ .$$

Being $\{u_j\} \subset SBD(\Omega, \mathbf{R}^2)$ then by Proposition 2.1 in [4] for every $\xi \in S^1$ and for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$ we have $(u_j)_{y,\xi} \in SBV(\Omega_{y,\xi})$ for every j and

$$\int_{\Omega_\xi} A_{y,\xi}(u_j)d\mathcal{H}^1(y) < +\infty\ . \tag{10}$$

Moreover for $\mathcal{H}^1$- a.e. $\xi \in S^1$ we have $J_{u_j,\xi} = J_{u_j}$. Now take $\xi \in S^1$ such that $J_{u_j,\xi} = J_{u_j}$ for every j, then

$$\int_{J_{u_j}} |\langle \nu, \xi \rangle| d\mathcal{H}^1(y) = \int_{\Omega_\xi} B_{y,\xi}(u_j) d\mathcal{H}^1(y)\,, \tag{11}$$

moreover, being $\mathcal{H}^1(J_{u_j}) < +\infty$, there exist a subsequence, denoted $\{u_k\}$, such that

$$\lim_{k\to+\infty} \int_{\Omega_\xi} B_{y,\xi}(u_k) d\mathcal{H}^1(y) = \liminf_{j\to+\infty} \int_{\Omega_\xi} B_{y,\xi}(u_j) d\mathcal{H}^1(y) < +\infty\,.$$

Take $\varepsilon \in (0, 1)$ and take a subsequence $\{u_l\}$ of $\{u_k\}$, such that

$$\begin{aligned} &\lim_{l\to+\infty} \int_{\Omega_\xi} \varepsilon A_{y,\xi}(u_j) + B_{y,\xi}(u_j) d\mathcal{H}^1(y) \\ &= \liminf_{j\to+\infty} \int_{\Omega_\xi} \varepsilon A_{y,\xi}(u_j) + B_{y,\xi}(u_j) d\mathcal{H}^1(y) < +\infty \end{aligned}$$

and such that for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$

$$u_{y,\xi} \in SBV(\Omega_{y,\xi}) \qquad (u_l)_{y,\xi} \to u_{y,\xi} \text{ strongly in } L^1_{loc}(\Omega_{y,\xi})\ .$$

By Fatou's Lemma we get that

$$\liminf_{l\to+\infty} \Big(\varepsilon A_{y,\xi}(u_l) + B_{y,\xi}(u_l) \Big) < +\infty$$

for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$. Let $y \in \Omega_\xi$ such that the previous inequality is satisfied, then there exists a further subsequence $\{u_n\}$ such that

$$\lim_{n\to+\infty} \Big(\varepsilon A_{y,\xi}(u_n) + B_{y,\xi}(u_n) \Big) = \liminf_{l\to+\infty} \Big(\varepsilon A_{y,\xi}(u_l) + B_{y,\xi}(u_l) \Big) < +\infty\,,$$

then by Ambrosio's compactness and lower semicontinuity Theorem in $SBV(\Omega_{y,\xi})$ [3] there exists a subsequence $\{u_h\}$ such that

$$\begin{aligned} B_{y,\xi}(u) &\le \liminf_{h\to+\infty} B_{y,\xi}(u_h) \\ &\le \liminf_{h\to+\infty} \Big(\varepsilon A_{y,\xi}(u_h) + B_{y,\xi}(u_h) \Big) \\ &\le \lim_{n\to+\infty} \Big(\varepsilon A_{y,\xi}(u_n) + B_{y,\xi}(u_n) \Big) \\ &\le \liminf_{l\to+\infty} \Big(\varepsilon A_{y,\xi}(u_l) + B_{y,\xi}(u_l) \Big) < +\infty\,. \end{aligned}$$

The previous inequality holds for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$ so by Fatou's Lemma

$$\begin{aligned}\int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1(y) &= \int_{\Omega_\xi} B_{y,\xi}(u) d\mathcal{H}^1(y)\\ &\le \liminf_{l\to+\infty} \int_{\Omega_\xi} \varepsilon A_{y,\xi}(u_l) + B_{y,\xi}(u_l) d\mathcal{H}^1(y)\\ &\le \varepsilon \limsup_{l\to+\infty} \int_{\Omega_\xi} A_{y,\xi}(u_l) d\mathcal{H}^1(y)\\ &\quad + \liminf_{l\to+\infty} \int_{\Omega_\xi} B_{y,\xi}(u_l) d\mathcal{H}^1(y)\\ &\le c\varepsilon + \lim_{k\to+\infty} \int_{\Omega_\xi} B_{y,\xi}(u_k) d\mathcal{H}^1(y)\\ &\le c\varepsilon + \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1 . \qquad (12)\end{aligned}$$

It remains to prove (12) for every $\xi \in S^1$. Let $\zeta \in S^1$ and let $\delta > 0$ then there exist ξ_δ such that $|\zeta - \xi_\delta| < \delta$ and

$$\int_{J_u} |\langle \nu, \xi\rangle| d\mathcal{H}^1 \le \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1 . \qquad (13)$$

It follows that

$$\begin{aligned}\int_{J_u} |\langle \nu, \zeta\rangle| d\mathcal{H}^1 &\le \int_{J_u} |\langle \nu, \xi_\delta\rangle| d\mathcal{H}^1 + c_1\delta\\ &\le \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi_\delta\rangle| d\mathcal{H}^1 + c_1\delta\\ &\le \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi_\delta - \zeta\rangle| + |\langle \nu, \zeta\rangle| d\mathcal{H}^1 + c_1\delta\\ &\le \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \zeta\rangle| d\mathcal{H}^1 + c_2\delta\end{aligned}$$

where c_1 and c_2 does no depend on δ.

At this point the lower semicontinuity inequality is proved for every function $\phi(v) = |\langle v, \zeta\rangle|$ for $\zeta \in S^1$. If ϕ is convex, 1-homogeneous and pair it can be written as

$$\phi(v) = \sup\{\psi(v) \,:\, \psi(u) = \langle u, \eta\rangle - c \;\text{ and }\; \psi(u) \le \phi(u)\} \qquad (14)$$

In (14) it is not restrictive to take ψ linear, indeed for every $\xi \in S^1$, being ϕ 1-homogeneous, if $\langle u, \eta\rangle - c \le \phi(u)$ for every u then $\langle u, \eta\rangle \le \phi(u)$. Thus we can define a set Θ such that

$$\phi(v) = \sup_{\xi\in\Theta} |\langle v, \xi\rangle| .$$

By the lower semicontinuity we have for every $\xi \in \Theta$

$$\int_{J_u} |\langle \nu, \xi \rangle| d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi \rangle| d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} \phi(\nu) d\mathcal{H}^1$$

and by a supremum of measures argument it is easy to deduce that

$$\int_{J_u} \phi(\nu) d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} \phi(\nu) d\mathcal{H}^1$$

which gives inequality (9). □

Moreover the proof of the Γ-limsup inequality relies on the following approximation result which is known to be true only for SBV function. We first introduce a set of regular functions with discontinuities.

Definition 1 *Let $\mathcal{W}(\Omega, \mathbf{R}^m)$ be the set of $u \in SBV(\Omega, \mathbf{R}^m)$ such that*

1. *S_u is essentially closed, i.e. $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$*
2. *$\overline{S_u}$ is the union of a finite number of $(n-1)$-simplexes*
3. *$u \in W^{k,\infty}(\Omega \setminus \overline{S_u}, \mathbf{R}^m) \forall k \in \mathbf{N}$.*

The following approximation result (proved under more general conditions in [14]) shows the importance of the space $\mathcal{W}(\Omega, \mathbf{R}^2)$.

Lemma 1 *Let $g(v)$ be a norm in $\mathbf{R}^2$ and let $u \in SBV^2(\Omega, \mathbf{R}^2) \cap L^\infty(\Omega, \mathbf{R}^2)$, then there exists a sequence $w_k \in \mathcal{W}(\Omega, \mathbf{R}^2)$ such that*

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega, \mathbf{R}^2)\,, \tag{15}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^2(\Omega, \mathbf{R}^2)\,, \tag{16}$$

$$\limsup_{k\to\infty} \|w_k\|_\infty \leq \|u\|_\infty\,. \tag{17}$$

For every open set $A \subset \Omega$ we have

$$\limsup_{k\to\infty} \int_{S_{w_k} \cap A} g(\nu)\, d\mathcal{H}^1 \leq \int_{S_u \cap \bar{A}} g(\nu)\, d\mathcal{H}^1\,. \tag{18}$$

Moreover we can assume that for every k the jump set S_{w_k} is a finite union of disjoint segments which do not intersect the boundary $\partial\Omega$.

An analogous result for boundary value problems is proved [21].

3 Statement of the convergence result

Let Ω be an open bounded polygonal set in $\mathbf{R}^2$. For a positive constant k, denote by $K(\Omega)$ the set of functions $u \in L^1(\Omega, \mathbf{R}^2)$ such that $|u(x)| \le k$ for a.e. $x \in \Omega$. The set Ω represents the reference configuration and K the constraint. Moreover let $\mathbf{M}^{2\times 2}$ be the space of 2×2 matrices with the norm $|M|^2 = \sum_{i,j} |m_{ij}|^2$ and let $\mathbf{M}^{2\times 2}_{sym}$ be the subspace of symmetric matrices.

For $i = 1, 2, 3$, let $\mathbf{T}^i_\varepsilon$ be the triangulations of $\mathbf{R}^2$ having the geometries represented in Figure 1. The corresponding finite element spaces, denoted by $V^i_\varepsilon(\Omega, \mathbf{R}^2)$, are the classical spaces of piecewise affine functions on $\mathbf{T}^i_\varepsilon$ restricted to Ω. Moreover given $\theta \in (0, \frac{\pi}{2})$ and an infinitesimal sequence $d_\varepsilon \ge 6\varepsilon$, let $\mathbf{T}^\theta_\varepsilon$ be the family of triangulations $\mathbf{T}_\varepsilon$ such that for every element the amplitude of the internal angles θ_i and the length of the edges ζ_i satisfy

$$\theta \le \theta_i \qquad \varepsilon \le \mathcal{H}^1(\zeta_j) \le d_\varepsilon\,. \tag{19}$$

The corresponding finite element *set*, given by the union of the spaces $V^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ defined on $\mathbf{T}_\varepsilon$, will be denoted by $\mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$.

