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Geometry of real polynomial mappings

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Abstract. In this paper we study the set of points at which a real polynomial mapping is not proper.

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1 Introduction

Let $f : X \to Y$ be a continuous map of locally compact spaces. We say that the mapping f is not proper at a point $y \in Y$, if there is no a neighborhood U of a point y such that the set $f^{-1}(cl(U))$ is compact.

The set S_f of points at which the map f is not proper indicates how the map f differs from a proper map. In particular f is proper if and only if this set is empty. Moreover, if $f(X)$ is open, then S_f contains the border of the set $f(X)$. The set S_f is the minimal set S with a property that the mapping $f : X \setminus f^{-1}(S) \to Y \setminus S$ is proper.

Further, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a generically finite polynomial mapping, then the mapping f has a constant topological degree over every connected component of the set $\mathbb{R}^n \setminus S_f$.

In our previous paper [8] we described the set S_f in the case of complex polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ and in the paper [9] we described this set in the case of real polynomial mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$.

The aim of this paper is to do the same in the general case of real polynomial non-constant mappings $f : \mathbb{R}^n \to \mathbb{R}^m$. Our main result is the following

Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial non-constant mapping. Then the set S_f is closed, semi-algebraic and for every non-empty connected*

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component $S \subset S_f$ we have $1 \leq \dim S \leq n-1$. Moreover, the set S_f is $\mathbb{R}$-uniruled. It means, that for every point $a \in S_f$ there is a non-constant polynomial mapping $\phi : \mathbb{R} \to S_f$ such that $\phi(0) = a$.

Corollary. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial non-constant mapping, then every non-empty connected component of S_f is unbounded.*

If the mapping f is generically finite, then we can say more:

Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial generically-finite mapping. Let $\deg f_i = d_i$ and assume that $d_1 \geq d_2 ... \geq d_m$. Then there is a real polynomial $P \in \mathbb{R}[x_1, ..., x_m]$ of degree at most $D = d_1 d_2 ... d_{n-1} - 1$, such that $S_f \subset P^{-1}(0)$ and $f(\mathbb{R}^n) \not\subset P^{-1}(0)$.*

In this paper we use methods from our recent papers [7] and [8]. The author wishes to express his gratitude to the referee for his useful remarks, which allow us to state our theorems in stronger versions and improve their proofs.

2 Terminology

Let $X \subset \mathbb{R}^n (\mathbb{P}^n(\mathbb{R}))$ be a semi-algebraic set. The Zariski closure of X will be denoted by $cl_Z(X)$. By $\mathbb{C}X$ we denote a complexification of X, i.e., $\mathbb{C}X$ is a Zariski closure of the set X in $\mathbb{C}^n$ $(\mathbb{P}^n(\mathbb{C}))$.

More generally, if $X \subset \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ is an algebraic set, then by its complexification $\mathbb{C}X$ we mean a Zariski closure of the set X in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ etc.

Let $X \subset \mathbb{R}^n$ be an algebraic set. A mapping $f : X \to \mathbb{R}^m$ is called polynomial, if $f = (f_1, ..., f_m)$, where each f_i is a restriction to X of a polynomial from $\mathbb{R}[x_1, ..., x_n]$.

More generally, if $X \subset \mathbb{P}^n(\mathbb{R})$, $Y \subset \mathbb{P}^m(\mathbb{R})$ are semialgebraic sets and $f : X \to Y$ is a regular mapping then f is called polynomial if it has an extension to a regular mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}Y$. Let us recall that a mapping $f : X \to Y$ of semi-algebraic subsets of a projective space is regular, if it can be written locally as a composition of quotients of homogeneous polynomials.

For an algebraic set $X \subset \mathbb{R}^n$ we consider its coordinate ring X: $\mathbb{R}[X] := \mathbb{R}[x_1, ..., x_n]/I(X)$, where $I(X) = \{f \in \mathbb{R}[x_1, ..., x_n] : f(X) = \{0\}\}$.

A polynomial mapping $f : X \to \mathbb{R}^m$ is called generically-finite if for a generic $x \in X$ the fiber $f^{-1}(f(x))$ is finite. It is easy to see that f is generically-finite if and only if the mapping $f : \mathbb{C}X \to \mathbb{C}^m$ is generically-finite. A polynomial mapping $f : X \to \mathbb{R}^m$ is called quasi-finite if all its fibers are finite.

If $X \subset \mathbb{R}^m$ is the empty set, then we put codim $X = \infty$.

