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Finding Einstein solvmanifolds by a variational method

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Abstract. We use a variational approach to prove that any nilpotent Lie algebra having a codimension-one abelian ideal, and anyone of dimension ≤ 5, admits a rank-one solvable extension which can be endowed with an Einstein left-invariant riemannian metric. A curve of 8-dimensional Einstein solvmanifolds is also given.

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1 Introduction

A *solvmanifold* is a solvable Lie group S endowed with a left invariant riemannian metric; S is called *standard* if $\mathfrak{a} = \mathfrak{n}^{\perp}$ is abelian, where $\mathfrak{s}$ is the Lie algebra of S and $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. Up to now, all known examples of noncompact homogeneous Einstein spaces are standard Einstein solvmanifolds. These spaces have been deeply investigated by Jens Heber in [H], who obtained very nice structure and uniqueness results. If S is Einstein, then for some distinguished element $H \in \mathfrak{a}$, the eigenvalues of $\operatorname{ad} H|_{\mathfrak{n}}$ are all positive integers without common divisors, say $k_1 < \cdots < k_r$. If $d_1, \ldots, d_r$ denote the corresponding multiplicities, then the tuple

$$(k; d) = (k_1 < \cdots < k_r; d_1, \ldots, d_r)$$

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is called the *eigenvalue type* of S. In every dimension, only finitely many eigenvalue types occur. Let $\mathcal{M}$ be the moduli space of all the isometry classes of N-dimensional Einstein solvmanifolds with scalar curvature equal to -1. The subspace $\mathcal{M}_{\rm st}$ of those which are standard is open in $\mathcal{M}$ (C^∞-topology). Each eigenvalue type $(k;d)$ determines a compact subset $\mathcal{M}_{(k;d)}$ of $\mathcal{M}_{\rm st}$, homeomorphic to a real semialgebraic set.

Most of the known examples of Einstein solvmanifolds are of eigenvalue type $(1 < 2; d_1, d_2)$, where necessarily the nilradical $\mathfrak{n}$ is two-step nilpotent (see [GK] and the references therein). It is difficult to find in the literature non-symmetric explicit examples, not only of other eigenvalue types, but also with nilradical of step of nilpotency greater than 2. Besides certain modifications of the non-compact symmetric spaces of rank ≥ 2 given in [GK], we only found just a few of such explicit examples falling into the following two special classes: bounded homogeneous domains endowed with the Bergman metric (or equivalently, non-compact homogeneous Kähler-Einstein spaces), which can be modeled on solvable normal j-algebras (see [PS]), and homogeneous quaternionic Kähler spaces, where the nilradical $\mathfrak{n}$ of the non-symmetric families is always 5-step or 7-step nilpotent (see [C]).

The aim of this paper is to approach the construction of new families of explicit examples of Einstein solvmanifolds of several different eigenvalue types, by using the variational method given in [L]: the $(n+1)$-dimensional rank-one ($\dim \mathfrak{a} = 1$) Einstein solvmanifolds are critical points of certain polynomial of degree 4 restricted to the sphere of a vector space which contains the set of all n-dimensional nilpotent Lie algebras as a real algebraic subset.

We prove in Sect. 4 that any nilpotent Lie algebra having a codimension-one abelian ideal admits a rank-one solvable extension which can be endowed with an Einstein left-invariant metric. This family provides rather exotic eigenvalue types, as well as examples of any step of nilpotency for the nilradical of an Einstein solvmanifold. In Sect. 5, we show that also any nilpotent Lie algebra of dimension ≤ 5 is the nilradical of a rank-one Einstein solvmanifold, and compute their eigenvalue types. Finally, we find in Sect. 6 a curve of pairwise non-isometric 8-dimensional Einstein solvmanifolds, which is the lowest possible dimension for the existence of such a curve. They are all of eigenvalue type $(1 < 2 < 3 < 4 < 5 < 6 < 7; 1, \dots, 1)$, and so 8 is the first dimension for which there is a space $\mathcal{M}_{(k;d)}$ which is different from a point.

2 Rank one Einstein solvmanifolds as critical points

In this section, we overview some results given in [L], which characterize the rank-one Einstein solvmanifolds as the critical points of certain natural functional. We fix an inner product vector space

$$(\mathfrak{s} = \mathbb{R}H \oplus \mathfrak{n}, \langle\cdot,\cdot\rangle), \qquad \text{with } \langle H, \mathfrak{n}\rangle = 0, \quad \langle H, H\rangle = 1,$$

where $\mathfrak{n}$ is a real vector space of dimension n. The metric Lie algebra of any $(n+1)$-dimensional rank-one solvmanifold S, can be modeled on $(\mathfrak{s} = \mathbb{R}H \oplus \mathfrak{n}, \langle\cdot,\cdot\rangle)$ for some nilpotent Lie algebra μ on $\mathfrak{n}$ and some $D \in \mathrm{Der}(\mu)$. Indeed, these data define a solvable Lie bracket $[\cdot,\cdot]$ on $\mathfrak{s}$ by

$$(1) \qquad [H, X] = DX, \qquad [X, Y] = \mu(X, Y), \qquad X, Y \in \mathfrak{n},$$

and S is then the corresponding simply connected Lie group with Lie algebra $(\mathfrak{s}, [\cdot,\cdot])$ endowed with the left invariant riemannian metric determined by $\langle\cdot,\cdot\rangle$. If D is symmetric, then S is Einstein if and only if

$$(2) \qquad c_\mu I + \mathrm{tr}(D)D = \mathrm{Ric}_\mu,$$

where Ric_μ is the Ricci operator of N_μ, the simply connected nilpotent Lie group with Lie algebra $(\mathfrak{n}, \mu)$ endowed with the left invariant riemannian metric determined by $\langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}}$, and $c_\mu = \mathrm{tr}\,\mathrm{Ric}_\mu^2 / \mathrm{tr}\,\mathrm{Ric}_\mu$ (see [L, Lemma 2]). Let $\mathfrak{p}_\mu = \mathrm{Der}(\mathfrak{n}, \mu) \cap \mathrm{sym}(\mathfrak{n})$ be the vector space of symmetric derivations. Since

$$(3) \qquad \mathrm{Ric}_\mu \perp \mathfrak{p}_\mu,$$

relative to the usual inner product $\mathrm{tr}\, AB$ on $\mathrm{sym}(\mathfrak{n})$ (see [L, (2)]), it follows from (2) that if S is Einstein then necessarily

$$(4) \qquad c_\mu I + \mathrm{tr}(D)D \perp \mathfrak{p}_\mu.$$

But a remarkable fact is that there exists a unique $D_\mu \in \mathfrak{p}_\mu$ satisfying (4) (think in the inner product space $\mathbb{R}I \oplus \mathfrak{p}_\mu$), thus we can associate to each nilpotent Lie algebra μ on $\mathfrak{n}$ a distinguished rank-one solvmanifold S_μ, defined by the data μ, D_μ (see (1)), which is the only one with possibilities of being Einstein among all those having nilradical μ.

