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On Schwarz-type Smoothers for Saddle Point Problems

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Summary. In this paper we consider additive Schwarz-type iteration methods for saddle point problems as smoothers in a multigrid method. Each iteration step of the additive Schwarz method requires the solutions of several small local saddle point problems. This method can be viewed as an additive version of a (multiplicative) Vanka-type iteration, well-known as a smoother for multigrid methods in computational fluid dynamics. It is shown that, under suitable conditions, the iteration can be interpreted as a symmetric inexact Uzawa method. In the case of symmetric saddle point problems the smoothing property, an important part in a multigrid convergence proof, is analyzed for symmetric inexact Uzawa methods including the special case of the additive Schwarz-type iterations. As an example the theory is applied to the Crouzeix-Raviart mixed finite element for the Stokes equations and some numerical experiments are presented.

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1 Introduction

Multigrid methods certainly belong to the fastest known methods for solving large systems of discretized partial differential equations. While the construction and convergence theory for primal (symmetric and elliptic) variational problems is well understood, the case of mixed variational problems is still a challenge, theoretically as well as computationally.

Roughly speaking, only one sort of variables, the primal variables, appear in a primal problem, while in a mixed problem two types of variables can be

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distinguished, the primal variables and the dual variables. In principle, there are two different approaches for mixed problems to take advantage of the multigrid idea.

One way is to use an outer iteration, typically a Uzawa-type iteration, in which the new iterates for the primal and the dual variables are computed separately by solving appropriate separated problems. In order to be efficient good preconditioners are needed for the separated problems. Multigrid methods applied to the separated problems as an inner iteration can be used to construct these efficient preconditioners. The construction of these methods rely on structural information of the separated problems. Typically, the crucial part is the construction of good preconditioners of some Schur complement. Selected from a large number of contributions to this approach, see, e.g., [13], [14] for block diagonal preconditioners and [3], [4] for block triangular preconditioners of this first type.

The other way is to use multigrid methods as an outer iteration combined with appropriate smoothers (as a sort of inner iteration). Particularily simple to implement are smoothers which are based on the solutions of small local problems in a Jacobi- or Gauss-Seidel-type manner. One of the best-known multigrid methods of this type was proposed by Vanka in [15] for solving the steady state incompressible Navier-Stokes equations in primitive variables. It is based on a finite volume discretization technique on a staggered grid. The computational domain is divided into non-overlapping cells with pressure nodes at the cell center and (component-wise) velocity nodes at the cell faces. The smoothing procedure is a so-called symmetric coupled Gauss-Seidel technique (SCGS). One iteration step of SCGS consists of solving local problems for each cell involving the pressure at the cell center and the neighboring velocity components at the cell faces. This is done cell by cell in a Gauss-Seidel-type manner and, therefore, can be viewed as a multiplicative Schwarz-type iteration.

This multigrid technique is not restricted to Navier-Stokes equations, it can easily be extended to other mixed variational problems if one is able to specify appropriate local sub-problems for the smoothing procedure. No structural information is need for the construction of the method, so it is a very flexible technique. It has been widely used in practice and has shown good convergence results. However, very little is known so far about convergence and smoothing properties of the underlying iterative method. The authors of this paper are aware of only one contribution, namely by Molenaar [11], where Fourier analysis for a simple model problem (a mixed finite element method of the Poisson equation in one dimension) was used.

Besides the multiplicative Schwarz-type iteration described above an additive version of the method is also at hand, where the same type of local sub-problems are solved independently of each other in a Jacobi-type manner.

The aim of this paper is to contribute to the analysis of this additive version of the iterative method for a general class of symmetric mixed variational problems, which includes e.g. the Stokes equations. The theory applies to a large variety of mixed finite element methods on structured and unstructured grids. The analysis of the multiplicative case remains an open problem and is not covered by this paper.

The paper is organized as follows: Section 2 contains the framework for describing the class of problems and the multigrid methods considered. In Section 3 it is shown that the proposed additive Schwarz-type methods belong to the more general class of symmetric inexact Uzawa methods. For this general class of symmetric inexact Uzawa methods new convergence and smoothing properties are derived, which can easily be applied to the considered additive Schwarz-type methods. In Section 4 the two major steps of the multigrid convergence analysis, the approximation property and the smoothing property, are discussed. Finally, in Section 5 the theoretical results are applied to the Crouzeix-Raviart element for the Stokes equations including some numerical experiments.

These numerical experiments are only of illustrative character. They confirm the theoretical results. However, as expected, the multiplicative version of the method shows a much better performance. So, this contribution is not so much a recommendation for the additive Schwarz method as a smoother in a multigrid method, but the presented analysis is ment to be a first and important step towards the understanding of the preferable multiplicative version.

2 The framework

Let V and Q be real Hilbert spaces, $a\colon V \times V \longrightarrow \mathbb{R}$, $b\colon V \times Q \longrightarrow \mathbb{R}$, $c\colon Q \times Q \longrightarrow \mathbb{R}$ continuous bilinear forms, and $F\colon V \longrightarrow \mathbb{R}$, $G\colon Q \longrightarrow \mathbb{R}$ continuous linear functionals. We consider the following mixed variational problem:

Find $u \in V$ and $p \in Q$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= \langle F, v \rangle \quad \text{for all } v \in V, \\ b(u, q) - c(p, q) &= \langle G, q \rangle \quad \text{for all } q \in Q. \end{aligned}$$

Here, $\langle F, v\rangle$ ($\langle G, q\rangle$) denotes the evaluation of the linear functional F (G) at the point v (q).

More concisely, the mixed variational problem can also be written as a variational problem on $V \times Q$:

Find $(u, p) \in V \times Q$ such that

$$\mathcal{B}((u, p), (v, q)) = \langle \mathcal{F}, (v, q) \rangle \quad \text{for all } (v, q) \in V \times Q \tag{1}$$

with the bilinear form

$$\mathcal{B}((w,r),(v,q)) = a(w,v) + b(v,r) + b(w,q) - c(r,q)$$

and the linear functional

$$\langle \mathcal{F}(v,q)\rangle = \langle F, v\rangle + \langle G, q\rangle.$$

It is assumed that a and b are symmetric and non-negative and that $\mathcal{B}$ is stable on $V \times Q$. Then the mixed variational problem (1) is well-posed and can be interpreted as a saddle point problem.

Typical examples of this type of problems are the Stokes problem from fluid mechanics, see Section 5, various problems from linear elasticity (nearly incompressible materials, mixed formulations based on the Hellinger-Reissner principle), or mixed formulations of boundary value problems for second order elliptic equations, see e.g. Brezzi, Fortin [7].

The Hilbert spaces V and Q are typically subspaces of Sobolev spaces on some domain Ω. Then, for discretizing the continuous problem (1), a sequence of finite element spaces V_k and Q_k are chosen for each level $k = 1, 2, \ldots,$ corresponding to a hierarchy of increasingly finer meshes on Ω, and symmetric bilinear forms $\mathcal{B}_k$ and linear functionals $\mathcal{F}_k$ on $V_k \times Q_k$.

