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Necklace Lie algebras and noncommutative symplectic geometry

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Abstract. Recently, V. Ginzburg proved that Calogero phase space is a coadjoint orbit for some infinite dimensional Lie algebra coming from noncommutative symplectic geometry, [12]. In this note we generalize his argument to specific quotient varieties of representations of (deformed) preprojective algebras. This result was also obtained independently by V. Ginzburg [13]. Using results of W. Crawley-Boevey and M. Holland [10], [8] and [9] we give a combinatorial description of all the relevant couples (α, λ) which are coadjoint orbits. We give a conjectural explanation for this coadjoint orbit result and relate it to different noncommutative notions of smoothness.

1 Introduction

In [18, § 9] M. Kontsevich gave a somewhat cryptic outline of possible applications of noncommutative (symplectic) geometry to representation theory. If A is a formally smooth algebra (such as free algebras or path algebras of quivers), then J. Cuntz and D. Quillen [11] have shown that the cohomology of the noncommutative deRham complex gives cyclic homology of algebras. Motivated by this, M. Kontsevich proposed to associate to A commutative affine schemes $\mathtt{rep}_n\ A$, the n-dimensional representations of A. For A formally smooth it follows that these schemes are smooth varieties. In this situation one assumes that noncommutative functions, noncommutative differential or symplectic forms on A induce ordinary GL_n-invariant functions, differential and symplectic forms on the varieties $\mathtt{rep}_n\ A$ and hence on the corresponding quotient varieties $\mathtt{iss}_n\ A$. If A is equipped with a

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noncommutative symplectic form, the noncommutative functions acquire a Lie algebra structure and one might expect that in ideal situations some subvarieties of the $\mathtt{iss}_n\ A$ will be coadjoint orbits for this Lie structure. In the paper [18] M. Kontsevich proved an acyclicity result for the noncommutative deRham cohomology for A a free associative algebra and computed the Lie structure on the functions when there is an even number of free generators.

As mentioned before, the path algebra $\mathbb{C}Q$ of a finite quiver Q is a formally smooth algebra. The representation varieties for $\mathbb{C}Q$ decompose as

$$\mathtt{rep}_n\ \mathbb{C}Q = \bigsqcup_{\alpha} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ \mathbb{C}Q$$

where $\alpha = (n_1, \ldots, n_k)$ runs over all dimension vectors with $\sum n_i = n$ and where $GL(\alpha) = GL_{n_1} \times \ldots \times GL_{n_k}$ is the basechange group of the vertex spaces. For this reason it is customary to consider the *quiver representation spaces* $\mathtt{rep}_\alpha\ \mathbb{C}Q$ rather than all n-dimensional representations. In order to apply Kontsevich's idea to the representation theory of quivers we need not to consider the usual deRham complex but rather the *relative* deRham complex with respect to the subalgebra V generated by the vertex-idempotents. In Sect. 3 we redo Kontsevich's computation of the cohomology groups of free algebras for these relative cohomology groups of $\mathbb{C}Q$ and prove

Theorem 1.1. *The noncommutative relative deRham cohomology groups of* $\mathbb{C}Q$ *are*

$$\begin{cases} H^0_{dR}\ \mathbb{C}Q & \simeq V \\ H^n_{dR}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

Next, we bring in the symplectic structure. We consider the double quiver $\mathbb{Q}$ of Q obtained by adjoining to every arrow a in Q an arrow in the opposite direction a^*. On the space of noncommutative functions

$$\mathbb{N}_Q = \frac{\mathbb{C}\mathbb{Q}}{[\mathbb{C}\mathbb{Q}, \mathbb{C}\mathbb{Q}]}$$

which is spanned by the necklace words in $\mathbb{Q}$ (that is, the oriented cycles in the quiver $\mathbb{Q}$ considered upto cyclic permutation of the arrows) we can define a Lie algebra structure see Fig. 1, which we call the *necklace Lie algebra* $\mathbb{N}_Q$. Using our results on deRham cohomology we are able in Sect. 4 to prove the existence of a central extension result

Theorem 1.2. *If V is equipped with the (trivial) commutator bracket, then there is a central extension of Lie algebras*

$$0 \longrightarrow V \longrightarrow \mathbb{N}_Q \longrightarrow Der_\omega\ \mathbb{C}Q \longrightarrow 0$$

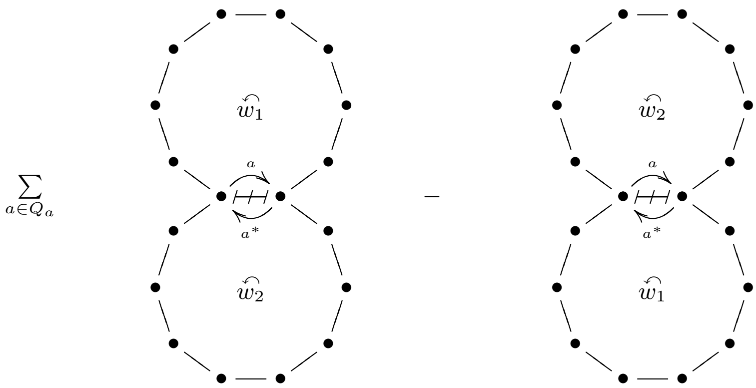


Fig. 1. Lie bracket $[w_1, w_2]$ in $\mathbb{N}_Q$

where the last term is the Lie algebra of symplectic derivations corresponding to the symplectic structure $\omega = \sum_{a \in Q_a} da^* da$.

The Lie algebra of symplectic derivations corresponds to the group of V-algebra automorphisms of $\mathbb{C}Q$ which preserve the *moment element* $m = \sum_{a \in Q_a} [a, a^*] \in \mathbb{C}\mathbb{Q}$. For this reason it is natural to expect that coadjointness results for the necklace Lie algebra $\mathbb{N}_Q$ come from representation schemes of (deformed) preprojective algebras as introduced by W. Crawley-Boevey and M. Holland in [10]

$$\Pi_\lambda = \frac{\mathbb{C}\mathbb{Q}}{(m - \lambda)}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$. However, as we will prove in Sect. 6 these deformed preprojective algebras are *never* formally smooth so usually their representation schemes $\mathtt{rep}_\alpha\, \Pi_\lambda$ will be highly singular as are their quotient schemes $\mathtt{iss}_\alpha\, \Pi_\lambda$. Still, extending the original approach of V. Ginzburg on the coadjointness of Calogero-Moser particles to this situation we are able in Sect. 5 to prove the following result.

Theorem 1.3. *If α is a dimension vector of a simple Π_λ-representation which is minimal, that is cannot be decomposed as a sum of two smaller dimension vectors of simples, then*

$$\mathtt{iss}_\alpha\, \Pi_\lambda$$

is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$.

For this result to be applicable we need a description of the set of dimension vectors of simple representations of Π_λ. Fortunately this (hard) problem was solved by W. Crawley-Boevey [8].

In the final section we try to give a conjectural explanation underlying these coadjoint orbit results. Consider the algebra $A_Q = \mathbb{C}[\mathbb{N}_Q] \otimes \mathbb{C}\mathbb{Q}$ with trace, mapping an oriented cycle to the corresponding necklace word and consider the group Aut_Q of trace preserving V-algebra automorphisms of A_Q preserving the moment element. Then, we conjecture that this group acts transitively on each stratum of the quotient variety $\mathtt{iss}_\alpha\ \Pi_\lambda = \mathtt{rep}_\alpha\ \Pi_\lambda / GL(\alpha)$ determined by a representation type of semisimple representations. The coadjoint orbit result would then be a consequence of the conjecture that for deformed preprojective algebras the noncommutative α-smooth locus (the subvariety of $\mathtt{iss}_\alpha\ \Pi_\lambda$ such that the inverse image of the quotient map is a smooth subscheme of $\mathtt{rep}_\alpha\ \Pi_\lambda$) coincides with the Azumaya algebra (the subvariety of $\mathtt{iss}_\alpha\ \Pi_\lambda$ where the quotient map is a principal $PGL(\alpha)$-fibration in the étale topology) of the α-dimensional approximation $\Pi_\lambda @ \alpha$ of the deformed preprojective algebra. For more details and for the relation with relative notions of noncommutative smoothness we refer to Sect. 6. Using the computation of the dimension of ext-groups of the preprojective algebra Π_0 by W. Crawley-Boevey [9] we are able to prove:

Theorem 1.4. *For α a dimension vector of a simple representation of Π_0, the α-smooth locus of the preprojective algebra Π_0 coincides with the Azumaya locus.*

We expect that the conjecture holds for arbitrary deformed preprojective algebras by a hyper-Kähler type argument and prove some partial results in this direction.

2 Necklace Lie algebras

In this section we introduce the main object of this note in a purely combinatorial way. Recall that a *quiver* Q is a finite directed graph on a set of vertices $Q_v = \{v_1, \ldots, v_k\}$, having a finite set $Q_a = \{a_1, \ldots, a_l\}$ of arrows, where we allow loops as well as multiple arrows between vertices. An arrow a with starting vertex $s(a) = v_i$ and terminating vertex $t(a) = v_j$ will be depicted as $(j) \xleftarrow{a} (i)$. The quiver information is encoded in the *Euler form* which is the bilinear form on $\mathbb{Z}^k$ determined by the matrix $\chi_Q \in M_k(\mathbb{Z})$ with

$$\chi_{ij} = \delta_{ij} - \#\ \{\ a \in Q_a\ \mid\ (j) \xleftarrow{a} (i)\ \}$$

The symmetrization $T_Q = \chi_Q + \chi_Q^{tr}$ of this matrix determines the *Tits form* of the quiver Q. An oriented cycle $c = a_{i_u} \ldots a_{i_1}$ of length $u \geq 1$ is a concatenation of arrows in Q such that $t(a_{i_j}) = s(a_{i_{j+1}})$ and $t(a_{i_u}) = s(a_{i_1})$. In addition to these there are k oriented cycles e_i of length 0 corresponding to the vertices of Q. All oriented cycles c' obtained from c by cyclically permuting the arrow components are said to be equivalent to c. A *necklace word* w for Q is an equivalence class of oriented cycles in the quiver Q.

