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Zeta distributions associated to a representation of a Jordan algebra

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Abstract. For a regular representation of a Euclidean Jordan algebra, we introduce multi-parameter zeta distributions with harmonic polynomial coefficient. Bernstein-Sato type identities are obtained and used to prove a functional equation. Examples are discussed in relation with Sato's general theory of zeta functions.

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0. Introduction

Let h be a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree k. For s a complex number, with $\Re s > -\frac{n}{2}$, the formula

$$Z(f, s\,; h) = \int_{\mathbb{R}^n} f(\xi)\, h(\xi)\, \|\xi\|^{2s}\, d\xi$$

clearly defines a tempered distribution. As a function of s, it can be meromorphically extended to the complex plane. The Fourier transform of this distribution can be expressed through the following formula, valid for any function f in the Schwartz class

$$Z(\hat{f}, -s\,; h) = \pi^{\frac{n}{2}}\, 4^{-s+\frac{n}{2}+\frac{k}{2}}\, (-i)^k\, \frac{\Gamma(-s+\frac{n}{2}+k)}{\Gamma(s)}\, Z(f, s-\frac{n}{2}-k\,; h) \quad .$$

This is the functional equation for the Epstein zeta distribution.

There are many extensions of this formula, and they culminate in the broad program developed by M. Sato, T. Shintani, F. Sato and others (see [Sa] for a presentation of this circle of ideas mainly based on the notion of prehomogeneous vector space). A different approach was initiated by Faraut and Korányi (see [F-K]), which is connected with the theory of Jordan algebras and their representations. See also [A1], [A2]. There are some cases *not* covered by the general theory developped by Sato and al. We follow this second approach (see section 5 for further discussion).

1. Preliminary results on Euclidean Jordan algebras

Let V be a real Euclidean Jordan algebra, which we assume for simplicity to be simple. A general reference for notations and results is [F-K]. $V^\times$ denotes the set of invertible elements in V, and Ω is the open component of $V^\times$ containing the neutral element e. Its closure is the set of squares in V. Let n be the dimension of V, r its rank. Then $n = r + d\frac{r(r-1)}{2}$, where d is an integer. Denote by tr (resp. $\det$) the trace and the generic norm of V. Let G be the neutral component of the group of linear transformations which preserve Ω, and let K the stabilizer of e in G. The group K is a maximal compact subgroup of G and it is the connected component of the group of automorphisms of the Jordan algebra V. As inner product on V, set $(x \mid y) = \operatorname{tr}(xy)$. It satisfies $(xz \mid y) = (z \mid xy)$ for any $x, y, z \in V$. Also the group K is the intersection of G and the orthogonal group $O(V)$ for this inner product.

Fix a Pierce decomposition

$$e = c_1 + c_2 + \cdots + c_r \qquad , \tag{1}$$

and let $\Delta_1, \ldots, \Delta_r = \det$ be the associated principal minors. Recall that for each $j, 1 \leq j \leq r$, Δ_j is a polynomial of degree j, and each Δ_j is strictly positive on Ω. For $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, let $|\mathbf{s}| = s_1 + s_2 + \cdots + s_r$. For $x \in \Omega$, define

$$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \ldots \Delta_r(x)^{s_r} \tag{2}$$

(generalized *power function*). For $\mathbf{m} \in \mathbb{N}^r$, and $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, observe that $\Delta_{\mathbf{m}}$ extends as a polynomial on V.

Let us recall some useful formulæ. First recall the function Γ_Ω, defined by

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} x} \Delta_{\mathbf{s}}(x) d^*x \tag{3}$$

where $d^*x = \det(x)^{-\frac{n}{r}} dx$ stands for the G-invariant measure on Ω. The integral converges absolutely when $\Re s_j > (j-1)\frac{d}{2}$, for $j = 1, 2, \ldots, r$

and it can be extended meromorphically to $\mathbb{C}^r$. In fact Γ_Ω is a product of classical Γ functions

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\Big(s_j - (j-1)\frac{d}{2}\Big) \quad . \tag{4}$$

Generalizing the classical notation $(s)_k = s(s+1)\dots(s+k-1)$, write for $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathbb{N}^r$

$$(\mathbf{s})_{\mathbf{m}} = \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{j=1}^{r} \Big(s_j - (j-1)\frac{d}{2}\Big)_{m_j} \quad .$$