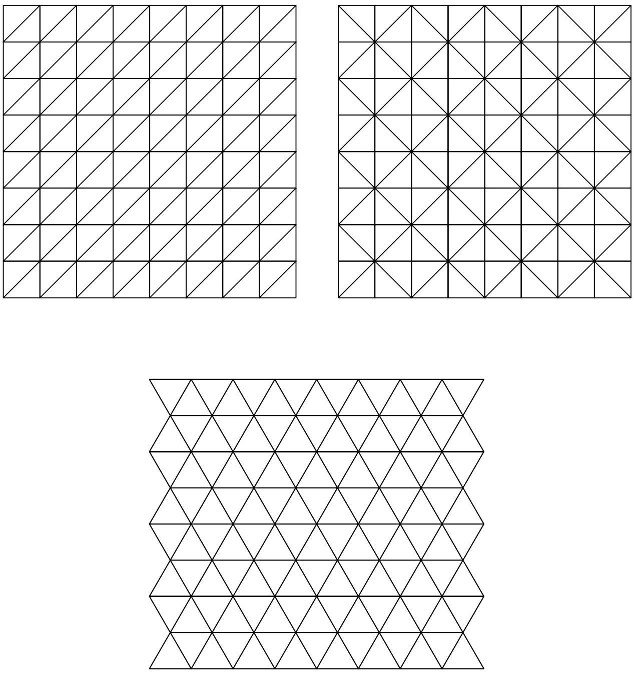


Fig. 1. Geometries of the triangulations $\mathbf{T}^i_\varepsilon$ for $i = 1, 2, 3$ respectively

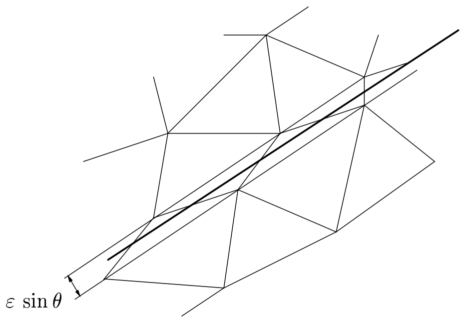


Fig. 2. An example of mesh $\mathbf{T}_\varepsilon \in \mathbf{T}_\varepsilon^\theta$ adapted along a straight line

Let $\psi : [0, +\infty) \to [0, 1]$ be a non decreasing function such that

(20) $$\psi(t) = o(t) \quad \text{for } t \to 0\,,$$

(21) $$(1 - \psi(t)) = o\Big(\frac{1}{t}\Big) \quad \text{for } t \to +\infty\,,$$

and such that for t large the function $(1 - \psi(t))t$ is non-increasing. Given $M \in \mathbf{M}^{2\times 2}$ let the strain energy density be defined as

(22) $$W(M^{sym}) = \mu|M^{sym}|^2 + \frac{\lambda}{2}|tr(M^{sym})|^2\,,$$

for $\mu > 0$ and $\lambda > 0$, and let

(23) $$f(\varepsilon, M) = \varepsilon W(M^{sym})(1 - \psi(\varepsilon|M|^2)) + \gamma\psi(\varepsilon|M|^2)\,.$$

Using the structured triangulations $\mathbf{T}_\varepsilon^i$ (for $i = 1, 2, 3$), the discrete functionals $F_\varepsilon^i(u)$ are defined as

(24) $$F_\varepsilon^i(u) = \sum_{T\in\mathbf{T}_\varepsilon^i} \frac{1}{\varepsilon}\int_{T\cap\Omega} f(\varepsilon, Dv_\varepsilon)dx$$

if $v_\varepsilon \in V_\varepsilon^i(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F_\varepsilon^i(u) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. The convergence result is the following.

Theorem 1 *For every mesh* $\mathbf{T}_\varepsilon^i$ *let* $\phi_i : \mathbf{R}^2 \to [0, +\infty)$ *be the anisotropy function (depending only on the geometry of the triangulation) defined in Section 4. Let the limit functional be given by*

(25) $$F^i(u) = \int_\Omega W(Eu)dx + \gamma\int_{J_u}\phi_i(\nu_u)d\mathcal{H}^1$$

if $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $F^i(u) = +\infty$ *otherwise in* $L^1(\Omega, \mathbf{R}^2)$. *Then for every* $u \in L^1(\Omega, \mathbf{R}^2)$ *and for every sequence* $v_{\varepsilon_j} \in V^i_{\varepsilon_j}(\Omega, \mathbf{R}^2)$, *converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$, *we have*

$$F^i(u) \le \liminf_{\varepsilon_j \to 0} F^i_{\varepsilon_j}(v_{\varepsilon_j}) .$$

Moreover for every $u \in SBV^2(\Omega, \mathbf{R}^2)$ *there exists a sequence* $v_{\varepsilon_j} \in V_{\varepsilon_j}(\Omega, \mathbf{R}^2)$, *converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$, *such that*

$$F^i(u) \ge \limsup_{\varepsilon_j \to 0} F^i_{\varepsilon_j}(v_{\varepsilon_j}) .$$

For the isotropic case, we consider the functional

$$\mathcal{F}^\theta_{\varepsilon_j}(v_{\varepsilon_j}) = \sum_{T \in \mathbf{T}^\theta_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx \tag{26}$$

if $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $\mathcal{F}^\theta_\varepsilon(v_\varepsilon) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. Then the convergence result is the following.

Theorem 2 *Let the limit functional be*

$$\mathcal{F}^\theta(u) = \int_\Omega W(Eu)dx + \gamma \sin\theta \, \mathcal{H}^1(J_u) \tag{27}$$

if $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $\mathcal{F}^\theta(u) = +\infty$ *otherwise in* $L^1(\Omega, \mathbf{R}^2)$. *Then for every* $u \in L^1(\Omega, \mathbf{R}^2)$ *and for every sequence* $v_{\varepsilon_j} \in \mathcal{V}^\theta_{\varepsilon_j}(\Omega, \mathbf{R}^2)$, *converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$, *we have*

$$\mathcal{F}^\theta(u) \le \liminf_{\varepsilon_j \to 0} \mathcal{F}^\theta_{\varepsilon_j}(v_{\varepsilon_j}) .$$

Moreover for every $u \in SBV^2(\Omega, \mathbf{R}^2)$ *there exists a sequence* $v_{\varepsilon_j} \in \mathcal{V}^\theta_{\varepsilon_j}(\Omega, \mathbf{R}^2)$, *converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$, *such that*

$$\mathcal{F}^\theta(u) \ge \limsup_{\varepsilon_j \to 0} \mathcal{F}^\theta_{\varepsilon_j}(v_{\varepsilon_j}) .$$

Remark 1 The easiest choice for the function ψ is given by

$$\psi(t) = \begin{cases} 0 \text{ if } t < \delta \\ 1 \text{ otherwise,} \end{cases}$$

nevertheless conditions (20) and (21) allow the use of smooth functions (such as $\frac{2}{\pi}\arctan(t^n)$ for $n \ge 2$) which are much better for the numerical implementation.

Note that the Γ-limsup inequality is not complete because the proof is based on a density argument which is not yet known for the case $u \in SBD^2(\Omega, \mathbf{R}^2) \setminus SBV^2(\Omega, \mathbf{R}^2)$. Complete convergence results can be stated as follows (but these formulations cannot ensure compactness for sequences of minima).

Remark 2 For $i = 1, 2, 3$ let the discrete functionals $F^i_\varepsilon(u)$ be defined as

$$F^i_\varepsilon(u) = \sum_{T \in \mathbf{T}^i_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx$$

if $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F^i_\varepsilon(u) = +\infty$ otherwise in $SBV^2(\Omega, \mathbf{R}^2)$. The functionals F^i_ε Γ-converge (as $\varepsilon \to 0$), respect to the strong topology of $L^1(\Omega, \mathbf{R}^2)$, to the functional

$$F^i(u) = \int_\Omega W(Eu) dx + \gamma \int_{J_u} \phi_i(\nu_u) d\mathcal{H}^1$$

if $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F^i(u) = +\infty$ otherwise in $SBV^2(\Omega, \mathbf{R}^2)$.

Remark 3 Let the isotropic limit functional be given by

$$\mathcal{F}^\theta_\varepsilon(v_\varepsilon) = \sum_{T \in \mathbf{T}^\theta_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx$$

if $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $\mathcal{F}^\theta_\varepsilon(v_\varepsilon) = +\infty$ otherwise in $SBV^2(\Omega, \mathbf{R}^2)$. The functionals $\mathcal{F}^\theta_\varepsilon$ Γ-converge (as $\varepsilon \to 0$), respect to the strong topology of $L^1(\Omega, \mathbf{R}^2)$, to the functional

$$\mathcal{F}(u) = \int_\Omega W(Eu) dx + \gamma \sin\theta \, \mathcal{H}^1(J_u)$$

if $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $\mathcal{F}(u) = +\infty$ otherwise in $SBV^2(\Omega, \mathbf{R}^2)$.

4 The anisotropy functions

The anisotropy functions appearing in (25) have been explicitly computed and studied in [20] for a similar functional, here we report only the main properties useful in the sequel. Let $S \subset \Omega$ be a segment and let ν be its unit normal, let

$$\mathbf{S}^i_\varepsilon = \{T \in \mathbf{T}^i_\varepsilon \, : \, T \cap S \neq \emptyset \text{ and } |T \cap (S + t\nu)| \neq \emptyset \text{ for } t \in (0, 1)\}$$

then we can define S^i_ε as the covering of S given by the union of the elements belonging to $\mathbf{S}^i_\varepsilon$. The functions ϕ_i are defined as a Minkowsky content of S

in the topology induced by $\mathbf{T}^i_\varepsilon$, indeed for every segment S with unit normal ν we have

$$\phi_i(\nu) = \limsup_{\varepsilon \to 0} \frac{|S^i_\varepsilon|}{\varepsilon \mathcal{H}^1(S)} \tag{28}$$

so that $\phi_i(\nu)$ measures in the limit the ratio between the measure of covering S^i_ε and the measure of the tubular neighborhood of S. Moreover it is not difficult to verify that the same relation is valid also for piecewise affine curves in the plane. In Figure 3 are reported the levels curves $\{\phi_i(v) = 1\}$, for $i = 1, 2, 3$ respectively. It's clear that ϕ_i reflects the symmetry properties of the mesh and that the more the level curve is close to a circle the more the triangulation is isotropic, thus the values

$$m_i = \inf\{\phi_i(\nu) \, : \, |\nu| = 1\} \qquad M_i = \sup\{\phi_i(\nu) \, : \, |\nu| = 1\}$$

suggests a possible measurement of the anisotropy, given by the ratio $a_i = M_i/m_i$. In our case $a_1 = \sqrt{2}$, $a_2 \approx 1.118$ and $a_3 = \frac{2\sqrt{3}}{3} = 1.154$.