3 Preliminaries

In the beginning we recall some basic facts about polynomial mappings of $\mathbb{C}^2$. We start with the theorem about surfaces which contain $\mathbb{C}^2$ as an open subset ([8]):

Theorem 3.1 *Let X be a complete normal surface. Assume, that X contains a plane $\mathbb{C}^2$ as an open, dense subset. Let $W_1, W_2 \subset X \setminus \mathbb{C}^2$ be two connected, complete curves without common components. Then the intersection $W_1 \cap W_2$ is either the empty set or it is a point.*

Definition 3.2 Let Γ be an affine curve such that there is a surjective polynomial mapping $\phi : \mathbb{C} \to \Gamma$. Then Γ is called to be *an affine parametric line* and the mapping ϕ is called to be *a parametrization of Γ.*

In the sequel we need also the following (see [8]):

Proposition 3.3 *Let X be an irreducible affine variety. The following conditions are equivalent:*

1. *for every point $x \in X$ there is a parametric affine line in X going through x;*
2. *there exists a Zariski-open, non-empty subset U of X, such that for every point $x \in U$ there is a parametric affine line in X going through x;*
3. *there exists a subset U of X of the second Baire's category, such that for every point $x \in U$ there is a parametric affine line in X going through x.*

Definition 3.4 An affine irreducible variety X is called $\mathbb{C} - uniruled$ if it satisfies one of equivalent conditions $1) - 3)$ listed in Proposition 3.3. More generally, an affine variety X is called $\mathbb{C} - uniruled$ if all its irreducible components are $\mathbb{C} - uniruled$.

The next result was proved in our papers [7], [8]:

Theorem 3.5 *Let $f = (f_1, ..., f_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a dominant polynomial map. Then the set S_f of points at which f is not proper is either empty, or it is a $\mathbb{C}$-uniruled hypersurface. Moreover, its degree is not greater than*

$$\frac{\prod_{i=1}^{n} \deg f_i - \mu(f)}{\min_{i=1,...,n} \deg\, f_i},$$

where $\mu(f)$ denotes the maximal number of points in (finite) fibers of f.

4 Mappings of the real plane

In this section we study non-constant polynomial mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^m$. We start with the following:

Definition 4.1 Let $\Gamma \subset \mathbb{R}^n$ be a semi-algebraic curve, such that there exists a polynomial surjective mapping $\phi : \mathbb{R} \to \Gamma$. Then Γ will be called *a parametric semi-line* and the mapping ϕ will be called *a parametrization of* Γ.

We come to our first result:

Theorem 4.2 *Let $f = (f_1, ..., f_m) : \mathbb{R}^2 \longrightarrow \mathbb{R}^m$ be a non-constant polynomial mapping. Then the set S_f of points at which f is not proper is a union of a finite family (possibly empty) of parametric semi-lines.*

Proof. First assume that the mapping f is generically-finite. Extend the mapping f to a rational mapping $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^m(\mathbb{R})$, also denoted by f. Now resolve points of indeterminacy of f by a sequence of blowup's. In this way we obtain a smooth surface Z and a regular mapping $F : Z \to \mathbb{P}^m(\mathbb{R})$, such that $res_{\mathbb{R}^2} F = f$.

We can also consider the surface $\mathbb{C}Z$ and the mapping $\mathbb{C}F$, which are canonical complexifications of Z and F (i.e. to obtain $\mathbb{C}Z, \mathbb{C}F$ we consider complex extension of real blowup's). Hence the mapping $\mathbb{C}F$ is regular in a neighborhood of Z. Without loss of generality we can assume that the mapping $\mathbb{C}F$ is regular on the whole of $\mathbb{C}Z$ (later we resolve only complex points of indeterminacy).

The set $R := Z \setminus \mathbb{R}^2$ is connected and it is a union $\bigcup R_i$ of circles $R_i \cong \mathbb{P}^1(\mathbb{R})$. Moreover, the complexification $\mathbb{C}R_i := S_i$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and it contains R_i as $\mathbb{P}^1(\mathbb{R})$, i.e., there is a (complex) biregular mapping $\Psi_i : \mathbb{P}^1(\mathbb{C}) \to S_i$ such that it induces a (real) biregular mapping $\psi_i : \mathbb{P}^1(\mathbb{R}) \to R_i$. It follows from elementary blowup properties.

Let us denote by L_∞ the hyperplane at infinity in $\mathbb{R}^m$ and take $Q := F^{-1}(L_\infty)$. Let R_j be a component of R which is not in Q. We show that R_j has at most one common point with Q. Indeed, the curve $Q' := \mathbb{C}F^{-1}(\mathbb{C}L_\infty)$ is connected as the complement of semi-affine surface $\mathbb{C}F^{-1}(\mathbb{C}^m)$ (see [8]). Hence our assertion follows from Theorem 3.1 (applied to the surface $\mathbb{C}Z$).