Another ingredient is the Laplace transform w.r.t. the cone Ω. For instance, the following formula holds for $\Re s_j > (j-1)\frac{d}{2},\ j = 1, 2, \dots, r$:

$$\int_\Omega e^{-(x|y)} \Delta_{\mathbf{s}}(x) d^*x = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}(y^{-1}), \quad \forall y \in \Omega \quad . \tag{5}$$

Introduce also the *opposite minors* $\Delta_j^*, 1 \le j \le r$, analoguous to the principal minors, but associated to the reverse Pierce decomposition $e = c_r + c_{r-1} + \dots + c_1$. If $\mathbf{s} = (s_1, s_2, \dots, s_r)$, let $\mathbf{s}^* = (s_r, s_{r-1}, \dots, s_1)$. There are two important formulæ. The first one is

$$\Delta_{\mathbf{s}}(x^{-1}) = \Delta^*_{-\mathbf{s}^*}(x) \quad . \tag{6}$$

For the second introduce any element m_0 in K such that $m_0\, c_j = c_{r-j+1}$ for $j = 1, 2, \dots, r$. Then

$$\Delta^*_{\mathbf{s}}(x) = \Delta_{\mathbf{s}}(m_0^{-1}x) \quad . \tag{7}$$

Notice also the following result which will be used later on

$$(-\mathbf{s}^*)_{\mathbf{m}} = (-1)^{|\mathbf{m}|}(\mathbf{s} - \mathbf{m}^* + \frac{n}{r})_{\mathbf{m}^*} \quad . \tag{8}$$

To any polynomial p on V one associates the constant coefficients differential operator $p(\frac{\partial}{\partial x})$, characterized by the following property:

$$p(\frac{\partial}{\partial x})\, e^{(x|y)} = p(y) e^{(x|y)},$$

for all $x, y \in V$.

For $\mathbf{m} = (m_1, m_2, \dots, m_r)$ in $\mathbb{N}^r$ with $m_1 \ge m_2 \ge \dots m_r$, the differential operator $\Delta^*_{\mathbf{m}}(\frac{\partial}{\partial x})$ satisfies the following Bernstein identity :

$$\Delta^*_{\mathbf{m}}(\frac{\partial}{\partial x})\, \Delta_{\mathbf{s}}\,(x) = \Big(\mathbf{s} - \mathbf{m}^* + \frac{n}{r}\Big)_{\mathbf{m}^*} \Delta_{\mathbf{s}-\mathbf{m}^*}(x) \tag{9}$$

(cf [F-K], Prop. VII.1.6).

Let us state the main result on the analytic continuation of the Riesz distributions (cf [F-K] Theorem VII.2.6.).

Proposition 1. *The distribution*

$$\varphi \mapsto \frac{1}{\Gamma_\Omega(\mathbf{s})}\int_\Omega \varphi(x)\Delta_{\mathbf{s}}(x)d^*x$$

has an analytic continuation to $\mathbb{C}^r$*.*

We also need some information on the generalized K*-Bessel functions*. For $x, y \in \Omega$, define

$$K_{\mathbf{s}}(x,y) = \int_\Omega e^{-(x|u)-(y|u^{-1})}\Delta_{\mathbf{s}}(u)d^*u \quad .$$

Proposition 2. *(i) The integral is absolutely convergent for all* $\mathbf{s} \in \mathbb{C}^r$*, and defines an entire function of* $\mathbf{s}$.

(ii) For $\Re s_j < -\frac{d}{2}(r-j), 1 \leq j \leq r$*, the integral extends continuously to* $x \in \overline{\Omega}, y \in \Omega$*, and satisfies the estimate*

$$|K_{\mathbf{s}}(x,y)| \leq \Gamma_\Omega(-\Re\mathbf{s}^*)\Delta_{\Re\mathbf{s}}(y), \quad \forall x \in \overline{\Omega}, y \in \Omega \quad . \tag{10}$$

Moreover, under the same conditions on $\mathbf{s}$

$$\lim_{\epsilon\downarrow 0} K_{\mathbf{s}}(\epsilon e, y) = \Gamma_\Omega(-\mathbf{s}^*)\,\Delta_{\mathbf{s}}(y) \quad . \tag{11}$$

For the first statement, see [F-K], Prop. XVI.3.1, and [C1]. For the second statement, observe that for $x \in \overline{\Omega}$, and $u \in \Omega$, $(x \mid u) \geq 0$, so that

$$|K_{\mathbf{s}}(x,y)| \leq \int_\Omega e^{-(y|u^{-1})}\Delta_{\Re\mathbf{s}}(u)d^*u \quad .$$

Use the change of variables $v = u^{-1}$, (6) and (5) to get the inequality of the statement. Now, the proof for getting the limit is just obtained from this inequality and the dominated convergence theorem.

