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A priori and a posteriori analysis of finite volume discretizations of Darcy's equations

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Summary This paper is devoted to the numerical analysis of some finite volume discretizations of Darcy's equations. We propose two finite volume schemes on unstructured meshes and prove their equivalence with either conforming or nonconforming finite element discrete problems. This leads to optimal a priori error estimates. In view of mesh adaptivity, we exhibit residual type error indicators and prove estimates which allow to compare them with the error in a very accurate way.

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1 Introduction

We are interested in the finite volume discretization of Darcy's equations in a bounded connected open set Ω in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary. These equations, introduced by H. Darcy [12] in a more general framework, model the flow of an incompressible and isothermal fluid in homogeneous porous media:

$$(1) \qquad \begin{cases} \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \textit{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \textit{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \textit{on } \partial\Omega. \end{cases}$$

The unknowns are the velocity $\boldsymbol{u}$ and the pressure p, while the data $\boldsymbol{f}$ represent a density of forces and $\boldsymbol{n}$ stands for the unit outward normal vector to the boundary of Ω. These equations also appear in several other models or algorithms. For instance, when written in the slightly modified form

$$\begin{cases} \boldsymbol{u} + \mathbf{grad}\, p = \mathbf{0} & \mathit{in}\ \Omega, \\ \operatorname{div} \boldsymbol{u} = g & \mathit{in}\ \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \mathit{on}\ \partial\Omega, \end{cases}$$

they provide a mixed formulation of the Laplace equation with Neumann type boundary conditions. They are also involved in the projection–diffusion algorithm for solving the time-dependent Stokes and Navier–Stokes equations, as suggested by A.J. Chorin [5] and R. Temam [17]. The aim of this paper is to propose and analyze accurate finite volume schemes for solving these equations on adaptive meshes.

A very interesting and unusual feature of Darcy's equations is that they admit two equivalent variational formulations, whether the space of velocities is more regular than the space of pressures or not. A third "nonsymmetric" formulation has been recently suggested by J.-M. Thomas [19] where the space for the solution is different from the space for the test functions. Applying the Galerkin method to these formulations leads to different discrete problems, and all of them do not give optimal a priori and/or a posteriori error estimates, see [3] for instance. This has determined the choice of the formulation we work with.

We propose two finite volume discretizations of problem (1), relying on a regular family of triangulations of the domain by triangles or tetrahedra. We refer to R. Eymard, T. Gallouët and R. Herbin [14] for a general description and analysis of the finite volume techniques. One of the important characteristics of the systems that result from our schemes is that the mass matrix is diagonal, which would not hold if working with the other formulations of Darcy's equations. We first check that the finite volume system is equivalent to a finite element problem, for some appropriate choices of the discrete spaces of velocities and pressures. In the first case, the discretization is conforming and relies on standard Lagrange elements, as described in [6](§ 6). In the second one, it is not conforming and involves the well-known finite element of M. Crouzeix and P.-A. Raviart [11]. From this equivalence, we derive in an easy way optimal a priori error estimates.

Next, we propose error indicators of residual type. We refer to R. Verfürth [20] for the definition and numerical analysis of this type of indicators in the finite element framework. We prove that the error is equivalent to the Hilbertian sum of the indicators in an optimal way (see [2] for the notion of optimal a posteriori estimates). Moreover, these indicators are local : they are associated either with one element of the triangulation or with an edge (or a face) of this element, so that they are fully appropriate for an adaptive refinement of the mesh where necessary. It can also be noted that their evaluation requires very low computational effort.

An outline of the paper is as follows.

- In Section 2, we describe the variational formulation of problem (1) and prove its well-posedness.
- The two finite volume schemes are described and analyzed in Sections 3 and 4, respectively.

2 A variational formulation of the continuous problem

We first write the variational formulation of problem (1). Next, we prove that the variational problem is well-posed and we recall some basic regularity properties of the solution.

In all that follows, on each connected open set $\mathcal{O}$ in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary and for all nonnegative real numbers s, we use the standard Sobolev spaces $H^s(\mathcal{O})$, provided with the norm $\|\cdot\|_{H^s(\mathcal{O})}$ and seminorm $|\cdot|_{H^s(\mathcal{O})}$. On the boundary $\partial\mathcal{O}$, we use the space $H^{\frac{1}{2}}(\partial\mathcal{O})$ defined as the space of traces on $\partial\mathcal{O}$ of functions in $H^1(\mathcal{O})$ and its dual space $H^{-\frac{1}{2}}(\partial\mathcal{O})$.

2.1 The variational formulation

Let $L^2_0(\Omega)$ stand for the space

$$L^2_0(\Omega) = \Big\{q \in L^2(\Omega);\ \int_\Omega q(\boldsymbol{x})\,d\boldsymbol{x} = 0\Big\}.$$

We set

$$X = L^2(\Omega)^d \qquad \text{and} \qquad M = H^1(\Omega) \cap L^2_0(\Omega). \tag{2}$$

We consider the following variational formulation of problem (1):

$$\begin{aligned} &\textit{Find } (\boldsymbol{u}, p) \textit{ in } X \times M \textit{ such that} \\ &\qquad \forall \boldsymbol{v} \in X, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \textstyle\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x}, \\ &\qquad \forall q \in M, \quad b(\boldsymbol{u}, q) = 0, \end{aligned} \tag{3}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot (\textbf{grad}\, q)(\boldsymbol{x})\,d\boldsymbol{x}.$$

If $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with a compact support in Ω, by using the density of $\mathcal{D}(\Omega)^d$ in $L^2(\Omega)^d$, we observe that the first equation of problem (3) is fully equivalent to the first line of (1) in the distribution sense. Similarly, letting q in (3) run through $\mathcal{D}(\Omega)$ yields the second line of (1) also in the distribution sense. Finally, we recall [15] (Chap. I, Thm 2.5) that a normal trace can be defined for all functions v in $L^2(\Omega)^d$ with a square–integrable divergence by the formula

$$(4) \qquad \forall \varphi \in H^1(\Omega), \quad \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle = \int_\Omega \big(\boldsymbol{v} \cdot \mathbf{grad}\, \varphi + \varphi \operatorname{div} \boldsymbol{v} \big)\, d\boldsymbol{x},$$

where the symbol $\langle \cdot, \cdot \rangle$ here denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Applying this formula to the velocity $\boldsymbol{u}$ of problem (3) yields the third line in (1) in the sense of $H^{-\frac{1}{2}}(\partial\Omega)$. Combining these arguments yields the following result.

Proposition 1 *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$, system (1) admits the equivalent variational formulation (3).*

Remark 1 Let $H_0(\operatorname{div}, \Omega)$ stand for the space of the functions in $L^2(\Omega)^d$ such that their divergence belongs to $L^2(\Omega)$ and that their normal trace, as defined in (4), vanishes. Then, another variational formulation of system (1) reads

$$(5) \qquad \begin{aligned} &\textit{Find } (\boldsymbol{u}, p) \textit{ dans } H_0(\operatorname{div}, \Omega) \times L^2_0(\Omega) \textit{ such that} \\ &\quad \forall \boldsymbol{v} \in H_0(\operatorname{div}, \Omega), \quad a(\boldsymbol{u}, \boldsymbol{v}) - \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x} \\ &\qquad\qquad\qquad\qquad = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\ &\quad \forall q \in L^2_0(\Omega), \quad - \int_\Omega (\operatorname{div} \boldsymbol{u})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned}$$

There also, it is readily checked that this problem is equivalent to system (1), however we have rather work with problem (3) in view of the discretization.

2.2 Existence and uniqueness of the solution

Problem (3) is of saddle-point type, so proving its well-posedness relies on some rather standard arguments.

Theorem 1 *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$, problem (3) has a unique solution $(\boldsymbol{u}, p)$ in $X \times M$. Moreover this solution satisfies the following estimate*

$$(6) \qquad \|\boldsymbol{u}\|_{L^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \le c_\Omega\, \|\boldsymbol{f}\|_{L^2(\Omega)^d},$$

for a constant c_Ω only depending on the geometry of Ω.

Proof. The form $a(\cdot, \cdot)$ coincides with the scalar product of $L^2(\Omega)^d$, so that its continuity and ellipticity are obvious. The form $b(\cdot, \cdot)$ is continuous on $L^2(\Omega)^d \times H^1(\Omega)$. Moreover, for any q in M, by taking $\boldsymbol{v}$ equal to $\mathbf{grad}\, q$, we have

$$b(\boldsymbol{v}, q) = \|\boldsymbol{v}\|_{L^2(\Omega)^d} |q|_{H^1(\Omega)}.$$

By using the standard Bramble–Hilbert (also called Poincaré–Wiertinger) inequality on M, we obtain the inf-sup condition

$$\forall q \in M, \quad \sup_{\boldsymbol{v} \in X} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{L^2(\Omega)^d}} \geq c\, \|q\|_{H^1(\Omega)}. \tag{7}$$

So, the desired result follows from [15] (Chap. I, Thm 4.1) for instance.