These spaces, linear and bilinear forms determine discrete problems at each level k:

Find $(u_k, p_k) \in V_k \times Q_k$ such that

$$\mathcal{B}_k((u_k, p_k),(v,q)) = \langle \mathcal{F}_k, (v,q)\rangle \quad \text{for all } (v,q) \in V_k \times Q_k. \tag{2}$$

A class of efficient solvers of these discrete problems are multigrid algorithms: We additionally need coarse-to-fine inter-grid transfer operators $I_{k-1}^k \colon V_{k-1} \times Q_{k-1} \longrightarrow V_k \times Q_k$. Then one iteration loop for solving (2) at level k is given in the following form:

Let $(u_k^0, p_k^0) \in V_k \times Q_k$ be a given approximation of the solution to (2). Then the iteration proceeds in two stages:

1. Smoothing: For $j = 0, 1, \ldots, m-1$ compute $(u_k^{j+1}, p_k^{j+1}) \in V_k \times Q_k$ by an iterative procedure of the form

$$(u_k^{j+1}, p_k^{j+1}) = \mathcal{S}_k(u_k^j, p_k^j).$$

2. Coarse grid correction: Set

$$\langle \tilde{\mathcal{F}}_{k-1}, (v,q)\rangle = \langle \mathcal{F}_k, I_{k-1}^k(v,q)\rangle - \mathcal{B}_k\left((u_k^m, p_k^m), I_{k-1}^k(v,q)\right)$$

for $(v,q) \in V_{k-1} \times Q_{k-1}$ and let $(\tilde{w}_{k-1}, \tilde{r}_{k-1}) \in V_{k-1} \times Q_{k-1}$ satisfy

$$\mathcal{B}_{k-1}((\tilde{w}_{k-1}, \tilde{r}_{k-1}),(v,q)) = \langle \tilde{\mathcal{F}}_{k-1}, (v,q)\rangle \tag{3}$$

for all $(v,q) \in V_{k-1} \times Q_{k-1}$.

If $k = 1$, compute the exact solution of (3) and set $(w_{k-1}, r_{k-1}) = (\tilde{w}_{k-1}, \tilde{r}_{k-1})$.
If $k > 1$, compute approximations (w_{k-1}, r_{k-1}) by applying $\mu \geq 2$ iteration steps of the multigrid algorithm applied to (3) on level $k-1$ with zero starting values.
Set

$$(u_k^{m+1}, p_k^{m+1}) = (u_k^m, p_k^m) + I_{k-1}^k (w_{k-1}, r_{k-1}).$$

In the next section the first stage of the multigrid iteration, the smoothing procedure, will be discussed in detail:

3 Additive Schwarz-type methods and symmetric inexact Uzawa methods

Let $v \in V_k$ and $q \in Q_k$. Then $\underline{v} \in \mathbb{R}^{n_k}$ and $\underline{q} \in \mathbb{R}^{m_k}$ denote their vector representations (i.e. the vectors of coefficients relative to some bases in V_k and Q_k). Furthermore, we introduce the matrix representation of the bilinear forms by

$$\mathcal{B}_k((w,r),(v,q)) = (A_k \underline{w}, \underline{v})_{\ell_2} + (B_k \underline{v}, \underline{r})_{\ell_2} + (B_k \underline{w}, \underline{q})_{\ell_2} - (C_k \underline{r}, \underline{q})_{\ell_2},$$

and the vector representation of the linear forms

$$\langle \mathcal{F}_k, (v,q) \rangle = (\underline{f}_k, \underline{v})_{\ell_2} + (\underline{g}_k, \underline{q})_{\ell_2}.$$

Here $(.,.)_{\ell_2}$ denotes the Euclidean scalar product, whose associated vector norm and matrix norm will both be denoted by $\|.\|_{\ell_2}$.

In matrix-vector notation the discrete problem (2) becomes:

$$\mathcal{K}_k \begin{pmatrix} \underline{u}_k \\ \underline{p}_k \end{pmatrix} = \begin{pmatrix} \underline{f}_k \\ \underline{g}_k \end{pmatrix} \quad \text{with} \quad \mathcal{K}_k = \begin{pmatrix} A_k & B_k^T \\ B_k & -C_k \end{pmatrix}.$$

Here, B_k^T denotes the transpose of the matrix B_k. We assume that A_k and C_k are symmetric positive semi-definite matrices, and that $\mathcal{K}_k$ is a nonsingular matrix.

Since the smoothing procedure involves only one level k of the hierarchy of spaces, we will simplify the notation for the rest of the section by dropping the subscript k and, additionally, omitting underlining the vectors. So, from now on, we discuss iterative methods (as smoothers) for linear systems of equations of the form:

$$\mathcal{K} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{with} \quad \mathcal{K} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix},$$

where $u \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, under the assumption that A is a symmetric positive semi-definite $n \times n$ matrix, B is a $m \times n$ matrix, and C is a symmetric positive semi-definite $m \times m$ matrix, and that $\mathcal{K}$ is nonsingular.

For setting up local sub-problems a set of linear operators is introduced:

$$P_i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}^n, \quad Q_i : \mathbb{R}^{m_i} \longrightarrow \mathbb{R}^m, \quad \text{for } i = 1, \dots, N,$$

where the dimensions n_i and m_i are typically much smaller than the dimensions n and m of the original spaces, respectively. The operators P_i and Q_i are interpreted as prolongation operators with associated restriction operators P_i^T and Q_i^T. We assume that

$$\sum_{i=1}^{N} Q_i Q_i^T \text{ is nonsingular and } \quad \sum_{i=1}^{N} P_i P_i^T = I, \tag{4}$$

where I denotes the identity matrix. Starting from some approximations u^j and p^j of the exact solutions u and p we consider iterative methods of form:

$$u^{j+1} = u^j + \sum_{i=1}^{N} P_i w_i^j, \quad p^{j+1} = p^j + \sum_{i=1}^{N} Q_i r_i^j,$$

where (w_i^j, r_i^j) solves the local saddle point problem

$$\begin{pmatrix} \hat{A}_i & B_i^T \\ B_i & B_i \hat{A}_i^{-1} B_i^T - \hat{S}_i \end{pmatrix} \begin{pmatrix} w_i^j \\ r_i^j \end{pmatrix} = \begin{pmatrix} P_i^T [f - Au^j - B^T p^j] \\ Q_i^T [g - Bu^j + Cp^j] \end{pmatrix}$$

with $\hat{S}_i = \tau^{-1}(C_i + B_i \hat{A}_i^{-1} B_i^T)$ for some relaxation parameter $\tau > 0$, $i = 1, \dots, N$.

That means, that the residuals of the approximations are first restricted to the smaller spaces, then a series of small saddle point problems must be solved, and, finally, the solutions are prolongated and determine the next iterate. This process can be viewed as an additive Schwarz method.

The introduction of an additional relaxation parameter τ will be necessary for the convergence analysis. In the case $\tau = 1$ the local saddle point problems completely resemble the global saddle point problem in shape.

So far, no conditions are yet fixed for choosing the matrices $\hat{A}_i$, B_i, and C_i. Two important conditions on the matrices of the local problems are introduced in the next theorem:

Theorem 1 *Assume that (4) is satisfied, the matrices $\hat{A}_i$ and $\hat{S}_i$ are symmetric and positive definite, and there is a symmetric positive definite $n \times n$-matrix $\hat{A}$ such that*

$$P_i^T \hat{A} = \hat{A}_i P_i^T \tag{5}$$

for all $i = 1, 2, \dots, N$. *Furthermore, assume that the matrices* B_i *obey the condition*

$$Q_i^T B = B_i P_i^T \tag{6}$$

for all $i = 1, 2, \dots, N$.