Now let us note that the set S_f is exactly the set $F(R \setminus Q)$. Let H be a connected (non-empty) component of the set S_f. Hence there are indices $j_1, ..., j_k$ such that $H = \bigcup_{i=1}^k F(R_{j_i} \setminus Q)$. On the other hand we know that for every j the curve $R_j \cong \mathbb{P}^1(\mathbb{R})$ has at most one common point with Q. Consequently, the restriction of the mapping F to $R_j \setminus Q$ gives a regular mapping $\phi_j : \mathbb{R} \to H \subset \mathbb{R}^m$. Here we identify $\mathbb{R}$ with $R_j \setminus Q$ or (if

$R_j \cap Q = \emptyset$) with R_j minus one point. Without restriction of generality we can assume that all mappings ϕ_j are non-constant. Indeed, since the set R is connected, it means that the set $F(R) \subset S_f \cup L_\infty$ is connected, too, hence H cannot be an isolated point. Consequently, we can exclude all constant mappings ϕ_j. Now, we show that the mapping ϕ_j must be polynomial.

Since the mapping ϕ_j is regular, we have $\phi_j = (P_1/Q_1, ..., P_m/Q_m)$, where P_i, Q_i are relatively prime real polynomials from $\mathbb{R}[t]$ with $Q_i \neq 0$ everywhere. (see [1], 3.1.9).

Moreover, the mapping ϕ_j extends uniquely to the mapping $\Phi_j : S_j \cong \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^m(\mathbb{C})$. If $\prod_{i=1}^m Q_i \neq const$, then S_j would have two or more (conjugate) complex common points with $Q' = \mathbb{C}F^{-1}(\mathbb{C}L_\infty)$ (namely zeros of some of Q_i). By Theorem 3.1 it is a contradiction.

Hence, the mapping ϕ_j is indeed a polynomial mapping. This also implies that $R_j \cap Q \neq \emptyset$ (i.e. $R_j \setminus Q \cong \mathbb{R}$), and consequently $H = \bigcup_{i=1}^k \phi_{j_i}(\mathbb{R})$.

Now assume that the mapping f is not generically finite. It can be easy deduced form the Lüroth theorem, that there are polynomial $\phi_1, ..., \phi_m \in \mathbb{R}[t]$ and a polynomial $h(x, y) \in \mathbb{R}[x, y]$, such that $f_i = \phi_i(h),\ i = 1, ..., m$. Let us consider the (non-constant) polynomial mapping $h : \mathbb{R}^2 \to \mathbb{R}$. Let A denote the point at infinity of the line $\mathbb{R} \subset \mathbb{P}^1(\mathbb{R})$. In the same way as in the first part of our proof we see that the set S_h is closed and that the set $S_h \cup \{A\}$ is connected. In particular, the set S_h is either the whole of the line $\mathbb{R}$, or a closed half-line or a union of two closed half-lines (or the empty set). In all cases it is a union of parametric semi-lines. But $S_f = \phi(S_h)$, where $\phi = (\phi_1, ..., \phi_m)$ and the theorem follows. □

Remark 4.3 Let $S_1, ..., S_k \subset \mathbb{R}^m$, where $m \geq 2$, be parametric semi-lines. We show in the section 7, Theorem 7.1, that there exists a polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^m$ with finite fibers for which $S_f = \bigcup_{i=1}^k S_i$. In particular this gives a full characterization of the set S_f for generically-finite polynomial mappings $f : \mathbb{R}^2 \to \mathbb{R}^m$:

A subset $S \subset \mathbb{R}^m$ ($m \geq 2$) is equal to the set S_f for some generically finite polynomial mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^m$ if and only if it is a union of a finite family (possibly empty) of parametric semi-lines.

5 Extensions of finite mappings

To prove the next result we need also some facts about extensions of finite mappings.

Definition 5.1 *Let $k = \mathbb{R}$ or $k = \mathbb{C}$. For algebraic sets $X \subset k^n$ and $Y \subset k^m$ let $f = (f_1, ..., f_m) : X \to Y$ be a polynomial mapping and let $f^* : k[Y] \ni h \to h \circ f \in k[X]$ denote the induced mapping. The mapping*

f is called finite if the ring $k[X]$ is integral over its subring $f^*(k[Y]) = k[f_1, ..., f_m]$.