These functions are convex, positively 1-homogeneous and pair. They have an easy representation in terms of scalar products. Indeed let

$$\xi_{1,1} = (1, 0) \quad \xi_{1,2} = (\sqrt{2}/2, \sqrt{2}/2) \quad \xi_{1,3} = (0, 1)\,,$$

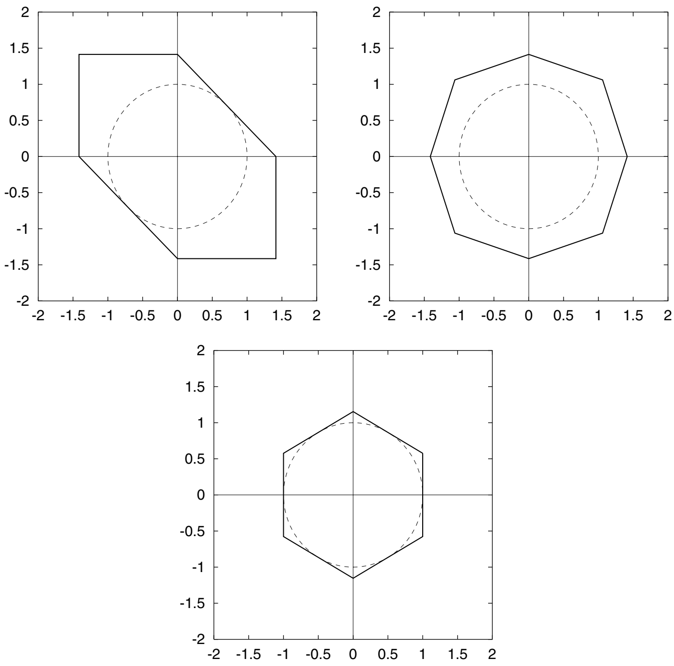


Fig. 3. The level curves compared with the unit circle

$$c_{1,1} = c_{1,3} = \sqrt{2}/2 \quad c_{1,2} = 1\,,$$

then we can write

$$\phi_1(\nu) = \max_{1 \le k \le 3} c_{1,k} |\langle \nu, \xi_{1,k} \rangle|\,. \tag{29}$$

Moreover let

$$\xi_{2,k} = \xi_{1,k} \text{ for } k = 1, 2, 3 \quad \xi_{2,4} = (-\sqrt{2}/2, \sqrt{2}/2) \quad \xi_{2,5} = (-1, 0)\,,$$

$$c_{2,k} = c_{1,k}/2 \text{ for } k = 1, 2, 3 \quad c_{2,4} = c_{2,2} \quad c_{2,5} = c_{2,1}\,,$$

then

$$\phi_2(\nu) = \max_{1 \le k \le 4} \{ c_{2,k} |\langle \nu, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu, \xi_{2,k+1} \rangle| \}\,. \tag{30}$$

Finally let

$$\xi_{3,1} = (-1, 0) \quad \xi_{3,2} = (1/2, \sqrt{3}/2) \quad \xi_{3,3} = (-1/2, \sqrt{3}/2)\,,$$

$$c_{3,k} = 1 \text{ for } k = 1, 2, 3\,,$$

then we have

$$\phi_3(\nu) = \max_{1 \le k \le 3} c_{3,k} |\langle \nu, \xi_{3,k} \rangle|\,. \tag{31}$$

On the contrary the idea for the isotropic approximation is to orient the elements along discontinuities (see Figure 2) in order to have a tubular neighborhood and consequently an isotropic approximation of the Hausdorff measure. This property is explained in the following lemma (for the proof see Appendix A in [11]).

Lemma 2 *Let S be the union of a finite number of disjoint segments S_m, then there exists a family of triangulations $\mathbf{T}_\varepsilon \in \mathbf{T}^\theta_\varepsilon$ such that*

$$\limsup_{\varepsilon \to 0} \frac{|S^\theta_\varepsilon|}{\varepsilon} = \sin\theta \, \mathcal{H}^1(S)\,, \tag{32}$$

where S^θ_ε is the covering of S in $\mathbf{T}_\varepsilon$.

5 Γ-liminf inequality

The proof of the Γ-liminf inequality is based on the measure theoretic argument (see [7] Proposition 1.16), which requires the localization of the functionals.

Definition 2 *Let $A \subset \Omega$ be an open set, the localized functionals are defined as*

$$F^i_\varepsilon(v_\varepsilon, A) = \sum_{T \in \mathbf{T}^i_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap A} f(\varepsilon, Dv_\varepsilon)\, dx$$

if $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F^i_\varepsilon(v_\varepsilon, A) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$.

We consider first the case of the structured triangulations.

Proposition 2 *For $i = 1, 2, 3$, denote by $\underline{F}^i(u)$ the Γ- $\liminf_{\varepsilon\to 0} F^i_\varepsilon(u)$ and take $u \in L^1(\Omega, \mathbf{R}^2)$ such that $\underline{F}^i(u) < +\infty$, then $u \in K(\Omega) \cap SBD^2(\Omega, \mathbf{R}^2)$ and*

$$\int_\Omega W(Eu)\,dx + \gamma \int_{J_u} \phi_i(\nu_u)\,d\mathcal{H}^1 \le \liminf_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon) \tag{33}$$

for every sequence $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$.

The proof of Proposition 2 requires some preliminary lemmas on the localized functionals.

Lemma 3 *For some positive constants α, β, for every open set $A \subset \Omega$ and for every function $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ there exists $v \in SBD^2(\Omega, \mathbf{R}^2)$ satisfying*

$$\int_{A_\varepsilon} W(Ev)\,dx + \alpha\,\mathcal{H}^1\,(J_v \cap A_\varepsilon) \le F^i_\varepsilon(v_\varepsilon, A)\,, \tag{34}$$

$$|\{x \in \Omega \;:\; v_\varepsilon(x) \ne v(x)\}| \le \beta\varepsilon F^i_\varepsilon(v_\varepsilon, A)\,, \tag{35}$$

where $A_\varepsilon = \{x \in A \;:\; d(x, \partial A) > \varepsilon\}$.

Proof. Let $\tau > 0$ such that

$$\sup\{\varepsilon W(M^{sym}) \text{ for } \varepsilon|M| < \tau\} \le \gamma$$

and define

$$\tilde\psi(t) = \begin{cases} 0 & \text{for } t < \tau \\ \psi(t) & \text{otherwise.} \end{cases} \tag{36}$$

Then the function

$$\tilde f(\varepsilon, M) = \varepsilon W(M^{sym}) = \varepsilon W(M^{sym})\Big(1 - \tilde\psi(\varepsilon|M|^2)\Big) + \gamma\tilde\psi(\varepsilon|M|^2) \tag{37}$$

satisfies $f(\varepsilon, M) \ge \tilde f(\varepsilon, M)$. Indeed if $\varepsilon|M|^2 \ge \tau$ then $\tilde\psi(\varepsilon|M|^2) = \psi(\varepsilon|M|^2)$, while for $\varepsilon|M|^2 < \tau$, being $\varepsilon W(M^{sym}) \le \gamma$ we have

$$\begin{aligned}\tilde f(\varepsilon, M) &= \varepsilon W(M^{sym})\Big(1 - \psi(\varepsilon|M|^2)\Big) + \varepsilon W(M^{sym})\psi(\varepsilon|M|^2) \\ &\le \varepsilon W(M^{sym})\Big(1 - \psi(\varepsilon|M|^2)\Big) + \gamma\psi(\varepsilon|M|^2) = f(\varepsilon, Du)\,.\end{aligned}$$

Moreover, being $\psi(t)$ non decreasing, $\tilde\psi(t) \ge \psi(\tau)$ for $t \ge \tau$ so that for $\varepsilon|M|^2 \ge \tau$

$$\tilde f(\varepsilon, M) = \varepsilon W(M^{sym})\Big(1 - \psi(\varepsilon|M|^2)\Big) + \gamma\psi(\varepsilon|M|^2) \ge \gamma\psi(\tau)\,.$$

Finally, given an open set $A \subset \Omega$ and given $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ let $\mathbf{A}^i_\varepsilon = \{T \in \mathbf{T}^i_\varepsilon \,:\, T \subset A\}$ and

$$\mathbf{A}^{i,\flat}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, \varepsilon|Dv_\varepsilon|^2 \le \tau\} \qquad \mathbf{A}^{i,\sharp}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, T \notin \mathbf{A}^{i,\flat}_\varepsilon\}\,.$$

Define $A^{i,\flat}_\varepsilon \subset \Omega$ and $A^{i,\sharp}_\varepsilon \subset \Omega$ as the union of the elements belonging to $\mathbf{A}^{i,\flat}_\varepsilon$ and $\mathbf{A}^{i,\sharp}_\varepsilon$ respectively. Then it follows by the previous inequalities that

$$\begin{aligned} F^i_\varepsilon(v_\varepsilon, A) &\ge \sum_{T\in\mathbf{A}^i_\varepsilon} \frac{1}{\varepsilon} f(\varepsilon, Dv_\varepsilon)|T| \\ &\ge \sum_{T\in\mathbf{A}^{i,\flat}_\varepsilon} \frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon)|T| + \sum_{T\in\mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon)|T| \\ &\ge \sum_{T\in\mathbf{A}^{i,\flat}_\varepsilon} W(Ev_\varepsilon)|T| + \sum_{T\in\mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon}\gamma\psi(\tau)|T|\,. \end{aligned} \tag{38}$$

Let $v \in SBD^2(\Omega, \mathbf{R}^2)$ be defined as

$$v = \begin{cases} v_\varepsilon & \text{in } A^{i,\flat}_\varepsilon \\ 0 & \text{in } \Omega \setminus A^{i,\flat}_\varepsilon \end{cases},$$

then from (38) follows

$$|\{x \in \Omega \,:\, v_\varepsilon(x) \ne v(x)\}| = |A^{i,\sharp}_\varepsilon| = \sum_{T\in\mathbf{A}^{i,\sharp}_\varepsilon} |T| \le \frac{\varepsilon}{\gamma\psi(\tau)} F^i_\varepsilon(v_\varepsilon, A)\,,$$

which proves (35) for $\beta = 1/(\gamma\psi(\tau))$. Moreover, being $\mathcal{H}^1(\partial T) \le c_i\varepsilon$, it easy to check that for a positive value of α we have

$$\frac{1}{\varepsilon}|T|\gamma\psi(\tau) \ge \alpha\mathcal{H}^1(\partial T)\,.$$

Thus from (38) follows

$$\begin{aligned} F^i_\varepsilon(v_\varepsilon, A) &\ge \sum_{T\in\mathbf{A}^{i,\flat}_\varepsilon} W(Ev)|T| + \sum_{T\in\mathbf{A}^{i,\sharp}_\varepsilon} \alpha\mathcal{H}^1(\partial T) \\ &\ge \int_{A_\varepsilon} W(Ev)\,dx + \alpha\mathcal{H}^1(J_v \cap A_\varepsilon)\,, \end{aligned}$$

which proves inequality (34). □

Lemma 4 *Let* $i = 1, 2, 3$*, for every* $\delta \in (0, 1)$ *there are some positive constants* α, β, η *(depending only on* δ*) such that for every* $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ *and for every vector* $\xi_{i,k}$ *(appearing in (29) (29) (29)) there exists* $v \in SBD^2(\Omega, \mathbf{R}^2)$ *satisfying*

$$\alpha\int_{A_\varepsilon} W(Ev)\,dx + \beta\mathcal{H}^1(J_v \cap A_\varepsilon) \le F^i_\varepsilon(v_\varepsilon, A)\,, \tag{39}$$

$$|\{x \in \Omega \,:\, v_\varepsilon(x) \neq v(x)\}| \leq \eta\, \varepsilon\, F^i_\varepsilon(v_\varepsilon, A)\,, \tag{40}$$

and for $i = 1$ *and* $i = 3$

$$(1-\delta)\gamma \int_{J_v \cap A_\varepsilon} c_{i,k} |\langle \nu_v, \xi_{i,k}\rangle| \, d\mathcal{H}^1 \leq F^i_\varepsilon(v_\varepsilon, A)\,, \tag{41}$$

while for $i = 2$

$$(1-\delta)\gamma \int_{J_v \cap A} \Big(c_{i,k}|\langle \nu_v, \xi_{i,k}\rangle| + c_{i,k+1}|\langle \nu_v, \xi_{i,k+1}\rangle|\Big) d\mathcal{H}^1 \leq F^i_\varepsilon(v_\varepsilon, A)\,. \tag{42}$$

Proof. **Step 1.** For a given $\delta \in (0,1)$ let τ_δ such that $(1-\delta) < \psi(t)$ for $t \geq \tau_\delta$. Let $\alpha_1 < 1$ such that for $\varepsilon|M|^2 < \tau_\delta$ we have $\alpha_1 \varepsilon W(M^{sym}) < \gamma$. Moreover define

$$\tilde{\psi}(t) = \begin{cases} 0 & \text{for } t \leq \tau_\delta \\ \psi(t) & \text{otherwise,} \end{cases} \tag{43}$$

and

$$\tilde{f}(\varepsilon, M) = \alpha_1 \varepsilon W(M^{sym})\Big(1 - \tilde{\psi}(\varepsilon|M|^2)\Big) + \gamma \tilde{\psi}(\varepsilon|M|^2)\,.$$