Then we have

$$u^{j+1} = u^j + w^j, \quad p^{j+1} = p^j + r^j, \tag{7}$$

where w^j, r^j *satisfy the equation*

$$\hat{\mathcal{K}} \begin{pmatrix} w^j \\ r^j \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{K} \begin{pmatrix} u^j \\ p^j \end{pmatrix} \quad \textit{with} \quad \hat{\mathcal{K}} = \begin{pmatrix} \hat{A} & B^T \\ B & B\hat{A}^{-1}B^T - \hat{S} \end{pmatrix}$$

and

$$\hat{S} = \left(\sum_{i=1}^{N} Q_i \hat{S}_i^{-1} Q_i^T \right)^{-1}.$$

Proof. From the local sub-problems it follows that

$$\hat{A}_i w_i^j + B_i^T r_i^j = P_i^T [f - Au^j - B^T p^j], \tag{8}$$

$$B_i w_i^j + [B_i \hat{A}_i^{-1} B_i^T - \hat{S}_i] r_i^j = Q_i^T [g - Bu^j + Cp^j]. \tag{9}$$

If (8) is multiplied by P_i and summed up, we obtain

$$\begin{aligned} \sum_{i=1}^{N} P_i \hat{A}_i w_i^j + \sum_{i=1}^{N} P_i B_i^T r_i^j &= \sum_{i=1}^{N} P_i P_i^T [f - Au^j - B^T p^j] \\ &= f - Au^j - B^T p^j. \end{aligned}$$

From (5) and (6) we obtain $P_i \hat{A}_i = \hat{A} P_i$ and $P_i B_i^T = B^T Q_i$, which immediately implies

$$\hat{A} w^j + B^T r^j = f - Au^j - B^T p^j \tag{10}$$

with $w^j = \sum_{i=1}^N P_i w_i^j$ and $r^j = \sum_{i=1}^N Q_i r_i^j$.

From (8) we have $w_i^j = \hat{A}_i^{-1}(P_i^T [f - Au^j - B^T p^j] - B_i^T r_i^j)$. Then (9), (5), and (6) imply

$$\begin{aligned} \hat{S}_i r_j^j &= B_i \hat{A}_i^{-1} P_i^T [f - Au^j - B^T p^j] - Q_i^T [g - Bu^j + Cp^j] \\ &= Q_i^T (B\hat{A}^{-1}[f - Au^j - B^T p^j] - [g - Bu^j + Cp^j]). \end{aligned}$$

Therefore,

$$r_i^j = \hat{S}_i^{-1} Q_i^T (\hat{A}^{-1}[f - Au^j - B^T p^j] - [g - Bu^j + Cp^j]).$$

If these equations are multiplied by Q_i and summed up, we obtain

$$r^j = \sum_{i=1}^{N} Q_i \hat{S}_i^{-1} Q_i^T (B\hat{A}^{-1}[f - Au^j - B^T p^j] - [g - Bu^j + Cp^j])$$

$$= \hat{S}^{-1}(B\hat{A}^{-1}[f - Au^j - B^T p^j] - [g - Bu^j + Cp^j]). \tag{11}$$

From (10) and (11) it follows that

$$\begin{pmatrix} \hat{A} & B^T \\ 0 & -\hat{S} \end{pmatrix} \begin{pmatrix} w^j \\ r^j \end{pmatrix} = \begin{pmatrix} I & 0 \\ -B\hat{A}^{-1} & I \end{pmatrix} \begin{pmatrix} f - Au^j - B^T p^j \\ g - Bu^j + Cp^j \end{pmatrix}.$$

If this equation is multiplied by the inverse of the matrix on the right hand side, the proof is completed. □

Remark 1. 1. The conditions (5) and (6) can also be written as commutative diagrams:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\hat{A}} & \mathbb{R}^n \\ {\scriptstyle P_i^T}\downarrow & & \downarrow{\scriptstyle P_i^T} \\ \mathbb{R}^{n_i} & \xrightarrow{\hat{A}_i} & \mathbb{R}^{n_i} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^n & \xrightarrow{B} & \mathbb{R}^m \\ {\scriptstyle P_i^T}\downarrow & & \downarrow{\scriptstyle Q_i^T} \\ \mathbb{R}^{n_i} & \xrightarrow{B_i} & \mathbb{R}^{m_i} \end{array}$$

1. The preconditioner $\hat{S}$ is of the typical form of an additive Schwarz preconditioner.

It can easily be seen that one step of the iteration (7) consists of three sub-steps:

$$\begin{aligned} \hat{A}(\hat{u}^{j+1} - u^j) &= f - Au^j - B^T p^j, \\ \hat{S}(p^{j+1} - p^j) &= B\hat{u}^{j+1} - Cp^j - g, \\ \hat{A}(u^{j+1} - u^j) &= f - Au^j - B^T p^{j+1}. \end{aligned}$$

In this sense the additive Schwarz method can be interpreted as a symmetric inexact Uzawa method. The convergence properties of this class of methods have been investigated in Bank, Welfert, Yserentant [1] from the point of view of inner and outer iterations, and, more generally, in Zulehner [17]. This class also contains the Braess-Sarazin smoothers and the inexact Braess-Sarazin smoothers, see Braess, Sarazin [2], and Zulehner [16]. However, none of the convergence results in these papers is helpful for discussing the so-called smoothing property, which is part of the multigrid convergence analysis, see the next section, in our situation.

Therefore, we will present a new convergence result for general symmetric inexact Uzawa methods (not only for the additive Schwarz-type iteration considered in Theorem 1) of the form

$$u^{j+1} = u^j + w^j, \quad p^{j+1} = p^j + r^j, \tag{12}$$

where w^j, r^j satisfy

$$\text{(13)} \quad \hat{\mathcal{K}} \begin{pmatrix} w^j \\ r^j \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{K} \begin{pmatrix} u^j \\ p^j \end{pmatrix} \quad \text{with} \quad \hat{\mathcal{K}} = \begin{pmatrix} \hat{A} & B^T \\ B & B\hat{A}^{-1}B^T - \hat{S} \end{pmatrix},$$

for general symmetric and positive definite matrices $\hat{A}$ and $\hat{S}$, which will be helpful in our case.

For this, we first introduce the iteration matrix

$$\mathcal{M} = I - \hat{\mathcal{K}}^{-1}\mathcal{K},$$

which controls the error propagation for the iterative method (12).

In the next lemma, which gives an important representation for $\mathcal{M}$, the following notations are used: $M < N$ ($N > M$) iff $N - M$ is positive definite, and $M \leq N$ ($N \geq M$) iff $N - M$ is positive semi-definite, for symmetric matrices M and N.