The *double quiver* $\mathbb{Q}$ of Q is the quiver obtained by adjoining to every arrow (or loop) $(j) \xleftarrow{a} (i)$ in Q an arrow in the opposite direction $(j) \xrightarrow{a^*} (i)$. That is, $\chi_{\mathbb{Q}} = T_Q - \mathbb{1}_k$.

The *necklace Lie algebra* $\mathbb{N}_Q$ for the quiver Q has as basis the set of all necklace words w for the *double* quiver $\mathbb{Q}$ and where the Lie bracket $[w_1, w_2]$ is determined as in Fig. 1. That is, for every arrow $a \in Q_a$ we look for an occurrence of a in w_1 and of a^* in w_2. We then open up the necklaces by removing these factors and regluing the open ends together to form a new necklace word. We repeat this operation for *all* occurrences of a (in w_1) and a^* (in w_2). We then replace the roles of a^* and a and redo this operation with a minus sign. Finally, we add up all these obtained necklace words for all arrows $a \in Q_a$. Using this graphical description the Jacobi identity for $\mathbb{N}_Q$ follows from Fig. 2.

3 An acyclicity result

The *path algebra* $\mathbb{C}Q$ of a quiver Q has as basis the set of all oriented paths $p = a_{i_u} \ldots a_{i_1}$ of length $u \geq 1$ in the quiver, that is $s(a_{i_{j+1}}) = t(a_{i_j})$ together with the vertex-idempotents e_i of length zero. Multiplication in $\mathbb{C}Q$ is induced by (left) concatenation of paths. More precisely, $1 = e_1 + \ldots + e_k$ is a decomposition of 1 into mutually orthogonal idempotents and further we define

– $e_j.a$ is always zero unless $(j) \xleftarrow{a} ()$ in which case it is the path a,
– $a.e_i$ is always zero unless $() \xleftarrow{a} (i)$ in which case it is the path a,
– $a_i.a_j$ is always zero unless $() \xleftarrow{a_i} () \xleftarrow{a_j} ()$ in which case it is the path $a_i a_j$.

Path algebras of quivers are the archetypical examples of *formally smooth algebras* as introduced and studied in [11].

In this section we will generalize Kontsevich's acyclicity result for the noncommutative deRham cohomology of the free algebra [18] to that of the path algebra $\mathbb{C}Q$. The crucial idea is to consider the *relative* differential forms (as defined in [11]) of $\mathbb{C}Q$ with respect to the semisimple subalgebra $V = \mathbb{C} \times \ldots \times \mathbb{C}$ generated by the vertex idempotents. The idea being that in considering quiver representations one works in the category of V-algebras rather than $\mathbb{C}$-algebras.

For a subalgebra B of A, let $\overline{A}_B$ denote the cokernel of the inclusion as B-bimodule. The space of relative differential forms of degree n of A with

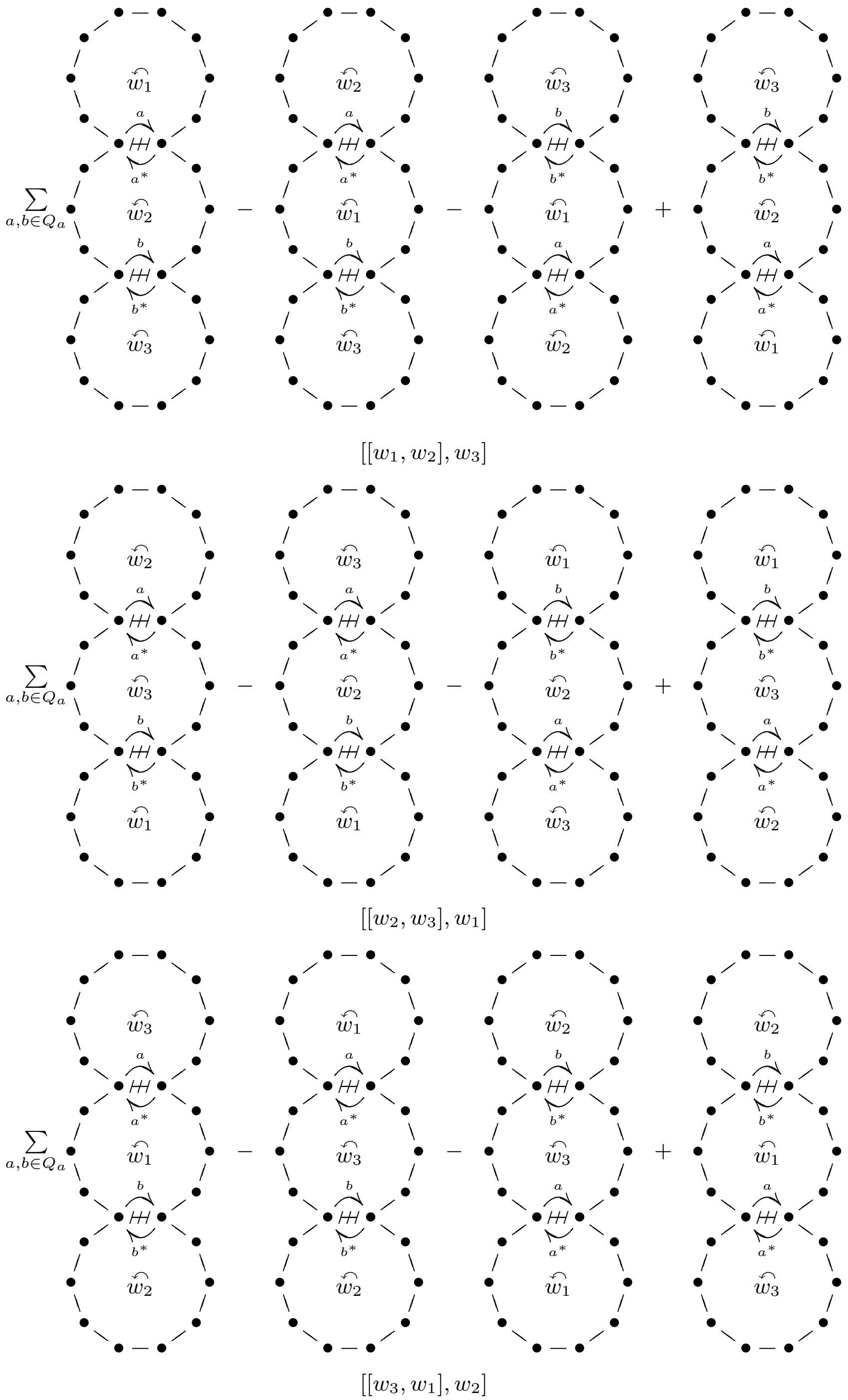


Fig. 2. Jacobi identity for the necklace Lie algebra $\mathbb{N}_Q$. Term $1a$ vanishes against $2c$, term $1b$ against $3d$, $1c$ against $3a$, $1d$ against $2b$, $2a$ against $3c$ and $2d$ against $3b$

respect to B is

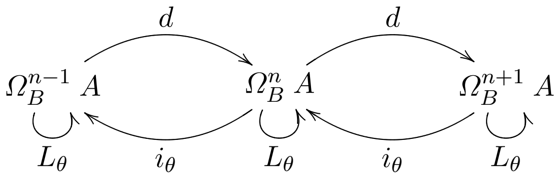
$$\Omega^n_B\ A = A \otimes_B \underbrace{\overline{A}_B \otimes_B \ldots \otimes_B \overline{A}_B}_{n}$$

The space $\Omega^\bullet_B\ A$ is given a differential graded algebra structure by taking the multiplication

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_m) = \sum_{i=0}^{n} (-1)^{n-i}(a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m)$$

and the differential $d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$, see [11]. Here, $(a_0, \ldots, a_n)$ is a representant of the class $a_0 da_1 \ldots da_n \in \Omega^n_B\ A$ and we recall that $\Omega^\bullet_B\ A$ s generated by the a and da for all $a \in A$. The *relative cohomology* $H^n_B\ A$ is defined as the cohomology of the complex $\Omega^\bullet_B\ A$.

For $\theta \in Der_B\ A$, the Lie algebra of B-derivations of A (that is θ is a derivation of A and $\theta(B) = 0$), we define a degree preserving derivation L_θ and a degree -1 super-derivation i_θ on $\Omega^\bullet_B\ A$ (that is, for all $\omega \in \Omega^i_B\ A$ we have that $i_\theta(\omega\omega') = i_\theta(\omega)\omega' + (-1)^i \omega i_\theta(\omega'))$



by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\,\theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in A$. We have the Cartan homotopy formula $L_\theta = i_\theta \circ d + d \circ i_\theta$ as both sides are degree preserving derivations on $\Omega^\bullet_B\ A$ and they agree on all the generators a and da for $a \in A$.