Being $f(\varepsilon, M)$ a convex combination of $\varepsilon W(M^{sym})$ and γ then $f(\varepsilon, M) \geq \min\{\varepsilon W(M^{sym}), \gamma\}$. By the choice of τ_δ and α_1 it follows that for $\varepsilon|M|^2 < \tau_\delta$ we have

$$\tilde{f}(\varepsilon, M) = \alpha_1 \varepsilon W(M^{sym}) \leq \min\{\varepsilon W(M^{sym}), \gamma\} \leq f(\varepsilon, M)\,.$$

Clearly $\tilde{f}(\varepsilon, M) \leq f(\varepsilon, M)$ also for $\varepsilon|M|^2 \geq \tau_\delta$, being $\tilde{\psi}(\varepsilon|M|^2) = \psi(\varepsilon|M|^2)$ and $\alpha_1 < 1$. As in the previous proof let $\mathbf{A}^i_\varepsilon = \{T \in \mathbf{T}^i_\varepsilon \,:\, T \subset A\}$. Let

$$\mathbf{A}^{i,\flat}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, \varepsilon|Dv_\varepsilon|^2 \leq \tau_\delta\}\,,$$

$$\mathbf{A}^{i,\sharp}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, T \notin \mathbf{A}^{i,\flat}_\varepsilon\}\,,$$

and define $A^{i,\flat}_\varepsilon$ and $A^{i,\sharp}_\varepsilon$ as the union of their elements. For $T \in \mathbf{A}^{i,\flat}_\varepsilon$ we have $\tilde{\psi}(\varepsilon|Dv_\varepsilon|^2) = 0$, then

$$\frac{1}{\varepsilon}\tilde{f}(\varepsilon, Dv_\varepsilon) \geq \alpha_1 W(Ev_\varepsilon)\,.$$

While for $T \in \mathbf{A}^{i,\sharp}_\varepsilon$ we have $\varepsilon|Dv_\varepsilon|^2 > \tau_\delta$ and then

$$\frac{1}{\varepsilon}\tilde{f}(\varepsilon, Dv_\varepsilon) \geq \frac{1}{\varepsilon}\gamma\psi(\varepsilon|Dv_\varepsilon|^2) \geq \frac{1}{\varepsilon}\gamma(1-\delta)\,.$$

Thus, arguing as in the previous Lemma we can write

$$
\begin{aligned}
F^i_\varepsilon(v_\varepsilon, A) &\geq \sum_{T\in \mathbf{A}^i_\varepsilon} \frac{1}{\varepsilon}\tilde f(\varepsilon, Dv_\varepsilon)|T| \\
&\geq \sum_{T\in \mathbf{A}^{i,\flat}_\varepsilon} \alpha_1 W(Ev_\varepsilon)|T| + \gamma \sum_{T\in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon}\psi(\varepsilon|Dv_\varepsilon|^2)|T| \\
&\geq \sum_{T\in \mathbf{A}^{i,\flat}_\varepsilon} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{T\in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon}|T| . \qquad (44)
\end{aligned}
$$

Step 2. Consider the case $i = 1$ and $i = 3$. The function v is defined as $v = v_\varepsilon$ on $A^{i,\flat}_\varepsilon$ thus by (44) follow inequality (40) for $\eta = 1/\gamma(1-\delta)$ and inequality

$$
\alpha_1 \int_{A^{i,\flat}_\varepsilon} W(Ev)\,dx \leq F^i_\varepsilon(v, A^{i,\flat}_\varepsilon) . \qquad (45)
$$

On $A^{i,\sharp}_\varepsilon$ we proceed element by element and component by component, defining v first on the boundary of the element and then in its interior, in such a way that, for a suitable choice of β_1 and α_2 the following inequalities hold

$$
\alpha_2 \int_T W(Ev)\,dx \leq \frac{|T|}{\varepsilon} , \qquad (46)
$$

$$
S_v \cap \partial T = \emptyset , \qquad (47)
$$

$$
\beta_1 \mathcal{H}^1(J_v \cap T) \leq \frac{|T|}{\varepsilon} , \qquad (48)
$$

$$
\int_{J_v\cap T} c_{i,k}|\langle \nu_v, \xi_{i,k}\rangle|\,d\mathcal{H}^1 \leq \frac{|T|}{\varepsilon} . \qquad (49)
$$

Then from (44), (47) and (49) follows (41) while from (46)–(48) follows

$$
\alpha_2\gamma(1-\delta) \int_{A^{i,\sharp}_\varepsilon} W(Ev)\,dx + \beta_1\mathcal{H}^1(J_v \cap A^{i,\sharp}_\varepsilon) \leq F^i_\varepsilon(u, A^{i,\sharp}_\varepsilon)
$$

and thus by (45) follows the existence of α and β such that (39) holds.

Let ζ_j denote the edges of ∂T and let a_j and b_j be the endpoints of ζ_j. We proceed by components. Let v^n_ε be the n$^{\text{th}}$ component of v_ε and let $\partial_j v^n_\varepsilon$ be the slope of v^n_ε along ζ_j. If $T \in \mathbf{A}^{i,\flat}_\varepsilon$ it's clear that

$$
\varepsilon|\partial_j v^n_\varepsilon|^2 \leq 2\tau_\delta . \qquad (50)
$$

Now, consider a triangle $T \in \mathbf{A}^{i,\sharp}_\varepsilon$ and an edge ζ_j then, proceeding by components, we set $v^n = v^n_\varepsilon$ on ζ_j if $\varepsilon|\partial_j v^n_\varepsilon|^2 \leq 2\tau_\delta$, otherwise we set

$$
v^n(ta_j + (1-t)b_j) = \begin{cases} v_\varepsilon(b_j) \text{ if } t < 1/2 \\ v_\varepsilon(a_j) \text{ if } t \geq 1/2. \end{cases} \qquad (51)
$$

In this way v^n is no longer continuous on ∂T but now its slope is uniformly controlled on $\partial T \setminus \{m_1, m_2, m_3\}$, where m_j denotes the middle point of the edges ζ_j.

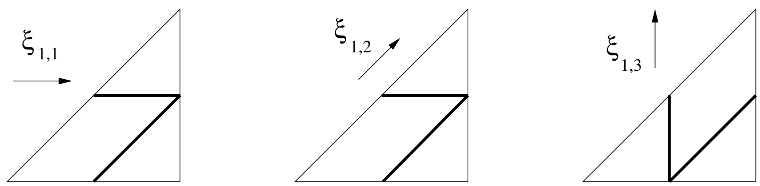
Given $\xi_{i,k}$ let $J_{i,k}$ be the bold set represented in Figure 4 for $\mathbf{T}_\varepsilon^1$ and in Figure 5 for $\mathbf{T}_\varepsilon^3$. It's easy to see by a simple trigonometric argument that for every k we have

$$\int_{J_{i,k}} c_{i,k} |\langle \nu, \xi_{i,k} \rangle| \, d\mathcal{H}^1 \leq \frac{|T|}{\varepsilon}, \tag{52}$$

and that for a sufficiently small parameter $\beta_1 > 0$ we get

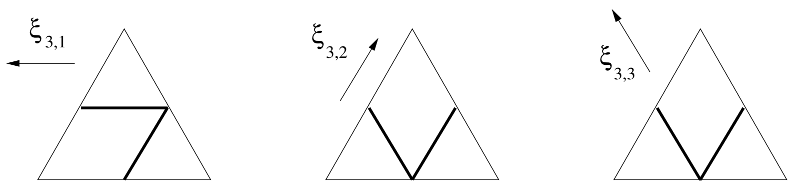
$$\beta_1 \mathcal{H}^1(J_{i,k} \cap T) \leq \frac{|T|}{\varepsilon}. \tag{53}$$

Given $\xi_{i,k}$, both the components of v are defined is such a way that the discontinuity set $J_{v^n} \subset J_{i,k}$, so that $J_v \subset J_{i,k}$ and consequently inequalities (48) and (49) hold as a consequence of (52) and (53). The construction is the same for all the choices of $\xi_{i,k}$. Let R_m for $m = 1, 2, 3$ be the regions of $T \setminus J_{i,k}$. On $\partial R_m \setminus J_{i,k} = \partial R \cap \partial T$ the component v^n is already assigned, by construction it is continuous because the middle points are all separated, and its slope is uniformly bounded by $\sqrt{2\tau_\delta/\varepsilon}$. As a consequence, its value on ∂T defines in each region an affine function v^n such that $|\nabla v^n|^2 \leq c/\varepsilon$ (see [20] for details), where c depends only on δ and on the mesh. Then, for a suitable constant α_2, inequality (46) holds and the proof is concluded.



(a) (b) (c)

Fig. 4. The sets of discontinuity for $\mathbf{T}_\varepsilon^1$



(a) (b) (c)

Fig. 5. The sets of discontinuity for $\mathbf{T}_\varepsilon^3$

Step 3. Consider now the case $i = 2$. The function v is defined again as $v = v_\varepsilon$ on $A_\varepsilon^{i,\flat}$ thus by (44) follows inequality (40) for $\eta = 1/\gamma(1-\delta)$ and (45) holds. As before we can proceed component by component but this time it is not possible to define v^n element by element because the anisotropy is the result of the orientation of all the triangles contained in the squares Q (see Figure 6 (a)) which represent the smallest periodic structure of the mesh. Thus let $\mathbf{Q}_\varepsilon$ be the set of squares $Q \subset A$. We partition $\mathbf{Q}_\varepsilon$ into the subsets $\mathbf{Q}_{\varepsilon,m}$ for $m = 0, \ldots, 4$, according to the number of triangles $T \subset Q$ belonging to $\mathbf{A}_\varepsilon^{i,\sharp}$. In particular (44) becomes

$$\begin{aligned} F_\varepsilon^i(v_\varepsilon, A) &\geq \sum_{T\in\mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{T\in\mathbf{A}_\varepsilon^{i,\sharp}} \frac{1}{\varepsilon}|T| \\ &\geq \sum_{T\in\mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{Q\in\mathbf{Q}_\varepsilon} \Big(\sum_{T\subset Q\,:\,T\in\mathbf{A}_\varepsilon^{i,\sharp}} \frac{|T|}{\varepsilon}\Big) \\ &\geq \sum_{T\in\mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{m=1}^{4} \Big(\sum_{Q\in\mathbf{Q}_{\varepsilon,m}} \frac{m|T|}{\varepsilon}\Big). \end{aligned} \tag{54}$$

Given $k = 1, \ldots, 4$ the function v^n will be defined in such a way that for $Q \in \mathbf{Q}_{\varepsilon,m}$

$$J_v \cap \partial Q = \emptyset\,, \tag{55}$$

$$\alpha_2 \int_Q W(Ev)dx \leq \frac{m|T|}{\varepsilon}\,, \tag{56}$$

$$\beta_1 \mathcal{H}^1(S_v \cap Q) \leq \frac{m|T|}{\varepsilon}\,, \tag{57}$$

$$\int_{S_v\cap Q} \Big(c_{2,k}|\langle \nu_v, \xi_{2,k}\rangle| + c_{2,k+1}|\langle \nu_v, \xi_{2,k+1}\rangle| \Big) d\mathcal{H}^1 \leq m\frac{|T|}{\varepsilon}\,. \tag{58}$$

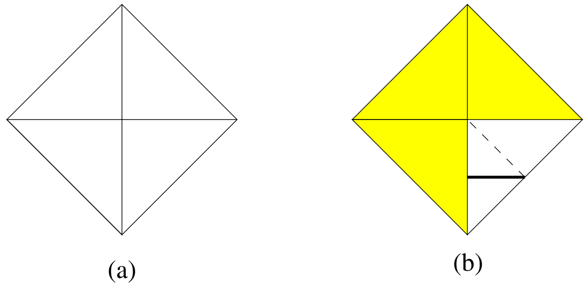


Fig. 6. The periodic structure of $\mathbf{T}_\varepsilon^2$. Discontinuity set J_1 for the case $Q \in \mathbf{Q}_{\varepsilon,1}$

If all the previous inequalities are satisfied then the proof is concluded, indeed, considering (54), from (55) and (58) follows (42) while from (55)–(57) follows the existence of α and β such that (39) holds.