Lemma 1 *Let $\hat{A}$ be a symmetric and positive definite $n \times n$ matrix, and $\hat{S}$ a symmetric positive definite $m \times m$ matrix, satisfying*

$$\text{(14)} \qquad \hat{A} > A \quad \text{and} \quad \hat{S} > C + B\hat{A}^{-1}B^T.$$

Then we have:

The iteration matrix $\mathcal{M} = I - \hat{\mathcal{K}}^{-1}\mathcal{K}$ can be written in the form

$$\mathcal{M} = \mathcal{Q}^{-1/2}\bar{\mathcal{M}}\mathcal{Q}^{1/2}$$

with the symmetric positive definite block diagonal matrix

$$\mathcal{Q} = \begin{pmatrix} \hat{A} - A & 0 \\ 0 & \hat{S} - C - B\hat{A}^{-1}B^T \end{pmatrix}$$

and

$$\bar{\mathcal{M}} = \mathcal{P}^T\mathcal{N}\mathcal{P},$$

where $\mathcal{N}$ is a normal matrix and $\mathcal{P}$ satisfies the conditions $\|\mathcal{P}\|_{\ell_2} \leq 1$. Moreover, for the spectrum $\sigma(\mathcal{N})$ we have:

$$\text{(15)} \qquad \sigma(\mathcal{N}) \subset \{z \in \mathbb{C} : \left| z - \frac{1}{2} \right| = \frac{1}{2}\}.$$

Proof. Simple calculations show that the iteration matrix can be written in the following form:

$$\begin{aligned}\mathcal{M} &= \hat{\mathcal{K}}^{-1}(\hat{\mathcal{K}} - \mathcal{K}) \\ &= \begin{pmatrix} \hat{A} & B^T \\ B & B\hat{A}^{-1}B^T - \hat{S} \end{pmatrix}^{-1} \begin{pmatrix} \hat{A} - A & 0 \\ 0 & C + B\hat{A}^{-1}B^T - \hat{S} \end{pmatrix} \\ &= \begin{pmatrix} \hat{A} & B^T \\ -B & \hat{S} - B\hat{A}^{-1}B^T \end{pmatrix}^{-1} \begin{pmatrix} \hat{A} - A & 0 \\ 0 & \hat{S} - C - B\hat{A}^{-1}B^T \end{pmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\bar{\mathcal{M}} &= \mathcal{Q}^{1/2}\mathcal{M}\mathcal{Q}^{-1/2} = \mathcal{Q}^{1/2} \begin{pmatrix} \hat{A} & B^T \\ -B & \hat{S} - B\hat{A}^{-1}B^T \end{pmatrix}^{-1} \mathcal{Q}^{1/2} \\ &= \mathcal{Q}^{1/2}\mathcal{D}^{-1/2}\mathcal{N}\mathcal{D}^{-1/2}\mathcal{Q}^{1/2}\end{aligned}$$

with

$$\mathcal{D} = \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{S} - B\hat{A}^{-1}B^T \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} I & \bar{B}^T \\ -\bar{B} & I \end{pmatrix}^{-1}$$

and

$$\bar{B} = (\hat{S} - B\hat{A}^{-1}B^T)^{-1/2}B\hat{A}^{-1/2}.$$

Straight forward computations show that $\mathcal{N}$ commutes with $\mathcal{N}^T$, i.e. $\mathcal{N}$ is normal, and that the eigenvalues λ of $\mathcal{N}^{-1}$ are of the form $1 \pm i\mu$ with $\mu^2 \in \{0\} \cup \sigma(\bar{B}\bar{B}^T)$. So, the eigenvalues lie on the straight line of all complex numbers with real part 1. By the transformation $z \longrightarrow 1/z$ this straight line is mapped to the circle (15), which, therefore, must contain all eigenvalues of $\mathcal{N}$.

With $\mathcal{P} = \mathcal{D}^{-1/2}\mathcal{Q}^{1/2}$ we obtain the required representation of the iteration matrix. Moreover, since $\mathcal{Q} \leq \mathcal{D}$ we have

$$\mathcal{P}\mathcal{P}^T = \mathcal{D}^{-1/2}\mathcal{Q}\mathcal{D}^{-1/2} \leq I,$$

which completes the proof. □

Next we consider the relaxed iterative method

$$u^{j+1} = u^j + \omega\, w^j, \quad p^{j+1} = p^j + \omega\, r^j, \tag{16}$$

where w^j, r^j satisfy

$$\hat{\mathcal{K}} \begin{pmatrix} w^j \\ r^j \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{K} \begin{pmatrix} u^j \\ p^j \end{pmatrix}$$

for some relaxation parameter $\omega > 0$.

The error propagation is now controlled by the iteration matrix $(1-\omega)I + \omega\mathcal{M}$.

The following convergence result for the relaxed method is a consequence of the representation of $\mathcal{M}$ in Lemma 1:

Theorem 2 *Let $\hat{A}$ be a symmetric and positive definite $n \times n$ matrix, and $\hat{S}$ a symmetric positive definite $m \times m$ matrix, satisfying (14).*

Then we have:

$$\|(1-\omega)I + \omega\mathcal{M}\|_{\mathcal{Q}} \le 1$$

for all relaxation factors $\omega \in [0, 2]$ and

$$\|(1-\omega)I + \omega\mathcal{M}\|_{\mathcal{Q}} < 1$$

for all relaxation factors $\omega \in (0, 2)$. Here $\|.\|_{\mathcal{Q}}$ denotes the matrix norm associated to the scalar product

$$((w, r), (v, q))_{\mathcal{Q}} = ((\hat{A} - A)w, v)_{\ell_2} + ((\hat{S} - C - B\hat{A}^{-1}B^T)r, q))_{\ell_2}.$$

Proof. We use the notations introduced in the proof of Theorem 1. Observe that

$$\|(1-\omega)I + \omega\mathcal{M}\|_{\mathcal{Q}} = \|(1-\omega)I + \omega\bar{\mathcal{M}}\|_{\ell_2}.$$

In a first step we show that

$$\|(1-\omega)I + \omega\bar{\mathcal{M}}\|_{\ell_2} \le 1, \tag{17}$$

or equivalently,

$$[(1-\omega)I + \omega\bar{\mathcal{M}}]^T[(1-\omega)I + \omega\bar{\mathcal{M}}] \le I.$$

We have

$$\begin{aligned}
&[(1-\omega)I + \omega\bar{\mathcal{M}}]^T[(1-\omega)I + \omega\bar{\mathcal{M}}] \\
&\quad = \omega^2\bar{\mathcal{M}}^T\bar{\mathcal{M}} + \omega(1-\omega)[\bar{\mathcal{M}}^T + \bar{\mathcal{M}}] + (1-\omega)^2 I \\
&\quad = \omega^2\mathcal{P}^T\mathcal{N}^T\mathcal{P}\mathcal{P}^T\mathcal{N}\mathcal{P} + \omega(1-\omega)\mathcal{P}^T[\mathcal{N}^T + \mathcal{N}]\mathcal{P} + (1-\omega)^2 I \\
&\quad \le \omega^2\mathcal{P}^T\mathcal{N}^T\mathcal{N}\mathcal{P} + \omega(1-\omega)\mathcal{P}^T[\mathcal{N}^T + \mathcal{N}]\mathcal{P} + (1-\omega)^2 I \\
&\quad = \mathcal{P}^T[(1-\omega)I + \omega\mathcal{N}]^T[(1-\omega)I + \omega\mathcal{N}]\mathcal{P} + (1-\omega)^2[I - \mathcal{P}^T\mathcal{P}].
\end{aligned}$$

Since $\mathcal{N}$ is a normal matrix, whose eigenvalues lie on the circle (15), it follows that

$$\|(1-\omega)I + \omega\mathcal{N}\|_{\ell_2} = \rho((1-\omega)I + \omega\mathcal{N}) \le 1$$

for $\omega \in [0, 2]$. Here $\rho(M)$ denotes the spectral radius of a matrix M. Therefore

$$[(1-\omega)I + \omega\mathcal{N}]^T[(1-\omega)I + \omega\mathcal{N}] \le I,$$

which allows to continue the chain of estimates from above:

$$\begin{aligned}
&[(1-\omega)I + \omega\bar{\mathcal{M}}]^T[(1-\omega)I + \omega\bar{\mathcal{M}}] \\
&\quad \le \mathcal{P}^T\mathcal{P} + (1-\omega)^2[I - \mathcal{P}^T\mathcal{P}] \\
&\quad = (1-\omega)^2 I + [1-(1-\omega)^2]\mathcal{P}^T\mathcal{P} \\
&\quad \le (1-\omega)^2 I + [1-(1-\omega)^2]I = I,
\end{aligned}$$

which completes the proof of (17).