Note that conversely, any $(n+1)$-dimensional rank-one Einstein solvmanifold is isometric to S_μ for some nilpotent μ. In fact, it follows from [H, 4.10] that we can assume, without any loss of generality, that $\mathrm{ad}\, H$ is symmetric. Thus the set $\mathcal{N}_n$ of all nilpotent Lie brackets on $\mathfrak{n}$ parametrizes a space of $(n+1)$-dimensional rank-one solvmanifolds

$$\{S_\mu : \mu \in \mathcal{N}_n\},$$

containing all those which are Einstein. $\mathcal{N}_n$ is an algebraic subset of $V = \Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$, the vector space of all bilinear skew-symmetric maps from $\mathfrak{n} \times \mathfrak{n}$

to $\mathfrak{n}$, since the Jacobi and nilpotency conditions are both polynomial. There is a natural action of $\mathrm{G}L(n)$ on V given by

$$\varphi.\mu(X,Y)=\varphi\mu(\varphi^{-1}X,\varphi^{-1}Y),\quad X,Y\in\mathfrak{n},\quad \varphi\in\mathrm{G}L(n),\quad \mu\in V,$$

and thus isomorphic Lie algebra structures lie in the same $\mathrm{G}L(n)$-orbit. Note that $\mathcal{N}_n$ is $\mathrm{G}L(n)$-invariant. It is easy to prove that two solvmanifolds S_μ and S_λ with $\mu,\lambda\in\mathcal{N}_n$ are isometric if and only if there exists $\varphi\in\mathrm{O}(\mathfrak{n})$ such that $\varphi.\mu=\lambda$ (see [L, Prop. 4]).

The Ricci operator $\mathrm{Ric}_\mu:\mathfrak{n}\longrightarrow\mathfrak{n}$ of N_μ is given by

$$\begin{aligned}\langle \mathrm{Ric}_\mu X,Y\rangle &= -\tfrac{1}{2}\sum_{ij}\langle\mu(X,X_i),X_j\rangle\langle\mu(Y,X_i),X_j\rangle\\ &\quad+\tfrac{1}{4}\sum_{ij}\langle\mu(X_i,X_j),X\rangle\langle\mu(X_i,X_j),Y\rangle,\end{aligned}\tag{5}$$

for all $X,Y\in\mathfrak{n}$, where $\{X_1,\dots,X_n\}$ is any orthonormal basis of $\mathfrak{n}$. The inner product $\langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}}$ determines naturally an inner product on V, denoted also by $\langle\cdot,\cdot\rangle$ and given by

$$\langle\mu,\lambda\rangle=\sum_{ijk}\langle\mu(X_i,X_j),X_k\rangle\langle\lambda(X_i,X_j),X_k\rangle.\tag{6}$$

Notice that for any $\mu\in\mathcal{N}_n$, the scalar curvature $\mathrm{tr}\,\mathrm{Ric}_\mu$ of N_μ equals $-\frac{1}{4}||\mu||^2$ (see (5)). Hence the natural algebraic normalization given by the sphere of V,

$$\mathsf{S}=\{\mu\in V:||\mu||=1\},\tag{7}$$

coincides on $\mathcal{N}_n$ with the following kind of geometric normalization,

$$\mathsf{S}\cap\mathcal{N}_n=\left\{\mu\in\mathcal{N}_n:\mathrm{sc}(N_\mu)=-\frac{1}{4}\right\}.\tag{8}$$

We lose nothing by restricting ourselves to $\mathsf{S}\cap\mathcal{N}_n$, since it is evident that for any $\mu\in\mathcal{N}_n$, S_μ is Einstein if and only if $S_{t\mu}$ is Einstein for any nonzero $t\in\mathbb{R}$. Consider the functional

$$F_n:V\longrightarrow\mathbb{R},\qquad F_n(\mu)=\mathrm{tr}\,\mathrm{Ric}_\mu^2.$$

Recall that Ric_μ can be formally defined as in (5) for every $\mu\in V$.

Theorem 2.1. *[L] For $\mu\in\mathcal{N}_n\cap\mathsf{S}$, the following statements are equivalent:*

(i) *S_μ is Einstein.*
(ii) *μ is a critical point of $F_n:\mathsf{S}\longrightarrow\mathbb{R}$.*
(iii) *μ is a critical point of $F_n:\mathrm{G}L(n).\mu\cap\mathsf{S}\longrightarrow\mathbb{R}$.*
(iv) $\mathrm{Ric}_\mu\in\mathbb{R}I\oplus\mathrm{Der}(\mu)$.

Condition (iv) is rather computable, and so it will be very useful throughout the paper to check if a certain candidate obtained by a variational approach is actually Einstein.

If S_μ is Einstein then for some $c > 0$, the eigenvalues of $c \operatorname{ad} H|_{\mathfrak{n}} = cD_\mu$ are all positive integers without common divisors, say $k_1 < \cdots < k_r$ with multiplicities $d_1, \dots, d_r$. We say that S_μ and the critical point $\mu/||\mu||$ are of *eigenvalue type* $(k_1 < \cdots < k_r; d_1, \dots, d_r)$ (see Sect. 1).

We note that given $\mu \in \mathcal{N}_n$, the existence of $\mu_0 \in \mathrm{G}L(n).\mu$ satisfying that S_{μ_0} is Einstein is equivalent to the existence of a left invariant metric $\langle \cdot, \cdot \rangle_0$ on N_μ such that there is a rank-one solvable extension S of N_μ for which the metric $\langle \cdot, \cdot \rangle_0$ is Einstein.

3 Critical values and abelian factors

In order to understand better the critical point behavior of F_n, we calculate in the following proposition their critical values.