First of all note that by symmetry it is sufficient to consider the case $k = 1$ ($\xi_{2,1} = (1, 0)$, $\xi_{2,2} = (\sqrt{2}/2, \sqrt{2}/2)$). As before, we proceed by components, defining v^n first in ∂Q and then the interior. Note that v is already defined in $Q \cap A_\varepsilon^{i,\flat}$. Let $T \subset Q$ such that $T \in \mathbf{A}_\varepsilon^{i,\sharp}$ and let ζ_j be the edge of $\partial T \cap \partial Q$. If $\varepsilon|\partial_j v_\varepsilon^n|^2 > 2\tau_\delta$ then v_ε^n is defined in ζ_j as in (51) otherwise we take $v^n = v_\varepsilon^n$. Let J_m be the sets represented in Figure 6–8. By a simple trigonometric argument it easy to check that

$$\int_{J_m \cap Q} \Big(c_{2,k} |\langle \nu, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu, \xi_{2,k+1} \rangle| \Big) d\mathcal{H}^1 \le m \frac{|T|}{\varepsilon}, \tag{59}$$

and clearly there exists β_1 such that

$$\beta_1 \mathcal{H}^1(J_m \cap Q) \le \frac{m|T|}{\varepsilon}. \tag{60}$$

Note that the sets J_m are defined in such a way that for every connected component C of $Q \setminus A_\varepsilon^{i,\flat}$ the slope of v^n is uniformly bounded by $\sqrt{2\tau_\delta/\varepsilon}$ on $\partial C \cap \partial Q$. Thus we can extend the values of v^n inside C in such a way that

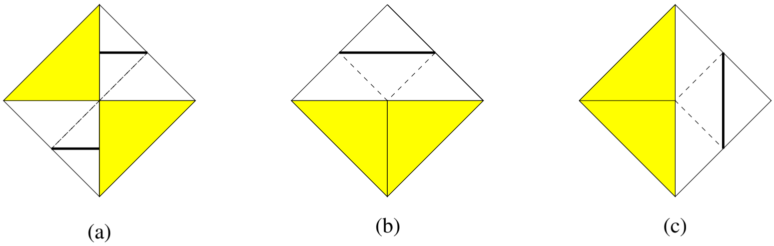


Fig. 7. Discontinuity set J_2 for the case $Q \in \mathbf{Q}_{\varepsilon,2}$

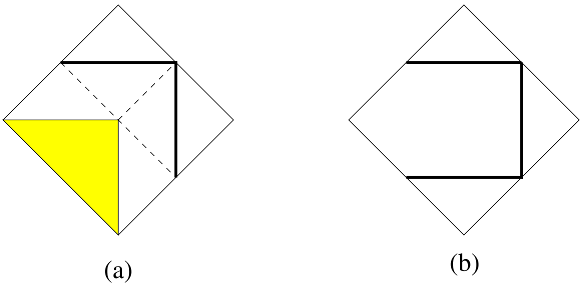


Fig. 8. Discontinuity sets J_3 and J_4 for the cases $Q \in \mathbf{Q}_{\varepsilon,3}$ and $Q \in \mathbf{Q}_{\varepsilon,4}$ respectively

$J_{v^n} \subset J_m$ and

$$\int_Q |\nabla v^n|^2 \, dx \le c \frac{m|T|}{\varepsilon} . \tag{61}$$

Once the components are defined in this way, property (55) is clearly satisfied, (56) and (57) follows easily and finally inequality (58) is proved from (59). □

Proposition 3 *Let $u \in L^1(\Omega, \mathbf{R}^2)$ and let A be an open set in Ω, if $\underline{F}^i(u, A) < +\infty$ then $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and*

$$\mathcal{H}^1(J_u \cap A) < +\infty , \tag{62}$$

$$\int_A W(Eu) \, dx \le \underline{F}^i(u, A) . \tag{63}$$

Moreover for $i = 1$ and $i = 3$ and for every $k = 1, \ldots, 3$ we have

$$\gamma \int_{J_u \cap A} c_{i,k} |\langle \nu_u, \xi_{i,k} \rangle| \, d\mathcal{H}^1 \le \underline{F}^i(u, A) . \tag{64}$$

Finally for $i = 2$ and for every $k = 1, \ldots, 4$ we have

$$\gamma \int_{J_u \cap A} \Big(c_{2,k} |\langle \nu_u, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu_u, \xi_{2,k+1} \rangle| \Big) \, d\mathcal{H}^1 \le \underline{F}^i(u, A) . \tag{65}$$

Proof. Let $\varepsilon_j \searrow 0$ and $v_{\varepsilon_j} \in V^i_{\varepsilon_j}(\Omega, \mathbf{R}^2)$ such that $v_{\varepsilon_j} \to u$ in $L^1(\Omega, \mathbf{R}^2)$ and $\liminf_{\varepsilon_j \to 0} F^i_{\varepsilon_j}(u_{\varepsilon_j}, A) < +\infty$. Up to taking a subsequence (denoted again as v_{ε_j}) it is not restrictive to suppose that $F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \le c < +\infty$.

Let us first prove inequalities (62) and (63). For every v_{ε_j} let $v_j \in SBD^2(\Omega, \mathbf{R}^2)$ be the function given by Lemma 3. From the convergence in $L^1(\Omega, \mathbf{R}^2)$ of v_{ε_j} and from (35) it follows that v_j converges to u and by (34) that

$$\int_{A_{\varepsilon_j}} W(Ev_j) \, dx + \alpha \mathcal{H}^1(J_{v_j} \cap A_{\varepsilon_j}) \le F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \le c .$$

Let $\eta > 0$, if ε_j is small enough then $A_\eta \subset A_{\varepsilon_j}$ and then

$$\int_{A_\eta} W(Ev_j) \, dx + \alpha \mathcal{H}^1(J_{v_j} \cap A_\eta) \le F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \le c .$$

Then by the compactness and lower semicontinuity result of Proposition 1 we have that $u \in SBD^2(A_\eta)$ and

$$\int_{A_\eta} W(Eu) \, dx + \alpha \mathcal{H}^1(J_u \cap A_\eta) \le \liminf_{j \to +\infty} F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) .$$

Since the previous inequality holds for every η we have

$$\int_A W(Eu)\,dx \le \liminf_{j\to+\infty} F^i_{\varepsilon_j}(v_{\varepsilon_j}, A)\,,$$

$$\mathcal{H}^1(J_u \cap A) < +\infty\,. \tag{66}$$

Applying the same reasoning for every sequence $\varepsilon_j \searrow 0$ and for every sequence v_{ε_j} it follows that

$$\int_A W(Eu)\,dx \le \underline{F}^i(u, A)\,. \tag{67}$$

Finally, to show inequalities (64) and (65), let this time v_j be the function given by Lemma 4 then, as for the previous inequalities, (64) and (65) will follow from the lower semicontinuity inequality (9). □

Proof of Proposition 2. The Γ-liminf inequality follows applying the usual supremum of measures argument and considering the representations (29)–(31) of the anisotropy functions. The constrain $u \in K(\Omega)$ follows by pointwise convergence. □

Consider now the isotropic case.

Proposition 4 *Denote by* $\underline{\mathcal{F}}^\theta(u)$ *the* $\Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}^\theta_\varepsilon(u)$ *and take* $u \in L^1(\Omega, \mathbf{R}^2)$ *such that* $\underline{\mathcal{F}}^\theta(u) < +\infty$. *Then* $u \in K(\Omega) \cap SBD^2(\Omega, \mathbf{R}^2)$ *and*

$$\int_\Omega W(Eu)\,dx + \gamma\mathcal{H}^1(S_u) \le \liminf_{\varepsilon\to 0} \mathcal{F}^\theta_\varepsilon(v_\varepsilon)\,, \tag{68}$$

for every sequence $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ *converging strongly to* u *in* $L^1(\Omega, \mathbf{R}^2)$.

Proof. Following the proof of Lemma 3 we can easily obtain (34) and (35). Arguing as in the proof of Proposition 2, we get (66) and (67). It remains to consider the $\mathcal{H}^1$-term. We prove first a result on the localized functional similar to Lemma 4. For every $\delta \in (0, 1)$ there are some positive constants α, β, η such that for every $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ and for every $\xi \in S^1$ there exists $v \in SBD^2(\Omega, \mathbf{R}^2)$ satisfying

$$\alpha \int_{A_\varepsilon} W(Ev)\,dx + \beta\mathcal{H}^1(J_v \cap A_\varepsilon) \le \mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,, \tag{69}$$

$$|\{x \in \Omega \,:\, v_\varepsilon(x) \neq v(x)\}| \le \eta\,\varepsilon\,\mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,, \tag{70}$$

$$\sin\theta\,(1-\delta)\gamma \int_{J_v\cap A_\varepsilon} |\langle \nu_v, \xi\rangle|\,d\mathcal{H}^1 \le \mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,. \tag{71}$$

Following exactly the proof of Proposition 4 we get again inequality (44). Define $v_\varepsilon = v$ on $A^{i,\flat}_\varepsilon$, so that from (44) we have (70) and

$$\alpha_1 \int_{A^{4,\flat}_\varepsilon} W(Ev)\,dx \le \mathcal{F}^\theta_\varepsilon(v, A^{\theta,\flat}_\varepsilon)\,.$$

We want to define v on $A^{i,\sharp}_\varepsilon$ in such a way that (46)–(48) and

$$\sin\theta \int_{J_v\cap T} |\langle \nu_v, \xi\rangle|\, d\mathcal{H}^1 \le \frac{|T|}{\varepsilon} \tag{72}$$

are satisfied. We can proceed element by element and component by component. Note that for a constant $c > 0$ if $T \in \mathbf{A}^{\theta,\flat}_\varepsilon$ then $\varepsilon|\partial_j v^n_\varepsilon|^2 \le c\tau_\delta$ for every edge $\zeta_j \subset \partial T$. Let now $T \in \mathbf{A}^{\theta,\sharp}_\varepsilon$ and let $\zeta_j \subset \partial T$. If $\varepsilon|\partial_j v^n_\varepsilon|^2 > c\tau_\delta$ then we define v^n on ζ_j as in (51) otherwise we set $v^n = v^n_\varepsilon$.