2. The Bernstein identity for $P_{\mathbf{s}}$

Let Φ be a representation of V in a Euclidean space E, of dimension N, equipped with an inner product $< .,. >$, and let $Q : E \longrightarrow V$ be the associated quadratic map defined by the formula

$$(Q(\xi), x) = < \Phi(x)\xi, \xi >, \ \forall x \in V \quad . \tag{12}$$

(see [F-K] ch. XVI, [C2] for the relevant definitions).

Notice that the image of Q is contained in $\overline{\Omega}$ and is stable by the action of G. We further assume that the representation is *regular*, that is we assume that the image of Q contains e (hence all the elements of Ω) (see [C2]). Let $E' = \{\xi \in E | Q(\xi) \in \Omega\}$, which is an open dense subset of E, whose complement has Lebesgue measure 0.

To any polynomial p on V we associate the polynomial P on E defined by

$$P(\xi) = p(Q(\xi)) \quad .$$

Specifically, for $j, 1 \leq j \leq r$, let $P_j(\xi) = \Delta_j(Q(\xi))$. The polynomial P_j is homogeneous of degree $2j$ and takes nonnegative values on E. Also use the notation $P_{\mathbf{m}} = \Delta_{\mathbf{m}} \circ Q = P_1^{m_1-m_2} P_2^{m_2-m_3} \dots P_r^{m_r}$. Note that $P_{\mathbf{m}}$ is homogeneous of degree $2\,|\mathbf{m}|$. Using the opposite Pierce decomposition, we define similarly P_j^* and $P_{\mathbf{m}}^*$.

For f a function in the Schwartz class $\mathcal{S}(E)$, define its Fourier $\hat{f}$ by

$$\hat{f}(\eta) = \int_E f(\xi) e^{-i<\xi,\eta>} d\xi \qquad \text{for } \eta \in E \quad .$$

This definition is then extended by duality to the space of tempered distributions on E.

As usual, we may associate to any polynomial R on E a constant coefficients differential operator denoted either by $\partial(R)$ or $R(\frac{\partial}{\partial\xi})$ characterized by

$$R(\frac{\partial}{\partial\xi})\, e^{<\eta,\,\xi\,>} = R(\eta) e^{<\eta,\,\xi>}, \quad \forall \eta \in E \quad .$$

If $\mathbf{s} \in \mathbb{C}^r$ satisfies $\Re s_1 > \Re s_2 > \cdots > \Re s_r > 0$, the function

$$P_{\mathbf{s}}(\xi) = P_1(\xi)^{s_1-s_2} P_2(\xi)^{s_2-s_3} \dots P_r(\xi)^{s_r}$$

is well defined on E and defines a tempered distribution on E. It is known that $\mathbf{s} \longmapsto P_{\mathbf{s}}$ has a meromorphic extension to $\mathbb{C}^r$ as a tempered distribution (the simplest way is to use the existence of generalized Bernstein polynomials, see [S]). We still denote by $P_{\mathbf{s}}$ this extension. The following result is a kind of Bernstein identity (which however would *not* be sufficient to prove the analytic continuation, see the observation in [B-R]).

Theorem 1. *For each* $\mathbf{m} \in \mathbb{N}^r$ *such that* $m_1 \geq m_2 \geq \cdots \geq m_r$,

$$P_{\mathbf{m}}^*(\frac{\partial}{\partial\xi}) P_{\mathbf{s}} = B_{\mathbf{m}}(\mathbf{s}) P_{\mathbf{s}-\mathbf{m}^*} \tag{13}$$

with

$$B_{\mathbf{m}}(\mathbf{s}) = 4^{|\mathbf{m}|}(-\mathbf{s}^* - \frac{N}{2r} + \frac{n}{r})_{\mathbf{m}}\, (-\mathbf{s}^*)_{\mathbf{m}} \qquad .$$

The proof is inspired by [F-K] Prop XVI.4.1. Introduce the *generalized heat kernel* $G(x,\xi)$ defined on $\Omega \times E'$ by

$$G(x,\xi) = (2\pi)^{-N} \int_E e^{-<\Phi(x)\eta,\eta>} e^{i<\xi,\eta>} d\eta$$
$$= (2\pi^{\frac{1}{2}})^{-N} \det(x)^{-\frac{N}{2r}} e^{-\frac{1}{4}<\Phi(x^{-1})\,\xi,\xi>} \quad .$$

For any polynomial h on V,

(14) $$h(\frac{\partial}{\partial x})G(x,\xi) = H(\frac{\partial}{\partial \xi})G(x,\xi) \quad .$$

where, in accordance with our general convention $H = h \circ Q$.