Proposition 3.1. *Let $\mu \in \mathsf{S}$ be a critical point of $F_n : \mathsf{S} \longrightarrow \mathbb{R}$ of eigenvalue type $(k_1 < \cdots < k_r; d_1, \dots, d_r)$. Then,*

$$F_n(\mu) = \frac{1}{16} \left(n - \frac{(k_1 d_1 + \cdots + k_r d_r)^2}{k_1^2 d_1 + \cdots + k_r^2 d_r} \right)^{-1}.$$

Proof. Assume that $\operatorname{Ric}_\mu = c_\mu I + D_\mu$ for some $c_\mu \in \mathbb{R}$ and $D_\mu \in \operatorname{Der}(\mu)$. Using that $\operatorname{tr} \operatorname{Ric}_\mu D_\mu = 0$ (see (3)) and $\operatorname{tr} \operatorname{Ric}_\mu = -\frac{1}{4}||\mu||^2 = -\frac{1}{4}$ we obtain that

$$F_n(\mu) = \operatorname{tr} \operatorname{Ric}_\mu^2 = -\frac{1}{4} c_\mu = \frac{1}{4} \frac{\operatorname{tr} D_\mu^2}{\operatorname{tr} D_\mu} = \frac{1}{4} \frac{\operatorname{tr} D^2}{c \operatorname{tr} D}, \tag{9}$$

where $cD_\mu = D$ is the derivation of μ with eigenvalues k_i, having multiplicities d_i. Multiplying equation $\operatorname{Ric}_\mu = c_\mu I + D_\mu$ by the identity map I and D and then taking trace we also get that

$$-\frac{1}{4} = c_\mu n + \frac{\operatorname{tr} D}{c}, \qquad 0 = c_\mu \operatorname{tr} D + \frac{\operatorname{tr} D^2}{c}.$$

We can easily deduce from these equations that $c = 4 \left(n \frac{\operatorname{tr} D^2}{\operatorname{tr} D} - \operatorname{tr} D \right)$, and by replacing in (9) we obtain that

$$F_n(\mu) = \frac{1}{16} \left(n \frac{\operatorname{tr} D^2}{\operatorname{tr} D} - \operatorname{tr} D \right)^{-1} \frac{\operatorname{tr} D^2}{\operatorname{tr} D} = \frac{1}{16} \left(n - \frac{(\operatorname{tr} D)^2}{\operatorname{tr} D^2} \right)^{-1},$$

concluding the proof. □

We obtain from the above proof the following

Corollary 3.2. *Let $\mu \in \mathcal{N}_n$ be a Lie bracket such that $\mu/||\mu||$ is a critical point of $F : \mathsf{S} \longrightarrow \mathbb{R}$ of eigenvalue type $(k_1 < \cdots < k_r; d_1, \ldots, d_r)$. Then the derivation D_μ which is used to define the Einstein solvmanifold S_μ by setting $\operatorname{ad} H|_{\mathfrak{n}} = D_\mu$ is given by*

$$D_\mu = \frac{||\mu||^2}{4}\left(n\frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{k_1 d_1 + \cdots + k_r d_r} - k_1 d_1 + \cdots + k_r d_r\right)^{-1} D,$$

where D is the derivation of μ with eigenvalues k_i of multiplicities d_i and $||\mu||^2$ is defined in (6).

We now describe what happens if we add to a critical point μ of F_n an abelian factor.

Proposition 3.3. *Let λ be the central extension of μ to $\tilde{\mathfrak{n}} = \mathfrak{n} \oplus \mathbb{R}^m$, that is, $\lambda|_{\mathfrak{n}\times\mathfrak{n}} = \mu$ and $\lambda(\mathbb{R}^m, \tilde{\mathfrak{n}}) \equiv 0$. Then $F_n(\mu) = F_{n+m}(\lambda)$ and λ is a critical point of the functional F_{n+m} of eigenvalue type*

$$\left(\alpha k_1 < \cdots < \frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{d} < \cdots < \alpha k_r; d_1, \ldots, m, \ldots, d_r\right),$$

where $d = \operatorname{mcd}(k_1 d_1 + \cdots + k_r d_r, k_1^2 d_1 + \cdots + k_r^2 d_r)$ and $\alpha = \frac{k_1 d_1 + \cdots + k_r d_r}{d}$. In case that $\frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{d} = \alpha k_i$ for some i, then the multiplicity is $m + d_i$.

Proof. It is easy to check that the corresponding $\operatorname{Ric}_\lambda : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$ is given by $\operatorname{Ric}_\lambda|_{\mathbb{R}^n} = \operatorname{Ric}_\mu$, $\operatorname{Ric}_\lambda|_{\mathbb{R}^m} \equiv 0$. Thus $\operatorname{Ric}_\lambda = c_\mu I + D_\lambda$, where $D_\lambda \in \operatorname{Der}(\lambda)$ is defined by $D_\lambda|_{\mathfrak{n}} = D_\mu$, $D_\lambda|_{\mathbb{R}^m} = -c_\mu I$. This implies that the eigenvalues of cD_λ are the k_i's along with

$$-c_\mu = \frac{\operatorname{tr} D^2}{\operatorname{tr} D} = \frac{k_1^2 d_1 + \cdots + k_r^2 d_r}{k_1 d_1 + \cdots + k_r d_r},$$

and so to get natural numbers with no common divisors, we have to multiply by α, obtaining in this way the type described in the proposition. We now use Proposition 3.1 to prove that $F_{n+m}(\lambda) = F_n(\mu)$ by the following direct computation:

$$
\begin{aligned}
&F_{n+m}(\lambda) = \\
&= \frac{1}{16}\left(n+m-\frac{(\alpha k_1 d_1+\cdots+\frac{k_1^2d_1+\cdots+k_r^2d_r}{d}m+\cdots+\alpha k_r d_r)^2}{\alpha^2k_1^2d_1+\cdots+(\frac{k_1^2d_1+\cdots+k_r^2d_r}{d})^2m+\cdots+\alpha^2k_r^2d_r}\right)^{-1}\\
&= \frac{1}{16}\left(n+m-\frac{(\alpha k_1 d_1+\cdots+k_r d_r+\frac{k_1^2d_1+\cdots+k_r^2d_r}{k_1d_1+\cdots+k_rd_r}m)^2}{k_1^2d_1+\cdots+k_r^2d_r+(\frac{k_1^2d_1+\cdots+k_r^2d_r}{k_1d_1+\cdots+k_rd_r})^2}\right)^{-1}\\
&= \frac{1}{16}\left(n+m-\frac{(k_1d_1+\cdots+k_rd_r)^2}{k_1^2d_1+\cdots+k_r^2d_r}\left(1+\frac{k_1^2d_1+\cdots+k_r^2d_r}{(k_1d_1+\cdots+k_rd_r)^2}m\right)\right)^{-1}\\
&= \frac{1}{16}\left(n+m-\frac{(k_1d_1+\cdots+k_rd_r)^2}{k_1^2d_1+\cdots+k_r^2d_r}-m)\right)^{-1}\\
&= F_n(\mu).
\end{aligned}
$$