Lemma 3.1. *Let $\theta, \gamma \in Der_B\ A$, then we have on $\Omega^\bullet_B\ A$ the identities of operators*

$$\begin{cases} L_\theta \circ i_\gamma - i_\gamma \circ L_\theta = [L_\theta, i_\gamma] & = i_{[\theta,\gamma]} = i_{\theta\circ\gamma-\gamma\circ\theta} \\ L_\theta \circ L_\gamma - L_\gamma \circ L_\theta = [L_\theta, L_\gamma] & = L_{[\theta,\gamma]} = L_{\theta\circ\gamma-\gamma\circ\theta} \end{cases}$$

Proof. Consider the first identity. By definition both sides are degree -1 super-derivations on $\Omega^\bullet_B\ A$ so it suffices to check that they agree on generators. Clearly, both sides give 0 when evaluated on $a \in A$ and for da we have

$$(L_\theta \circ i_\gamma - i_\gamma \circ L_\theta)da = L_\theta\,\gamma(a) - i_\gamma\, d\,\theta(a) = \theta\,\gamma(a) - \gamma\theta(a) = i_{[\theta,\gamma]}(da)$$

A similar argument proves the second identity. □

Specialize to the quiver-case with $A = \mathbb{C}Q$ the path algebra and $B = V = \mathbb{C}^k$ the vertex algebra.

Lemma 3.2. *Let Q be a quiver on k vertices, then a basis for $\Omega^n_V\ \mathbb{C}Q$ is given by the elements*

$$p_0 dp_1 \dots dp_n$$

where p_i is an oriented path in the quiver such that $length\ p_0 \geq 0$ and $length\ p_i \geq 1$ for $1 \leq i \leq n$ and such that the starting point of p_i is the endpoint of p_{i+1} for all $1 \leq i \leq n-1$.

Proof. Clearly $l(p_i) \geq 1$ when $i \geq 1$ or p_i would be a vertex-idempotent whence in V. Let v be the starting point of p_i and w the end point of p_{i+1} and assume that $v \neq w$, then

$$p_i \otimes_V p_{i+1} = p_i v \otimes_V w p_{i+1} = p_i v w \otimes_V p_{i+1} = 0$$

from which the assertion follows. □

Proposition 3.3. *Let Q be a quiver on k vertices, then the relative differential form-complex have the following cohomology*

$$\begin{cases} H^0_V\ \mathbb{C}Q & \simeq \mathbb{C} \times \dots \times \mathbb{C} \ (k \text{ factors}) \\ H^n_V\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

Proof. Define the *Euler derivation* E on $\mathbb{C}Q$ by the rules that

$$E(e_i) = 0\ \forall\ 1 \leq i \leq k \qquad \text{and} \qquad E(a) = a\ \forall a \in Q_a$$

By induction on the length $l(p)$ of an oriented path p in the quiver Q one easily verifies that $E(p) = l(p)p$. By induction one can also proof that $L_E(p_0 dp_1 \dots dp_n) = (l(p_0) + \dots + l(p_n)) p_0 dp_1 \dots dp_n$. This implies that L_E is a bijection on each $\Omega^i_V\ \mathbb{C}Q$, where $i > 1$ and on $\Omega^0_V\ \mathbb{C}Q$, L_E has V as its kernel. By applying the Cartan homotopy formula for L_E, we obtain that the complex is acyclic. □

The complex $\Omega^\bullet_V\ \mathbb{C}Q$ induces the *relative Karoubi complex*

$$\mathtt{dR}^0_V\ \mathbb{C}Q \xrightarrow{d} \mathtt{dR}^1_V\ \mathbb{C}Q \xrightarrow{d} \mathtt{dR}^2_V\ \mathbb{C}Q \xrightarrow{d} \dots$$

with

$$\mathtt{dR}^n_V\ \mathbb{C}Q = \frac{\Omega^n_V\ \mathbb{C}Q}{\sum_{i=0}^n [\ \Omega^i_V\ \mathbb{C}Q, \Omega^{n-i}_V\ \mathbb{C}Q\ ]}$$

In this expression the brackets denote supercommutators with respect to the grading on $\Omega^\bullet_V\ \mathbb{C}Q$. In the commutative case, $\mathtt{dR}^0$ are the functions on the manifold and $\mathtt{dR}^1$ the 1-forms.

Lemma 3.4. *A $\mathbb{C}$-basis for the noncommutative functions*

$$\mathtt{dR}^0_V \, \mathbb{C}Q \simeq \frac{\mathbb{C}Q}{[\, \mathbb{C}Q, \mathbb{C}Q \,]}$$

are the necklace words in the quiver Q.

Proof. Let $\mathbb{W}$ be the $\mathbb{C}$-space spanned by all necklace words w in Q and define a linear map

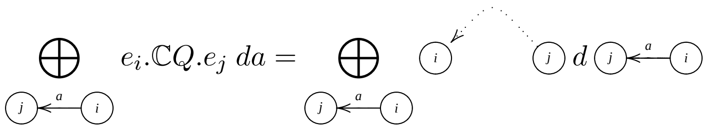
$$\mathbb{C}Q \xrightarrow{n} \mathbb{W} \qquad \begin{cases} p \mapsto w_p & \text{if } p \text{ is a cycle} \\ p \mapsto 0 & \text{if } p \text{ is not} \end{cases}$$

for all oriented paths p in the quiver Q, where w_p is the necklace word in Q determined by the oriented cycle p. Because $w_{p_1p_2} = w_{p_2p_1}$ it follows that the commutator subspace $[\mathbb{C}Q, \mathbb{C}Q]$ belongs to the kernel of this map. Conversely, let

$$x = x_0 + x_1 + \ldots + x_m$$

be in the kernel where x_0 is a linear combination of non-cyclic paths and x_i for $1 \leq i \leq m$ is a linear combination of cyclic paths mapping to the same necklace word w_i, then $n(x_i) = 0$ for all $i \geq 0$. Clearly, $x_0 \in [\mathbb{C}Q, \mathbb{C}Q]$ as we can write every noncyclic path $p = a.p' = a.p' - p'.a$ as a commutator. If $x_i = a_1p_1 + a_2p_2 + \ldots + a_lp_l$ with $n(p_i) = w_i$, then $p_1 = q.q'$ and $p_2 = q'.q$ for some paths q, q' whence $p_1 - p_2$ is a commutator. But then, $x_i = a_1(p_1 - p_2) + (a_2 - a_1)p_2 + \ldots + a_lp_l$ is a sum of a commutator and a linear combination of strictly fewer elements. By induction, this shows that $x_i \in [\mathbb{C}Q, \mathbb{C}Q]$. □

Lemma 3.5. $\mathtt{dR}^1_V \, \mathbb{C}Q$ *is isomorphic as $\mathbb{C}$-space to*



Proof. If $p.q$ is not a cycle, then $pdq = [p, dq]$ and so vanishes in $\mathtt{dR}^1_V \, \mathbb{C}Q$ so we only have to consider terms pdq with $p.q$ an oriented cycle in Q. For any three paths p, q and r in Q we have the equality

$$[p.qdr] = pqdr - qd(rp) + qrdp$$

whence in $\mathtt{dR}^1_V \, \mathbb{C}Q$ we have relations allowing to reduce the length of the differential part

$$qd(rp) = pqdr + qrdp$$

so $\mathtt{dR}^1_V\ \mathbb{C}Q$ is spanned by terms of the form pda with $a \in Q_a$ and $p.a$ an oriented cycle in Q. Therefore, we have a surjection

$$\Omega^1_V\ \mathbb{C}Q \longrightarrow\!\!\!\!\to \bigoplus_{(j) \xleftarrow{a} (i)} e_i.\mathbb{C}Q.e_j\ da$$

By construction, it is clear that $[\Omega^0_V\ \mathbb{C}Q, \Omega^1_V\ \mathbb{C}Q]$ lies in the kernel of this map and using an argument as in the lemma above one shows also the converse inclusion. □

Using the above descriptions of $\mathtt{dR}^i_V\ \mathbb{C}Q$ for $i = 0, 1$ and the differential $\mathtt{dR}^0_V\ \mathbb{C}Q \xrightarrow{d} \mathtt{dR}^1_V\ \mathbb{C}Q$ we can define *partial differential operators* associated to any arrow $(j) \xleftarrow{a} (i)$ in Q.

$$\frac{\partial}{\partial a}\ :\ \mathtt{dR}^0_V\ \mathbb{C}Q \longrightarrow e_i\mathbb{C}Qe_j \qquad \text{by} \qquad df = \sum_{a \in Q_a} \frac{\partial f}{\partial a} da$$

To take the partial derivative of a necklace word w with respect to an arrow a, we run through w and each time we encounter a we open the necklace by removing that occurrence of a and then take the sum of all the paths obtained.