2.3 Some regularity properties

When the function $\boldsymbol{f}$ is such that its curl belongs to $L^2(\Omega)^{2d-3}$ (the curl of a vector field is a scalar function in dimension $d = 2$, a vector field in dimension $d = 3$), system (1) implies

$$\begin{cases} \mathbf{curl}\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{f} & in\ \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & in\ \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & on\ \partial\Omega. \end{cases} \tag{8}$$

So, proving the regularity properties of the velocity $\boldsymbol{u}$ is achieved by identifying the largest number s such that the space of function in $L^2(\Omega)^d$ with square–integrable divergence and curl and zero normal trace is imbedded in $H^s(\Omega)^d$. The evaluation of such an s has been performed for several classes of domains Ω in [8], [15] (Chap. I, § 3.5) and [1] (§ 2).

Theorem 2 *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$ such that* **curl** $\boldsymbol{f}$ *belongs to $L^2(\Omega)^{2d-3}$, the solution $(\boldsymbol{u}, p)$ of system (1) is such that the velocity $\boldsymbol{u}$ belongs to $H^s(\Omega)^d$, with*
(i) s equal to $\frac{1}{2}$ without further assumption on Ω,
(ii) s equal to 1 when either the boundary of Ω is of class $\mathcal{C}^{1,1}$ or Ω is convex.
If moreover the function $\boldsymbol{f}$ is in $H^s(\Omega)^d$, the solution $(\boldsymbol{u}, p)$ belongs to $H^s(\Omega)^d \times H^{s+1}(\Omega)$.

These results can be extended to smoother domains Ω, for instance with boundary of class $\mathcal{C}^{m,1}$ for an integer $m \geq 2$. However, in view of the applications, we are more interested in the following results.

- When Ω is a convex polygon or polyhedron, it can be derived from [13] that there exist real numbers $p > 2$ and $s > 1$ such that, if **curl** $\boldsymbol{f}$ belongs to $L^p(\Omega)^{2d-3}$ or $H^{s-1}(\Omega)^{2d-3}$, the velocity $\boldsymbol{u}$ belongs to $W^{1,p}(\Omega)^d$ or $H^s(\Omega)^d$, respectively.

- When Ω is a nonconvex polygon or polyhedron, it is proved in [9] (Thm 3.5) that the velocity $\boldsymbol{u}$ admits the expansion

$$\boldsymbol{u} = \boldsymbol{w} + \mathbf{grad}\, \varphi,$$

where $\boldsymbol{w}$ belongs to $X \cap H^1(\Omega)^d$ and φ is a linear combination of singular functions of the Laplace equation with homogeneous Neumann boundary conditions (this means that φ belongs to $H^1(\Omega)$ but not to $H^2(\Omega)$, while $\Delta\varphi$ belongs to $L^2(\Omega)$). When Ω is a polygon with largest angle ω, $\pi < \omega < 2\pi$, these singular functions are explicitly known and the velocity $\boldsymbol{u}$ belongs to $H^s(\Omega)^2$, $s < \frac{\pi}{\omega}$.

3 The first finite volume scheme

We first recall some useful notation. Next we describe the finite volume scheme. Its equivalence with a finite element problem is checked, which leads to a priori error estimates on both the velocity and the pressure. Finally, we propose a family of error indicators and compare them with the error.

3.1 Some notation

In what follows, we assume for simplicity that Ω is either a polygon or a polyhedron with a Lipzchitz–continuous boundary. Let $(\mathcal{T}_h)_h$ be a family of triangulations of the domain Ω, in the usual sense: each $\mathcal{T}_h$ is a finite set of triangles ($d = 2$) or tetrahedra ($d = 3$) such that $\overline{\Omega}$ is the union of these triangles or tetrahedra and the intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge or a whole face of both of them. As usual, h denotes the maximal diameter of the elements of $\mathcal{T}_h$. We make the further assumption that this family is regular, *i.e.* there exists a positive constant σ such that, for all h and for all K in $\mathcal{T}_h$, the ratio of the diameter h_K of K to the diameter of its inscribed circle or sphere is smaller than σ.

Let $\mathcal{V}_h$ stand for the set of all vertices of elements in $\mathcal{T}_h$ and, for any K in $\mathcal{T}_h$, let $\mathcal{V}_K$ be the set of the $d+1$ vertices of K. For each $\boldsymbol{a}$ in $\mathcal{V}_h$, we denote by $\lambda_{\boldsymbol{a}}$ the Lagrange function associated with $\boldsymbol{a}$, i.e. the continuous function on $\overline{\Omega}$ which is affine on each element of $\mathcal{T}_h$, is equal to 1 in $\boldsymbol{a}$ and vanishes at the other points of $\mathcal{V}_h$. We define $\omega_{\boldsymbol{a}}$ as the support of $\lambda_{\boldsymbol{a}}$, $\mathcal{T}_{\boldsymbol{a}}$ as the set of elements of $\mathcal{T}_h$ that contain $\boldsymbol{a}$ and $\mathcal{E}_{\boldsymbol{a}}$ the set of edges in the case $d = 2$ or faces in the case $d = 3$, of elements of $\mathcal{T}_h$ that contain $\boldsymbol{a}$.

With any K in $\mathcal{T}_h$, we associate the characteristic function χ_K of K and also the set $\mathcal{E}_K$ of edges in the case $d = 2$, faces in the case $d = 3$, of K. For any vertex $\boldsymbol{a}$ of K, let $e_{K,\boldsymbol{a}}$ be the edge ($d = 2$) or face ($d = 3$) of K that does not contain $\boldsymbol{a}$. Then $\boldsymbol{n}_{K,\boldsymbol{a}}$ stands for the unit outward normal vector to

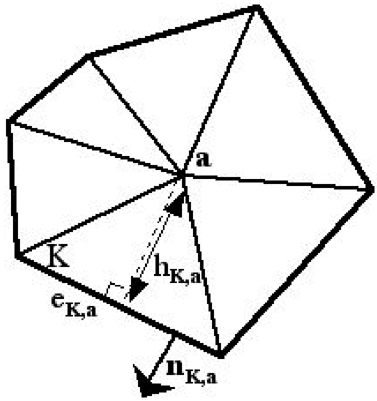

Fig. 1. The subdomain $\omega_{\boldsymbol{a}}$

K on $e_{K,\boldsymbol{a}}$ and $h_{K,\boldsymbol{a}}$ for the length of the height of K issued from $\boldsymbol{a}$. This is illlustrated in Figure 1.

For any $\boldsymbol{a}$ in $\mathcal{V}_h$ and any e in $\mathcal{E}_{\boldsymbol{a}}$, let $\boldsymbol{n}_e$ be one of the unit normal vectors to e, which is oriented from an element K of $\mathcal{T}_h$ toward either another element K' or outside Ω. For any family $(\varphi_K)_{K\in\mathcal{T}_h}$, we agree to denote by $[\varphi]_e$

- the quantity $\varphi_K - \varphi_{K'}$ if the interior part of e is contained in Ω,
- the quantity φ_K if e is contained in $\partial\Omega$.

For all K in $\mathcal{T}_h$ and any nonnegative integer k, let $\mathcal{P}_k(K)$ stand for the space of the restrictions to K of polynomials with d variables and total degree $\leq k$.

3.2 Description of the finite volume scheme

We consider the following problem

$$\begin{aligned}
&\textit{Find } (\boldsymbol{u}_K)_{K\in\mathcal{T}_h} \textit{ and } (p_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{V}_h} \textit{ such that}\\
&\forall K \in \mathcal{T}_h, \quad \boldsymbol{u}_K - \sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}}\, h_{K,\boldsymbol{a}}^{-1}\, \boldsymbol{n}_{K,\boldsymbol{a}} = \frac{1}{\mathrm{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x},\\
&\forall \boldsymbol{a} \in \mathcal{V}_h, \quad \sum_{e\in\mathcal{E}_{\boldsymbol{a}}} [\boldsymbol{u}\cdot\boldsymbol{n}_e]_e\, \mathrm{meas}(e) = 0.
\end{aligned} \tag{9}$$

It is readily checked that problem (9) is a square linear system. Its number of unknowns and equations is equal to d times the number of elements in $\mathcal{T}_h$ plus the number of vertices in $\mathcal{V}_h$.

3.3 Equivalence with a finite element problem

In view of the variational formulation (3) of problem (1), we introduce a finite-dimensional subspace X_h of X, a finite-dimensional subspace M_h of

M, and we consider the following problem, obtained by the standard Galerkin method,

$$\text{(10)}\qquad \begin{array}{l} \textit{Find } (\boldsymbol{u}_h, p_h) \textit{ in } X_h \times M_h \textit{ such that} \\ \quad \forall \boldsymbol{v}_h \in X_h, \quad a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x}, \\ \quad \forall q_h \in M_h, \quad b(\boldsymbol{u}_h, q_h) = 0. \end{array}$$

The idea is to prove that system (9) is equivalent to problem (10), for an appropriate choice of the spaces X_h and M_h, and also for appropriate definitions of $\boldsymbol{u}_h$ and p_h in these spaces.