It is easy to check that everything is the same if $\frac{k_1^2d_1+\cdots+k_r^2d_r}{d} = \alpha k_i$ for some i. □

4 Nilradical with a codimension-one abelian ideal

In this section we shall consider solvmanifolds S_μ such that the Lie algebra μ has a codimension-one abelian ideal. In this case we will be able to carry out the calculations of critical points explicitely. Such a μ is determined by the map $\mathrm{ad}_\mu X$ for a single $X \in \mathfrak{n}$. We can fix a decomposition $\mathfrak{n} = \mathbb{R}X \oplus \mathfrak{m}$, and assume, up to isomorphism, that $\mathfrak{m}$ is the required abelian ideal for any μ. It is easy to see that μ is isomorphic to λ if and only if $\mathrm{ad}_\mu X$ and $\mathrm{ad}_\lambda X$ are $\mathrm{G}L(\mathfrak{m})$-conjugate. Therefore the isomorphism classes of n-dimensional nilpotent Lie algebras having a codimension-one abelian ideal are in a one-to-one correspondence with the finitely many conjugacy classes of $(n-1)$-dimensional nilpotent matrices.

The objective of this section is to show, by using Theorem 2.1, that each isomorphism class (or $\mathrm{G}L(n)$-orbit) of a nilpotent Lie algebra having a codimension-one abelian ideal contains an element μ for which S_μ is Einstein. We first prove a technical lemma.

Lemma 4.1. *The function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$, $k = n_1 + \cdots + n_r$, defined by*

$$f(\{x_{ij}\}) = \sum_{i=1}^{r} x_{i1}^2 + (x_{i1}-x_{i2})^2 + \cdots + (x_{i(n_i-1)} - x_{in_i})^2 + x_{in_i}^2$$

has only one critical point restricted to the leaf

$$\sum_{ij} x_{ij} \equiv b, \qquad x_{ij} > 0,$$

which is given by

$$x_{ij} = x_{i(n_i+1-j)} = c(jn_i - j(j-1)), \qquad i = 1, \dots, r, \quad j = 1, \dots, n_i,$$

for some $c > 0$.

Proof. The Lagrange multiplier method says that if $\{x_{ij}\}$ is a critical point of f restricted to that leaf, then there exists $d \in \mathbb{R}$ such that for any $i = 1, \dots, r$

$$\frac{\partial f}{\partial x_{ij}} = 4x_{ij} - 2x_{i(j-1)} - 2x_{i(j+1)} = d,$$
$$j = 1, \dots, n_i, \quad x_{i0} = x_{i(n_i+1)} := 0.$$

Thus each subset $\{x_{ij}\}_{j=0,\dots,n_i+1}$ satisfies the following recurrence formula

$$x_{i(j+1)} = 2x_{ij} - x_{i(j-1)} - \frac{d}{2},$$

for which $x_{ij} = \frac{d}{4}(jn_i - j(j-1))$ is the unique solution. □

Theorem 4.2. *Let* $\mu \in \mathcal{N}_n$ *be an* n*-dimensional nilpotent Lie algebra which has a codimension-one abelian ideal* $\mathfrak{m}$*, and assume that* $\mathfrak{n} = \mathbb{R}X \oplus \mathfrak{m}$*. Let* $\{X_{ij} : 1 \le i \le r,\ 1 \le j \le n_i + 1\}$ *denote a Jordan basis for* $\operatorname{ad}_\mu X|_\mathfrak{m}$*, that is,* $n_1 \ge \dots \ge n_r$*,* $(n_1+1) + \dots + (n_r+1) = n-1$*, and*

$$\mu(X, X_{ij}) = X_{i(j+1)}, \qquad X_{i(n_i+2)} := 0.$$

Then $GL(n).\mu \cap \mathsf{S}$ *contains a critical point* $\mu_0/||\mu_0||$ *of* $F_n : \mathsf{S} \longrightarrow \mathbb{R}$ *given by*

$$\mu_0(X, X_{ij}) = (jn_i - j(j-1))X_{i(j+1)}.$$

Proof. For any choice of positive numbers $\{a_{ij}\}$, consider the Lie algebras $\mu = \mu(\{a_{ij}\})$, for which $\operatorname{ad}_\mu X|_\mathfrak{m}$ in terms of the basis $\{X_{ij}\}$ is given by

$$\operatorname{ad}_\mu X|_\mathfrak{m} = J = \begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_r} \end{bmatrix}, \qquad J_{n_i} = \begin{bmatrix} 0 & & & \\ a_{i1} & \ddots & & \\ & \ddots & \ddots & \\ & & a_{in_i} & 0 \end{bmatrix}$$

with $n_1 \ge \dots \ge n_r$, $(n_1+1) + \dots + (n_r+1) = n-1$. Recall that the Lie algebra μ in the theorem correspond to put all the a_{ij} equal to 1, and the $\mu(\{a_{ij}\})$ are all isomorphic to each other. We are looking for a critical point of $F_n|_{GL(n).\mu\cap\mathsf{S}}$. Our plan will be to find first a critical point of F_n restricted to the subset given by $\{\mu(\{a_{ij}\}) : a_{ij} > 0\} \cap \mathsf{S}$, and then consider condition (iv) in Theorem 2.1.