Defining the *relative deRham cohomology* $H^n_{dR}\ \mathbb{C}Q$ to be the cohomology of the Karoubi complex and observing that the operators L_θ and i_θ on $\Omega^\bullet_V\ \mathbb{C}Q$ induce operators on the Karoubi complex, we have the *acyclicity result*

Theorem 3.6. *The relative Karoubi complex has the following cohomology*

$$\begin{cases} H^0_{dR}\ \mathbb{C}Q & \simeq\ V \\ H^n_{dR}\ \mathbb{C}Q & \simeq\ 0 \qquad \forall n \geq 1 \end{cases}$$

Proof. Define $K = \oplus_{m,n}[\Omega^n_V\ \mathbb{C}Q, \Omega^m_V\ \mathbb{C}Q]$ then one verifies for the Euler derivation that

$$L_E(K) \subset K \quad i_E(K) \subset K \quad L_E = i_E \circ d + d \circ i_E$$

The length of a path induces a graded algebra structure on $\Omega_V\ \mathbb{C}Q$ and clearly K and $d^{-1}K$ are spanned by homogeneous elements. The differential of a homogeneous element is either zero or an element of the same length. Writing $x = \sum_i x_i \in d^{-1}K$ in homogeneous components we have $dx = \sum_i dx_i$ is a homogeneous decomposition. Hence, all $dx_i \in K$ whence

$x_i \in d^{-1}K$. Assume that ω is a homogeneous element of length $l > 1$ in $d^{-1}K$, then

$$\begin{aligned}\omega + K &= \frac{1}{l} L_E(\omega) + K \\ &= \frac{1}{l}(i_E(d\omega) + d(i_E(\omega))) + K \\ &= d(i_E(\omega)) + K\end{aligned}$$

From these facts the result follows by mimicking the proof for the cohomology of the relative differential form complex above. □

4 Symplectic interpretation

In this section we use the acyclicity result to give a Poisson interpretation to the Lie bracket in $\mathbb{N}_Q$. This generalizes the *Kontsevich bracket* [18] in the free case to path algebras of doubles of quivers. If Q is a quiver with double quiver $\mathbb{Q}$, then we can define a canonical *symplectic structure* on the path algebra of the double $\mathbb{C}\mathbb{Q}$ determined by the element

$$\omega = \sum_{a \in Q_a} da^* da \in \mathtt{dR}_V^2\ \mathbb{C}\mathbb{Q}$$

As in the commutative case, ω defines a bijection between the noncommutative 1-forms $\mathtt{dR}_V^1\ \mathbb{C}\mathbb{Q}$ and the *noncommutative vectorfields* which are defined to be the V-derivations of $\mathbb{C}\mathbb{Q}$. This correspondence is

$$Der_V\ \mathbb{C}\mathbb{Q} \xrightarrow{\tau} \mathtt{dR}_V^1\ \mathbb{C}\mathbb{Q} \qquad \text{given by} \qquad \tau(\theta) = i_\theta(\omega)$$

In analogy with the commutative case we define a derivation $\theta \in Der_V\ \mathbb{C}\mathbb{Q}$ to be *symplectic* if and only if $L_\theta\omega = 0 \in \mathtt{dR}_V^2\ \mathbb{C}\mathbb{Q}$ and denote the subspace of symplectic derivations by $Der_\omega\ \mathbb{C}\mathbb{Q}$. It follows from the homotopy formula and the fact that ω is a closed form, that $\theta \in Der_\omega\ \mathbb{C}\mathbb{Q}$ implies $L_\theta\omega = di_\theta\omega = d\tau(\theta) = 0$. That is, $\tau(\theta)$ is a closed form which by the acyclicity of the Karoubi complex shows that it must be an exact form. That is we have an isomorphism of exact sequences of $\mathbb{C}$-vectorspaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & \mathtt{dR}_V^0\ \mathbb{C}\mathbb{Q} & \xrightarrow{d} & (\mathtt{dR}_V^1\ \mathbb{C}\mathbb{Q})_{exact} & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{\simeq} & & \Big\downarrow{\tau^{-1}} & & \\ 0 & \longrightarrow & V & \longrightarrow & \dfrac{\mathbb{C}\mathbb{Q}}{[\mathbb{C}\mathbb{Q}, \mathbb{C}\mathbb{Q}]} & \longrightarrow & Der_\omega\ \mathbb{C}\mathbb{Q} & \longrightarrow & 0 \end{array}$$

The symplectic structure ω defines a Poisson bracket on the noncommutative functions.

Definition 4.1. *Let Q be a quiver and $\mathbb{Q}$ its double. The* Kontsevich bracket *on the necklace words in $\mathbb{Q}$, $\mathtt{dR}^0_V\ \mathbb{C}\mathbb{Q}$ is defined to be*

$$\{w_1, w_2\}_K = \sum_{a \in Q_a} \left(\frac{\partial w_1}{\partial a} \frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*} \frac{\partial w_2}{\partial a} \right) \mathtt{mod}\ [\mathbb{C}\mathbb{Q}, \mathbb{C}\mathbb{Q}]$$

By the description of the partial differential operators it is clear that $\mathtt{dR}^0_V\ \mathbb{C}\mathbb{Q}$ with this bracket is isomorphic to the necklace Lie algebra $\mathbb{N}_Q$.

The symplectic derivations $Der_\omega\ \mathbb{C}\mathbb{Q}$ have a natural Lie algebra structure by commutators of derivations. We will show that $\tau^{-1} \circ d$ is a Lie algebra morphism.

For every necklace word w we have a symplectic derivation $\theta_w = \tau^{-1} dw$ defined by

$$\begin{cases} \theta_w(a) & = -\frac{\partial w}{\partial a^*} \\ \theta_w(a^*) & = \frac{\partial w}{\partial a} \end{cases}$$

With this notation we get the following interpretations of the Kontsevich bracket

$$\{w_1, w_2\}_K = i_{\theta_{w_1}}(i_{\theta_{w_2}}\omega) = L_{\theta_{w_1}}(w_2) = -L_{\theta_{w_2}}(w_1)$$

where the next to last equality follows because $i_{\theta_{w_2}}\omega = dw_2$ and the fact that $i_{\theta_{w_1}}(dw) = L_{\theta_{w_1}}(w)$ for any w. More generally, for any V-derivation θ and any necklace word w we have the equation

$$i_\theta(i_{\theta_w}\omega) = L_\theta(w).$$

When we look at the image of the Kontsevich bracket under $\tau^{-1}d$, we obtain the following

$$\begin{aligned} \tau^{-1}d\{w_1, w_2\}_K &= \tau^{-1} d L_{\theta_{w_1}} w_2 \\ &= \tau^{-1} L_{\theta_{w_1}} dw_2 \\ &= \tau^{-1} L_{\theta_{w_1}} i_{\theta_{w_2}} \omega \\ &= \tau^{-1}([L_{\theta_{w_1}}, i_{\theta_{w_2}}] + i_{\theta_{w_2}} L_{\theta_{w_1}})\omega \\ &= \tau^{-1} i_{[\theta_{w_1}, \theta_{w_2}]}\omega \\ &= [\theta_{w_1}, \theta_{w_2}] \end{aligned}$$

Above we made use of the fact that L_θ commutes with d, and the defining equation $dw_2 = i_{\theta_{w_2}}\omega$. In the fourth line we omitted the last term because θ_{w_1} is a symplectic derivation. Finally Lemma 3.1 enabled us to transform the commutator in i and L to of commutator of the derivations θ_{w_1} and θ_{w_2}. This calculation concluded the proof of:

Theorem 4.2. *With notations as before,* $\mathtt{dR}^0_V\ \mathbb{C}\mathbb{Q}$ *with the Kontsevich bracket is isomorphic to the necklace Lie algebra* $\mathbb{N}_Q$*, and the sequence*

$$0 \longrightarrow V \longrightarrow \mathbb{N}_Q \xrightarrow{\tau^{-1}d} Der_\omega\ \mathbb{C}\mathbb{Q} \longrightarrow 0$$

is an exact sequence (hence a central extension) of Lie algebras.

5 Coadjoint orbits

Consider a dimension vector $\alpha = (n_1, \ldots, n_k)$, that is, a k-tuple of natural numbers, then the space of α-dimensional representations of the double quiver $\mathbb{Q}$, $\mathtt{rep}_\alpha\ \mathbb{Q}$ can be identified via the trace pairing with the cotangent bundle $T^*\ \mathtt{rep}_\alpha\ Q$ of the space of α-dimensional representations of the quiver Q, see for example [8], and as such acquires a natural symplectic structure. The natural action of the basechange group $GL(\alpha) = GL_{n_1} \times \ldots \times GL_{n_k}$ on $\mathtt{rep}_\alpha\ \mathbb{Q}$ is symplectic and induces a Poisson structure on the coordinate ring as well as on the ring of polynomial quiver invariants, which are generated by traces along oriented cycles by [21].

The symplectic derivations $Der_\omega\ \mathbb{C}\mathbb{Q}$ correspond to the V-automorphisms of the path algebra of the double $\mathbb{C}\mathbb{Q}$ preserving the *moment element*

$$m = \sum_{a \in Q_a} [a, a^*] \in \mathbb{C}\mathbb{Q}$$

For this reason it is natural to consider the *complex moment map*

$$\mathtt{rep}_\alpha\ \mathbb{Q} \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C}) \qquad V \mapsto \sum_{a \in Q_a} [V_a, V_{a^*}]$$

where $M^0_\alpha(\mathbb{C})$ is the subspace of k-tuples $(m_1, \ldots, m_k) \in M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ such that $\sum_i tr(m_i) = 0$, that is $M^0_\alpha(\mathbb{C}) = Lie\ PGL(\alpha)$ where $PGL(\alpha) = GL(\alpha)/\mathbb{C}^*(\mathbb{1}_{n_1}, \ldots, \mathbb{1}_{n_k})$.

For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that $\sum_i n_i \lambda_i = 0$ we consider the element $\underline{\lambda} = (\lambda_1 \mathbb{1}_{n_1}, \ldots, \lambda_k \mathbb{1}_{n_k})$ in $M^0_\alpha(\mathbb{C})$. The inverse image $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is a $GL(\alpha)$-closed affine subvariety of $\mathtt{rep}_\alpha\ \mathbb{Q}$.

In [13] V. Ginzburg proved the following coadjointness result using the results of the preceding sections.

Theorem 5.1 (Ginzburg). *Assume that* $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ *is smooth and irreducible and that* $PGL(\alpha)$ *acts freely on* $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$*, then the quotient variety (the orbit space)*

$$\mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$$

is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$*.*

Using results of W. Crawley-Boevey [8] we will identify the situations (α, λ) satisfying the conditions of the theorem. For $\lambda \in \mathbb{C}^k$ as above, W. Crawley-Boevey and M. Holland introduced and studied the *deformed preprojective algebra*

$$\Pi_\lambda = \frac{\mathbb{C}\mathbb{Q}}{(m - \lambda)}$$

where $\lambda = \lambda_1 e_1 + \ldots + \lambda_k e_k \in \mathbb{C}\mathbb{Q}$. From [10] we recall that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is the scheme of α-dimensional representations $\mathtt{rep}_\alpha\ \Pi_\lambda$ of the deformed preprojective algebra Π_λ.