Given $\xi \in S^1$, following the idea of [11], let the edges of ∂T be ordered according to

$$\langle m_1, \xi\rangle \le \langle m_2, \xi\rangle \le \langle m_3, \xi\rangle\,,$$

where m_j denotes the middle point of ζ_j. We define the discontinuity set J as the union of the segments $[m_1, m_2]$ and $[m_2, m_3]$. Then by [11] we have

$$\mathcal{H}^1(J) \le \frac{2|T|}{\varepsilon \sin\theta}\,, \tag{73}$$

$$\int_J |\langle \nu_v, \xi\rangle|\, d\mathcal{H}^1 \le \frac{|T|}{\varepsilon\sin\theta}\,. \tag{74}$$

Note that J contains all the middle points m_j and that the components v^n are continuous on $\partial T \setminus \{m_1, m_2, m_3\}$ and that their slope is uniformly bounded by $\sqrt{c\tau_\delta/\varepsilon}$. Consequently for every connected component of $T \setminus J$ the value of v^n on ∂T defines an affine function whose gradient is controlled by c/ε (see [11] Remark 3.5) and for a suitable choice of α_1 we have (69). Moreover in this way $J_{v^n} \subset J$ and then (72) follows from (74). Then the Γ-liminf inequality is obtained following the proof of Proposition 2. □

6 Γ-limsup inequality

Proposition 5 *Let $i = 1, 2, 3$ then for every $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ there exists a sequence $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ such that*

$$v_\varepsilon \longrightarrow u \textit{ strongly in } L^1(\Omega, \mathbf{R}^2)\,, \tag{75}$$

$$\limsup_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon) \le F^i(u)\,. \tag{76}$$

Proof. **Step 1.** Consider first $u \in K(\Omega) \cap \mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support, then, by Definition 1, $\overline{S_u}$ is the union of the disjoint segments S_m, for $1 \leq m \leq k$. Let $S^i_{m,\varepsilon}$ be the coverings of S_m and consider ε sufficiently small in such a way that they are pairwise disjoint. Let $S^i_{u,\varepsilon}$ be their union and $\Omega^i_\varepsilon = \Omega \setminus S^i_{u,\varepsilon}$. Being $\overline{S_u} \subset S^i_{u,\varepsilon}$ by regularity we have $u \in C^\infty(\overline{\Omega^i_\varepsilon}, \mathbf{R}^2)$, thus v_ε can be defined in $\overline{\Omega^i_\varepsilon}$ as the Lagrange interpolation of u. Moreover $\overline{\Omega^i_\varepsilon}$ contains all the knots of the mesh $\mathbf{T}^i_\varepsilon$ because the sets $S^i_{m,\varepsilon}$ are disjoint and their interior do not contain any vertex by definition. Thus the function v_ε is defined in the whole set Ω, it clearly belongs to $V^i_\varepsilon(\Omega, \mathbf{R}^2)$ and it satisfies also the constraint $v_\varepsilon \in K(\Omega)$.

By a standard result on finite elements (see [12]) there exists a constant C, which does not depend on ε and u, such that

$$\|u - v_\varepsilon\|_{m,q,T} \leq C\, |T|^{\frac{1}{q}-\frac{1}{p}} \varepsilon^{2-m}\, |u|_{2,p,T}\,. \tag{77}$$

Then for every triangle $T \not\subset S^i_{u,\varepsilon}$, for $m = 0$, $q = 1$ and $p = \infty$ we have

$$\int_T |v_\varepsilon - u|\, dx \leq C\, |T|\, \varepsilon^2\, |u|_{2,\infty,(\Omega\setminus\overline{S_u})}\;.$$

Considering that $\|v_\varepsilon\|_\infty \leq \|u\|_\infty$ and that $|S^i_{u,\varepsilon}| \to 0$ it follows easily that v_ε converges strongly to u in $L^1(\Omega, \mathbf{R}^2)$. Moreover for $m = 1$, $q = 2$ and $p = \infty$ for every triangle $T \not\subset S^i_{u,\varepsilon}$ we have

$$\int_T |Dv_\varepsilon - Du|^2\, dx \leq C\, |T|\, \varepsilon^2\, |u|^2_{2,\infty,(\Omega\setminus\overline{S_u})}\;.$$

Denote by D_ε the function $Dv_\varepsilon 1_{\Omega^i_\varepsilon}$ (where $1_{\Omega^i_\varepsilon}$ is the characteristic function of Ω^i_ε). Then D_ε converges strongly to Du in $L^2(\Omega, \mathbf{M}^{2\times 2})$, indeed from $|S^i_{u,\varepsilon}| = O(\varepsilon)$, the regularity of u and the previous inequality it follows that

$$\begin{aligned}\int_\Omega |D_\varepsilon - Du|^2\, dx &\leq \int_{\Omega^i_\varepsilon} |Dv_\varepsilon - Du|^2\, dx + \int_{S^i_{u,\varepsilon}} |Du|^2\, dx \\ &\leq c\varepsilon^2\, |u|^2_{2,\infty,(\Omega\setminus\overline{S_u})} + |S^i_{u,\varepsilon}|\, |u|^2_{1,\infty,(\Omega\setminus\overline{S_u})}\,.\end{aligned} \tag{78}$$

To prove (76) we must consider separately the behavior in Ω^i_ε and $S^i_{u,\varepsilon}$. We start with Ω^i_ε. By the previous inequality it follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_{\Omega^i_\varepsilon} W(Dv_\varepsilon^{sym})\, dx &= \limsup_{\varepsilon\to 0} \int_\Omega W(D_\varepsilon^{sym})\, dx \\ &= \int_\Omega W(Eu)\, dx\,.\end{aligned} \tag{79}$$

Moreover, being $(1-\psi(t)) \leq 1$, we have

$$\limsup_{\varepsilon\to 0} \int_{\Omega^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx \leq \limsup_{\varepsilon\to 0} \int_{\Omega^i_\varepsilon} W(Dv_\varepsilon^{sym})\,dx$$

$$\leq \int_\Omega W(Eu)\,dx\,. \tag{80}$$

As $u \in W^{1,\infty}(\Omega \setminus \overline{S_u}, \mathbf{R}^2)$ then $|Dv_\varepsilon| \leq c$ uniformly in Ω^i_ε and thus by (20)

$$\limsup_{\varepsilon\to 0} \int_{\Omega^i_\varepsilon} \frac{\gamma}{\varepsilon}\psi(\varepsilon|Dv_\varepsilon|^2)\,dx \leq \limsup_{\varepsilon\to 0} \int_{\Omega^i_\varepsilon} \frac{\gamma}{\varepsilon}\psi(\varepsilon c^2)\,dx = 0\,. \tag{81}$$

Let us consider now the behavior in $S^i_{u,\varepsilon}$. If $S \subset \Omega$ is a segment and S^i_ε is its covering, then, being $(1-\psi(t))t$ bounded in $[0,+\infty)$, it follows that

$$\limsup_{\varepsilon\to 0} \int_{S^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx$$

$$\leq \limsup_{\varepsilon\to 0} \int_{S^i_\varepsilon} \frac{c}{\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)\varepsilon|Dv_\varepsilon|^2\,dx$$

$$\leq c\limsup_{\varepsilon\to 0} \frac{|S^i_\varepsilon|}{\varepsilon} \leq c\mathcal{H}^1(S)\,. \tag{82}$$

Let now $\delta > 0$, let $S^\delta_u = \{x \in S_u \,:\, |u^+ - u^-| \geq \delta\}$ and $(S^\delta_u)^i_\varepsilon$ be its covering. Being $u \in W^{1,\infty}(\overline{\Omega \setminus S_u}, \mathbf{R}^2)$, for ε sufficiently small we have $|Dv_\varepsilon| \geq \frac{\delta}{4\varepsilon}$ for $T \subset (S^\delta_u)^i_\varepsilon$. Then, considering that $\varepsilon|Dv_\varepsilon|^2$ diverges in $(S^\delta_u)^i_\varepsilon$ and that $(1-\psi(t))t$ is decreasing for t large we deduce that

$$\limsup_{\varepsilon\to 0} \int_{(S^\delta_u)^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx$$

$$\leq \limsup_{\varepsilon\to 0} \int_{(S^\delta_u)^i_\varepsilon} \frac{1}{\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)c\varepsilon|Dv_\varepsilon|^2\,dx$$

$$\leq \limsup_{\varepsilon\to 0} \frac{|(S_u)^i_\varepsilon|}{\varepsilon}\Big(1-\psi\Big(\frac{\delta^2}{16\varepsilon}\Big)\Big)\frac{\delta^2}{16\varepsilon}c$$

$$\leq c\mathcal{H}^1(S^\delta_u)\limsup_{\varepsilon\to 0}\Big(1-\psi\Big(\frac{\delta^2}{16\varepsilon}\Big)\Big)\frac{\delta^2}{16\varepsilon} = 0\,. \tag{83}$$

Then, for every $\delta > 0$, by inequalities (82) and (83), we have

$$\limsup_{\varepsilon\to 0} \int_{(S_u)^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx$$

$$\leq \limsup_{\varepsilon\to 0} \int_{(S^\delta_u)^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx$$

$$+ \limsup_{\varepsilon\to 0} \int_{(S_u)^i_\varepsilon \setminus (S^\delta_u)^i_\varepsilon} \Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx$$

$$\leq C\mathcal{H}^1(S_u \setminus S^\delta_u)\,,$$

which proves (for $\delta \to 0$) that

(84) $$\limsup_{\varepsilon\to 0} \int_{(S_u)^i_\varepsilon} \Big(1 - \psi(\varepsilon|Dv_\varepsilon|^2)\Big) W(Dv_\varepsilon^{sym})\,dx = 0\,.$$

Finally, for every segment S_m, from (28) follows

$$\limsup_{\varepsilon\to 0} \int_{S^i_{m,\varepsilon}} \gamma \frac{\psi(\varepsilon|Dv_\varepsilon|^2)}{\varepsilon}\,dx \le \gamma \limsup_{\varepsilon\to 0} \frac{|(S_m)^i_\varepsilon|}{\varepsilon} = \gamma \int_{S_m} \phi_i(\nu)d\mathcal{H}^1\,.$$
(85)

Then by inequalities (80)–(81) and (84)–(85) it follows that

(86) $$\limsup_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon) \le \int_\Omega W(Eu)dx + \gamma \int_{J_u} \phi_i(\nu_u)d\mathcal{H}^1 = F^i(u)\,.$$

So the Γ-limsup inequality is proved for $u \in K(\Omega)\cap\mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support. Here the compact support is not strictly necessary, it just prevents technical problems near the boundary $\partial\Omega$.