We now denote the basis $\{X_{ij}\}$ by $\{X_1, \dots, X_{n-1}\}$, just for notational convenience. If we assume that $X \perp \mathfrak{m}$ and $||X|| = 1$, then it follows from (5) that

$$\langle \operatorname{Ric}_\mu X, X\rangle = -\frac{1}{2}\sum_{ij}\langle \mu(X, X_i), X_j\rangle^2 = -\frac{1}{2}\sum_{ij}\langle JX_i, X_j\rangle^2 = -\frac{1}{2}\operatorname{tr} JJ^*.$$

On the other hand, it is easy to see that $\langle \operatorname{Ric}_\mu X, X_i\rangle = 0$ for all $i = 1, \dots, n-1$, and for $Z, Y \in \mathfrak{m}$ we have that

$$\begin{aligned}\langle \operatorname{Ric}_\mu Z, Y\rangle &= -\tfrac{1}{2}\sum_i \langle \mu(Z, X), X_i\rangle\langle \mu(Y, X), X_i\rangle \\ &\quad +\tfrac{1}{2}\sum_i \langle \mu(X, X_i), Z\rangle\langle \mu(X, X_i), Y\rangle \\ &= -\tfrac{1}{2}\langle \mu(Z, X), \mu(Y, X)\rangle + \tfrac{1}{2}\langle (\operatorname{ad}_\mu X)^* Z, (\operatorname{ad}_\mu X)^* Y\rangle \\ &= -\tfrac{1}{2}\langle JX, JY\rangle + \tfrac{1}{2}\langle J^*Z, J^*Y\rangle \\ &= \tfrac{1}{2}\langle (JJ^* - J^*J)Z, Y\rangle.\end{aligned}$$

We therefore obtain that

$$\operatorname{Ric}_\mu = \left[\begin{array}{c|ccc} -\frac{1}{2}\operatorname{tr} JJ^* & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \frac{1}{2}[J, J^*] & \\ 0 & & & \end{array}\right], \tag{10}$$

thus $F_n(\mu) = \frac{1}{4}(\operatorname{tr} JJ^*)^2 + \frac{1}{4}\operatorname{tr}[J, J^*]^2$ and $||\mu||^2 = -4\operatorname{tr}\operatorname{Ric}_\mu = 2\operatorname{tr} JJ^*$. By a straightforward calculation we obtain for each $i = 1, \dots, r$ that,

$$J_{n_i}J_{n_i}^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{i1}^2 & & \\ \vdots & & \ddots & \\ 0 & & & a_{in_i}^2 \end{bmatrix},$$

and

$$[J_{n_i}, J_{n_i}^*] = \begin{bmatrix} -a_{i1}^2 & & & & \\ & a_{i1}^2 - a_{i2}^2 & & & \\ & & \ddots & & \\ & & & a_{i(n_i-1)}^2 - a_{in_i}^2 & \\ & & & & a_{in_i}^2 \end{bmatrix}. \tag{11}$$

This implies that

$$1 = ||\mu||^2 = 2\sum_{ij} a_{ij}^2$$

and so

$$F_n(\mu) = \frac{1}{16} + \frac{1}{4}\sum_i (a_{i1}^2)^2 + (a_{i1}^2 - a_{i2}^2)^2 + \dots + (a_{i(n_i-1)}^2 - a_{in_i}^2)^2 + (a_{in_i}^2)^2. \tag{12}$$

By Lemma 4.1, we know that there is a critical point $\{b_{ij}^2\}$ of $F_n(\mu) = f(\{a_{ij}^2\})$ restricted to the leaf $\sum a_{ij}^2 = \frac{1}{2}$, and it is given by

$$b_{ij}^2 = c(jn_i - j(j-1)), \qquad i = 1, \dots, r, \quad j = 1, \dots, n_i, \tag{13}$$

for a suitable $c > 0$. We now consider the corresponding $\mu_0 = \mu(\{b_{ij}\})$, and we will show that such a μ_0 satisfies $\mathrm{Ric}_{\mu_0} \in \mathbb{R}I \oplus \mathrm{Der}(\mu_0)$. It follows from (13) that for any i, j,

$$b_{ij}^2 - b_{i(j+1)}^2 = c(2j - n_i), \tag{14}$$

and then by (10) and (11) we get that

$$(\mathrm{Ric}_{\mu_0} + (c + \frac{1}{4})I)X = cX \tag{15}$$

and $\mathrm{Ric}_{\mu_0} + (c + \frac{1}{4})I|_{\mathfrak{m}}$ is the direct sum of r blocks of the form

$$c \begin{bmatrix} (\frac{c+\frac{1}{4}}{c} - \frac{n_i}{2}) & & & \\ & (\frac{c+\frac{1}{4}}{c} - \frac{n_i}{2}) + 1 & & \\ & & \ddots & \\ & & & (\frac{c+\frac{1}{4}}{c} - \frac{n_i}{2}) + n_i \end{bmatrix}. \tag{16}$$

Using that $\mu_0(X, \mathbb{R}X_{ij}) \subset \mathbb{R}X_{i(j+1)}$ for all i, j, it is evident that $\mathrm{Ric}_{\mu_0} + (c + \frac{1}{4})I$ is a derivation of μ_0, and thus $\mu_0/||\mu_0||$ is a critical point of $F_n : \mathsf{S} \longrightarrow \mathbb{R}$ (see Theorem 2.1). □

It follows from Theorem 2.1 that for each μ_0 obtained in Theorem 4.2, the solvmanifold S_{μ_0} is Einstein. We then obtain that, by taking different decompositions $n - 1 = (n_1 + 1) + \dots + (n_r + 1)$, any step of nilpotency is possible for the nilradical of an Einstein solvmanifold since μ_0 is $(n_1 + 1)$-step nilpotent. The spaces S_μ are modeled on completely solvable Lie groups, therefore if μ is not isomorphic to λ, then S_μ and S_λ can never be isometric (see [A] or [H, 2.7]). Thus in each dimension $n+1$, we have found as many explicit examples of Einstein solvmanifolds as $(n-1) \times (n-1)$

nilpotent Lie matrices up to conjugation (see the first paragraph of this section), or equivalently, as many as the number $P(n-1)$ of decompositions $n-1 = (n_1+1)+\dots+(n_r+1)$ with integers $n_1 \geq \dots \geq n_r$. The number $P(k)$ of decompositions of k into integer summands without regard to order is asymptotic to

$$\frac{e^{\pi\sqrt{\frac{2}{3}}\sqrt{k}}}{4k\sqrt{3}},$$

and one can find in [Gp] the value of $P(k)$ for every $k \leq 500$. For instance, this family provides 11 8-dimensional Einstein solvmanifolds and 385 of dimension 20.