We recall the characterization due to V. Kac [14] of the dimension vectors of indecomposable representations of the quiver Q. To a vertex v_i in which Q has no loop, we define a *reflection* $\mathbb{Z}^k \xrightarrow{r_i} \mathbb{Z}^k$ by

$$r_i(\alpha) = \alpha - T_Q(\alpha, \epsilon_i)\epsilon_i$$

where $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$. The *Weyl group of the quiver* Q $Weyl_Q$ is the subgroup of $GL_k(\mathbb{Z})$ generated by all reflections r_i.

A *root* of the quiver Q is a dimension vector $\alpha \in \mathbb{N}^k$ such that $\mathtt{rep}_\alpha\ Q$ contains indecomposable representations. All roots have connected support. A root is said to be

$$\begin{cases} \textit{real} & \text{if } \chi_Q(\alpha, \alpha) = 1 \\ \textit{imaginary} & \text{if } \chi_Q(\alpha, \alpha) \leq 0 \end{cases}$$

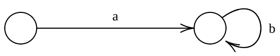
For a fixed quiver Q we will denote the set of all roots, real roots and imaginary roots respectively by Δ, Δ_{re} and Δ_{im}. With Π we denote the set $\{\epsilon_i \mid v_i \text{ has no loops }\}$. The *fundamental set of roots* is defined to be the following set of dimension vectors

$$F_Q = \{\alpha \in \mathbb{N}^k - \underline{0} \mid T_Q(\alpha, \epsilon_i) \leq 0 \text{ and } supp(\alpha) \text{ is connected }\}$$

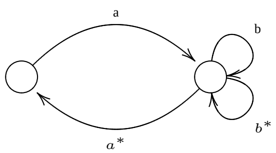
Kac's result asserts that

$$\begin{cases} \Delta_{re} & = Weyl_Q.\Pi \cap \mathbb{N}^k \\ \Delta_{im} & = Weyl_Q.F_Q \cap \mathbb{N}^k \end{cases}$$

Example 5.2. The quiver Q and double quiver $\mathbb{Q}$ appearing in the study of Calogero phase space (see [26] and [12]) which stimulated the above generalizations are



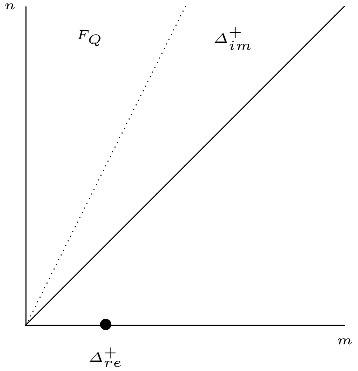
and



The Euler- and Tits form of the quiver Q are determined by the matrices

$$\chi_Q = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad T_Q = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

The root-system for Q is easy to work out. We have



$$\begin{cases} F_Q & = \{(m,n) \mid n \geq 2m\} \\ \Delta^+_{im} & = \{(m,n) \mid n \geq m\} \\ \Pi = \Delta^+_{re} & = \{(1,0)\} \end{cases}$$

Fix $\lambda \in \mathbb{C}^k$ and denote Δ^+_λ to be the set of positive roots $\beta = (b_1, \ldots, b_k)$ for Q such that $\lambda.\beta = \sum_i \lambda_i b_i = 0$. With S_λ (resp. Σ_λ) we denote the subsets of dimension vectors α which are roots for Q such that

$$1 - \chi_Q(\alpha,\alpha) \quad \geq \text{ (resp. } >) \quad r - \chi_Q(\beta_1,\beta_1) - \ldots - \chi_Q(\beta_r,\beta_r)$$

for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with the $\beta_i \in \Delta^+_\lambda$. The main results of [8] can be summarized into:

Theorem 5.3 (W. Crawley-Boevey).

(1) $\alpha \in S_0$ *if and only if* $\mu_{\mathbb{C}}$ *is a flat morphism. In this case,* $\mu_{\mathbb{C}}$ *is also surjective.*
(2) $\alpha \in \Sigma_\lambda$ *if and only if* Π_λ *has a simple* α*-dimensional representation. In this case,* $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ *is a reduced and irreducible complete intersection of dimension* $1 + \alpha.\alpha - 2\chi_Q(\alpha,\alpha)$.

Using the results of [21] one verifies that the set of dimension vectors of simple representations of $\mathbb{Q}$ coincides with the fundamental set F_Q. As any simple Π_λ-representation is a simple $\mathbb{Q}$-representations it follows that $\Sigma_\lambda \hookrightarrow F_Q$.

Example 5.4. For the Calogero-example above, we have

(1) The set S_0 consisting of all (m,n) such that the complex moment map $\mu_{\mathbb{C}}$ is surjective and flat is the set of roots

$$S_0 = \{(m,n) \mid n \geq 2m-1\} \sqcup \{(1,0)\}$$

(2) The set Σ_0 of dimension vectors (m,n) of simple representations of the preprojective algebra Π_0 is the set of roots

$$\Sigma_0 = \{(m,n) \mid n \geq 2m\} \sqcup \{(1,0)\}$$

which is $F_Q \sqcup \{(1,0)\}$.

(3) For $\lambda = (-n,m)$ with $gcd(m,n)=1$, the set Σ_λ of dimension vectors of simple representations of the deformed preprojective algebra is the set of roots

$$\Sigma_\lambda = \{k.(m,n) \mid k \in \mathbb{N}_+\}$$

with unique minimal element (m,n).

For the first two parts the essential calculation is to verify the conditions on the decomposition $(m,n) = (m-1,n) + (1,0)$.

We obtain the following combinatorial description of the couples (α, λ) for which Ginzburg's criterium applies.

Theorem 5.5. *$\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is smooth and irreducible with a free action of $PGL(\alpha)$ (and hence $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$ is a coadjoint orbit for $\mathbb{N}_Q$) if and only if α is a minimal non-zero element of Σ_λ.*

Proof. We know that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) = \mathtt{rep}_\alpha\ \Pi_\lambda$. By a result of M. Artin [1] one knows that the geometric points of the quotient scheme $\mathtt{rep}_\alpha\ \Pi_\lambda/GL(\alpha)$ are the isomorphism classes of α-dimensional semi-simple representations of Π_λ. Moreover, the $PGL(\alpha)$-stabilizer of a point in $\mathtt{rep}_\alpha\ \Pi_\lambda$ is trivial if and only if it determines a simple α-dimensional representation of Π_λ. The result follows from this and the results recalled above. The fact that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is smooth if α is a minimal non-zero element of Σ_λ follows from computing the differential of the complex moment map, see also [8, Lemma 5.5]. □

Example 5.6. Consider the special case when $\lambda = (-n,1)$ and $\alpha = (1,n)$ the unique minimal element in Σ_λ, then it follows from [26] that we have canonical identifications of the quotient varieties

$$\mathtt{iss}_\alpha\ \Pi_\lambda \simeq Calo_n$$

where $Calo_n$ is the phase space of n Calogero particles. In particular, $Calo_n$ is a coadjoint orbit. Wilson [26] has shown that

$$Gr^{ad} = \bigsqcup_n Calo_n$$

where Gr^{ad} is the adelic Grassmannian which can be thought of as the space parametrizing isomorphism classes of right ideals in the first Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ by [7]. In [3] it is shown that there is a non-differentiable action of the automorphism group of $A_1(\mathbb{C})$ on Gr^{ad} having a transitive action on each of the $Calo_n$. It was then conjectured by Y. Berest and G. Wilson that $Calo_n$ might be a coadjoint orbit for a central extension of the automorphism group. (Added may 2001: for more information on these connections as well as to related papers [7], [19] and [15] we refer to the recent preprints of Yu. Berest and G. Wilson [3] and [4].)

Example 5.7. M. Holland and W. Crawley-Boevey have a conjectural extension of the foregoing example. Let Q' be an extended Dynkin quiver on k vertices $\{v_1, \ldots, v_k\}$ with minimal imaginary root $\delta = (d_1, \ldots, d_k)$. A vertex v_i is said to be an extending vertex provided $d_i = 1$. Consider the quiver Q on $k+1$ vertices $\{v_0, v_1, \ldots, v_k\}$ which is Q' on the last k vertices and there is one extra arrow from v_o to an extending vertex v_i. For a generic $\lambda' = (\lambda_1, \ldots, \lambda_k)$ they defined a noncommutative algebra $\mathcal{O}^{\lambda'}$ extending the role of the Weyl algebra in the previous example. They conjecture that there is a bijection between the isomorphism classes of stably free right ideals in $\mathcal{O}^{\lambda}$ and points in

$$\sqcup_n \, \mu_{\mathbb{C}}^{-1}(\lambda_n)/GL(\alpha_n)$$

where $\alpha_n = (1, n\delta)$ and $\lambda_n = (-n\lambda'.\delta, \lambda')$. This remains to be seen but from our theorem we deduce that each of the quotient varieties $\mu_{\mathbb{C}}^{-1}(\lambda_n)/GL(\alpha_n)$ is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$. (Note added may 2001: recently the Crawley-Boevey and Holland conjecture was proved by V. Baranovsky, V. Ginzburg and A. Kuznetsov see [2].)