Proposition 2 *The families $(\boldsymbol{u}_K)_{K\in\mathcal{T}_h}$ and $(p_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{V}_h}$ are a solution of problem (9) if and only if the pair $(\boldsymbol{u}_h, p_h)$ defined by*

$$\text{(11)}\qquad \boldsymbol{u}_h = \sum_{K\in\mathcal{T}_h} \boldsymbol{u}_K\, \chi_K, \qquad p_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} p_{\boldsymbol{a}}\, \lambda_{\boldsymbol{a}},$$

is, up to a constant on the pressure, solution of the discrete problem (10) for the spaces X_h and M_h defined by

$$\begin{array}{l} X_h = \left\{\boldsymbol{v}_h \in L^2(\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \boldsymbol{v}_{h\,|K} \in \mathcal{P}_0(K)^d\right\}, \\ M_h = \left\{q_h \in M;\ \forall K \in \mathcal{T}_h,\ q_{h\,|K} \in \mathcal{P}_1(K)\right\}. \end{array}$$

Proof. It can be noted that the functions $(\chi_K, 0)$ and $(0, \chi_K)$ in the case $d = 2$, $(\chi_K, 0, 0)$, $(0, \chi_K, 0)$ and $(0, 0, \chi_K)$ in the case $d = 3$, $K \in \mathcal{T}_h$, form a basis of X_h. So taking $\boldsymbol{v}_h$ in (10) equal to $\mu\chi_K$ for any vector μ of $\mathbb{R}^d$ yields

$$\text{meas}(K)\, \boldsymbol{u}_K + \int_K \textbf{grad}\ p_h\, d\boldsymbol{x} = \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x},$$

whence

$$\text{(12)}\qquad \boldsymbol{u}_K + \frac{1}{\text{meas}(K)} \int_K \textbf{grad}\ p_h\, d\boldsymbol{x} = \frac{1}{\text{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x}.$$

By integrating by parts and noting that $p_{h\,|K}$ admits the expansion $\sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}}\lambda_{\boldsymbol{a}}$, we obtain

$$\begin{aligned} \tfrac{1}{\text{meas}(K)} \textstyle\int_K \textbf{grad}\ p_h\, d\boldsymbol{x} &= \tfrac{1}{\text{meas}(K)} \textstyle\int_{\partial K} p_h\, \boldsymbol{n}_K\ d\tau \\ &= \tfrac{1}{\text{meas}(K)} \textstyle\sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}} \int_{\partial K} \lambda_{\boldsymbol{a}}\, \boldsymbol{n}_K\ d\tau, \end{aligned}$$

where $\boldsymbol{n}_K$ denotes the unit outward normal vector to K. First, in dimension $d = 2$, for any edge e of K, if $\boldsymbol{m}$ denotes the midpoint of e, we have

$$\int_e \lambda_{\boldsymbol{a}}\, \boldsymbol{n}_K\ d\tau = \text{meas}(e)\, \lambda_{\boldsymbol{a}}(\boldsymbol{m})\, \boldsymbol{n}_K,$$

moreover $\lambda_{\boldsymbol{a}}(\boldsymbol{m})$ is equal to zero if e is the opposite edge to $\boldsymbol{a}$ and to $\frac{1}{2}$ on the other two edges. In dimension $d = 3$, for any face e of K, if $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ denote the midpoints of the edges of e, we have

$$\int_e \lambda_{\boldsymbol{a}}\, \boldsymbol{n}_K \; d\tau = \frac{1}{3}\, \text{meas}(e) \sum_{i=1}^{3} \lambda_{\boldsymbol{a}}(\boldsymbol{m}_i)\, \boldsymbol{n}_K,$$

and it can be noted that $\lambda_{\boldsymbol{a}}(\boldsymbol{m}_i)$ is equal to zero if $\boldsymbol{m}_i$ belongs to the face that does not contain $\boldsymbol{a}$, to $\frac{1}{2}$ otherwise. Moreover, each midpoint of an edge appears twice when summing up on the faces. Consequently, in both dimensions $d = 2$ and $d = 3$, we derive

$$\int_K \textbf{grad}\; p_h\, d\boldsymbol{x} = \frac{1}{d} \sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}} \sum_{e\in\mathcal{E}_K - \{e_{K,\boldsymbol{a}}\}} \text{meas}(e)\, \boldsymbol{n}_K.$$

Next, we observe that

$$\sum_{e\in\mathcal{E}_K} \text{meas}(e)\, \boldsymbol{n}_K = \int_{\partial K} \boldsymbol{n}_K \, d\tau = \int_K \textbf{grad}\; 1\, d\boldsymbol{x} = 0,$$

whence

$$\frac{1}{\text{meas}(K)} \int_K \textbf{grad}\; p_h\, d\boldsymbol{x} = -\frac{1}{d\,\text{meas}(K)} \sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}}\, \text{meas}(e_{K,\boldsymbol{a}})\, \boldsymbol{n}_{K,\boldsymbol{a}}.$$

By inserting this into (12), we obtain

$$\boldsymbol{u}_K - \frac{1}{d\,\text{meas}(K)} \sum_{\boldsymbol{a}\in\mathcal{V}_K} p_{\boldsymbol{a}}\, \text{meas}(e_{K,\boldsymbol{a}})\, \boldsymbol{n}_{K,\boldsymbol{a}} = \frac{1}{\text{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x}.$$

We also have

$$\text{meas}(K) = \frac{1}{d}\, \text{meas}(e_{K,\boldsymbol{a}})\, h_{K,\boldsymbol{a}},$$

which implies the first line of (9). Conversely, the first line of (9) implies that the first equation in (10) is satisfied for all $\boldsymbol{v}_h$ with one component equal to χ_K and the other ones equal to zero, $K \in \mathcal{T}_h$, hence for all $\boldsymbol{v}_h$ in X_h.

On the other hand, since the $\lambda_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h$, form a basis of the space of continuous and piecewise affine functions, taking q_h equal to $\lambda_{\boldsymbol{a}}$ in the second line of (10) gives

$$0 = \int_{\omega_{\boldsymbol{a}}} \boldsymbol{u}_h \cdot \textbf{grad}\, \lambda_{\boldsymbol{a}}\, d\boldsymbol{x} = \sum_{K\subset\omega_{\boldsymbol{a}}} \boldsymbol{u}_K \cdot \int_K \textbf{grad}\, \lambda_{\boldsymbol{a}}\, d\boldsymbol{x}.$$

The same arguments as above lead to

$$0 = \frac{1}{d} \sum_{K\subset\omega_{\boldsymbol{a}}} \boldsymbol{u}_K \cdot \sum_{e\in\mathcal{E}_{\boldsymbol{a}}, e\subset\partial K} \text{meas}(e)\, \boldsymbol{n}_K,$$

whence the second line of (9). The converse property is obvious.

The continuity of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $X_h \times X_h$ and $X_h \times M_h$ and the ellipticity of $a(\cdot,\cdot)$ on X_h are a special case of the analogous properties on X and M which are recalled in Section 2. Moreover, for any q_h in M_h, by taking $\boldsymbol{v}_h$ equal to $\mathbf{grad}\, q_h$, we derive the following inf-sup condition: there exists a constant β independent of h such that

$$\forall q_h \in M_h, \quad \sup_{\boldsymbol{v}_h \in X_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}} \geq \beta \, \|q_h\|_{H^1(\Omega)}. \tag{13}$$

Combining all these properties yields the well-posedness result.

Theorem 3 *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$, problem (9) has a unique solution $(\boldsymbol{u}_K)_{K\in\mathcal{T}_h}$ and $(p_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{V}_h}$, up to an additive constant on the $(p_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{V}_h}$.*

Note that the constant on the $(p_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{V}_h}$ can be chosen either by enforcing one of the $p_{\boldsymbol{a}}$ to be zero or by enforcing the condition

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} p_{\boldsymbol{a}} \operatorname{meas}(\omega_{\boldsymbol{a}}) = 0. \tag{14}$$

If (14) is used, the function p_h defined in (11) is exactly the pressure of the finite element problem, which allows for comparing it with the pressure p of problem (3).

3.4 A priori error estimates

Thanks to (13), the following estimates concerning problem (10) are standard, see [4], [15] (Chap. II, Thm 1.1) or [16] (Thm 10.4) for instance:

(15)
$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} &\leq 2(1+\beta^{-1}) \inf\nolimits_{\boldsymbol{w}_h\in X_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^2(\Omega)^d} \\ &\quad + \inf\nolimits_{q_h\in M_h} \|p - q_h\|_{H^1(\Omega)}, \\ \|p - p_h\|_{H^1(\Omega)} &\leq \beta^{-1}\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + (1+\beta) \inf\nolimits_{q_h\in M_h} \|p - q_h\|_{H^1(\Omega)}\big). \end{aligned}$$

So evaluating the error is a simple consequence of the approximation properties of the spaces X_h and M_h.