Remark 5.2

a) A finite mapping is proper and it has finite fibers. Conversely, for $k = \mathbb{C}$, the proper mapping of affine varieties is finite.
b) If a real mapping f is finite then the mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}Y$ is also finite and conversely.
c) A composition of finite mappings is a finite mapping.
d) A non-constant polynomial mapping $\phi : \mathbb{R} \to \mathbb{R}^m$ is finite.

Theorem 5.3 *Let $X \subset \mathbb{R}^m$ be a real affine algebraic variety and let Γ be a closed algebraic subvariety of X. Let $f : \Gamma \to \mathbb{R}^n$ be a finite polynomial map and assume $n \geq \dim X$. Then there exists a finite polynomial map $F : X \to \mathbb{R}^n$, such that $\mathrm{res}_\Gamma F = f$.*

Proof. We follow closely our method from [7]. Let $I(\Gamma)$ denote the ideal of Γ in $\mathbb{R}[X]$. Let $g = (g_1, \ldots, g_n)$ be a polynomial extension of f to X. Let $x_1, \ldots, x_m$ be coordinate polynomial functions in $\mathbb{R}^m$. By our assumptions there exist polynomial functions $a^i_j \in \mathbb{R}[X_1, \ldots, X_n]$, such that $H_i = \sum_{j=0}^{n_i} a^i_j(g) x_i^{n_i - j} = 0 \mod I(\Gamma)$, and $a^i_0(g) = \mathrm{const} \neq 0$ for $i = 1, \ldots, m$.

Consider the map $H = (g_1, \ldots, g_n, H_1, \ldots, H_m) : X \to \mathbb{R}^{n+m}$. By the construction, the mapping H is finite, $H(\Gamma) \subset \mathbb{R}^n \times \{0\}$ and dim $H(X)$ =dim $X \leq n$. Of course, the set $H(X)$ is a closed (in the Euclidian topology) subset of $\mathbb{R}^{n+m}$. Let $Y := cl_Z(H(X))$ be the Zariski closure of $H(X)$ in $\mathbb{R}^{n+m}$. Then dim $Y =$ dim X. It is easy to see that there exists a finite linear projection $\pi : Y \to \mathbb{R}^n \times \{0\}$.

Indeed, we have the canonical inclusion $\mathbb{R}^{n+m} \subset \mathbb{P}^{n+m}(\mathbb{R})$ and the set W of points at infinity of Y has dimension $\leq n - 1$. Hence we can find a linear subspace L of the hyperplane at infinity of dimension $m - 1$ which is disjoint from $W \cup cl(\mathbb{R}^n \times \{0\})$. Let $\pi_L : \mathbb{P}^{n+m}(\mathbb{R}) \to cl(\mathbb{R}^n \times \{0\})$ be the projection determined by L. The restriction of π_L to Y is a finite mapping of Y into $\mathbb{R}^n \times \{0\}$ and it is the projection π we were looking for. Now, to obtain a finite extension of f to the whole of X, it is enough to take $F = p \circ H$. □

It is worth to note (although we do not need this in the sequel) that we also have:

Theorem 5.4 *Let $X \subset \mathbb{R}^m$ be a real affine variety and $Y \subset X$ be a closed algebraic subvariety. Let $f : Y \to \mathbb{R}^n$ be a polynomial mapping. Assume, that* dim $X \leq n$. *Then there exists a real polynomial mapping* $F : X \to \mathbb{R}^n$ *such that*

1. $res_Y F = f$
2. *the mapping* $res_{X \setminus Y} F : X \setminus Y \to \mathbb{R}^n$ *is quasi-finite.*

In particular if the mapping f *is quasi-finite, then the mapping* F *is quasi-finite, too.*

Proof. Let $I(Y) = (h_1, ..., h_r)$ be the ideal of Y in $\mathbb{R}[X]$. Denote by $g = (g_1, \ldots, g_n)$ a polynomial extension of f to X and by $x_1, \ldots, x_m$ coordinate polynomials in $\mathbb{R}^m$. Take $H_{ij} := h_i \cdot x_j$; $i = 1, ..., r$, $j = 1, ..., m$; and consider the map $H = (g_1, \ldots, g_n, h_1, \ldots, h_r, H_{11}, ..., H_{1m}, ..., H_{r1}, \ldots, H_{rm}) : X \to \mathbb{R}^{n+r+mr}$. By the construction, the mapping H is injective outside Y, $H(Y) \subset \mathbb{R}^n \times \{0\}$ and dim $H(X) =$ dim $X \leq n$. Let Γ denote the Zariski closure of $H(X)$ in $\mathbb{R}^{n+r+mr}$. It is easy to see (as in the proof of Theorem 5.3) that there exists a finite linear projection $\pi : \Gamma \to \mathbb{R}^n \times \{0\}$, such that $\pi(H(X)) \subset \mathbb{R}^n \times \{0\}$. Now it is enough to take $F = \pi \circ H$. $\square$