If $\alpha \in \Sigma_\lambda$ but not minimal, there are several *representation types* $\tau = (m_1, \beta_1; \ldots, m_v, \beta_v)$ of semi-simple α-dimensional representations of Π_λ with the $\beta_i \in \Sigma_\lambda$ and $\sum m_i\beta_i = \alpha$ and the m_i determine the multiplicities of the simple components. With $\mathtt{iss}_\alpha(\tau)$ we denote the subvariety of the quotient variety $\mathtt{iss}_\alpha\, \Pi_\lambda = \mathtt{rep}_\alpha\, \Pi_\lambda/GL(\alpha)$ consisting of all semi-simple representations of type τ.

Consider the algebra $A_Q = \mathbb{C}[\mathbb{N}_Q] \otimes_{\mathbb{C}} \mathbb{C}\mathbb{Q}$ which has a natural *trace map* $tr : A_Q \longrightarrow \mathbb{C}[\mathbb{N}_Q]$ mapping an oriented cycle in $\mathbb{Q}$ to the corresponding necklace word and all open paths to zero. With Aut_Q we denote the automorphism group of trace preserving $\mathbb{C}$-algebra automorphisms of A_Q which preserve the moment element $m = \sum_{a \in Q_a} [a, a^*]$. A natural extension of the above coadjoint orbit result would be a positive solution to the following problem.

Conjecture 5.8. Aut_Q acts transitively on every stratum $\mathtt{iss}_\alpha(\tau)$.

6 Smoothness and deformed preprojective algebras

In this section we will relate the coadjoint orbit result to different notions of smoothness in noncommutative geometry.

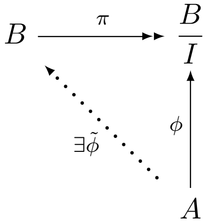
The path algebra $\mathbb{C}\mathbb{Q}$ of the double quiver $\mathbb{Q}$ is formally smooth in the sense of [11], that is, it has the lifting property with respect to nilpotent ideals. Hence, $\mathbb{C}\mathbb{Q}$ is the coordinate ring of a noncommutative affine manifold and has a good theory of differential forms (acyclicity).

On the other hand, we will see that the deformed preprojective algebras Π_λ are *never* formally smooth. For this reason, the differential forms of $\mathbb{C}\mathbb{Q}$ when restricted to Π_λ may have rather unpredictable behavior.

Still, it may be possible that certain representation spaces $\mathtt{rep}_\alpha\ \Pi_\lambda$ are smooth and we need a notion of noncommutative (formal) smoothness relative to the dimension vector α. Recall that if α is a minimal dimension vector in Σ_λ, then $\mathtt{rep}_\alpha\ \Pi_\lambda = \mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is smooth. We will now investigate whether there are other examples of smooth fibers $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ using the relative notion of smoothness introduced by C. Procesi in [24] and studied in detail in [20]. First, we will recall its ringtheoretical characterization.

Let $\alpha = (n_1, \ldots, n_k)$ and set $n = \sum_i n_i$. With $\mathtt{alg@}\alpha$ we denote the category of all V-algebras A which are equipped with a trace map, that is a linear map $tr : A \longrightarrow A$ such that for all $a, b \in A$ we have $tr(a)b = btr(a)$, $tr(ab) = tr(ba)$ and $tr(tr(a)b) = tr(a)tr(b)$ satisfying the following properties. First, we must have that $tr(1) = n$, the trace map must satisfy the formal Cayley-Hamilton identity of degree n, see [24] and finally the trace values of the vertex-idempotents are given by $tr(e_i) = n_i$, the components of the dimension vector α.

Morphisms in the category $\mathtt{alg@}\alpha$ are trace preserving V-algebra morphisms. An algebra A in in $\mathtt{alg@}\alpha$ is said to be α-*smooth* if it satisfies the lifting property with respect to nilpotent ideals in $\mathtt{alg@}\alpha$. That is, every diagram



with B, $\frac{B}{I}$ in $\mathtt{alg@}\alpha$, I a nilpotent ideal and π and ϕ trace preserving maps, can be completed with a trace preserving algebra map $\tilde{\phi}$.

Observe that if $n = 1$ and $\alpha = (1)$ we have that $\mathtt{alg@}\alpha = \mathtt{commalg}$ the category of commutative $\mathbb{C}$-algebras and by Grothendieck's characterization of regular algebras one has in this case that an algebra is α-smooth if and only if it is regular.

In general, a geometric characterization of this lifting property is that an algebra A is α-smooth if and only if the scheme of α-dimensional trace preserving representations of A is a smooth $GL(\alpha)$-variety, see [24] or [20].

There is a partial functor $\mathtt{alg} \longrightarrow \mathtt{alg@}\alpha$ which assigns to an affine V-algebra B the algebra of $GL(\alpha)$-equivariant maps

$$\mathtt{rep}_\alpha\, A \longrightarrow M_n(\mathbb{C})$$

(where $GL(\alpha)$ acts on $M_n(\mathbb{C})$ by conjugation via the obvious embedding along the diagonal $GL(\alpha) \hookrightarrow GL_n$) which is an object in $\mathtt{alg@}\alpha$. We will denote this algebra of equivariant maps by $B@\alpha$. Clearly, the scheme $\mathtt{rep}_\alpha\, B$ of α-dimensional representations of B coincides with the scheme of α-dimensional trace preserving representations of $B@\alpha$.

For this reason, the fiber $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is a smooth affine variety if and only if the algebra $\Pi_\lambda@\alpha$ is α-smooth. As we have seen before $\Pi_\lambda@\alpha$ is α-smooth if (λ, α) is such that $\lambda.\alpha = 0$ and α is a minimal non-zero vector in Σ_λ. In this case, $\Pi_\lambda@\alpha$ is even an Azumaya algebra over the coadjoint orbit, that is, the quotient map

$$\mathtt{rep}_\alpha\, \Pi_\lambda = \mu_{\mathbb{C}}^{-1}(\underline{\lambda}) \longrightarrow\!\!\!\!\!\rightarrow \mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$$

is a principal $PGL(\alpha)$-fibration in the étale topology. For more details on Azumaya algebras and their relation to étale cohomology we refer to the book by J.S. Milne [23].

Noncommutative geometry, as propagated by M. Kontsevich in [18] is based on the fact that noncommutative functions and noncommutative (relative) differential forms associated to a formally smooth $\mathbb{C}$-algebra A (resp. a formally smooth V-algebra A) induce ordinary functions and differential forms on the smooth representations schemes $\mathtt{rep}_n\, A$ (resp. $\mathtt{rep}_\alpha\, A$) of n-dimensional (resp. α-dimensional) representations and their corresponding quotient varieties $\mathtt{iss}_n\, A$ resp. $\mathtt{iss}_\alpha\, A$. For this reason one expects that the closed subscheme $\mathtt{iss}_\alpha\, \Pi_\lambda$ behaves well with respect to noncommutative symplectic forms (in particular, is a coadjoint orbit for the necklace algebra $\mathbb{N}_Q$) if and only if $\Pi_\lambda@\alpha$ is α-smooth.

On the other hand, if the coadjoint orbit result follows from the conjectural transitive action of the group Aut_Q as stated in Conjecture 5.8, this can only happen if there is just one stratum. That is, if and only if $\Pi_\lambda@\alpha$ is an Azumaya algebra, or equivalently, that α is a minimal element of Σ_λ.

These conjectural equivalences of (1) $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})/GL(\alpha)$ coadjoint orbit, (2) $\Pi_\lambda@\alpha$ an α-smooth algebra and (3) α a minimal element of Σ_λ follow from a stronger conjecture on deformed preprojective algebras formulated below.

Consider the algebraic quotient map

$$\mathtt{rep}_\alpha\, \Pi_\lambda \xrightarrow{\pi_\alpha} \mathtt{iss}_\alpha\, \Pi_\lambda = \mathtt{rep}_\alpha\, \Pi_\lambda/GL(\alpha)$$

By Artin's result [1], a $\mathbb{C}$-point ξ of $\mathtt{iss}_\alpha\, \Pi_\lambda$ corresponds to an isomorphism class of an α-dimensional semisimple representation

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z}$$

of Π_λ. Here, S_i is an α_i-dimensional simple representation of Π_λ occurring with multiplicity e_i in M_ξ. In particular we have that

$$\text{for all } i : \alpha_i \in \Sigma_\lambda \quad \text{and} \quad \sum_i e_i\alpha_i = \alpha$$

Fix a point M_ξ of the closed $GL(\alpha)$-orbit $\mathcal{O}(M_\xi)$ in $\mathtt{rep}_\alpha\, \Pi_\lambda$. We will say that $\xi \in \mathtt{iss}_\alpha\, \Pi_\lambda$ belongs to the *noncommutative smooth locus* $Sm_\alpha\, \Pi_\lambda$ of Π_λ (or of $\Pi_\lambda@\alpha$) if $\mathtt{rep}_\alpha\, \Pi_\lambda$ is smooth in M_ξ. Because the singular locus is a closed subvariety of $\mathtt{rep}_\alpha\, \Pi_\lambda$ is a closed subvariety we have that $\Pi_\lambda@\alpha$ is α-smooth iff $Sm_\alpha\, \Pi_\lambda = \mathtt{iss}_\alpha\, \Pi_\lambda$.

Now we restrict to $\alpha \in \Sigma_\lambda$ and consider the Zariski open subscheme $Az\, \Pi_\lambda@\alpha$ of points $\xi \in \mathtt{iss}_\alpha\, \Pi_\lambda$ such that M_ξ is a simple representation of Π_λ, then the restriction of the quotient map π_α to $\pi_\alpha^{-1}(Az\, \Pi_\lambda@\alpha)$ is a principal $PGL(\alpha)$-fibration in the étale topology. We call $Az\, \Pi_\lambda@\alpha$ the *Azumaya locus* of $\Pi_\lambda@\alpha$. The above conjectural equivalences follow from an affirmative answer to the following conjecture.