Theorem 4 *Assume that the solution $(\boldsymbol{u}, p)$ of problem (3) belongs to $H^s(\Omega)^d \times H^{s+1}(\Omega)$, $0 < s \leq 1$. There exists a constant c independent of h such that the following a priori error estimate holds between this solution $(\boldsymbol{u}, p)$ and the solution $(\boldsymbol{u}_h, p_h)$ defined in (11) from problem (9) and satisfying (14)*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H^1(\Omega)} \leq c\, h^s \left(\|\boldsymbol{u}\|_{H^s(\Omega)^d} + \|p\|_{H^{s+1}(\Omega)}\right). \tag{16}$$

This estimate is optimal and takes into account all the a priori knowledge of the regularity of the solution $(\boldsymbol{u}, p)$, see Theorem 2.

3.5 Error indicators and a posteriori error estimates

We first define a piecewise constant approximation $\boldsymbol{f}_h$ of $\boldsymbol{f}$ by the equation

$$\boldsymbol{f}_{h\,|K} = \frac{1}{\text{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x}, \quad K \in \mathcal{T}_h. \tag{17}$$

Indeed, it follows from the first line in (9) that, with this choice, the residual of the first line of (1), which is constant on each K and equal to $(\boldsymbol{f}_h - \boldsymbol{u}_h - \mathbf{grad}\, p_h)_{|K}$, vanishes. So we are led to the next definition of the error indicator

$$\eta_K = \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \, \| \, [\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e \, \|_{L^2(e)}. \tag{18}$$

Since the jump $[\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e$ is constant on each e, this can be equivalently written as

$$\eta_K = \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \, \text{meas}(e)^{\frac{1}{2}} \, | \, [\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e \, |. \tag{19}$$

Proving the a posteriori error estimate relies on the following idea. We define the bilinear form

$$\mathcal{A}(U, V) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q), \quad \text{with} \quad U = (\boldsymbol{u}, p), \; V = (\boldsymbol{v}, q),$$

and observe from the ellipticity of $a(\cdot, \cdot)$ and the inf-sup condition (7) on $b(\cdot, \cdot)$ that the following "global" inf-sup condition holds [15] (Chap. I, Lemma 4.1) for a positive constant γ (see also [4] or [16])

$$\forall U \in X \times M, \qquad \sup_{V \in X \times M} \frac{\mathcal{A}(U, V)}{\|V\|_{X \times M}} \geq \gamma \, \|U\|_{X \times M}. \tag{20}$$

Applying this to the quantity $U = (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$ gives the inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H^1(\Omega)} \leq \gamma^{-1} \sup_{V = (\boldsymbol{v}, q) \in X \times M} \frac{\mathcal{A}(U, V)}{\|\boldsymbol{v}\|_{L^2(\Omega)^d} + \|q\|_{H^1(\Omega)}}. \tag{21}$$

Theorem 5 *There exists a constant c independent of h such that the following a posteriori error estimate holds between the solution $(\boldsymbol{u}, p)$ of problem (3) and the solution $(\boldsymbol{u}_h, p_h)$ defined in (11) from problem (9) and satisfying (14)*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H^1(\Omega)} \leq c \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 + \|\boldsymbol{f} - \boldsymbol{f}_h\|^2_{L^2(\Omega)^d} \Big)^{\frac{1}{2}}. \tag{22}$$

Proof. Thanks to (21), we must evaluate for any $V = (\boldsymbol{v}, q)$ in $X \times M$ the quantity

$$\mathcal{A}(U, V) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) + b(\boldsymbol{u} - \boldsymbol{u}_h, q). \tag{23}$$

From the first line of problem (3), we have

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) &= \int_\Omega (\boldsymbol{f} - \boldsymbol{u}_h - \mathbf{grad}\, p_h) \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ &= \int_\Omega (\boldsymbol{f}_h - \boldsymbol{u}_h - \mathbf{grad}\, p_h) \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ &\quad + \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h) \cdot \boldsymbol{v}\, d\boldsymbol{x}. \end{aligned}$$

As already noted, the quantity $\boldsymbol{f}_h - \boldsymbol{u}_h - \mathbf{grad}\, p_h$ vanishes on each K, whence

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h) \le \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\Omega)^d} \|\boldsymbol{v}\|_{L^2(\Omega)^d}. \tag{24}$$

To handle the last term of (23), we note that, for any q_h in M_h,

$$\begin{aligned} b(\boldsymbol{u} - \boldsymbol{u}_h, q) = -b(\boldsymbol{u}_h, q) &= -b(\boldsymbol{u}_h, q - q_h) \\ &= -\sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{u}_h \cdot \mathbf{grad}\,(q - q_h)\, d\boldsymbol{x}, \end{aligned}$$

Integrating by parts gives

$$b(\boldsymbol{u} - \boldsymbol{u}_h, q) = -\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{u}_h \cdot \boldsymbol{n}_K\, (q - q_h)\, d\tau.$$

This last expression can be written as (with λ_e equal to 1 or $\frac{1}{2}$ according as e is contained in $\partial\Omega$ or not)

$$\begin{aligned} b(\boldsymbol{u} - \boldsymbol{u}_h, q) &= -\sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \lambda_e \int_e [\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, (q - q_h)\, d\tau \\ &\le \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \|\, [\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, \|_{L^2(e)} \|q - q_h\|_{L^2(e)}. \end{aligned}$$

By going to a reference triangle and taking q_h equal to the image of q by Clément's regularization operator [7] (for instance), it is readily checked that

$$\|q - q_h\|_{L^2(e)} \le c\, h_e^{\frac{1}{2}}\, \|q\|_{H^1(\Delta_K)},$$

where Δ_K stands for the union of the elements of $\mathcal{T}_h$ that share at least a vertex with K. This yields

$$b(\boldsymbol{u} - \boldsymbol{u}_h, q) \le c \sum_{K \in \mathcal{T}_h} \eta_K\, \|q\|_{H^1(\Delta_K)},$$

whence

$$b(\boldsymbol{u} - \boldsymbol{u}_h, q) \le c \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}} \|q\|_{H^1(\Omega)}. \tag{25}$$

Inserting (24) and (25) into (21) yields the desired estimate.

Proving an upper bound for the error indicator relies on rather standard arguments. For a reference element $\hat{K}$ and one of its edges $\hat{e}$, we introduce a lifting operator $\hat{R}$ from the space of polynomials on $\hat{e}$ vanishing at the endpoints of $\hat{e}$ ($d = 2$) or on $\partial\hat{e}$ ($d = 3$) into the space of polynomials on $\hat{K}$ vanishing on $\partial\hat{K} \setminus \hat{e}$ such that

$$\forall \hat{\varphi} \in \mathcal{P}_d(\hat{e}), \quad |\hat{R}\hat{\varphi}|_{H^1(\hat{K})} \le \hat{c}\, \|\hat{\varphi}\|_{L^2(\hat{e})}. \tag{26}$$

Then, by affine transformation, a lifting operator $R_{K,e}$ is constructed for all K in $\mathcal{T}_h$ and all e in $\mathcal{E}_K$.

Theorem 6 *There exists a constant c independent of h such that the following estimate holds for all K in $\mathcal{T}_h$ and for the indicator η_K introduced in (18)*

$$\eta_K \le c\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\omega_K)^d}, \tag{27}$$

where ω_K denotes the union of the at most $d+2$ elements of $\mathcal{T}_h$ that share at least an edge ($d = 2$) or a face ($d = 3$) with K.