In what follows, we will calculate the eigenvalue type of the Einstein solvmanifolds S_{μ_0} obtained in Theorem 4.2. The derivation D_{μ_0} is described in (15) and (16), thus we should first analyze in which cases we have that $1 + \frac{1}{4c} - \frac{n_j}{2} \in \mathbb{N}$ for every $j = 1, \dots, r$. The number c can be calculated from the proof of Theorem 4.2 by using that

$$\sum_{ij} b_{ij}^2 = \operatorname{tr} JJ^* = \frac{1}{2}||\mu_0||^2 = \frac{1}{2},$$

obtaining that

$$c = \frac{3}{n_1(n_1+1)(n_1+2) + \dots + n_r(n_r+1)(n_r+2)}.$$

Thus the question is whether

$$(17) \qquad 1 + \frac{1}{4c} - \frac{n_j}{2} = 1 + \frac{\sum_i n_i(n_i+1)(n_i+2)}{12} - \frac{n_j}{2} \in \mathbb{N},$$

for every $j = 1, \dots, r$. It is easy to see that (17) holds only in the following two cases:

(i) n_i is even for every $i = 1, \dots, r$.
(ii) n_i is odd for every $i = 1, \dots, r$ and $\sharp\{i : n_i \equiv 1 \mod (4)\}$ is odd.

It follows from (15) and (16) that in both cases the eigenvalues are $\{1, \theta - \frac{n_i}{2}, \theta - \frac{n_i}{2} + 1, \dots, \theta + \frac{n_i}{2} : i = 1, \dots, r\}$, where

$$\theta = 1 + \frac{1}{4c}.$$

Thus the eigenvalues are actually $\{1, \theta - \frac{n_1}{2}, \theta - \frac{n_1}{2} + 1, \dots, \theta + \frac{n_1}{2}\}$ and the corresponding multiplicities can be easily computed. We obtain that the eigenvalue type of S_{μ_0} is

(i) $\big(1 < \theta - \frac{n_1}{2} < \dots < \theta - 1 < \theta < \theta + 1 < \dots < \theta + \frac{n_1}{2};$
$1, d_1, \dots, d_{\frac{n_1}{2}}, d_{\frac{n_1+1}{2}}, d_{\frac{n_1}{2}}, \dots, d_1\big),$

(ii) $\big(1 < \theta - \frac{n_1}{2} < \cdots < \theta - \frac{1}{2} < \theta + \frac{1}{2} < \cdots < \theta + \frac{n_1}{2};$
$1, d_1, \ldots, d_{\frac{n_1+1}{2}}, d_{\frac{n_1+1}{2}}, \ldots, d_1\big),$

where $d_k = \sharp\{i : n_1 \leq n_i + 2(k-1)\}$.

Otherwise, if

(iii) $\theta - \frac{n_j}{2} \notin \mathbb{N}$ for some $j \in \{1, \ldots, r\}$,

then the eigenvalues are $\{2, 2\theta - n_e, 2\theta - n_e + 2, \ldots, 2\theta - 2, 2\theta, \theta + 2, \ldots, 2\theta + n_e - 2, 2\theta + n_e, 2\theta - n_o, 2\theta - n_o + 2, \ldots, 2\theta - 1, 2\theta + 1, \ldots, 2\theta + n_o - 2, 2\theta + n_o\}$, where n_e and n_o are the greatest even and odd numbers among the $n_i's$ respectively. The corresponding multiplicities are easy to obtain, but somewhat difficult to write.

In particular, for $r = 1$ we have that the eigenvalue type of S_{μ_0} is given by

$$\begin{cases} (1 < \theta - \frac{n_1}{2} < \cdots < \theta < \cdots < \theta + \frac{n_1}{2}; 1, 1, \ldots, 1) & n_1 \text{ even}; \\ (1 < \theta - \frac{n_1}{2} < \cdots < \theta - \frac{1}{2} < \theta + \frac{1}{2} < \cdots < \theta + \frac{n_1}{2}; 1, 1, \ldots, 1) & \\ & n_1 \equiv 1 \ (4); \\ (2 < 2\theta - n_1 < \cdots < 2\theta - 1 < 2\theta + 1 < \cdots < 2\theta + n_1; 1, 1, \ldots, 1) & \\ & n_1 \equiv 3 \ (4); \end{cases}$$

where $\theta = 1 + \frac{n_1(n_1+1)(n_1+2)}{12}$. The only exception is when $n_1 = 1$ (3-dimensional Heisenberg Lie algebra), since this is the only case where $1 = \theta - \frac{n_1}{2}$, and so the eigenvalue type is $(1 < 2; 2, 1)$.

Finally, it is easy to prove using Proposition 3.1, (12) and (14), that the critical value of the critical point $\mu_0/||\mu_0||$ given in Theorem 4.2 is

$$F_n(\mu_0/||\mu_0||) = \frac{1}{16} + \frac{3/4}{n_1(n_1+1)(n_1+2) + \cdots + n_r(n_r+1)(n_r+2)}. \tag{18}$$

5 Low dimensional Einstein solvmanifolds

The goal of this section is, by applying our variational method and some previous results, to understand the low dimensional rank-one Einstein solvmanifolds. We shall obtain a complete picture for dimension ≤ 6.

Cases $n = 3, 4$ are covered by the results obtained in Sect. 4. Indeed, the variety $\mathcal{N}_3$ consists of only one non-abelian $\mathrm{G}L(3)$-orbit, corresponding to the Heisenberg Lie algebra $\mu_H(x_1, x_2) = x_3$, which by Theorem 4.2 is a critical point of $F_3 : \mathsf{S} \longrightarrow \mathbb{R}$ of eigenvalue type $(1 < 2; 2, 1)$. $\mathcal{N}_4$ is the union of two non-abelian orbits $GL(4).\lambda_1$ and $GL(4).\lambda_2$, where $\lambda_1 = \mu_H \oplus \mathbb{R}$ (i.e. $\mathbb{R} = \mathbb{R}x_4$ is an abelian factor), and the only nonzero brackets of λ_2 are

$$\lambda_2(x_1, x_2) = x_3, \quad \lambda_2(x_1, x_3) = x_4.$$

It follows from Proposition 3.3 and Theorem 4.2 that $\lambda_1/||\lambda_1||$ and $\lambda_2/||\lambda_2||$ themselves are critical points of F_4 of type $(2 < 3 < 4; 2, 1, 1)$ and $(1 < 2 < 3 < 4; 1, 1, 1, 1)$, respectively.

Therefore, we will concentrate on the case $n = 5$. The variety $\mathcal{N}_5$ consists of 8 non-abelian orbits (see [M]), represented by Table 1.

Table 1.