Conjecture 6.1. For $\alpha \in \Sigma_\lambda$ we have

$$Sm_\alpha\, \Pi_\lambda = Az\, \Pi_\lambda@\alpha$$

We will give an affirmative solution to this conjecture in the special case of the preprojective algebra Π_0. By a result of W. Crawley-Boevey [9], we can control the Ext^1-spaces of representations of Π_0. Let V and W be representations of Π_0 of dimension vectors α and β, then we have

$$\begin{aligned} dim_{\mathbb{C}}\, Ext^1_{\Pi_0}(V,W) = dim_{\mathbb{C}}\, Hom_{\Pi_0}(V,W) + dim_{\mathbb{C}}\, Hom_{\Pi_0}(W,V) \\ -T_Q(\alpha,\beta) \end{aligned}$$

For $\xi \in \mathtt{iss}_\alpha\, \Pi_0$ to belong to the smooth locus $\xi \in Sm_\alpha\, \Pi_0$ it is necessary and sufficient that $\mathtt{rep}_\alpha\, \Pi_0$ is smooth along the orbit $\mathcal{O}(M_\xi)$ where M_ξ is the semi-simple α-dimensional representation of Π_0 corresponding to ξ.

Assume that ξ is of type $\tau = (e_1,\alpha_1;\ldots;e_z,\alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z}$$

with S_i a simple Π_0-representation of dimension vector α_i. Then, the normal space to the orbit $\mathcal{O}(M_\xi)$ is determined by $Ext^1_{\Pi_o}(M_\xi, M_\xi)$ and can be depicted by a local quiver setting (Q_ξ, α_ξ) where Q_ξ is a quiver on z vertices having as many arrows from vertex i to vertex j as the dimension of $Ext^1_{\Pi_0}(S_i,S_j)$ and where $\alpha_\xi = \alpha_\tau = (e_1,\ldots,e_z)$.

As $\mathtt{rep}_\alpha\ \Pi_0$ is assumed to be smooth in M_ξ we can apply the strong form of the Luna slice theorem, see [22] or [25] which asserts that the action morphism and corresponding quotient maps

$$\begin{array}{ccc} GL(\alpha) \times^{GL(\alpha_\xi)} N_\xi & \longrightarrow & \mathtt{rep}_\alpha\ \Pi_0 \\ \downarrow & & \downarrow \\ N_\xi/GL(\alpha_\xi) & \longrightarrow & \mathtt{iss}_\alpha\ \Pi_0 \end{array}$$

where N_ξ is the normal space to the orbit in M_ξ, are étale in M_ξ (resp. in ξ) and that the upper map is $GL(\alpha)$-equivariant. With the above quiver-theoretic interpretation of the normal space N_ξ we deduce

Lemma 6.2. *With notations as above, $\xi \in Sm_\alpha\ \Pi_0$ if and only if*

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = dim_{M_\xi}\ \mathtt{rep}_\alpha\ \Pi_0$$

As we have enough information to compute both sides, we can prove:

Theorem 6.3. *If $\xi \in \mathtt{iss}_\alpha\ \Pi_0$ with $\alpha = (a_1, \ldots, a_k) \in S_{\underline{0}}$, then $\xi \in Sm_\alpha\ \Pi_0$ if and only if M_ξ is a simple n-dimensional representation of Π_0. That is, the smooth locus of Π_0 coincides with the Azumaya locus.*

Proof. Assume that ξ is a point of semi-simple representation type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z} \qquad \text{with} \qquad dim(S_i) = \alpha_i$$

and S_i a simple Π_0-representation. We have

$$\begin{cases} dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_j) & = -T_Q(\alpha_i, \alpha_j) \qquad i \neq j \\ dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_i) & = 2 - T_Q(\alpha_i, \alpha_i) \end{cases}$$

But then, the dimension of $Ext^1_{\Pi_0}(M_\xi, M_\xi)$ is equal to

$$\sum_{i=1}^{z}(2 - T_Q(\alpha_i, \alpha_i))e_i^2 + \sum_{i \neq j} e_i e_j(-T_Q(\alpha_i, \alpha_j)) = 2\sum_{i=1}^{z} e_i^2 - T_Q(\alpha, \alpha)$$

from which it follows immediately that

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = \alpha.\alpha + \sum_{i=1}^{z} e_i^2 - T_Q(\alpha, \alpha)$$

On the other hand, as $\alpha \in S_{\underline{0}}$ we know that

$$\begin{aligned} dim\, \mathtt{rep}_\alpha\, \Pi_0 &= \alpha.\alpha - 1 + 2p_Q(\alpha) \\ &= \alpha.\alpha - 1 + 2 - 2\chi_Q(\alpha,\alpha) = \alpha.\alpha + 1 - T_Q(\alpha,\alpha) \end{aligned}$$

But then, equality occurs if and only if $\sum_i e_i^2 = 1$, that is, $\tau = (1,\alpha)$ or M_ξ is a simple n-dimensional representation of Π_0. □

In particular it follows that the preprojective algebra Π_0 is *never* formally smooth as this implies that all the representation varieties must be smooth. Further, as $\vec{v_i} = (0,\ldots,1,0,\ldots,0)$ are dimension vectors of simple representations of Π_0 it follows that Π_0 is α-smooth if and only if $\alpha = \vec{v_i}$ for some i.

Example 6.4. Let Q be an extended Dynkin diagram and δ the minimal imaginary root, then $\delta \in S_0$. The dimension of the quotient variety

$$\begin{aligned} dim\, \mathtt{iss}_\delta\, \Pi_0 &= dim\, \mathtt{rep}_\delta\, \Pi_0 - \delta.\delta + 1 \\ &= 2 \end{aligned}$$

so it is a surface. The only other semi-simple δ-dimensional representation of Π_0 is the trivial representation. By the theorem, this must be an isolated singular point of $\mathtt{iss}_\delta\, Q$. In fact, one can show that $\mathtt{iss}_\delta\, \Pi_0$ is the Kleinian singularity corresponding to the extended Dynkin diagram Q.

The proof of Theorem 6.3 can be repeated verbatim for the deformed preprojective algebras Π_λ provided we would have an analogue of Crawley-Boevey's formula for the dimension of the extension groups $Ext^1_{\Pi_\lambda}(M,N)$. Unfortunately, no such formula is known at present. Observe that an affirmative answer to Conjecture 6.1 follows from

Conjecture 6.5. Let S and T be (isomorphism classes of) simple Π_λ representations of dimension vector α resp. β, then

$$dim_{\mathbb{C}}\, Ext^1_{\Pi_\lambda}(S,T) = 2\delta_{ST} - T_Q(\alpha,\beta)$$

In particular, the extension form on semisimple Π_λ-representations is symmetric.

Before we can prove some partial results for deformed preprojective algebras we need to recall that $\mathtt{rep}_\alpha\, \mathbb{Q}$ admits a hyper-Kähler structure (that is, an action of the quaternion algebra $\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k$) defined for all arrows $a \in Q_a$ and all arrows $b \in \mathbb{Q}_a$ by the formulae, see for example [9]

$$\begin{aligned} &(i.V)_b = iV_b \\ &(j.V)_a = -V_{a^*}^\dagger \quad (j.V)_{a^*} = V_a^\dagger \\ &(k.V)_a = -iV_{a^*}^\dagger \quad (k.V)_{a^*} = iV_a^\dagger \end{aligned}$$

where this time we denote the Hermitian adjoint of a matrix M by $M^\dagger$ to distinguish it from the star-operation on the arrows of the double quiver $\mathbb{Q}$. Let $U(\alpha)$ be the product of unitary groups $U_{n_1} \times \ldots \times U_{n_k}$ and consider the *real moment map*

$$\mathtt{rep}_\alpha\, \mathbb{Q} \xrightarrow{\mu_{\mathbb{R}}} Lie\, U(\alpha) \qquad V \mapsto \sum_{\substack{\bigcirc \xleftarrow{b} \bigcirc \\ b \in \mathbb{Q}_a}} \frac{i}{2}[V_b, V_b^\dagger]$$

For $\lambda \in \mathbb{R}^k$, multiplication by the quaternion-element $h = \frac{i+k}{\sqrt{2}}$ gives a homeomorphism between the real varieties

$$\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}) \xrightarrow{h.} \mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\underline{\lambda})$$

Moreover, the hyper-Kähler structure commutes with the base-change action of $U(\alpha)$, whence we have a natural one-to-one correspondence between the quotient spaces

$$(\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}))/U(\alpha) \xrightarrow{h.} (\mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\underline{\lambda}))/U(\alpha)$$

see [9] for more details. By results of Kempf and Ness [16] we can identify the left hand side as the quotient variety $\mathtt{iss}_\alpha\, \Pi_\lambda$ and by results of A. King [17] we can identify the right hand side as the moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$ of λ-semistable α-dimensional representations of the preprojective algebra Π_0, at least if λ has rational components.

Recall that a representation $V \in \mathtt{rep}_\alpha\, \mathbb{Q}$ is said to be λ-(semi)stable if and only if for every proper subrepresentation W of V say with dimension vector β we have $\lambda.\beta > 0$ (resp. $\lambda.\beta \geq 0$). The scheme $\mathtt{rep}_\alpha^{ss}(\Pi_0, \lambda)$ of λ-semistable α-dimensional representations of Π_0 is the intersection of $\mu_{\mathbb{C}}^{-1}(\underline{0})$ with the subvariety of λ-semistable representations in $\mathtt{rep}_\alpha\, \mathbb{Q}$. The corresponding moduli space $M_\alpha^{ss}(\Pi_0, \lambda)$ classifies isomorphism classes of direct sums of λ-stable representations of Π_0 of total dimension α.