Proof. For each e in $\mathcal{E}_K$, we assume without restriction that e is also an edge or a face of another element K' of $\mathcal{T}_h$ (indeed, if e is contained in $\partial\Omega$, K' is chosen to be empty). Let ψ_e denote the bubble function associated with e, i.e. the product of the d barycentric coordinates associated with the vertices of e. We define the function q on Ω by

$$q = \begin{cases} R_{K,e}([\boldsymbol{u}_h \cdot \boldsymbol{n}]_e\, \psi_e) & \text{on } K, \\ R_{K',e}([\boldsymbol{u}_h \cdot \boldsymbol{n}]_e\, \psi_e) & \text{on } K', \\ 0 & \text{on } \Omega \setminus (K \cup K'). \end{cases}$$

We have

$$\begin{aligned} \|\,[\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, \psi_e^{\frac{1}{2}}\|^2_{L^2(e)} &= \int_e [(\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}_e]_e\, q\, d\tau \\ &= \int_{\partial K} (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}_K\, q\, d\tau \\ &\quad + \int_{\partial K'} (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}_{K'}\, q\, d\tau. \end{aligned}$$

By integration by parts, since div $(\boldsymbol{u} - \boldsymbol{u}_h)$ is zero in both K and K', this yields

$$\begin{aligned} \|\,[\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, \psi_e^{\frac{1}{2}}\|^2_{L^2(e)} &= \int_{K \cup K'} (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \mathbf{grad}\, q\, d\boldsymbol{x} \\ &\le \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(K \cup K')^d} |q|_{H^1(K \cup K')}. \end{aligned}$$

To conclude, we derive from (26) that

$$|q|_{H^1(K \cup K')} \le c\, h_e^{-\frac{1}{2}}\, \|\,[\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, \psi_e\|_{L^2(e)} \le c\, h_e^{-\frac{1}{2}}\, \|\,[\boldsymbol{u}_h \cdot \boldsymbol{n}_e]_e\, \psi_e^{\frac{1}{2}}\|_{L^2(e)},$$

and we recall from [6] (form. (25.14)) that

$$\|\psi_e^{\frac{1}{2}}\|_{L^2(e)} = \sqrt{\frac{2\,d!}{(2d)!}}\, h_e^{\frac{1}{2}},$$

whence

$$\|\,[\boldsymbol{u}_h\,\cdot\,\boldsymbol{n}_e]_e\,\psi_e^{\frac{1}{2}}\|_{L^2(e)} = \sqrt{\frac{2\,d!}{(2d)!}}\,\|\,[\boldsymbol{u}_h\,\cdot\,\boldsymbol{n}_e]_e\|_{L^2(e)}.$$

Estimates (22) and (27) are fully optimal, in the sense of [2]. Moreover, estimate (27) is local, so that the η_K, $K \in \mathcal{T}_h$, provide an appropriate tool for automatic refinement of the mesh.

4 The second finite volume scheme

We now work with another type of finite volume discretization, which leads to the introduction of a nonconforming finite element. This section is organized as the previous one.

4.1 Some further notation

We introduce the set $\mathcal{E}_h$ of all edges ($d = 2$) or faces ($d = 3$) of elements in $\mathcal{T}_h$, and the subset $\mathcal{E}_h^0$ made of all elements e in $\mathcal{E}_h$ that are not contained in $\partial\Omega$. For any e in $\mathcal{E}_h$, we denote by h_e the length or diameter of e and $\boldsymbol{m}_e$ the midpoint ($d = 2$) or barycenter ($d = 3$) of e. With this e, we associate the function φ_e which is affine on each element of $\mathcal{T}_h$, is equal to 1 in $\boldsymbol{m}_e$ and vanishes on all other $\boldsymbol{m}_{e'}$, $e' \in \mathcal{E}_h$, $e' \neq e$. Let also $\mathcal{T}_e$ be the set of elements of $\mathcal{T}_h$ that contain e, and ω_e the union of the elements in $\mathcal{T}_e$. Note that ω_e is the support of φ_e.

We use the same definition as in Section 3 for the jump $[\cdot]_e$.

4.2 Description of the finite volume scheme

We consider the following problem

(28)

Find $(\boldsymbol{u}_K)_{K\in\mathcal{T}_h}$ *and* $(p_e)_{e\in\mathcal{E}_h}$ *such that*

$$\forall K \in \mathcal{T}_h, \quad \boldsymbol{u}_K - \frac{1}{\text{meas}(K)} \sum_{e\in\mathcal{E}_K} p_e \text{meas}(e)\, \boldsymbol{n}_K = \frac{1}{\text{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x},$$

$$\forall e \in \mathcal{E}_h, \quad [\boldsymbol{u}\,\cdot\,\boldsymbol{n}_e]_e = 0.$$

Problem (28) results into a square linear system, where the number of equations and unknowns is equal to d times the number of triangles (or tetrahedra) in $\mathcal{T}_h$ plus the number of edges (or faces) in $\mathcal{E}_h$.

4.3 Equivalence with a finite element problem

The problem that we introduce now, in comparison with (10), is slightly different: indeed, we still employ a finite-dimensional subspace X_h of X but now the finite-dimensional subspace $\widetilde{M}_h$ is no longer contained in M, which means that we are working with a nonconforming discretization of problem (3). We consider the following problem

$$\text{(29)}\qquad \begin{aligned} &\textit{Find } (\boldsymbol{u}_h, p_h) \textit{ in } X_h \times \widetilde{M}_h \textit{ such that} \\ &\forall \boldsymbol{v}_h \in X_h, \quad a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x}, \\ &\forall q_h \in \widetilde{M}_h, \quad b_h(\boldsymbol{u}_h, q_h) = 0, \end{aligned}$$

where the bilinear form $b_h(\cdot,\cdot)$ is now defined by

$$b_h(\boldsymbol{v}_h, q_h) = \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{v}_h \cdot \mathbf{grad}\, q_h \, d\boldsymbol{x}.$$

Here also, we check that system (28) is equivalent to problem (29), for an appropriate choice of the spaces X_h and $\widetilde{M}_h$ and appropriate definitions of $\boldsymbol{u}_h$ and p_h.

Proposition 3 *The families $(\boldsymbol{u}_K)_{K \in \mathcal{T}_h}$ and $(p_e)_{e \in \mathcal{E}_h}$ are a solution of problem (28) if and only if the pair $(\boldsymbol{u}_h, p_h)$ defined by*

$$\text{(30)}\qquad \boldsymbol{u}_h = \sum_{K \in \mathcal{T}_h} \boldsymbol{u}_K\, \chi_K, \qquad p_h = \sum_{e \in \mathcal{E}_h} p_e\, \varphi_e,$$

is, up to a constant on the pressure, solution of the discrete problem (29) for the space X_h introduced in Proposition 2 and the space $\widetilde{M}_h$ of functions of $L^2_0(\Omega)$

- *such that their restrictions to each element K of $\mathcal{T}_h$ belong to $\mathcal{P}_1(K)$,*
- *which are continuous at each point $\boldsymbol{m}_e$, $e \in \mathcal{E}_h^0$.*

It must be observed that the space X_h coincides with that in Section 3 while the nonconforming space $\widetilde{M}_h$ coincides with the one introduced in [11], up to the boundary conditions.

Proof. The solution $(\boldsymbol{u}_h, p_h)$ of problem (29) admits the expansion (30). We now prove the equivalence in two steps.
1) As in the proof of Proposition 2, since, in dimension $d = 2$ for instance, the functions $(\chi_K, 0)$ and $(0, \chi_K)$, $K \in \mathcal{T}_h$, form a basis of X_h, the first line of (29) is equivalent to the system of equations, for all K in $\mathcal{T}_h$,

$$\operatorname{meas}(K)\, \boldsymbol{u}_K + \int_K \mathbf{grad}\, p_h \, d\boldsymbol{x} = \int_K \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x}.$$

To evaluate the integral, we note that

$$\int_K \mathbf{grad}\, p_h\, d\boldsymbol{x} = \sum_{e\in\mathcal{E}_K} \int_e p_h\, \boldsymbol{n}_K\, d\tau = \sum_{e\in\mathcal{E}_K} p_e \operatorname{meas}(e)\, \boldsymbol{n}_K.$$

This yields the equivalence with the first line of (28).
2) Next, we observe that the φ_e, $e \in \mathcal{E}_h$, form a basis of M_h. Taking q_h equal to one of these φ_e in (29) leads to

$$0 = \sum_{K\in\mathcal{T}_e} \boldsymbol{u}_K \cdot \int_K \mathbf{grad}\, \varphi_e\, d\boldsymbol{x} = \sum_{K\in\mathcal{T}_e} \boldsymbol{u}_K \cdot \boldsymbol{n}_K \operatorname{meas}(e),$$

whence the result. Here also, the converse property is obvious.

Since problem (29) is a nonconforming discretization of problem (3), its analysis is a little more complex. However, the continuity of the form $a(\cdot,\cdot)$ on $X_h \times X_h$ and its ellipticity are still derived from the same properties on $X \times X$. To study the form $b_h(\cdot,\cdot)$, we introduce the mesh-dependent norm

$$\|q\|_{H_h^1(\Omega)} = \left(\sum_{K\in\mathcal{T}_h} \|q\|_{H^1(K)}^2\right)^{\frac{1}{2}}. \tag{31}$$

Indeed, when $\widetilde{M}_h$ is provided with this norm, the form $b_h(\cdot,\cdot)$ is continuous on $X_h \times \widetilde{M}_h$, with norm bounded independently of h. Moreover, since the range of $\widetilde{M}_h$ by the gradient operator is contained in X_h, the following inf-sup condition is derived by taking $\boldsymbol{v}_h$ equal to $\mathbf{grad}\, q_h$:

$$\forall q_h \in \widetilde{M}_h, \quad \sup_{\boldsymbol{v}_h\in X_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}} \geq \left(\sum_{K\in\mathcal{T}_h} |q_h|_{H^1(K)}^2\right)^{\frac{1}{2}}. \tag{32}$$

We now intend to prove that the quantity in the right-hand side of this inequality is equivalent to the norm $\|\cdot\|_{H_h^1(\Omega)}$ defined in (31).