6 General case

In the beginning we compare the real set S_f with the complex one. We have the following basic fact, whose proof is an easy exercise from the general topology:

Proposition 6.1 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a generically-finite polynomial map and let* $\mathbb{C}f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *be its complexification. Then the set* S_f *of points at which* f *is not proper is contained in the set* $S_{\mathbb{C}f}$ *of points at which the mapping* $\mathbb{C}f$ *is not proper.*

In the sequel the following definition will be useful:

Definition 6.2 Let $S \subset \mathbb{R}^n$ be a semialgebraic set. The set S is called $\mathbb{R}$-uniruled if for every point $a \in S$ there is a parametric semi-line in S through the point a. Moreover, the set S is called generically $\mathbb{R}$-uniruled if there is an open and dense subset $U \subset S$, such that for every point $a \in U$ there is a parametric semi-line in S through this point.

Remark 6.3 Of course, every $\mathbb{R}$-uniruled set is generically $\mathbb{R}$-uniruled. Moreover, if a set S is generically $\mathbb{R}$-uniruled, then the set $\mathbb{C}S$ is $\mathbb{C}$-uniruled. It follows immediately from Proposition 3.3.

Now we are ready to prove our main theorem:

Theorem 6.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then the set S_f is closed, semi-algebraic and for every non-empty connected component $S \subset S_f$ we have $1 \le \dim S \le n-1$. Moreover, the set S_f is $\mathbb{R}$-uniruled.*

Proof. Of course, we can assume that $n > 1$. Let $\overline{X}$ be the closure of the set $X := graph(f) \subset \mathbb{R}^n \times \mathbb{R}^m$ in $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^m$. Let us note that X is polynomially isomorphic to $\mathbb{R}^n$. The set $\overline{X}$ is semi-algebraic, hence the set $R := \overline{X} \setminus X$ is semialgebraic, too. Of course, dim $R < n$. Moreover, if $\pi : \mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m$ is the canonical projection we have $S_f = \pi(R)$ and hence the set S_f is closed, semi-algebraic and dim $S_f < n$.

Let $b \in S_f$. By the curve selecting lemma and the classical Puisex Theorem, there is a meromorphic function $\phi(t) = a_s t^s + ... + a_{-k} t^{-k} + ... \in \mathbb{R}^n$ (where $a_j \in \mathbb{R}^n$), with a pole at infinity, which is defined for $|t| > R$, such that $lim_{t=\infty} f(\phi(t)) = b$. In particular for a sufficiently large k we have $f(a_s t^s + ... + a_{-k} t^{-k}) = b + w_1/t + ... + w_r/t^r$, where $w_j \in \mathbb{R}^m$. Let $\psi(t) = a_s t^s + ... + a_{-k} t^{-k}$.

There are two possible cases; either ψ is a polynomial, or ψ has a pole for $t = 0$. Let us define the set $X \subset \mathbb{R}^2$ in the following way: if ψ is a polynomial then X is the line $\{x(t) = (t, 0); t \in \mathbb{R}\}$, if ψ has a pole for $t = 0$, then X is the hyperbola $\{x(t) = (t, 1/t); t \in \mathbb{R}\}$.

Let us note that the function ψ induces a finite polynomial mapping $g : X \ni x(t) \to \psi(t) \in \mathbb{R}^m$. Indeed, the mapping $\mathbb{C}g$ is proper, hence finite and this implies that the mapping g is finite. By Theorem 5.3 we can extend the mapping g to a finite polynomial mapping $G : \mathbb{R}^2 \to \mathbb{R}^n$.

Since the mapping f is non-constant, we can assume that the mapping $F := f \circ G$ is also non-constant (indeed, it is enough to choose G in this way that for some $x \notin X$, we have $G(x) \notin f^{-1}(b)$). Further, $lim_{t=\infty} F(x(t)) = lim_{t=\infty} f(a_s t^s + ... + a_{-k} t^{-k}) = lim_{t=\infty} b + w_1/t + ... + w_r/t^r = b$, hence $b \in S_F$. Moreover, since the mapping G is finite, we have $S_F \subset S_f$. Now by Theorem 4.2 we get that there is a parametric semi-line in S_f through b. In particular for every non-empty connected component $S \subset S_f$ we have $1 \le \dim S$. □

If the mapping f is generically finite, then we can say more:

Theorem 6.5 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial generically-finite mapping. Let $\deg f_i = d_i$ and assume that $d_1 \ge d_2 ... \ge d_m$. Then there is a real polynomial $P \in \mathbb{R}[x_1, ..., x_m]$ of degree at most $D = d_1 d_2 ... d_{n-1} - 1$, such that $S_f \subset P^{-1}(0)$ and $f(\mathbb{R}^n) \not\subset P^{-1}(0)$.*

Proof. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a sufficiently general projection of the type $T(x_1, ..., x_m) = (\sum_{i=1}^m a_i^1 x_i, \sum_{i=2}^m a_i^2 x_i, ..., \sum_{i=n}^m a_i^n x_i)$. Take $h = T \circ f$. Then $h : \mathbb{R}^n \to \mathbb{R}^n$ is a generically-finite polynomial mapping and

$T(S_f) \subset S_h$. Let us consider the mapping $\mathbb{C}h : \mathbb{C}^n \to \mathbb{C}^n$. We know by Proposition 6.1 that $S_h \subset S_{\mathbb{C}h}$. By Theorem 3.5 the set $S_{\mathbb{C}h}$ is described by a polynomial Q of degree at most $D' = \deg h_1 ... \deg h_{n-1} - 1 \leq d_1 d_2 ... d_{n-1} - 1 = D$. Moreover, since the mapping h is real, we have that the polynomial Q is also real (it can be deduced from [4]). Now, the set S_f is contained in the zero set of a polynomial $P := Q \circ T$ and the theorem follows. □

Corollary 6.6 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then every non-empty connected component of S_f is $\mathbb{R}$-uniruled, in particular unbounded.*

Corollary 6.7 *Let $n > 1$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant polynomial mapping. Let Q be a non-empty connected component of $\mathbb{R}^n \setminus S_f$. Then $H_{n-1}(Q, \mathbb{Z}) = 0$. In particular, for $n = 2$ every such component is homeomorphic to an open disc, consequently it is simply connected.*

Proof. Let $\mathbb{S}^n$ be a one point compactification of $\mathbb{R}^n$ and $\overline{S_f}$ denote the closure of S_f in $\mathbb{S}^n$. Since all components of S_f are unbounded, the set $\overline{S_f}$ is connected. Further we have $H_{n-1}(\mathbb{R}^n \setminus S_f, \mathbb{Z}) = H_{n-1}(\mathbb{S}^n \setminus \overline{S_f}, \mathbb{Z}) \cong H^1(\mathbb{S}^n, \overline{S_f}, \mathbb{Z}) \cong \tilde{H}^0(\overline{S_f}, \mathbb{Z}) = 0$, where $\tilde{H}^0$ denotes the reduced module of cohomology. Consequently $H_{n-1}(Q, \mathbb{Z}) = 0$. Hence for $n = 2$ we have $H_1(Q, \mathbb{Z}) = 0$ and it is well-known that it implies that Q is homeomorphic to an open disc, in particular it is simply connected (it can be easily deduced e.g. from [11] Theorem 13.11). □

Remark 6.8 In particular, the last statement of Corollary 6.7 gives a full geometric picture for polynomial mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a non-zero Jacobian (e.g. for the Pinchuk mapping, see [10]). Indeed, let $S_1, ..., S_r$ be all connected components of the set $\mathbb{R}^2 \setminus S_f$ with non-empty preimages. Take $f^{-1}(S_i) = \bigcup_{j=1}^{r_i} S_{i,j}$, where $S_{i,j}$ are connected components of the set $f^{-1}(S_i)$. Mappings $res_{S_{i,j}} f : S_{i,j} \to S_i$ are proper and unramified, hence they are topological coverings. Since all sets S_i are simply connected, the mappings $res_{S_{i,j}} f$ are diffeomorphisms, and consequently the mapping f is a glueing of a finite number of diffeomorphisms of discs.

Corollary 6.9 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Assume that S is an irreducible algebraic component of the (real) Zariski closure of S_f. Then the set S is unbounded and the algebraic variety $\mathbb{C}S$ is $\mathbb{C}$-uniruled. Moreover, if $S \subset S_f$ and either S is smooth and connected or S is smooth and has pure dimension $n - 1$, then S is generically $\mathbb{R}$-uniruled.*

Proof. Let $cl_Z(S_f) = \bigcup_{i=1}^r S_i$ be a decomposition of $cl_Z(S_f)$ into irreducible algebraic components. We can assume that $S = S_1$. The set