Step 2. Denote by $\overline{F^i}(u)$ the Γ-limsup. By the previous step, if $u \in K(\Omega)\cap \mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support then $\overline{F^i}(u) \le F^i(u)$. Consider now $u \in K(\Omega) \cap SBV^2(\Omega, \mathbf{R}^2)$ with compact support. By Proposition 1 there exists a sequence of functions $w_k \in K(\Omega)\cap\mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support such that (15)–(18) hold. Then by the lower semicontinuity of $\overline{F^i}$ it follows that

$$\overline{F^i}(u) \le \liminf_{k\to+\infty} \overline{F^i}(w_k) \le \limsup_{k\to+\infty} \overline{F^i}(w_k) \le \limsup_{k\to+\infty} F^i(w_k) \le F^i(u)\,.$$

It remains to remove the hypothesis on the compact support. Let $u \in K(\Omega)\cap SBV^2(\Omega, \mathbf{R}^2)$, from Lemma 4.2 in [11] follows the existence of a function $u' \in SBV^2(\mathbf{R}^2, \mathbf{R}^2)$ with compact support such that $u' = u$ in Ω, $\|u'\|_\infty = \|u\|_\infty$ and $\mathcal{H}^1(S_{u'} \cap \partial\Omega) = \emptyset$. Let Ω' be a rectangle containing the support of u', then there exists a sequence $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ such that $\limsup_{\varepsilon\to 0} F^i(v_\varepsilon, \Omega') \le F^i(u, \Omega')$. Considering the Γ-liminf inequality, we have

$$\begin{aligned} F^i\varepsilon(u, \Omega') &\ge \limsup_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon, \Omega') \\ &\ge \limsup_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon, \Omega) + \liminf_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon, \Omega' \setminus \overline{\Omega}) \\ &\ge \overline{F^i}(u, \Omega) + F^i(u, \Omega' \setminus \overline{\Omega})\,. \end{aligned}$$

Then

$$\overline{F^i}(u) = \overline{F^i}(u, \Omega) \le F^i(u, \Omega') - F^i(u, \Omega' \setminus \overline{\Omega}) = F^i(u, \Omega) = F^i(u)\,,$$

which completes the proof. □

Consider now the isotropic approximation.

Proposition 6 *For every $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ there exists a "sequence" $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega)$ such that*

$$v_\varepsilon \longrightarrow u \text{ strongly in } L^1(\Omega, \mathbf{R}^2)\ , \tag{87}$$

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\theta_\varepsilon(v_\varepsilon) \le \mathcal{F}^\theta(u)\,. \tag{88}$$

Proof. **Step 1.** Let $u \in K(\Omega) \cap \mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support, and let $\overline{S_u}$ be the union of the disjoint segments S_m. By Lemma 2 there exists a mesh $\mathbf{T}_\varepsilon \in \mathbf{T}^\theta_\varepsilon$ such that (32) holds. Using this mesh we can repeat the proof of Proposition 5 and we get (80),(81) and (84). Finally, by (32) we have

$$\limsup_{\varepsilon\to 0} \int_{S^i_{m,\varepsilon}} \gamma \frac{\psi(\varepsilon|Dv_\varepsilon|^2)}{\varepsilon}\, dx \le \gamma \limsup_{\varepsilon\to 0} \frac{|(S_m)^i_\varepsilon|}{\varepsilon} = \gamma\ \sin\theta\, \mathcal{H}^1(S_u)\,,$$

and then the Γ-liminf inequality is proved for $u \in K(\Omega) \cap \mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support.

Step 2. See the proof of Proposition 5. □

7 Boundary value problems

Let Ω be an open, bounded, connected set in $\mathbf{R}^2$ having Lipschitz boundary and let Ω_0 be an open bounded set. Suppose that $\Omega \cap \Omega_0 \neq \emptyset$ and that $\partial\Omega_D = \partial\Omega \cap \Omega_0 \neq \emptyset$ is polygonal. Let $g \in W^{2,\infty}(\Omega_0, \mathbf{R}^2)$. Finally let $\Omega' = \Omega \cup \Omega_0$, $\Omega'_0 = \Omega_0 \setminus \Omega$.

For a constant $k \geq \|g\|_\infty$, define the constrain $K(\Omega') = \{\|u\|_\infty\,, \le k \;\; u = g \text{ in } \Omega'_0\}$. Being $g \in W^{2,\infty}(\Omega_0, \mathbf{R}^2)$ we can clearly define its Lagrange interpolation and we denote it by g_ε. Denote by $\Omega'_{0,\varepsilon}$ the union of the elements $T \in \mathbf{T}^i_\varepsilon$ such that $T \subset \Omega'_0$. Then we can introduce the discretized constrain $K_\varepsilon(\Omega) = \{\|u\|_\infty\,, \le k \quad u = g_\varepsilon \text{ in } \Omega'_{0,\varepsilon}\}$.

The discrete functional is now defined as

$$G^i_\varepsilon(v_\varepsilon) = \frac{1}{\varepsilon} \sum_{T\in\mathbf{T}_\varepsilon i} \int_{T\cap\Omega'} f(\varepsilon, Dv_\varepsilon)\, dx\ , \tag{89}$$

if $v_\varepsilon \in V^i_\varepsilon(\Omega', \mathbf{R}^2) \cap K_\varepsilon(\Omega')$ and $G^i_\varepsilon(v_\varepsilon) = +\infty$ otherwise in $L^1(\Omega', \mathbf{R}^2)$, while the limit functional is

$$G^i(u) = \int_{\Omega'} W(Eu)\, dx + \int_{S_u} \phi_i(\nu_u)\, d\mathcal{H}^1\ , \tag{90}$$

if $u \in SBD^2(\Omega', \mathbf{R}^2) \cap K(\Omega')$ and $G^i(u) = +\infty$ otherwise in $L^1(\Omega', \mathbf{R}^2)$.

Proposition 7 *For every mesh* $\mathbf{T}^i_\varepsilon$ *let* $\phi_i : \mathbf{R}^2 \to [0, +\infty)$ *be the corresponding anisotropy function. Then for every* $u \in L^1(\Omega, \mathbf{R}^2)$ *and for every sequence* $v_{\varepsilon_j} \in V^i_{\varepsilon_j}(\Omega, \mathbf{R}^2)$*, converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$*, we have*

$$G^i(u) \leq \liminf_{\varepsilon \to 0} G^i_\varepsilon(v_\varepsilon)\,. \tag{91}$$

Proof. Up to extracting a subsequence we can suppose that $G^i_\varepsilon(v_\varepsilon) \leq c < \infty$ and then by L^1-convergence follows $u \in K(\Omega)$. Inequality (91) is proved as in Proposition 2. □

Proposition 8 *For every* $i = 1, 2, 3$ *and for every* $u \in SBV^2(\Omega, \mathbf{R}^2)$ *there exists a sequence* $v_{\varepsilon_j} \in V^i_{\varepsilon_j}(\Omega, \mathbf{R}^2)$*, converging strongly to u in* $L^1(\Omega, \mathbf{R}^2)$*, such that*

$$G^i(u) \geq \limsup_{\varepsilon \to 0} G^i_\varepsilon(v_\varepsilon)\,.$$

Proof. It follows by a density result with respect to boundary value problems, proved in [21], and by Proposition 5 . □

Remark 4 Note that this formulation takes into account the case of a fracture along the set $\partial\Omega_D$ where the boundary condition is assigned. Indeed we can write

$$\int_{\Omega'} W(Eu)\,dx = \int_\Omega W(Eu)dx + \int_{\Omega'_0} W(Eg)dx = \int_\Omega W(Eu)dx + C\,,$$

$$\int_{S_u} \phi_i(\nu_u)\,d\mathcal{H}^1 = \int_{S_u \cap \Omega} \phi_i(\nu_u)\,d\mathcal{H}^1 + \int_{S_u \cap \partial\Omega_D} \phi_i(\nu_u)\,d\mathcal{H}^1\,.$$

Then the minimum problem $\min\{F^i(u)\}$, for $u \in SBV^2(\Omega', \mathbf{R}^2) \cap K(\Omega')$, becomes

$$\min\left\{F^i(u, \Omega) + \gamma \int_{S_u \cap \partial\Omega_D} \phi_i(\nu)d\mathcal{H}^1\right\}$$

for $u \in SBV^2(\Omega', \mathbf{R}^2) \cap K(\Omega')$.

Proposition 9 *Let* $\varepsilon_j \searrow 0$ *and* w_{ε_j} *be a sequence of minima of* $G_{\varepsilon_j}(v_{\varepsilon_j})$*, then there exists a subsequence which converges strongly in* $L^1(\Omega', \mathbf{R}^2)$ *to a function* $u \in SBD^2(\Omega', \mathbf{R}^2)$.

Proof. Using the technique of Lemma 3, we know that for every ε_j there is a function $v_j \in SBD^2(\Omega', \mathbf{R}^2)$ such that

$$\int_{\Omega'_{\varepsilon_j}} W(Ev_j)\,dx + \alpha\mathcal{H}^1\,(J_{v_j} \cap \Omega'_{\varepsilon_j}) \leq G^i_{\varepsilon_j}(w_{\varepsilon_j})$$

$$|\{x \in \Omega' \,:\, w_{\varepsilon_j}(x) \neq v_j(x)\}| \leq \beta\,\varepsilon\,G^i_{\varepsilon_j}(w_{\varepsilon_j})\,.$$

By Proposition 1 v_j is compact in $SBD^2(\Omega'_{\varepsilon_j}, \mathbf{R}^2)$ and then (up to a subsequence) it converges to a function $u_j \in SBD^2(\Omega'_{\varepsilon_j}, \mathbf{R}^2)$. The convergence result follows by a diagonal procedure for an increasing sequence of sets Ω'_{ε_j} converging to Ω'. □

By a standard argument in the theory of Γ-convergence [15] we have the following result on the convergence of minima.

Proposition 10 *Let w_{ε_j} and u as in Proposition 9. If $u \in SBV^2(\Omega', \mathbf{R}^2)$ then it is a minimum of the functional $G^i(u)$ restricted to $SBV^2(\Omega' R^2) \cap K(\Omega')$.*

Remark 5 Clearly, using the same arguments, we can prove similar convergence and compactness results for the isotropic case.

8 Numerical results for a quasi-static evolution of a pre-existing fracture

This section presents some numerical results obtained for a model problem. A detailed analysis of the numerical implementation and a comparison between benchmark experimental results is out the purposes and will be the subject for a further investigation [5].

Considering the notations of Section 7 and following [18] let the boundary condition be given by a monotonically increasing function $g(t, x) = t\hat{g}(x)$ for $\hat{g}(x) \in W^{2,\infty}(\Omega_0, \mathbf{R}^2)$ and let $S \subset \Omega$ be a segment representing the initial fracture. Moreover let $0 = t_0 < t_1 < \ldots < t_n = T$ be a uniform discretization of the time interval $[0, T]$ and let $\hat{g}_\varepsilon(x) \in V^i_\varepsilon(\Omega_0, \mathbf{R}^2)$ be the Lagrange interpolation of $\hat{g}(x)$. Finally for a given $k > 0$, let $K_{\varepsilon,t}(\Omega') = \{u \in L^1(\Omega', \mathbf{R}^2) \,:\, \|u\|_\infty \le k\,,\ u = t\,\hat{g}_\varepsilon(x) \ \text{in } \Omega'_{0,\varepsilon}\}$ and S_{ε,t_0} be the covering of S in the mesh $\mathbf{T}^i_\varepsilon$. Then for $t = t_0$ our discrete functional becomes

$$G^i_{\varepsilon,t_0}(v_\varepsilon) = \frac{1}{\varepsilon} \sum_{T \in \mathbf{T}^i_\varepsilon \setminus S_{\varepsilon,t_0}} \int_{T\cap\Omega'} f(\varepsilon, Dv_\varepsilon)\, dx\,. \tag{92}$$

Given

$$w_{\varepsilon,t_0} \in \operatorname{argmin}\{G^i_{\varepsilon,t_0}(u_\varepsilon) \text{ for } u_\varepsilon \in V^i_\varepsilon(\Omega', \mathbf{R}^2) \cap K_{\varepsilon,t_0}(\Omega')\}\,, \tag{93}$$

the discrete set of crack, denoted by J_{ε,t_0}, is given by the elements $T \in \mathbf{T}^i_\varepsilon$ such that the local Griffith's criterion $\varepsilon W(Ew_{\varepsilon,t_0}) > \gamma$ is satisfied. In order to ensure the irreversibility of the fracture we define $S_{\varepsilon,t_1} = S_{\varepsilon,t_0} \cup J_{\varepsilon,t_0}$ and consequently G_{ε,t_1} as in (92). Proceeding by induction, the evolution will be given by the sequence of functions w_{ε,t_i} for $i = 0, \ldots, n$.