Furthermore, in view of the a posteriori analysis of problem (29), we want to prove this equivalence property on the "non discrete" space

$$\widetilde{M}(\mathcal{T}_h) = \Big\{q \in L_0^2(\Omega);\ \forall K \in \mathcal{T}_h,\ q_{|K} \in H^1(K) \\ \text{and } \forall e \in \mathcal{E}_h^0,\ \int_e [q]_e\, d\tau = 0\Big\}, \tag{33}$$

the main idea being that $\widetilde{M}(\mathcal{T}_h)$ contains both the spaces $\widetilde{M}_h$ and M. The next lemma states a generalized Bramble–Hilbert inequality. Its proof relies on an argument due to M. Crouzeix [10] (see also [18] (Chap. V, Th. 4.3) for an analogous result concerning a similar problem with homogeneous Dirichlet boundary conditions).

Lemma 1 *There exists a constant c independent of h such that*

$$\forall q \in \widetilde{M}(\mathcal{T}_h), \quad \|q\|_{H_h^1(\Omega)} \leq c \left(\sum_{K \in \mathcal{T}_h} |q|^2_{H^1(K)} \right)^{\frac{1}{2}}. \tag{34}$$

Proof. Let q be any function of $\widetilde{M}(\mathcal{T}_h)$. We must check that

$$\|q\|_{L^2(\Omega)} \leq c \left(\sum_{K \in \mathcal{T}_h} |q|^2_{H^1(K)} \right)^{\frac{1}{2}}. \tag{35}$$

We start from the formula

$$\|q\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{\int_\Omega q(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}}{\|g\|_{L^2(\Omega)}}. \tag{36}$$

For any g in $L^2(\Omega)$, we denote by g_0 the function g minus its mean value on Ω and we consider the Laplace problem with homogeneous boundary conditions

$$\begin{cases} -\Delta\varphi = g_0 & \text{on } \Omega, \\ \partial_n \varphi = 0 & \text{on } \partial\Omega, \\ \int_\Omega \varphi(\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{cases}$$

The regularity properties of the Laplace operator in a polygon or a polyhedron yield the existence of a real number $s > \frac{1}{2}$ such that its solution φ belongs to $H^{s+1}(\Omega)$ and satisfies

$$\|\varphi\|_{H^{s+1}(\Omega)} \leq c\, \|g_0\|_{L^2(\Omega)} \leq c'\, \|g\|_{L^2(\Omega)}. \tag{37}$$

We have

$$\int_\Omega q(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega q(\boldsymbol{x}) g_0(\boldsymbol{x})\, d\boldsymbol{x} = -\sum_{K \in \mathcal{T}_h} \int_K q(\boldsymbol{x}) (\Delta\varphi)(\boldsymbol{x})\, d\boldsymbol{x}.$$

So we derive by integration by parts

$$\begin{aligned} \int_\Omega q(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x} = \textstyle\sum_{K \in \mathcal{T}_h} \Big(\int_K &\mathbf{grad}\, q(\boldsymbol{x}) \cdot \mathbf{grad}\, \varphi(\boldsymbol{x})\, d\boldsymbol{x} \\ &- \frac{1}{2} \sum_{e \in \mathcal{E}_K} \int_e [q]_e(\tau) (\partial_n \varphi)(\tau)\, d\tau \Big). \end{aligned} \tag{38}$$

Bounding the first term in the right-hand side is easy:

$$\left| \int_K \mathbf{grad}\, q(\boldsymbol{x}) \cdot \mathbf{grad}\, \varphi(\boldsymbol{x})\, d\boldsymbol{x} \right| \leq |q|_{H^1(K)} |\varphi|_{H^1(K)}. \tag{39}$$

To handle the second term, since $\partial_n\varphi$ vanishes on $\partial\Omega$, we consider an edge or face e in $\mathcal{E}_h^0$ which is shared by two elements K and K' of $\mathcal{T}_h$. It follows from the definition of $\widetilde{M}(\mathcal{T}_h)$ that $q_{|K}$ and $q_{|K'}$ have an equal mean value on e, that we denote by $\overline{q}_e$. By going to the reference subdomain made of two elements $\hat{K}$ and $\hat{K}'$ that share a common edge or face $\hat{e}$, we obtain with the usual notation

$$\begin{aligned} &|\textstyle\int_e [q]_e(\tau)(\partial_n\varphi)(\tau)\,d\tau| \\ &\quad\le c\,\mathrm{meas}(e) \sum_{\kappa\in\{K,K'\}} h_\kappa^{-1}\,\|\hat{q}_{|\hat{\kappa}} - \overline{q}_e\|_{L^2(\hat{e})} \|\mathbf{grad}\,\hat{\varphi}_{|\hat{\kappa}}\|_{L^2(\hat{e})}. \end{aligned}$$

Note that

$$\|\hat{q}_{|\kappa} - \overline{q}_e\|_{L^2(\hat{e})} \le \hat{c}\,|\hat{q}|_{H^1(\hat{\kappa})}, \qquad \|\mathbf{grad}\,\hat{\varphi}_{|\hat{\kappa}}\|_{L^2(\hat{e})} \le \hat{c}\,\|\mathbf{grad}\,\hat{\varphi}\|_{H^s(\hat{\kappa})}.$$

So, going back to the triangles or tetrahedra K and K', we derive

$$\begin{aligned} &\frac{1}{\mathrm{meas}(e)}\Big|\int_e [q]_e(\tau)(\partial_n\varphi)(\tau)\,d\tau\Big| \\ &\le c \sum_{\kappa\in\{K,K'\}} h_\kappa^{-1}\,\mathrm{meas}(\kappa)^{-\frac{1}{2}}\,h_\kappa\,|q|_{H^1(\kappa)}\,\mathrm{meas}(\kappa)^{-\frac{1}{2}}\,h_\kappa\,\|\mathbf{grad}\,\varphi\|_{H^s(\kappa)}, \end{aligned}$$

whence

$$\Big|\int_e [q]_e(\tau)(\partial_n\varphi)(\tau)\,d\tau\Big| \le c \sum_{\kappa\in\{K,K'\}} |q|_{H^1(\kappa)}\,\|\varphi\|_{H^{s+1}(\kappa)}. \tag{40}$$

The desired inequality follows by inserting (37) to (40) into (36).

By combining (32) and Lemma 1, we derive the inf-sup condition

$$\forall q_h \in \widetilde{M}_h, \quad \sup_{\boldsymbol{v}_h\in X_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}} \ge \tilde{\beta}\,\|q\|_{H_h^1(\Omega)}, \tag{41}$$

where the constant $\tilde{\beta}$ is independent of h. This yields the well-posedness of problem (28).

Theorem 7 *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$, problem (28) has a unique solution $(\boldsymbol{u}_K)_{K\in\mathcal{T}_h}$ and $(p_e)_{e\in\mathcal{E}_h}$, up to an additive constant on the $(p_e)_{e\in\mathcal{E}_h}$.*

Here also, the best way for choosing the constant on the $(p_e)_{e\in\mathcal{E}_h}$ in order to prove a priori and a posteriori estimates is to enforce the condition

$$\sum_{e\in\mathcal{E}_h} p_e\,\mathrm{meas}(\omega_e) = 0. \tag{42}$$

Remark 2 It can be observed from (28) and the definition of the space X_h that the velocity $\boldsymbol{u}_h$ defined in (29) is exactly divergence–free. Furthermore, the space $\widetilde{V}_h$ defined by

$$\widetilde{V}_h = \left\{ \boldsymbol{v}_h \in X_h;\ \forall q_h \in \widetilde{M}_h,\ b_h(\boldsymbol{v}_h, q_h) = 0 \right\}, \tag{43}$$

is imbedded into the space

$$V = \left\{ H_0(\mathrm{div}, \Omega);\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega \right\}. \tag{44}$$

So the second and third lines in (1) are exactly satisfied by the discrete solution $\boldsymbol{u}_h$.

4.4 A priori error estimates

Even with the present nonconforming discretization, the a priori error estimate is easy to derive thanks to Remark 2.