In order to exclude the equality sign in (17) for $\omega \in (0,2)$, it remains to show that there is no vector z with $\|z\|_{\ell_2} = 1$ and $\|[(1-\omega)I + \omega\bar{\mathcal{M}}]z\|_{\ell_2} = 1$ for $\omega \in (0,2)$. Assume now that such a vector exists. Using the same chain of inequalities as before, one easily shows that

$$\begin{aligned}
1 &= z^T[(1-\omega)I + \omega\bar{\mathcal{M}}][(1-\omega)I + \omega\bar{\mathcal{M}}]z \\
&\le z^T\mathcal{P}^T[(1-\omega)I + \omega\mathcal{N}]^T[(1-\omega)I + \omega\mathcal{N}]\mathcal{P}z \\
&\quad + (1-\omega)^2 z^T[I - \mathcal{P}^T\mathcal{P}]z \\
&\le z^T\mathcal{P}^T\mathcal{P}z + (1-\omega)^2 z^T[I - \mathcal{P}^T\mathcal{P}]z \\
&= (1-\omega)^2 z^T z + [1-(1-\omega)^2]z^T\mathcal{P}^T\mathcal{P}z \\
&\le z^T z = 1.
\end{aligned}$$

But this can only happen if

$$z^T\mathcal{P}^T[(1-\omega)I + \omega\mathcal{N}]^T[(1-\omega)I + \omega\mathcal{N}]\mathcal{P}z = z^T\mathcal{P}^T\mathcal{P}z$$

and

$$z^T\mathcal{P}^T\mathcal{P}z = z^T z,$$

which imply

$$[(1-\omega)I + \omega\mathcal{N}]^T[(1-\omega)I + \omega\mathcal{N}]\mathcal{P}z = \mathcal{P}z$$

and

$$\mathcal{P}^T\mathcal{P}z = z. \tag{18}$$

Since $(1-\omega)I + \omega\mathcal{N}$ is a normal matrix, whose only eigenvalue with modulus 1 is equal to 1, we can further deduce that

$$[(1-\omega)I + \omega\mathcal{N}]\mathcal{P}z = \mathcal{P}z,$$

therefore

$$\mathcal{N}\mathcal{P}z = \mathcal{P}z. \tag{19}$$

The relations (18) and (19) lead to

$$\mathcal{P}\mathcal{P}^T\mathcal{P}z = \mathcal{P}z \quad \text{and} \quad \mathcal{N}^{-1}\mathcal{P}z = \mathcal{P}z,$$

from which we obtain

$$(\mathcal{D} - \mathcal{Q})w = 0 \quad \text{and} \quad (\mathcal{D}^{1/2}\mathcal{N}^{-1}\mathcal{D}^{1/2} - \mathcal{D})w = 0$$

for $w = \mathcal{D}^{-1/2}\mathcal{P}z$. That means

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} w = 0 \quad \text{and} \quad \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix} w = 0.$$

This, however, implies $\mathcal{K}w = 0$. Therefore, $w = 0$ and $z = 0$, in contradiction to the assumption $\|z\|_{\ell_2} = 1$. □

Theorem 2 states that the relaxed method (16) converges for all relaxation parameters $\omega \in (0, 2)$. Of particular interest is the limiting case $\omega \to 2$, for which Theorem 2 guarantees at least that the iterates do not blow up, i.e.:

$$\|2\mathcal{M} - I\|_{\mathcal{Q}} \leq 1.$$

This property leads to an important estimate, formulated in the next theorem and needed in the forthcoming multigrid convergence analysis:

Theorem 3 *Let $\hat{A}$ be a symmetric and positive definite $n \times n$ matrix, and $\hat{S}$ a symmetric positive definite $m \times m$ matrix, satisfying*

$$\hat{A} \geq A \quad \textit{and} \quad \hat{S} \geq C + B\hat{A}^{-1}B^T.$$

Then

$$\|\mathcal{K}\mathcal{M}^m\|_{\ell_2} \leq \eta_0(m)\, \|\hat{\mathcal{K}} - \mathcal{K}\|_{\ell_2}$$

with $\mathcal{K}$ given by (13) and

$$\eta_0(m) = \frac{1}{2^{m-1}} \binom{m-1}{[m]/2} \leq \begin{cases} \sqrt{\dfrac{2}{\pi(m-1)}} & \text{for even } m, \\ \sqrt{\dfrac{2}{\pi m}} & \text{for odd } m, \end{cases}$$

where $\binom{n}{k}$ denotes the binomial coefficient and $[x]$ denotes the largest integer smaller than or equal to $x \in \mathbb{R}$.

Proof. We first assume that the strict inequalities $\hat{A} > A$ and $\hat{S} > C + B\hat{A}^{-1}B^T$ hold. Then we have with the notations used in the proofs of the last two theorems:

$$\begin{aligned} \mathcal{K}\mathcal{M}^m &= (\hat{\mathcal{K}} - \mathcal{K})(I - \mathcal{M})\mathcal{M}^{m-1} \\ &= (\hat{\mathcal{K}} - \mathcal{K})\mathcal{Q}^{-1/2}(I - \bar{\mathcal{M}})\bar{\mathcal{M}}^{m-1}\mathcal{Q}^{1/2} \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \mathcal{Q}^{1/2}(I - \bar{\mathcal{M}})\bar{\mathcal{M}}^{m-1}\mathcal{Q}^{1/2} \end{aligned}$$

Therefore,

$$\|\mathcal{K}\mathcal{M}^m\|_{\ell_2} \le \|(I - \bar{\mathcal{M}})\bar{\mathcal{M}}^{m-1}\|_{\ell_2}\, \|\mathcal{Q}\|_{\ell_2}.$$

From

$$Q = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (\hat{\mathcal{K}} - \mathcal{K})$$

we obtain

$$\|Q\|_{\ell_2} = \|\hat{\mathcal{K}} - \mathcal{K}\|_{\ell_2}.$$

From Theorem 2 it follows that

$$\|2\bar{\mathcal{M}} - I\|_{\ell_2} \le 1.$$

Then Reusken's lemma, see Reusken [12], Hackbusch [10], implies

$$\|(I - \bar{\mathcal{M}})\bar{\mathcal{M}}^{m-1}\|_{\ell_2} \le \eta_0(m)$$

with

$$\eta_0(m) = \frac{1}{2^{m-1}} \binom{m-1}{[m]/2} \le \begin{cases} \sqrt{\dfrac{2}{\pi(m-1)}} & \text{for even } m, \\ \sqrt{\dfrac{2}{\pi m}} & \text{for odd } m. \end{cases}$$

A simple closure argument for the case $\hat{A} \ge A$ and $\hat{S} \ge C + B\hat{A}^{-1}B^T$ completes the proof. □

4 Multigrid convergence analysis

A classical technique for analyzing the convergence of multigrid methods relies on two properties: the approximation property and the smoothing property, see Hackbusch [9], which will be discussed in this section.

First we need mesh-dependent norms on $V_k \times Q_k$. Let $|||(.,.)|||_{0,k}$ be an L^2-like norm on $V_k \times Q_k$, for which we assume that

$$|||(v,q)|||_{0,k} \sim \left(\|\underline{v}\|_{\ell_2}^2 + \|\underline{q}\|_{\ell_2}^2\right)^{1/2} = \left\| \begin{pmatrix} \underline{v} \\ \underline{q} \end{pmatrix} \right\|_{\ell_2} \tag{20}$$

for $v \in V_k$, $q \in Q_k$ with vector representations $\underline{v} \in \mathbb{R}^{n_k}$, $\underline{q} \in \mathbb{R}^{m_k}$. The symbol $\sim$ denotes the equivalence of norms.