By an obvious rescaling argument it is not restrictive to suppose that $\gamma = 1$. Let $c = (\mu + \frac{\lambda}{2})$ and $s > 1$, then a good choice for the control function $\psi(z)$ is

$$\psi(z) = \frac{2}{\pi} \arctan\left((cz)^s\right)$$

which gives

$$\psi(\varepsilon|Du_\varepsilon|^2) = \frac{2}{\pi} \arctan\left(\varepsilon^s \left(\mu + \frac{\lambda}{2}\right)^s |Du_\varepsilon|^{2s}\right).$$

Indeed, considering $(\mu+\frac{\lambda}{2})|Du_\varepsilon|^2$ as an approximation of the energy $W(Eu_\varepsilon)$, the local Griffith's criterion becomes

$$\varepsilon(\mu + \frac{\lambda}{2})|Du_\varepsilon|^2 > 1 = \gamma ,$$

suggesting that the function $\psi(z)$ should change its behavior around $z = 1$.

From the numerical point of view the difficulties come from the minimization of the function G^i_{ε,t_i}. Indeed in order to reproduce accurately Griffith's criterion we should use a function $\psi(z)$ having a fast transition between 0 and 1. This is clearly obtained taking s large, indeed in this way the bulk and surface energies are computed carefully, because the function $f(\varepsilon, M)$ has the following behavior

$$f(\varepsilon, M) \simeq \begin{cases} \varepsilon W(M^{sym}) & \text{for } \varepsilon c|M|^2 < 1 \\ \gamma & \text{otherwise.} \end{cases}$$

Unfortunately the numerical minimization for s large is very difficult due to the sharp layer at $z = 1$ (the algorithm seems to be unable to overcome the layer and thus the solution does not exhibit any motion of the crack). For this reason we need a graduated-non-convexity strategy (in short GNC). For every time step t_j let $1.5 = s_1 < \ldots < s_8 = 8.5$ with $s_{n+1} - s_n = 1$ and let $G^{i,s_k}_{\varepsilon,t_j}$ be the corresponding discrete functional. Starting from s_1 the idea consists in computing a solution of $G^{i,s_k}_{\varepsilon,t_j}$ taking as initial guess the solution of $G^{i,s_{k-1}}_{\varepsilon,t_j}$. Clearly for $G^{i,s_1}_{\varepsilon,t_j}$ the initial guess is the solution at time t_{j-1}, while for $G^{s_0}_{\varepsilon,t_0}$ it is $u_\varepsilon \equiv 0$, being $g(0, x) \equiv 0$.

For every time t_j and every value s_k the minimization is performed by a quasi-Newton algorithm for non-convex functions using a quadratic backtracking as line search strategy (we refer to [22] and to the references therein for the details).

Our model problem is defined in the set $\Omega = (0, 2) \times (0, 1)$, the initial fracture is the segment with extrema $(0, 0.5)$ and $(0.55, 0.5)$. The boundary condition $\hat{g}(x)$ is given on the sets $\partial\Omega_D^{up} = \{(x_1, 1) \text{ for } x_1 \in (0,2)\}$ and $\partial\Omega_D^{down} = \{(x_1, 0) \text{ for } x_1 \in (0,2)\}$ and it is defined as

$$\hat{g}(x) = \begin{cases} (0, 0.5) & \text{for } x \in \partial\Omega_D^{up} \\ \\ (0, -0.5) & \text{for } x \in \partial\Omega_D^{down}. \end{cases}$$

The Lamé constants are $\mu = 9$ and $\lambda = 12$ and the fracture toughness is $\gamma = 1$. Let us try to give a rough estimate of the critical time t_c when the crack should start to move. Suppose that the crack tip is located at the point $(L, 0.5)$, we expect the fracture to evolve horizontally from left to right. We will restrict our analysis to the set $(L, 2) \times (0, 1)$ because the value of the energy in $(0, L) \times (0, 1)$ remains basically constant until loss of cohesion occurs. Considering the geometrical symmetries of the problem we can approximate the elastic energy by

$$(\mu + \frac{\lambda}{2}) \left| \frac{\partial \hat{u}}{\partial x_2} \right|^2 t^2 (2 - L)\,,$$

where $\hat{u}(x)$ is the affine function in Ω having boundary condition $\hat{g}(x)$ on $\partial\Omega_D$. Let $c_e = (\mu + \frac{\lambda}{2})|\frac{\partial \hat{u}}{\partial x_2}|^2$ and let dl, the increase in fracture length, be the unknown. Then the energy is

$$G_t(dl) = c_e t^2 (2 - L - dl) + \gamma dl = dl(\gamma - c_e t^2) + c_e t^2 (2 - L)\,,$$

which is a linear function of dl. The minimum problem becomes

$$\min_{0 \le dl \le (2-L)} dl(\gamma - c_e t^2) + c_e t^2 (2 - L)\,.$$

Thus for $(\gamma - c_e t^2) > 0$ the minimum is attained in $dl = 0$ (the crack does not move) while for $(\gamma - c_e t^2) < 0$ it is attained at $dl = (2 - L)$ (loss of

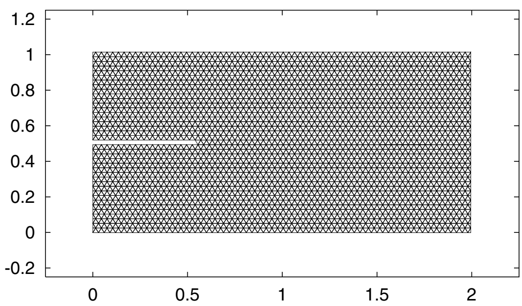


Fig. 9. Initial configuration

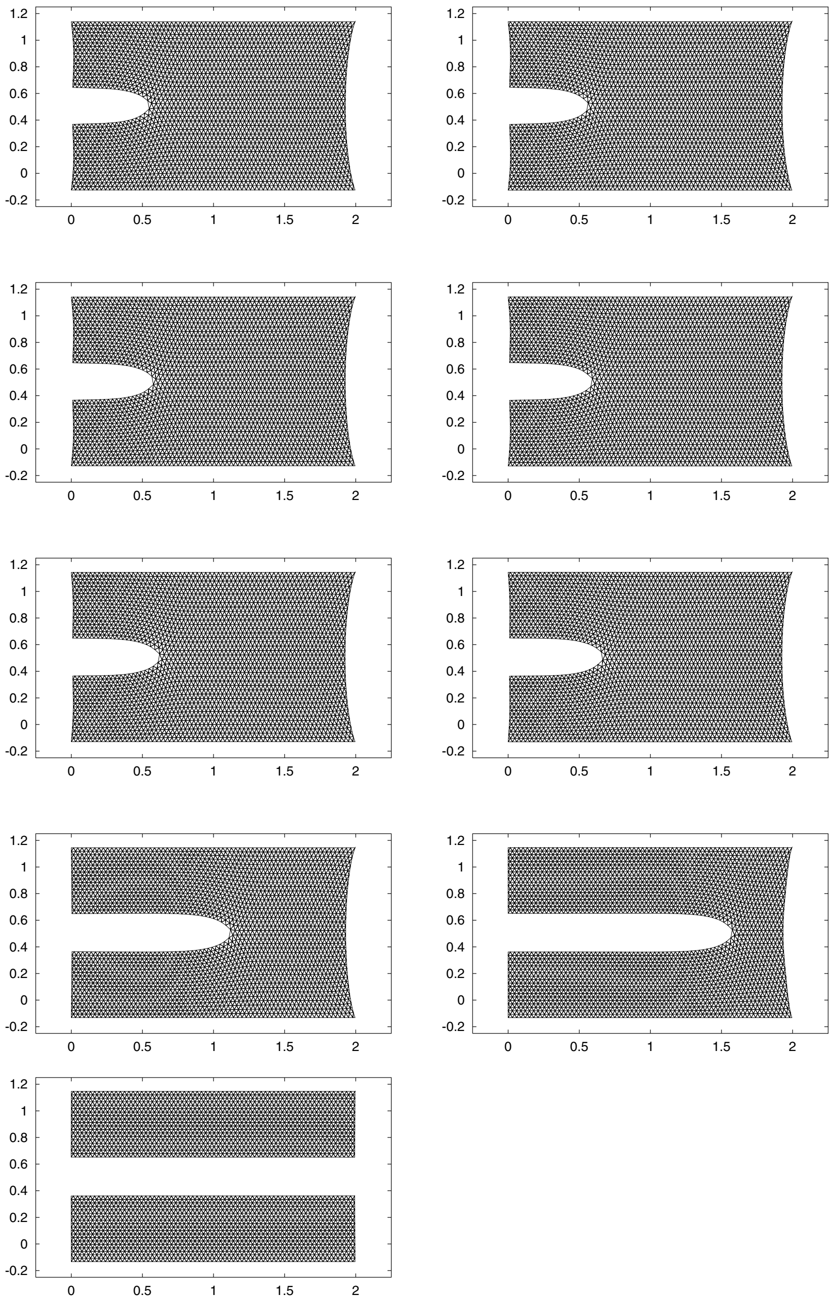


Fig. 10. Configurations from $t = 0.250$ to $t = 0.266$ with $dt = 0.002$

cohesion). Even if the real behavior is not so simple, because of the influence of lower order terms, we can take as critical time the value t_c such that

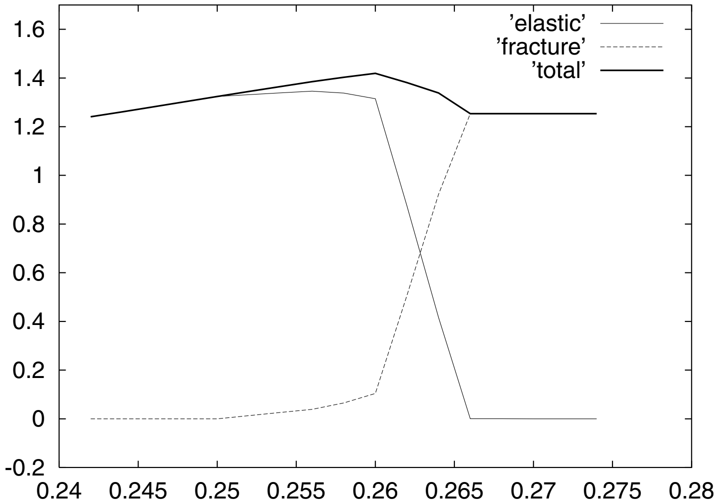


Fig. 11. Comparison of elastic, fracture and total energies

$\gamma = c_e t_c^2$. In our case $t_c \simeq 0.258$. The computed results seem to obey to this estimate, being the numerical critical time $t_c^n = 0.252$.

Finally, Figure 10 and 11 show the evolution of the fracture and the behavior of the energies. It is clear that at time $t = 0.262$ the fracture starts "running" in order to reach the global minimum corresponding to the loss of connection.

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