Theorem 8 *Assume that the solution $(\boldsymbol{u}, p)$ of problem (3) belongs to $H^s(\Omega)^d \times H^{s+1}(\Omega)$, $0 < s \leq 1$. There exists a constant c independent of h such that the following a priori error estimate holds between this solution $(\boldsymbol{u}, p)$ and the solution $(\boldsymbol{u}_h, p_h)$ defined in (30) from problem (28) and satisfying (42)*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H_h^1(\Omega)} \leq c\, h^s \left(\|\boldsymbol{u}\|_{H^s(\Omega)^d} + \|p\|_{H^{s+1}(\Omega)} \right). \tag{45}$$

Proof. As explained in Remark 2, the space $\widetilde{V}_h$ is contained in V, so letting $\boldsymbol{v}$ run through $\widetilde{V}_h$ in (3) leads to

$$\forall \boldsymbol{v}_h \in \widetilde{V}_h, \quad a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0.$$

This yields, for any $\boldsymbol{v}_h$ in $\widetilde{V}_h$,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(\Omega)^d} = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{v}_h),$$

whence

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} \leq \inf_{\boldsymbol{v}_h \in \widetilde{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)^d}.$$

By combining this estimate with the inf-sup condition (41) and [15] (Chap. II, form. (1.16)) (see also [4] or [16]), we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} \leq c \inf_{\boldsymbol{w}_h \in X_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^2(\Omega)^d},$$

whence the error estimate on the velocity. Using once more the inf-sup condition (41) gives, for any q_h in $\widetilde{M}_h \cap H^1(\Omega)$,

$$\|p_h - q_h\|_{H_h^1(\Omega)} \leq \tilde{\beta}^{-1} \sup_{\boldsymbol{v}_h \in X_h} \frac{b_h(\boldsymbol{v}_h, p_h - q_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}}.$$

We have

$$b_h(\boldsymbol{v}_h, p_h - q_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, q_h)$$
$$= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, q_h).$$

Since the bilinear forms $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ coincide on $L^2(\Omega)^d \times H^1(\Omega)$, we derive

$$\|p - p_h\|_{H_h^1(\Omega)} \le \tilde{\beta}^{-1}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + (1 + \tilde{\beta}^{-1}) \inf_{q_h \in \widetilde{M}_h \cap H^1(\Omega)} \|p - q_h\|_{H^1(\Omega)}.$$

The error estimate on the pressure is thus an easy consequence of that on the velocity and on the approximation properties of the space $\widetilde{M}_h \cap H^1(\Omega)$ which coincides with the space M_h of Section 3.

4.5 Error indicators and a posteriori error estimates

We keep working with the approximation $\boldsymbol{f}_h$ of $\boldsymbol{f}$ defined in (17). Here, the family of error indicators is indexed by the elements e of $\mathcal{E}_h^0$. Indeed, we introduce the family of indicators η_e, $e \in \mathcal{E}_h^0$, defined by

(46) $$\eta_e = h_e^{-\frac{1}{2}}\,\|\,[p_h]_e\|_{L^2(e)}.$$

From the previous line, these indicators are only linked to the nonconformity of the method, this comes from the fact that the residuals of the last two lines in (1) vanish, see Remark 2, and also that the residual of the first line vanishes thanks to the choice of $\boldsymbol{f}_h$.

Remark 3 Let $\boldsymbol{a}_i$, $1 \le i \le d$, denote the endpoints or vertices of e. Moreover, if e is shared by two elements K and K', let e_i and e_i' denote the opposite edge or face to $\boldsymbol{a}_i$ in K and K', respectively (see Figure 2). Then it can be checked from [6] (form. (25.14)) (or also by using an appropriate quadrature formula) that, in the case of dimension $d = 2$,

(47) $$\eta_e = \frac{1}{\sqrt{3}}\,|\,[[p_h]]_e|,$$

where $[[\,p_h]]_e$ stands for the alternate sum $p_{e_1} - p_{e_1'} + p_{e_2'} - p_{e_2}$. A similar formula holds in dimension $d = 3$, it reads

(48) $$\eta_e = \frac{1}{2}\,h_e^{-\frac{1}{2}}\,\operatorname{meas}(e)^{\frac{1}{2}} \left(\sum_{j=1}^{3} [[\,p_h]]_j^2\right)^{\frac{1}{2}},$$

where $[[\,p_h]]_1$, $[[\,p_h]]_2$ and $[[\,p_h]]_3$ now stand for the alternate sums $p_{e_2} - p_{e_2'} + p_{e_3'} - p_{e_3}$, $p_{e_3} - p_{e_3'} + p_{e_1'} - p_{e_1}$ and $p_{e_1} - p_{e_1'} + p_{e_2'} - p_{e_2}$, respectively. So in both cases the quantity η_e is easy to compute since it involves $2d$ nodal values of the discrete pressure.

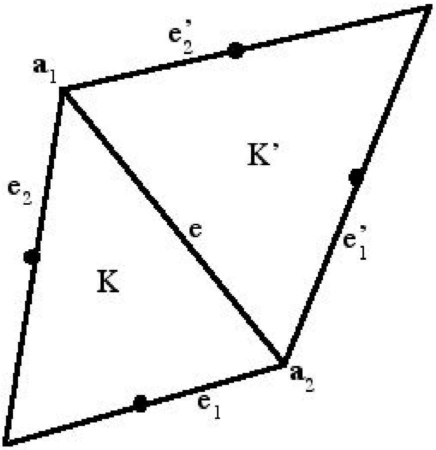

Fig. 2. The degrees of freedom involved by η_e

We now prove the a posteriori estimate; it seems that the analogue of (21) cannot be used in the present situation, due to the lack of conformity of the method.

Theorem 9 *There exists a constant c independent of h such that the following a posteriori error estimate holds between the solution $(\boldsymbol{u}, p)$ of problem (3) and the solution $(\boldsymbol{u}_h, p_h)$ defined in (30) from problem (28) and satisfying (42)*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H^1_h(\Omega)} \leq c \Big(\sum_{e \in \mathcal{E}_h^0} \eta_e^2 + \|\boldsymbol{f} - \boldsymbol{f}_h\|^2_{L^2(\Omega)^d} \Big)^{\frac{1}{2}}. \tag{49}$$

Proof. The estimate is established in two steps.
1) We have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^2(\Omega)^d} = a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) = \int_\Omega (\boldsymbol{f}-\boldsymbol{u}_h)\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\, d\boldsymbol{x} - b(\boldsymbol{u}-\boldsymbol{u}_h, p).$$

However, as already observed, both $\boldsymbol{u}$ and $\boldsymbol{u}_h$ belong to V, so that the last term vanishes. By adding and subtracting $b_h(\boldsymbol{u} - \boldsymbol{u}_h, p_h)$, we obtain

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(\Omega)^d} = \sum_{K \in \mathcal{T}_h} \int_K (\boldsymbol{f} - \boldsymbol{u}_h - \mathbf{grad}\, p_h) \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\, d\boldsymbol{x} \\ + b_h(\boldsymbol{u} - \boldsymbol{u}_h, p_h). \end{aligned} \tag{50}$$

Bounding the first term in the right-hand side of (50) relies on a Cauchy–Schwarz inequality

$$\begin{aligned} &\int_K (\boldsymbol{f} - \boldsymbol{u}_h - \mathbf{grad}\, p_h) \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\, d\boldsymbol{x} \\ &\quad \leq \big(\|\boldsymbol{f}_h - \boldsymbol{u}_h - \mathbf{grad}\, p_h\|_{L^2(K)^d} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(K)^d} \big) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(K)^d}. \end{aligned}$$

Since the quantity $\boldsymbol{f}_h - \boldsymbol{u}_h - \mathbf{grad}\, p_h$ vanishes on each K, this gives

$$\int_K (\boldsymbol{f} - \boldsymbol{u}_h - \mathbf{grad}\, p_h) \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\, d\boldsymbol{x} \leq \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(K)^d} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(K)^d}.$$

To estimate the second term in (50), we introduce a conforming approximation p_h^* of p_h, i.e. in $\widetilde{M}_h \cap H^1(\Omega)$, and we use Cauchy–Schwarz inequality

$$\begin{aligned} b_h(\boldsymbol{u} - \boldsymbol{u}_h, p_h) &= b_h(\boldsymbol{u} - \boldsymbol{u}_h, p_h - p_h^*) \\ &\leq \Big(\sum_{K \in \mathcal{T}_h} |p_h - p_h^*|^2_{H^1(K)} \Big)^{\frac{1}{2}} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d}. \end{aligned}$$

The idea is now to choose the function p_h^* such that

$$\forall \boldsymbol{a} \in \mathcal{V}_h, \quad p_h^*(\boldsymbol{a}) = \frac{1}{N_{\boldsymbol{a}}} \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} p_{h\,|K}(\boldsymbol{a}),$$

where $N_{\boldsymbol{a}}$ denotes the number of elements in $\mathcal{T}_{\boldsymbol{a}}$. Since $N_{\boldsymbol{a}}$ is bounded by a constant that only depends on the regularity parameter σ, we obtain, for some constants c_e bounded as a function of $N_{\boldsymbol{a}}$, hence of σ,

$$(p_h - p_h^*)(\boldsymbol{a}) = \sum_{e \in \mathcal{E}_{\boldsymbol{a}}^0} c_e\, [p_h]_e(\boldsymbol{a}),$$

where $[p_h]_e$ denotes the jump of p_h through e and $\mathcal{E}_{\boldsymbol{a}}^0$ denotes the set $\mathcal{E}_{\boldsymbol{a}} \cap \mathcal{E}_h^0$ (so that none of these edges or faces e is contained in $\partial\Omega$). By an inverse inequality, this leads to

$$|(p_h - p_h^*)(\boldsymbol{a})| \leq \sum_{e \in \mathcal{E}_{\boldsymbol{a}}^0} |c_e|\, \|\,[p_h]_e\|_{L^\infty(e)} \leq c \sum_{e \in \mathcal{E}_{\boldsymbol{a}}} h_e^{-\frac{d-1}{2}}\, \|\,[p_h]_e\|_{L^2(e)}.$$