Next we introduce a second discrete norm on $V_k \times Q_k$ by

$$|||(w,r)|||_{2,k} = \sup_{(v,q)\in V_k\times Q_k} \frac{|\mathcal{B}_k((w,r),(v,q))|}{|||(v,q)|||_{0,k}}.$$

Then, under reasonable assumptions on the continuous problem and its discretization, it can be shown for the two-grid algorithm (i.e. exact solution of the coarse grid correction equation (3) at level $k-1$) that

$$|||\mathcal{B}_k|||_{0,k}\ |||(u_k^{m+1}-u_k,\ p_k^{m+1}-p_k)|||_{0,k} \le c_A\ |||(u_k^m-u_k,\ p_k^m-p_k)|||_{2,k},$$

for some constant c_A which is independent of k. See e.g. Brenner [5], [6] for several cases of mixed variational problems and appropriate finite element spaces, for which this property, the so-called approximation property, could be shown. In the next section the approximation property for the Crouzeix-Raviart element for the Stokes problem is discussed in detail.

The missing part to complete the proof of the two-grid convergence is the smoothing property:

$$|||(u_k^m-u_k,\ p_k^m-p_k)|||_{2,k} \le \eta(m)\ |||\mathcal{B}_k|||_{0,k}\ |||(u_k^0-u_k,\ p_k^0-p_k)|||_{0,k}$$

for some function $\eta(m)$ which is independent of k, and

$$\eta(m) \to 0 \quad \text{for } m \to \infty.$$

The convergence of the two-grid method for a sufficiently large number m of smoothing steps easily follows by combining the approximation property and the smoothing property. From this the convergence of the multigrid method can be derived by standard arguments, see, e.g., Hackbusch [9].

From the matrix representation of the bilinear form $\mathcal{B}_k$ and the scaling (20) we obtain

$$|||(v,q)|||_{2,k} \sim \left\| \mathcal{K}_k \begin{pmatrix} \underline{v} \\ \underline{q} \end{pmatrix} \right\|_{\ell_2}.$$

From this, it easily follows that the smoothing property translates to the following conditions in matrix-notation:

$$\|\mathcal{K}_k\mathcal{M}_k^m\|_{\ell_2} \le \eta(m)\ \|\mathcal{K}_k\|_{\ell_2}. \tag{21}$$

Summarizing the results of Section 3 (see, in particular, Theorem 3) one immediately obtains:

Theorem 4 *Let*

$$\hat{\mathcal{K}}_k = \begin{pmatrix} \hat{A}_k & B_k^T \\ B_k & B_k\hat{A}_k^{-1}B_k^T - \hat{S}_k \end{pmatrix}$$

with $\hat{A}_k$ a symmetric and positive definite $n \times n$ matrix, $\hat{S}_k$ a symmetric and positive definite $m \times m$ matrix, satisfying

$$\hat{A}_k \ge A_k \quad \text{and} \quad \hat{S}_k \ge C_k + B_k\hat{A}_k^{-1}B_k^T.$$

Futhermore, assume that

$$\|\hat{\mathcal{K}}_k - \mathcal{K}_k\|_{\ell_2} \le c_R \,\|\mathcal{K}_k\|_{\ell_2}$$

for some constant c_R.

Then the smoothing property

$$\|\mathcal{K}_k \mathcal{M}_k^m\|_{\ell_2} \le \eta(m)\,\|\mathcal{K}_k\|_{\ell_2}$$

is satisfied with smoothing rate $\eta(m) = c_R\,\eta_0(m) = O(1/\sqrt{m})$.

5 Application to the Crouzeix–Raviart element for the Stokes problem

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ and f a given function in $L^2(\Omega)$. The Stokes problem with homogeneous Dirichlet boundary conditions is given by:

$$\begin{aligned} -\Delta u + \operatorname{grad} p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega p\,dx &= 0. \end{aligned}$$

The weak formulation of this problem leads to a mixed variational problem:

Find $u \in V = H_0^1(\Omega)^2$ and $p \in Q = L_0^2(\Omega)$, the subspace $L^2(\Omega)$ with vanishing mean value, such that

$$\begin{aligned} a(u, v) + b(v, p) &= \langle F, v\rangle \quad \text{for all } v \in V, \\ b(u, q) &= 0 \qquad\quad\ \text{for all } q \in Q, \end{aligned}$$

with

$$\begin{aligned} a(w, v) &= \int_\Omega \operatorname{grad} w : \operatorname{grad} v\,dx, \\ b(v, q) &= -\int_\Omega q\ \operatorname{div} v\,dx, \\ \langle F, v\rangle &= \int_\Omega f \cdot v\,dx. \end{aligned}$$

Let $(\mathcal{T}_k)$ be a sequence of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of edges of the triangles in $\mathcal{T}_k$. We denote $\max\{\operatorname{diam} T : T \in \mathcal{T}_k\}$ by h_k.

Then the Crouzeix–Raviart element, see [8], is determined by the following non-conforming finite element spaces:

$$\begin{aligned} V_k = \{ & v \in L^2(\Omega)^2 : v\big|_T \text{ is linear for all } T \in \mathcal{T}_k, \\ & v \text{ is continuous at the midpoints of interelement boundaries} \\ & \text{and } v = 0 \text{ at the midpoints of edges along } \partial\Omega\} \\ Q_k = \{ & q \in L_0^2(\Omega) : q\big|_T \text{ is constant for all } T \in \mathcal{T}_k\} \end{aligned}$$

The finite element discretization is given by the discrete variational problem:

Find $u_k \in V_k$ and $p_k \in Q_k$ such that

$$\begin{aligned} a_k(u_k, v) + b_k(v, p_k) &= \langle F, v\rangle \quad \text{for all } v \in V_k, \\ b_k(u_k, q) &= 0 \qquad\quad \text{for all } q \in Q_k \end{aligned}$$

with

$$a_k(w, v) = \sum_{T \in \mathcal{T}_k} \int_T \operatorname{grad} w : \operatorname{grad} v \, dx,$$

$$b_k(v, q) = -\sum_{T \in \mathcal{T}_k} \int_T q \, \operatorname{div} v \, dx,$$

which eventually leads to a linear system

$$\begin{pmatrix} A_k & B_k^T \\ B_k & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_k \\ \underline{p}_k \end{pmatrix} = \begin{pmatrix} \underline{f}_k \\ \underline{g}_k \end{pmatrix},$$

where the unknowns $\underline{u}_k$ are ordered pointwise.

Next a multigrid method is formulated by specifying the inter-grid transfer operators and the smoothing procedure.

Following Brenner [5] the inter-grid transfer operators $I_{k-1}^k : V_{k-1} \times Q_{k-1} \longrightarrow V_k \times Q_k$ are given by

$$I_{k-1}^k(v, q) = (J_{k-1}^k v, q)$$

with

$$J_{k-1}^k v(m_e) = \begin{cases} v(m_e) & \text{if } m_e \in \operatorname{int} T \text{ for some } T \in \mathcal{T}_{k-1} \\ \frac{1}{2}\left[v|_{T_1} + v|_{T_2}\right] & \text{if } e \subset T_1 \cap T_2 \text{ for some } T_1, T_2 \in \mathcal{T}_{k-1} \end{cases}$$

at midpoints m_e of internal edges e in $\mathcal{T}_k$.