If $V \in \mathtt{rep}_\alpha\, \Pi_\lambda$ belongs to $\mu_{\mathbb{R}}^{-1}(\underline{0})$ we have that V is a semisimple Π_λ-representation

$$V = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

with the S_i a simple Π_λ-representation of dimension vector β_i. If $W \in \mathtt{rep}_\alpha\, \Pi_0$ belongs to $\mu_{\mathbb{R}}^{-1}(\underline{\lambda})$, then W is the direct sum of λ-stable representations of Π_0

$$W = T_1^{\oplus f_1} \oplus \ldots \oplus T_s^{\oplus f_s}$$

with T_i a λ-stable Π_0-representation of dimension vector γ_i. Because the hyper-Kähler correspondence preserves blockdecomposition of matrices we deduce from $W = h.V$ that $r = s$, $e_i = f_i$, $\beta_i = \gamma_i$ and $T_i \simeq h.S_i$.

Proposition 6.6. *The deformed preprojective algebra Π_λ has semisimple representations of representation type $\tau = (e_1, \beta_1; \ldots ; e_r, \beta_r)$ if and only if the preprojective algebra Π_0 has λ-stable representations of dimension vector β_i for all $1 \leq i \leq r$.*

In particular, Π_λ has a simple representation of dimension vector α if and only if Π_0 has a λ-stable representation of dimension vector α.

With Φ_λ we denote the set of dimension vectors $\alpha \in \Sigma_\lambda$ such that $\Pi_\lambda @\alpha$ is α-smooth (that is, $\mathtt{rep}_\alpha\ \Pi_\lambda$ is smooth) and moreover the quotient variety $\mathtt{iss}_\alpha\ \Pi_\lambda$ is smooth. Our conjecture is that Φ_λ is the set of minimal elements of Σ_λ. The following result provides some partial support for this.

Proposition 6.7. *(1) If $\alpha \in \Sigma_\lambda$ such that $2\alpha \in \Sigma_\lambda$, then $2\alpha \notin \Phi_\lambda$.*
(2) Let $\alpha, \beta, \alpha + \beta \in \Sigma_\lambda$ such that $T_Q(\alpha, \beta) < -2$, then $\alpha + \beta \notin \Phi_\lambda$.

Proof. (1): As $\alpha \in \Sigma_\lambda$ we know that the local quiver Q_ξ in a simple representation S corresponding to ξ is a one vertex quiver having $2 - T_Q(\alpha, \alpha)$ loops (because $\mathtt{rep}_\alpha\ \Pi_\lambda$ is smooth in S by [8, Lemma 5.5]). That is,

$$dim\ Ext^1_{\Pi_\lambda}(S, S) = 2 - T_Q(\alpha, \alpha)$$

But then, for $\xi \in \mathtt{iss}_{2\alpha}\ \Pi_\lambda$ a point corresponding to $S \oplus S$, the local quiver is still Q_ξ but this time the local dimension vector $\alpha_\xi = 2$. If ξ lies in the smooth locus, then by the Luna slice theorem we must have

$$dim\ GL(2\alpha) \times^{GL_2} \mathtt{rep}_{\alpha_\xi}\ Q_\xi = dim\ \mathtt{rep}_{2\alpha}\ \Pi_\lambda$$

The left hand side is $4\alpha.\alpha + 4 - 4T_Q(\alpha, \alpha)$ whereas the right hand side is equal to (because $2\alpha \in \Sigma_\lambda$) $4\alpha.\alpha + 1 - 4T_Q(\alpha, \alpha)$, a contradiction.

(2): Let V resp. W be a λ-stable representation of Π_0 of dimension vector α resp. β. The normal space to the orbit of $V \oplus W$ in $\mathtt{rep}^{ss}_{\alpha+\beta}\ \Pi_0$ is the representation space of dimension vector $(1, 1)$ for the quiver Γ on two vertices having $2 - T_Q(\alpha, \alpha)$ loops in the first, $2 - T_Q(\beta, \beta)$ loops in the second and $-T_Q(\alpha, \beta)$ arrows in both directions between the vertices. By Knop's generalization of the Luna slice result, see [25], and a computation of dimensions we see that the image of the slice map in the principal fibration

$$GL(\alpha + \beta) \times^{\mathbb{C}^* \times \mathbb{C}^*} \mathtt{rep}_{(1,1)}\ \Gamma$$

is of codimension one. Because $-T_Q(\alpha, \beta) \geq 3$ every codimension one subvariety of the quotient contains a singularity in the trivial representation. Therefore, the moduli space $M^{ss}_{\alpha+\beta}(\Pi_0, \lambda)$ is singular in the point corresponding to $V \oplus W$. But then, by the hyper-Kähler correspondence, the quotient variety $\mathtt{iss}_{\alpha+\beta}\ \Pi_\lambda$ is singular in a point of representation type $(1, \alpha; 1, \beta)$, whence $\alpha + \beta \notin \Phi_\lambda$. □

Observe that W. Crawley-Boevey has proved that $T_Q(\alpha,\beta) \leq -2$ for $\alpha,\beta,\alpha+\beta \in \Sigma_\lambda$ see [8, Thm 4.6]. (Added may 2001: the first author has recently given a complete classification of quiver settings with a smooth quotient variety, see [5] and [6]. We believe that a combination of this result and the method of proof of the previous proposition will provide a characterization of Φ_λ. We hope to come back to this problem in a future publication.)

We end this paper by proving that α-smoothness of a closely related sheaf of algebras is equivalent to α being a minimal element of Σ_λ.

Taking locally the algebras of $GL(\alpha)$-equivariant maps from $\mathtt{rep}^{ss}_\alpha(\Pi_0, \lambda)$ to $M_n(\mathbb{C})$ defines a sheaf of algebras in $\mathtt{alg@}\alpha$, $\mathcal{A}_{\lambda,\alpha}$ on the moduli space $M^{ss}_\alpha(\Pi_0,\lambda)$.

Theorem 6.8. *With notations as above, for $\alpha \in \Sigma_\lambda$ the following are equivalent:*

(1) $\mathcal{A}_{\lambda,\alpha}$ is a sheaf of α- smooth algebras on the moduli space $M^{ss}_\alpha(\Pi_0,\lambda)$.
(2) α is a minimal non-zero vector in Σ_λ (and hence the quotient variety $\mathtt{iss}_\alpha\ \Pi_\lambda$ is a coadjoint orbit for the necklace Lie algebra $\mathbb{N}_Q$).

Proof. As $\alpha \in \Sigma_\lambda$ we know that $\mathtt{iss}_\alpha\ \Pi_\lambda$ has dimension $1+\alpha.\alpha - 2\chi_Q(\alpha,\alpha) - dim\ PGL(\alpha)$ which is equal to $2 - T_Q(\alpha,\alpha)$. By the hyper-Kähler correspondence so is the dimension of $M^{ss}_\alpha(\Pi_0,\lambda)$, whence the open subset of $\mu^{-1}_{\mathbb{C}}(\underline{0})$ consisting of λ-semistable representations has dimension

$$1+\alpha.\alpha - 2\chi_Q(\alpha,\alpha)$$

as there are λ-stable representations in it (again via the hyper-Kähler correspondence). Take a $GL(\alpha)$-closed orbit $\mathcal{O}(V)$ in this open set. That is, V is the direct sum of λ-stable subrepresentations

$$V = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

with S_i a λ-stable representation of Π_0 of dimension vector β_i occurring in V with multiplicity e_i whence $\alpha = \sum_i e_i\beta_i$.

Again, the normal space in V to $\mathcal{O}(V)$ can be identified with $Ext^1_{\Pi_0}(V, V)$. As all S_i are Π_0-representations we can determine this space by the knowledge of all $Ext^1_{\Pi_0}(S_i,S_j)$.

$$Ext^1_{\Pi_0}(S_i,S_j) = 2\delta_{ij} - T_Q(\beta_i,\beta_j)$$

But then the dimension of the normal space to the orbit is

$$dim\ Ext^1_{\Pi_0}(V,V) = 2\sum_{i=1}^r e_i - T_Q(\alpha,\alpha)$$

By the Luna slice theorem [22], the étale local structure in the smooth point V is of the form $GL(\alpha) \times^{GL(\tau)} Ext^1(V,V)$ where $\tau = (e_1, \ldots, e_r)$ and is therefore of dimension

$$\alpha.\alpha + \sum_{i=1}^{2} e_i^2 - T_Q(\alpha,\alpha)$$

This number must be equal to the dimension of the subvariety of λ-semistable representations of Π_0 which has dimension $1+\alpha.\alpha - T_Q(\alpha,\alpha)$ if and only if $r = 1$ and $e_1 = 1$, that is if and only if V is λ-stable. Hence, if $\mathtt{rep}^{ss}_\alpha(\Pi_0,\lambda)$ is smooth, then α must be a minimal non-zero vector in the set of dimension vectors of λ-stable representations of Π_0 and hence by the hyper-Kähler correspondence, α is a minimal non-zero vector in Σ_λ.

Conversely, if α is a minimal vector in Σ_λ, then $\mathtt{iss}_\alpha\ \Pi_\lambda$ is a coadjoint orbit, whence smooth and hence so is $M^{ss}_\alpha(\Pi_0,\lambda)$ by the correspondence. Moreover, all α-dimensional λ-semistable representations must be λ-stable by the minimality assumption and so $\mathtt{rep}^{ss}_\alpha(\Pi_0,\lambda)$ is a principal $PGL(\alpha)$-fibration over $M^{ss}_\alpha(\Pi_0,\lambda)$ whence smooth. Therefore, $\mathcal{A}_{\lambda,\alpha}$ is a sheaf of α-Cayley smooth algebras. □

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