On the other hand, we have

$$p_h - p_h^* = \sum_{\boldsymbol{a} \in \mathcal{V}_h} (p_h - p_h^*)(\boldsymbol{a})\, \lambda_{\boldsymbol{a}},$$

which implies, for any K in $\mathcal{T}_h$,

$$|p_h - p_h^*|_{H^1(K)} \leq c \sum_{\boldsymbol{a} \in \mathcal{V}_K} |(p_h - p_h^*)(\boldsymbol{a})|\, h_K^{\frac{d}{2}-1}.$$

By combining the two previous estimates, we obtain

$$|p_h - p_h^*|_{H^1(K)} \leq c \sum_{\boldsymbol{a} \in \mathcal{V}_K} \sum_{e \in \mathcal{E}_{\boldsymbol{a}}} h_e^{-\frac{1}{2}}\, \|\,[p_h]_e\|_{L^2(e)}.$$

Summing up the square of this inequality on the K and noting that each $\|\,[p_h]_e\|_{L^2(e)}$ only appears a finite number of times in the sum (where "finite" means bounded as a function of σ), we derive a bound for the second term in (50):

$$b_h(\boldsymbol{u}-\boldsymbol{u}_h, p_h) \le c \left(\sum_{\boldsymbol{a}\in\mathcal{V}_h} \sum_{e\in\mathcal{E}_{\boldsymbol{a}}} h_e^{-1} \, \| [p_h]_e \|^2_{L^2(e)} \right)^{\frac{1}{2}} \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d},$$

which gives the estimate on the velocity.
2) Thanks to Lemma 1 and since $p - p_h$ belongs to $\widetilde{M}(\mathcal{T}_h)$, we have

$$\|p-p_h\|^2_{H^1_h(\Omega)} \le c \sum_{K\in\mathcal{T}_h} \int_K \mathbf{grad}\,(p-p_h) \cdot \mathbf{grad}\,(p-p_h)\,d\boldsymbol{x}.$$

We introduce the function $\boldsymbol{v}$ such that, for any K in $\mathcal{T}_h$, its restriction to any K in $\mathcal{T}_h$ is equal to $\boldsymbol{v}_{|K} = \big(\mathbf{grad}\,(p-p_h)\big)_{|K}$. Since the forms $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ coincide on $L^2(\Omega)^d \times H^1(\Omega)$, we obtain

$$\|p-p_h\|^2_{H^1_h(\Omega)} \le c\, b_h(\boldsymbol{v}, p-p_h) = c\Big(\int_\Omega f(\boldsymbol{x}) \cdot v(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u},\boldsymbol{v}) - \sum_{K\in\mathcal{T}_h} (\mathbf{grad}\,p_h)_{|K} \cdot \int_K v(\boldsymbol{x})\,d\boldsymbol{x}\Big).$$

Let now $\boldsymbol{v}_h$ denote the image of $\boldsymbol{v}$ by the orthogonal projection operator from $L^2(\Omega)$ onto X_h. Clearly, on each element K of $\mathcal{T}_h$, $\boldsymbol{v}_h$ is equal to the mean value of $\boldsymbol{v}$ on K. This yields

$$\|p-p_h\|^2_{H^1_h(\Omega)} \le c\big(\textstyle\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u},\boldsymbol{v}) - \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{v}_h(\boldsymbol{x}) \cdot \mathbf{grad}\,p_h\,d\boldsymbol{x}\big),$$

whence

$$\|p-p_h\|^2_{H^1_h(\Omega)} \le c \left(\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{v}-\boldsymbol{v}_h)(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u}_h,\boldsymbol{v}_h) \right).$$

It follows from the definitions of $\boldsymbol{v}_h$ and $\boldsymbol{f}_h$, see (17), that

$$\begin{aligned}\|p-p_h\|^2_{H^1_h(\Omega)} &\le c\big(\textstyle\int_\Omega (\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{x}) \cdot (\boldsymbol{v}-\boldsymbol{v}_h)(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})\big)\\ &= c\big(\textstyle\int_\Omega (\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x} - a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})\big),\end{aligned}$$

which gives

$$\|p-p_h\|^2_{H^1_h(\Omega)} \le c\big(\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d}\big)\|\boldsymbol{v}\|_{L^2(\Omega)^d}.$$

Owing to the definition of $\boldsymbol{v}$, we derive

$$\|p-p_h\|_{H^1_h(\Omega)} \le c\big(\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d}\big),$$

which, when combined with the first part of the proof, leads to the desired estimate.

The arguments for checking the converse inequality are very similar to those used in Section 3.

Theorem 10 *There exists a constant c independent of h such that the following estimate holds for all e in $\mathcal{E}_h^0$ and for the indicator η_e introduced in (46)*

$$\eta_e \le c \sum_{K\in\mathcal{T}_e} |p - p_h|_{H^1(K)}. \tag{51}$$

Proof. For a fixed e in $\mathcal{E}_h^0$, let K and K' denote the two elements of $\mathcal{T}_h$ that contain e. We solve the Neumann problems

$$\begin{cases} -\Delta\varphi = 0 & \text{in } K, \\ \partial_n\varphi = [p_h]_e & \text{on } e, \\ \partial_n\varphi = 0 & \text{on } \partial K \setminus e, \\ \int_K \varphi(\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{cases} \qquad \begin{cases} -\Delta\varphi' = 0 & \text{in } K', \\ \partial_n\varphi' = [p_h]_e & \text{on } e, \\ \partial_n\varphi' = 0 & \text{on } \partial K' \setminus e, \\ \int_{K'} \varphi'(\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{cases}$$

and we take the function $\boldsymbol{v}$ equal to $\mathbf{grad}\,\varphi$ on K, to $\mathbf{grad}\,\varphi'$ on K' and to zero elsewhere. By integration by parts, we have

$$b(\boldsymbol{v}, p - p_h) = \frac{1}{2} \sum_{K\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_K} \int_e \boldsymbol{v} \cdot \boldsymbol{n}_K [p_h]_e \, d\tau = \| [p_h]_e \|^2_{L^2(e)},$$

which leads to

$$\| [p_h]_e \|^2_{L^2(e)} \le \sum_{\kappa\in\{K,K'\}} \|\boldsymbol{v}\|_{L^2(\kappa)^d} |p - p_h|_{H^1(\kappa)}.$$

To estimate the norm of $\boldsymbol{v}$, we note that, on the element K for instance,

$$\|\boldsymbol{v}\|^2_{L^2(K)^d} = |\varphi|^2_{H^1(K)} = \int_e [p_h]_e(\tau)\varphi(\tau)\, d\tau \le \| [p_h]_e \|_{L^2(e)} \|\varphi\|_{L^2(e)}.$$

Since the mean value of the solution φ on K is zero, by going to the reference element, we obtain

$$\begin{aligned} \|\varphi\|_{L^2(e)} &\le \hat{c}\, \mathrm{meas}(e)^{\frac{1}{2}}\, \|\hat{\varphi}\|_{L^2(\hat{e})} \\ &\le \hat{c}\, \mathrm{meas}(e)^{\frac{1}{2}}\, \|\hat{\varphi}\|_{H^1(\hat{K})} \\ &\le \hat{c}'\, \mathrm{meas}(e)^{\frac{1}{2}}\, |\hat{\varphi}|_{H^1(\hat{K})}. \end{aligned}$$

So, when going back to K, we derive

$$\|\boldsymbol{v}\|_{L^2(K)^d} \le c\, h_K^{\frac{1}{2}}\, \| [p_h] \|_{L^2(e)}.$$

Using a similar estimate on K' gives

$$h_e^{-\frac{1}{2}} \, \| [p_h] \|_{L^2(e)} \le c \sum_{\kappa \in \{K, K'\}} |p - p_h|_{H^1(\kappa)}.$$

This is the desired estimate.

4.6 Some conclusions

Even if the proofs are more technical, the discretization proposed in this section leads to the same optimal a priori and a posteriori error estimates as in Section 3. Also, as in Section 3, the mass matrix corresponding to the bilinear form $a(\cdot, \cdot)$ is fully diagonal.

Furthermore and in contrast to Section 3, the discrete velocity $\boldsymbol{u}_h$ introduced in (30) has the further property to be exactly divergence–free, so that the scheme proposed in (28) is fully conservative, in the sense of [14] (Chap. I). For this last reason and even if the size of the global linear system in (28) is a little larger than in (9), we prefer (and even preconize) the use of the scheme described in (28).

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