$R := S_f \setminus \bigcup_{i=2}^r S_i$ is nonempty. Let $a \in R$. By Theorem 6.4 there is a parametric semi-line l in S_f through the point a. Since the parametric semi-line is irreducible we have that in fact $l \subset S$, in particular the set S is unbounded. If additionally $S \subset S_f$ and either S is smooth and connected or S is smooth and has pure dimension $n-1$, then the set R is dense in S and S is generically $\mathbb{R}$-uniruled. Finally, the algebraic variety $\mathbb{C}S$ is $\mathbb{C}$-uniruled by Proposition 3.3 p. 3). □

Remark 6.10 If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a generically finite polynomial mapping, then the real algebraic closure of the set S_f can be not generically $\mathbb{R}$-uniruled. In fact Gwoździewicz show in [5], that if f is the Pinchuk mapping, then the algebraic closure of the set S_f has a point as an isolated component.

7 Examples

In this section we give some examples of the set S_f, as well as some applications of our results.

Theorem 7.1 *Let $S_1, ..., S_r \subset \mathbb{R}^m$ be semi-algebraic sets and assume that there are finite and surjective polynomial mapping $\phi_i : \mathbb{R}^{k_i} \to S_i$ ($n > k_i \geq 1$). Assume that $m \geq n$. Then there exists a polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, with finite fibers for which $S_F = \bigcup_{i=1}^r S_i$.*

Proof. Let $L_1, ..., L_r$ be a family of linear subspaces of $\mathbb{R}^n$, such that

1) dim $L_i = k_i$,

2) L_i is given by linear equations $l_j(x_1, ..., x_{n-1}) = 0,\ j = 1, ..., p_i = n - k_i$,

3) $L_i \cap L_j = \emptyset$ for $i \neq j$.

Note that L_i is given by the equation $e_i(x_1, ..., x_{n-1}) = \sum_{j=1}^{p_i} (l_{ij})^2 = 0$. Let $G : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-1}, (\prod_{i=1}^r e_i)x_n^2 + x_n)$. It is easy to see that G has finite fibers and $S_G = \{x \in \mathbb{R}^n : \prod_{i=1}^r e_i(x) = 0\} = \bigcup_{i=1}^r L_i$.

Now let $\phi_i : L_i \ni x \to \phi(x) \in S_i \subset \mathbb{R}^m$ be a finite polynomial parametrization of S_i. By Theorem 5.3, the mapping $\phi = \bigcup_{i=1}^r \phi_i$ can be extended to a finite polynomial mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$. It is easy to check that if $F = \Phi \circ G$, then F has finite fibers and $S_F = \Phi(S_G) = \phi(\bigcup_{i=1}^r L_i) = \bigcup_{i=1}^r S_i$. □

Remark 7.2 a) Let us consider a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. If $n = 2$, then the set S_f (if non-empty) has codimension 1. Theorem 7.1 shows that for $n > 2$ it is not longer true even for generically-finite mappings. In fact, for every $0 < k < n$, we can construct a generically-finite mapping $f_k : \mathbb{R}^n \to \mathbb{R}^n$ such that codim $S_{f_k} = n - k$.

b) On the other hand, in the case of the (not generically-finite) projection $f : \mathbb{R}^n \ni (x_1, ..., x_k, ..., x_n) \to (x_1, ..., x_k) \in \mathbb{R}^k$, where $n > k$, we have $S_f = \mathbb{R}^k$.

Example 7.3

a) There is no a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_2 > 0, x_3 > 0, ..., x_n > 0,\ x_1 \cdot ... \cdot x_n > 1\}$. Indeed, let $U = f(\mathbb{R}^n)$ and $S = bd(U)$ (the border of U). Since dim $S = n-1$ we have that $S' := cl_Z(S)$ is an algebraic component of the set $cl_Z(S_f)$. By Corollary 6.9 the set $\mathbb{C}S'$ must be $\mathbb{C}$-uniruled. It means that the hypersurface $\{(x_1, ..., x_n) \in \mathbb{C}^n :\ x_1 \cdot ... \cdot x_n = 1\}$ is $\mathbb{C}$-uniruled. Since the last hypersurface does contain no affine parametric curves, we get a contradiction.
b) There is no polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = \{(x_1, ..., x_n) \in \mathbb{R}^n :\ \sum_{i=1}^n x_i^2 > 1\}$. Indeed, let $U = f(\mathbb{R}^n)$ and $S = bd(U) = \{(x_1, ..., x_n) \in \mathbb{R}^n :\ \sum_{i=1}^n x_i^2 = 1\}$. Then S is a real algebraic set of dimension $n - 1$ and consequently it must be an irreducible component of the set $cl_Z(S_f)$. In particular it must be generically $\mathbb{R}$-uniruled, hence unbounded, a contradiction.
c) More generally, let $U \subset \mathbb{R}^n$ be an open domain. Assume that $bd(U)$ contains an algebraic set Γ of dimension $n - 1$, such that either Γ is bounded, or Γ is smooth, connected and not generically $\mathbb{R}$-uniruled. Then there is no polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = U$.