The mesh-dependent L^2-like norm on $V_k \times Q_k$ is given by

$$|||(v, q)|||_{0,k} = \left[\|v\|_{L^2(\Omega)^2}^2 + h_k^2 \, \|q\|_{L^2(\Omega)}^2\right]^{1/2}.$$

By an appropriate scaling we can achieve that

$$\|v\|_{L^2(\Omega)^2} \sim \|\underline{v}\|_{\ell_2}, \quad h_k \, \|q\|_{L^2(\Omega)} \sim \|\underline{q}\|_{\ell_2}$$

Then condition (20) is satisfied.

Lemma 3.3 and Lemma 5.3 in Brenner [5] contain the approximation property

$$|||\mathcal{B}_k|||_{0,k}\, |||(u_k^{m+1} - u_k,\, p_k^{m+1} - p_k)|||_{0,k} \le c_A\, |||(u_k^m - u_k,\, p_k^m - p_k)|||_{2,k}$$

for some constants c_A, independent of the grid level k.

For the smoothing procedure we have to define appropriate local sub-problems at grid level k.

Let $\mathcal{N}_{k,e} = \{1, 2, \dots, N_{k,e}\}$ denote the index set of all midpoints of inter-element boundaries and $\mathcal{N}_{k,T} = \{1, 2, \dots, N_{k,T}\}$ the index set of all triangles in $\mathcal{T}_k$. With these notations we have for the dimension of the saddle point problem: $n_k = 2N_{k,e}$ and $m_k = N_{k,T}$.

For each $i \in \mathcal{N}_{k,T}$ let $\mathcal{N}_{k,i} \subset \mathcal{N}_{k,e}$ be the subset of all indices which correspond to the midpoints of all interior edges of the triangle with index i. So, $\mathcal{N}_{k,i}$ consists of 3 indices for interior triangles, the number of indices reduces to 2 or 1 near the boundary of the domain.

To each triangle with index i we assign a local sub-problem of dimension $n_{k,i} = 2\,|\mathcal{N}_{k,i}|$ and $m_{k,i} = 1$ by defining the following canonical prolongations:

$Q_{k,i}$ is the $m_k \times 1$ matrix whose entry of the i-th row is equal to 1, all other entries are 0.

$\hat{P}_{k,i}$ is the $n_k \times n_{k,i}$ matrix, or better the $n_k/2 \times n_{k,i}/2$ block matrix of 2×2 blocks, whose block columns correspond to the indices $l \in \mathcal{N}_{k,i}$. The l-th block position of the l-th block column is equal to the 2×2 identity matrix, all other entries are 0.

It is easy to see that

$$\sum_{i=1}^{N_{k,T}} \hat{P}_{k,i} \hat{P}_{k,i}^T = 2\, I,$$

because each midpoint of an edge is contained in exactly 2 different sets $\mathcal{N}_{k,i}$. Therefore, for

$$P_{k,i} = \frac{1}{\sqrt{2}} \hat{P}_{k,i}$$

the correct scaling condition

$$\sum_{i=1}^{N_{k,T}} P_{k,i} P_{k,i}^T = I$$

of Theorem 1 is satisfied.

Next we choose for $\hat{A}_k$ the scaled Jacobi preconditioner of A_k:

$$\hat{A}_k = \frac{1}{\sigma} \operatorname{diag}(A_k) \tag{22}$$

with σ small enough to ensure $\hat{A}_k \geq A_k$, i.e.:

$$\frac{1}{\sigma}\,\mathrm{diag}(A_k) \geq A_k.$$

Since A_k has at most 5 non-zero entries per row, it suffices to have $\sigma \leq 1/5$.

For the local sub-problems we choose just the restriction of $\hat{A}_k$ to those components of $\underline{u}_k$ whose indices are in $\mathcal{N}_{k,i}$:

$$\hat{A}_{k,i} = \hat{P}_{k,i}^T \hat{A}_k \hat{P}_{k,i}. \tag{23}$$

Since the matrices $\hat{A}_k$ and $\hat{A}_{k,i}$ are diagonal the condition (5) is satisfied.

The other matrices of the local sub-problems are specified similarly: For

$$\hat{B}_{k,i} = Q_{k,i}^T B_k \hat{P}_{k,i}$$

one can easily verify the relation

$$Q_{k,i}^T B_k = \hat{B}_{k,i} \hat{P}_{k,i}^T.$$

The argument is, that the i-th component of $B_k v$, which corresponds to the value of $B_k v$ on the triangle with index i, depends only on the velocities in the neighboring midpoints of that triangle, whose indices are collected in the set $\mathcal{N}_{k,i}$. On this index set $P_{k,i}^T$ acts like the identity.

From this identity the condition (6) immediately follows if we set

$$B_{k,i} = \sqrt{2}\,\hat{B}_{k,i}. \tag{24}$$

Finally, we set

$$C_{k,i} = 0. \tag{25}$$

With the notations of Theorem 1 the definitions (23), (24) and (25) lead to

$$\hat{S}_{k,i} = \frac{2}{\tau}\, Q_{k,i}^T B_k \hat{A}_k^{-1} B_k^T Q_{k,i}.$$

Hence

$$\begin{aligned}\hat{S}_k &= \frac{2}{\tau}\left(\sum_{i=1}^{N_{k,T}} Q_{k,i} (Q_{k,i}^T B_k \hat{A}_k^{-1} B_k^T Q_{k,i})^{-1} Q_{k,i}^T\right)^{-1} \\ &= \frac{2}{\tau}\,\mathrm{diag}(B_k \hat{A}_k^{-1} B_k^T). \end{aligned} \tag{26}$$

According to the conditions of Theorem 3 the relaxation parameter τ has to chosen such that $\hat{S}_k \geq B_k \hat{A}_k^{-1} B_k^T$, i.e.:

$$\frac{2}{\tau}\,\mathrm{diag}(B_k \hat{A}_k^{-1} B_k^T) \geq B_k \hat{A}_k^{-1} B_k^T$$

Because of the sparsity pattern of $B_k \hat{A}_k^{-1} B_k^T$ it can easily be shown that it suffices to choose $\tau \leq 1/2$.

Finally, the last missing part for the smoothing property is the estimate

$$\|\hat{\mathcal{K}}_k - \mathcal{K}_k\|_{\ell_2} \leq c_R \, \|\mathcal{K}_k\|_{\ell_2}.$$

Using the simple estimates $\|\operatorname{diag}(M)\|_{\ell_2} \leq \|M\|_{\ell_2}$ for any matrix M and $\|M\|_{\ell_2} \leq \|N\|_{\ell_2}$ for symmetric matrices M, N with $0 \leq M \leq N$ we have

$$\begin{aligned}
\|\hat{\mathcal{K}}_k - \mathcal{K}_k\|_{\ell_2} &= \max(\|\hat{A}_k - A_k\|_{\ell_2}, \|\hat{S}_k - B_k \hat{A}_k^{-1} B_k^T\|_{\ell_2}) \\
&\leq \max(\|\hat{A}_k\|_{\ell_2}, \|\hat{S}_k\|_{\ell_2}) \\
&\leq \max(\sigma^{-1}\|A_k\|_{\ell_2}, 2\tau^{-1}\|B_k \hat{A}_k^{-1} B_k^T\|_{\ell_2}).
\end{aligned}$$

The entries of $\hat{A}_k^{-1} B_k^T$ are of the form $b_k(v, q)/a_k(v, v)$, where v and q are basis functions with local support. A standard scaling argument shows that these entries are bounded independently of k, say by some constant c_S. Considering the sparsity pattern of B_k^T, one obtains

$$\|\hat{A}_k^{-1} B_k^T\|_{\ell_2} \leq 3c_S.$$

Therefore

$$\begin{aligned}
\|\hat{\mathcal{K}}_k - \mathcal{K}_k\|_{\ell_2} &\leq \max(\sigma^{-1}\|A_k\|_{\ell_2}, 6c_S\tau^{-1}\|B_k\|_{\ell_2}) \\
&\leq \max(\sigma^{-1}, 6c_S\tau^{-1}) \max(\|A_k\|_{\ell_2}, \|B_k\|_{\ell_2}) \\
&\leq \max(\sigma^{-1}, 6c_S\tau^{-1}) \, \|\mathcal{K}_k\|_{\ell_2} \\
&= c_R \, \|\mathcal{K}_k\|_{\ell_2}
\end{aligned}$$

with $c_R = \max(\sigma^{-1}, 6c_S\tau^{-1})$.