$\mu_1(x_1, x_2) = x_3,\ \mu_1(x_1, x_3) = x_4,\ \mu_1(x_1, x_4) = x_5$	$(4{-}\text{step});$
$\mu_2(x_1, x_2) = x_3,\ \mu_2(x_1, x_3) = x_4,\ \mu_2(x_1, x_4) = x_5,$ $\mu_2(x_2, x_3) = x_5$	$(4{-}\text{step});$
$\mu_3(x_1, x_2) = x_4,\ \mu_3(x_2, x_3) = x_5,\ \mu_3(x_1, x_4) = x_5$	$(3{-}\text{step});$
$\mu_4(x_1, x_2) = x_5,\ \mu_4(x_3, x_4) = x_5$	$(2{-}\text{step});$
$\mu_5(x_1, x_2) = x_3,\ \mu_5(x_1, x_3) = x_4,\ \mu_5(x_2, x_3) = x_5$	$(3{-}\text{step});$
$\mu_6(x_1, x_2) = x_4,\ \mu_6(x_1, x_3) = x_5$	$(2{-}\text{step});$
$\mu_7 = \mu_H \oplus \mathbb{R}^2,$	$(2{-}\text{step});$
$\mu_8 = \lambda_2 \oplus \mathbb{R},$	$(3{-}\text{step}).$

We will find explicitly a critical point of $F_5 : \mathsf{S} \longrightarrow \mathbb{R}$ in each of the 8 orbits, and will calculate their eigenvalue types and critical values. For μ_1 and μ_6 we can apply Theorem 4.2, and cases μ_7 and μ_8 follow from Proposition 3.3. μ_4 is the 5-dimensional Heisenberg Lie algebra and so S_{μ_4} is the complex hyperbolic space, which is well-known to be Einstein. The variational approach will be applied to the remaining cases. We will consider in detail only case μ_2, being the most difficult one. The proof in the cases μ_3 and μ_5 is completely analogous.

In order to show that $GL(5).\mu_2$ contains a critical point of $F_5 : \mathsf{S} \longrightarrow \mathbb{R}$, we consider for each positive numbers $a, b, c, d \in \mathbb{R}$ the Lie bracket $\mu =$

$\mu(a, b, c, d)$ isomorphic to μ_2 defined by

$$\mu(x_1, x_2) = ax_3,\ \mu(x_1, x_3) = bx_4,\ \mu(x_1, x_4) = cx_5,\ \mu(x_2, x_3) = dx_5.$$

As in the proof of Theorem 4.2, our plan is to find first a critical point of F_5 restricted to the set $\{\mu(a, b, c, d) : a, b, c, d \in \mathbb{R}_{>0}\} \cap \mathsf{S}$, and after that, to show using the characterization given in Theorem 2.1 (iv) that such a point is really a critical point of F_5. Using (5), we can see by simple computations that Ric_μ equals the diagonal matrix

$$\mathrm{Ric}_\mu = \frac{1}{2}\begin{bmatrix} -a^2 - b^2 - c^2 & & & & \\ & -a^2 - d^2 & & & \\ & & a^2 - b^2 - d^2 & & \\ & & & b^2 - c^2 & \\ & & & & c^2 + d^2 \end{bmatrix}.$$

We then obtain that $||\mu||^2 = -4\,\mathrm{tr}\,\mathrm{Ric}_\mu = 2(a^2 + b^2 + c^2 + d^2)$ and

$$\begin{aligned} F_5(\mu) &= \mathrm{tr}\,\mathrm{Ric}_\mu^2 \\ &= f(a, b, c, d) \\ &= \tfrac{1}{4}\Big((a^2 + b^2 + c^2)^2 + (a^2 + d^2)^2 + (a^2 - b^2 - d^2)^2 \\ &\quad +(b^2 - c^2)^2 + (c^2 + d^2)^2\Big). \end{aligned}$$

By standard methods, it is easy to see that $a^2 = b^2 = 3$, $c^2 = d^2 = 2$, define a critical point of f restricted to the leaf $a^2 + b^2 + c^2 + d^2 = 10$. The corresponding $\mu_0 = \mu(3^{\frac{1}{2}}, 3^{\frac{1}{2}}, 2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ then satisfies

$$\mathrm{Ric}_{\mu_0} = \frac{1}{2}\begin{bmatrix} -8 & & & & \\ & -5 & & & \\ & & -2 & & \\ & & & 1 & \\ & & & & 4 \end{bmatrix} = -\frac{11}{2}I + \frac{1}{2}\begin{bmatrix} 3 & & & & \\ & 6 & & & \\ & & 9 & & \\ & & & 12 & \\ & & & & 15 \end{bmatrix}.$$

We then obtain that $\mathrm{Ric}_{\mu_0} \in \mathbb{R}I \oplus \mathrm{Der}(\mu_0)$, which implies that S_{μ_0} is Einstein (see Theorem 2.1). Moreover, it is evident that the eigenvalue type of S_{μ_0} equals $(1 < 2 < 3 < 4 < 5; 1, \ldots, 1)$.

We now summarize the results obtained in this section in the following

Theorem 5.1. *For any $i = 1, \dots, 8$ there exists $\mu_i' \in \mathrm{GL}(5).\mu_i$ such that $\mu_i'/||\mu_i'||$ is a critical point of $F_5 : \mathsf{S} \longrightarrow \mathbb{R}$. Their eigenvalue types and critical values are as follows:*

critical point	eigenvalue type	F_5
$\mu_1'(x_1,x_2) = 3x_3,\ \mu_1'(x_1,x_3) = 4x_4,$ $\mu_1'(x_1,x_4) = 3x_5,$	$(2 < 9 < 11 < 13 < 15; 1, \dots, 1)$	$\frac{1}{16} \cdot \frac{6}{5}$
$\mu_2'(x_1,x_2) = 3^{\frac{1}{2}}x_3,\ \mu_2'(x_1,x_3) = 3^{\frac{1}{2}}x_4,$ $\mu_2'(x_1,x_4) = 2^{\frac{1}{2}}x_5,\ \mu_2'(x_2,x_3) = 2^{\frac{1}{2}}x_5,$	$(1 < 2 < 3 < 4 < 5; 1, \dots, 1)$	$\frac{1}{16} \cdot \frac{11}{10}$
$\mu_3'(x_1,x_2) = x_4,\ \mu_3'(x_2,x_3) = 2^{\frac{1}{2}}x_5,$ $\mu_3'(x_1,x_4) = 2^{\frac{1}{2}}x_5$	$(3 < 4 < 6 < 7 < 10; 1, \dots, 1)$	$\frac{1}{16} \cdot \frac{7}{5}$
$\mu_4' = \mu_4$	$(1 < 2; 4, 1)$	$\frac{1}{16} . 2$
$\mu_5'(x_1,x_2) = 4x_3,\ \mu_5'(x_1,x_3) = 3x_4,$ $\mu_5'(x_2,x_3) = 3x_5.$	$(1 < 2 < 3; 2, 1, 2)$	$\frac{1}{16} \cdot \frac{6}{5}$
$\mu_6' = \mu_6$	$(2 < 3 < 5; 1, 2, 2)$	$\frac{1}{16} . 2$
$\mu_7' = \mu_7$	$(2 < 3 < 4; 2, 2, 1)$	$\frac{1}{16} . 3$
$\mu_8' = \mu_8$	$(1 < 2 < 3 < 4; 1, 1, 2, 1)$	$\frac{1}{16} \cdot \frac{3}{2}$