8 Real Jacobian conjecture

At the end of this paper we consider the case of a real polynomial mapping with the nowhere vanishing Jacobian. We start with the following:

Lemma 8.1 *Let S be a closed semialgebraic subset of $\mathbb{R}^n$ If $\operatorname{codim} S \geq 2$, then the set $\mathbb{R}^n \setminus S$ is connected. If $\operatorname{codim} S \geq 3$, then the set $\mathbb{R}^n \setminus S$ is simply connected.*

Proof. If S is smooth, in particular finite it follows directly from [4, Th.3.1 p. 22 and Th.2.3 p. 146]. In the general case we use induction with respect to dim S. Let $r = \dim S$ and let S_1 be a r-dimensional stratum in a finite semialgebraic stratification of S [1, Proposition 2.5.1]. Then $S = S_1 \cup E$, where dim $E \leq r - 1$ and E is a closed semialgebraic subset of $\mathbb{R}^n$. Now assume that codim $S \geq 2$. Since the set S_1 is smooth in $\mathbb{R}^n \setminus E$ and codim $S_1 \geq 2$, we have by [4, Th.3.1, p. 22] that the set $\mathbb{R}^n \setminus S$ is connected if the set $\mathbb{R}^n \setminus E$ is, and we conclude this part of the proof by the induction principle.

Now let codim $S \geq 3$. Since the set S_1 is smooth in $\mathbb{R}^n \setminus E$, we have by [4, Th.2.3, p. 146] that $\pi_1(\mathbb{R}^n \setminus S) = \pi_1(\mathbb{R}^n \setminus E)$ and we again conclude our proof by the induction principle. □

We have the following:

Theorem 8.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with the Jacobian which nowhere vanishes. If $\operatorname{codim} S_f \geq 3$ then f is a bijection (and consequently $S_f = \emptyset$).*

Proof. Let us consider sets $X := \mathbb{R}^n \setminus S_f$ and $Y := \mathbb{R}^n \setminus f^{-1}(S_f)$. Since the mapping f is a local homeomorphism, we have that codim $f^{-1}(S_f) =$ codim $S_f \geq 3$. In particular it implies that the set Y is connected. Moreover, since $\operatorname{codim} S_f \geq 3$ we have that the set X is simply connected. Since the mapping $f : Y \to X$ is proper and unramified, we have that f is a topological covering. In particular it is a homeomorphism. Consequently, the general fiber of the mapping f is a one point. Since the mapping f is unramified, this implies that f is an injection. Now it suffices to use theorem of Białynicki-Rosenlicht - see [2]. □

On the other hand, the example of Pinchuk (see [10]) shows that there are real polynomial mappings with the Jacobian which nowhere vanishes and with codim $S_f = 1$. Hence the only interesting case is that of codim $S_f = 2$ and we can state:

Real Jacobian Conjecture. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with the Jacobian which nowhere vanishes. If $\operatorname{codim} S_f \geq 2$ then f is a bijection (and consequently $S_f = \emptyset$).*

By Theorem 4.2 this conjecture is true in dimension two. Consequently, the first interesting case is $n = 3$ and dim $S_f = 1$.

The Real Jacobian Conjecture is closely connected with the following famous Jacobian Conjecture:

Jacobian Conjecture. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with the Jacobian which nowhere vanishes. Then f is an isomorphism.*

In fact we have:

Proposition 8.3 *The Real Jacobian Conjecture in dimension $2n$ implies the Jacobian Conjecture in (complex) dimension n.*

Proof. Indeed, let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with a non-zero Jacobian. We can treat the mapping f as a real polynomial mapping $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Assume that f is not an isomorphism. In particular the mapping f can not be finite. Thus by [7], [8] we get that the set S_f has a complex codimension 1, hence it has a real codimension 2. Consequently, if the Real Jacobian Conjecture holds in dimension $2n$, this gives a contradiction. □

Remark 8.4 From a topological point of view our Real Jacobian Conjecture is a real counterpart of the (complex) Jacobian Conjecture. In particular if we find a counterexample to the Real Jacobian Conjecture we show that there is no topological obstruction to find a counterexample to the (complex) Jacobian Conjecture. From this point of view the example of Pinchuk, which is a glueing of finite number of diffeomorphisms of discs (see Remark 6.8) gives nothing.

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