Remark 2. The construction of local sub-problems satisfying (4), (5) and (6) can easily be extended to general finite element discretizations. Assume for simplicity there is a nodal basis for the finite element spaces, the u-nodes determine V_k and the p-nodes determine Q_k. First the space $\mathbb{R}^{m_k}$ for the dual variable p is split into a direct sum of subspaces corresponding to disjoint index-sets $\mathcal{M}_{k,i}$, $i = 1, 2, \ldots, N_k$ of p-nodes. In the simplest case $\mathcal{M}_{k,i}$ consists of just one index representing one individual p-node. The prolongation $Q_{k,i}$ is the corresponding canonical embedding into $\mathbb{R}^{m_k}$. All u-nodes which are connected to some p-node with index in $\mathcal{M}_{k,i}$ (i.e. the corresponding entry $b(v, q)$ is non-zero) determine an index-set $\mathcal{N}_{k,i}$. The prolongation $\hat{P}_{k,i}$ is the corresponding canonical embedding into $\mathbb{R}^{n_k}$.

For $\hat{A}_k$ one can choose any block-diagonal matrix with 2×2 diagonal blocks. The local sub-problems are given by the matrices $\hat{A}_{k,i} = \hat{P}_{k,i}^T \hat{A}_k \hat{P}_{k,i}$, $\hat{B}_{k,i} = Q_{k,i}^T B_k \hat{P}_{k,i}$ and $C_{k,i} = Q_{k,i}^T C_k Q_{i,k}$.

Table 1. Convergence rates for the additive Schwarz smoother

	Unknowns		Smoothing steps					
Level k	n_k	m_k	14	16	18	20	24	32
4	416	128	0.790	0.590	0.373	0.237	0.130	0.259
5	1 600	512	0.768	0.571	0.395	0.356	0.331	0.306
6	6 272	2 048	0.773	0.578	0.403	0.368	0.349	0.300
7	24 832	8 192	0.772	0.577	0.404	0.370	0.354	0.302
8	98 816	32 768	0.772	0.577	0.410	0.378	0.358	0.304

All entries of value 1 in $\hat{P}_{k,i}$ correspond to some u-node with index $j \in \mathcal{N}_{k,i}$. In order to guarantee (4) these entries have to be replaced by $1/\sqrt{\mu_{k,j}}$ where $\mu_{k,j}$ is the number of index sets $\mathcal{N}_{k,l}$ with $j \in \mathcal{N}_{k,l}$ (the local overlap depth at that u-node). This gives the corrected prolongation operators $P_{k,i}$. The scaling of $\hat{B}_{k,i}$ has to be changed accordingly, resulting in the corrected matrices $B_{k,i}$.

Next we present some numerical results for the example $\Omega = (0, 1) \times (0, 1)$, $f = 0$. The initial grid (level $k = 1$) consists of two triangles by connecting the vertices (0, 0) and (1, 1).

Randomly chosen starting values for u_k^0 and p_k^0 for the exact solution $u_k = 0$ and $p_k = 0$ were used.

The discretized equations on grid level k were solved by a multigrid iteration with the W-cycle and $m/2$ pre- and $m/2$ post-smoothing steps. The preconditioners $\hat{A}_k$ and $\hat{S}_k$ were chosen according to (22) and (26). Using the corresponding maximum eigenvalues, which were numerically determined by the Lanczos method, the parameters σ and τ were adjusted such that $\hat{A}_k \geq A_k$ and $\hat{S}_k \geq B_k A_k^{-1} B_k^T$. Table 1 contains the number of unknowns n_k and m_k depending on the level k and the (averaged) convergence rates q depending on the level k and the number m of smoothing steps.

The convergence rates show the typical multigrid behavior: asymptotic independence of the grid level and improvement of the rates with an increasing number of smoothing steps. No convergence could be obtained with less than 7 pre- and 7 post-smoothing steps.

The next table 2 shows the rates with the multiplicative version of the smoother. In this case no particular scaling was performed, i.e. the local saddle point problems are given by restricting the global saddle point problem to the corresponding local variables.

As expected, the rates for the multiplicative Schwarz smoother (Vanka-smoother) are significantly better than the rates for the additive Schwarz smoother. Convergence occurred for less smoothing steps: only 2 pre- and 2 post-smoothing steps are required to guarantee convergence.

Table 2. Convergence rates for the multiplicative Schwarz smoother

	Unknowns		Smoothing steps					
Level k	n_k	m_k	4	6	8	10	12	14
4	416	128	0.580	0.346	0.213	0.182	0.138	0.079
5	1 600	512	0.590	0.351	0.206	0.167	0.159	0.136
6	6 272	2 048	0.579	0.351	0.207	0.177	0.157	0.138
7	24 832	8 192	0.589	0.347	0.208	0.177	0.158	0.139
8	98 816	32 768	0.601	0.345	0.209	0.180	0.160	0.142

Table 3. Relative work factors for the additive Schwarz smoother

Smoothing steps	14	16	18	20	24	32
W_r	54.1	29.1	20.2	20.6	23.4	26.9

Table 4. Relative work factors for the multiplicative Schwarz smoother

smoothing steps	4	6	8	10	12	14
W_r	7.9	5.6	5.1	5.8	6.5	7.2

A reasonable measure for comparing the two different methods with a varying number m of smoothing steps is the total computational work W_t necessary for reducing an initial error by a factor $1/e$.

The number of iterations It for achieving this reduction is asymptotically given by $It = -1/\ln q$, where q is the convergence rate of the method.

If we assume that the total amount of computational work W_t is dominated by the computational work for performing the smoothing steps and that the computational work for one smoothing step is the same for the additive as well as for the multiplicative version, say W_0, we obtain $W_t = It * m * W_0 = W_r * W_0$, where the relative work factor W_r is given by $W_r = -m/\ln q$.

The next two tables 3 and 4 show the relative work factors for the additive and the multiplicative smoothers, based on the convergence rates at level 8.

One can see that the most efficient case for the additive Schwarz smoother, the W-cycle with 9 pre- and 9 post-smoothing steps, requires about four times more work than the optimal case for the multiplicative Schwarz smoother, the W-cycle with 4 pre- and 4 post-smoothing steps.

In summary, the numerical experiments confirm the theoretical results on the additive Schwarz smoother. The multiplicative Schwarz smoother leads to significantly better rates, however, a theoretical analysis for the convergence and smoothing properties of this iteration is still missing.

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