We recall that, in order to construct the Einstein solvmanifold S_μ from the knowledge of the critical point $\mu/||\mu||$ and its eigenvalue type, one has to use Corollary 3.2. It should be noticed that, independently from how the critical points μ_i' was found, one can show by a very simple computation that the solvmanifolds $S_{\mu_i'}$ are Einstein spaces, by using for instance [H, Lemma 4.4].

Remark. In dimension 7 appear the first examples of characteristically nilpotent Lie algebras (i.e. $\mathrm{Der}(\mu)$ nilpotent), which can never be critical points of $F_n : \mathsf{S} \longrightarrow \mathbb{R}$ since they do not admit non-zero semisimple derivations. On the other hand, we do not know if each of the 34 nilpotent Lie algebras of dimension 6 has a critical point of $F_6 : \mathsf{S} \longrightarrow \mathbb{R}$ in its orbit.

6 A curve of 8-dimensional Einstein solvmanifolds

We have seen in Sect. 5 that each eigenvalue type $\mathcal{M}_{(k,d)}$ for $n=5$ consists of only one point. In order to get a curve of $(n+1)$-dimensional rank-one Einstein solvmanifolds of the same eigenvalue type, it is necessary to have a curve of pairwise non-isomorphic n-dimensional nilpotent Lie algebras (see [A], [H, 2.7] or [L, Prop. 4]). The lowest dimension where there is such a curve is $n=7$ (see [M, S]). We will show now, as another application of our variational method, that one of such curve in dimension 7, give rise to a curve of 8-dimensional rank-one Einstein solvmanifolds of eigenvalue type $(1<2<3<4<5<6<7;1,\ldots,1)$.

Consider for each set of real numbers $\{a_{ij}\}$, the bilinear form $\mu = \mu(\{a_{ij}\}) \in \Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$ defined by $\mu(X_i,X_j) = a_{ij}X_{i+j}$, where $\{X_1,\ldots,X_7\}$ is a basis of $\mathfrak{n}$. We calculate Ric_μ, obtaining that $||\mu||^2 = 2\sum a_{ij}^2$. Then, by applying the Lagrange multiplier theorem and solving a 9×9 linear system, it is not hard to prove that the critical points of $F_7(\mu) = \operatorname{tr}\operatorname{Ric}_\mu^2$ restricted to $\{\mu(\{a_{ij}\}) : \sum a_{ij}^2 \equiv \text{const}\}$ depend on three parameters μ_{t_1,t_2,t_3}. Setting two of the parameters equal to 1 we obtain the following curve:

$$
\begin{array}{ll}
\mu_t(X_1,X_2) = (1-t)^{\frac{1}{2}}X_3, & \mu_t(X_2,X_3) = X_5,\\
\mu_t(X_1,X_3) = X_4, & \mu_t(X_2,X_4) = X_6,\\
\mu_t(X_1,X_4) = t^{\frac{1}{2}}X_5, & \mu_t(X_2,X_5) = t^{\frac{1}{2}}X_7,\\
\mu_t(X_1,X_5) = X_6, & \mu_t(X_3,X_4) = (1-t)^{\frac{1}{2}}X_7.\\
\mu_t(X_1,X_6) = X_7, &
\end{array}
$$

It is easy to check using (5) that for any $0<t<1$ we have that

$$
\operatorname{Ric}_{\mu_t} = -\frac{1}{2}\begin{bmatrix} 4 &&&&&& \\ & 3 &&&&& \\ && 2 &&&& \\ &&& 1 &&& \\ &&&& 0 && \\ &&&&& -1 & \\ &&&&&& -2 \end{bmatrix} = -\frac{5}{2}I + \frac{1}{2}\begin{bmatrix} 1 &&&&&& \\ & 2 &&&&& \\ && 3 &&&& \\ &&& 4 &&& \\ &&&& 5 && \\ &&&&& 6 & \\ &&&&&& 7 \end{bmatrix}.
$$

Thus $\operatorname{Ric}_{\mu_t} \in \mathbb{R}I \oplus \operatorname{Der}(\mu_t)$, that is, $\mu_t/||\mu_t||$ is a critical point of $F_7 : \mathsf{S} \longrightarrow \mathbb{R}$ of eigenvalue type $(1<2<3<4<5<6<7;1,\ldots,1)$, and so S_{μ_t} is Einstein for all $0<t<1$ (see Theorem 2.1). Each Lie algebra μ_t is isomorphic to the Lie algebra λ_t, denoted by $1,2,3,4,5,7_I : t$ in [S, pp.494] (see also the curve $\tilde{g}(0,t,1,0,1,0,0,0)$ in [M, 5.2.3]). The isomorphism is given by $\lambda_t = \varphi_t \mu(\varphi_t^{-1}\cdot, \varphi_t^{-1}\cdot)$, where

$$\varphi_t^{-1} = \begin{bmatrix} 1 & & & & & & \\ & t^{\frac{1}{2}} & & & & & \\ & & t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} & & & & \\ & & & t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} & & & \\ & & & & t(1-t)^{\frac{1}{2}} & & \\ & & & & & t(1-t)^{\frac{1}{2}} & \\ & & & & & & t(1-t)^{\frac{1}{2}} \end{bmatrix}.$$

This proves that μ_t is really a curve of pairwise non-isomorphic Lie algebras, and consequently S_{μ_t} is a curve of pairwise non-isometric Einstein solvmanifolds. It follows from Proposition 3.1 that the critical value is $F_7(\mu_t) = \frac{1}{16}\frac{5}{7}$ for all $0 < t < 1$.

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