Let $\xi \in E'$, and consider the following integral :

(15) $$F_{\mathbf{s}}(\xi) = \int_\Omega G(x,\xi)\, \Delta_{\mathbf{s}+\frac{N}{2r}}(x)\, d^*x \quad .$$

The integral converges for $\Re s_j < -(r-j)\frac{d}{2}$, $1 \leq j \leq r$. In fact, using the change of variable $y = x^{-1}$ (which preserves the measure d^*x), (6) and (7):

$$F_{\mathbf{s}}(\xi) = (2\pi^{\frac{1}{2}})^{-N} \int_\Omega e^{-\frac{1}{4}(x^{-1}\,|\,Q(\xi))} \Delta_{\mathbf{s}}(x) d^*x$$
$$= (2\pi^{\frac{1}{2}})^{-N} \int_\Omega e^{-\frac{1}{4}(y\,|\,Q(\xi))} \Delta^*_{-\mathbf{s}^*}(y) d^*y$$
$$= (2\pi^{\frac{1}{2}})^{-N} \int_\Omega e^{-\frac{1}{4}(y\,|\,Q(\xi))} \Delta_{-\mathbf{s}^*}(m_0^{-1}y) d^*y$$
$$= (2\pi^{\frac{1}{2}})^{-N} \int_\Omega e^{-\frac{1}{4}(z\,|\,m_0^{-1}Q(\xi))} \Delta_{-\mathbf{s}^*}(z) d^*z \quad .$$

Now use (5) to get

$$F_{\mathbf{s}}(\xi) = (2\pi^{\frac{1}{2}})^{-N} \Gamma_\Omega(-\mathbf{s}^*) \Delta_{-\mathbf{s}^*}\big((m_0^{-1}\frac{Q(\xi)}{4})^{-1}\big)$$

and use again (6) and (7) to get

(16) $$F_{\mathbf{s}}(\xi) = (2\pi^{\frac{1}{2}})^{-N} 4^{-|\mathbf{s}|} \Gamma_\Omega(-\mathbf{s}^*) P_{\mathbf{s}}(\xi) \quad .$$

Rewrite

$$F_{\mathbf{s}}(\xi) = \int_\Omega G(x,\xi)\, \Delta_{\mathbf{s}+\frac{N}{2r}-\frac{n}{r}}(x)\, dx \quad ,$$

apply (14) with $h = \Delta^*_{\mathbf{m}}$, and integrate by parts $|\mathbf{m}|$ times to get

$$P^*_{\mathbf{m}}(\frac{\partial}{\partial \xi}) F_{\mathbf{s}}(\xi) = (-1)^{|\mathbf{m}|} \int_\Omega G(x,\xi)\, \Big(\Delta^*_{\mathbf{m}}(\frac{\partial}{\partial x}) \Delta_{\mathbf{s}+\frac{N}{2r}-\frac{n}{r}}\Big)(x) dx$$

Hence from (9)

$$P^*_{\mathbf{m}}(\tfrac{\partial}{\partial\xi})F_{\mathbf{s}}(\xi) = (-1)^{|\mathbf{m}|}\Big(\mathbf{s}-\mathbf{m}^*+\frac{N}{2r}\Big)_{\mathbf{m}^*}F_{\mathbf{s}-\mathbf{m}^*}(\xi)$$
$$= (-\mathbf{s}^*-\frac{N}{2r}+\frac{n}{r})_{\mathbf{m}}\,F_{\mathbf{s}-\mathbf{m}^*}(\xi) \quad .$$

Now use (16) to get the formula (13), for $\xi \in E'$. This argument is valid under the restriction $\Re s_j < -(r-j)\frac{d}{2}$, $1 \le j \le r$, but both sides can be continued analytically on E'. For $\Re s_j \gg 0$, both sides of (13) are continuous functions. As they coincide on a dense open set, they must coincide everywhere on E. Then the existence of a meromorphic continuation of both sides as tempered distributions imply that the result is true for all values of $\mathbf{s}$.

3. The functional equation for the zeta integral

Introduce the *Stiefel manifold* $\Sigma = \{\xi \in E | Q(\xi) = e\}$. Every element in E' can be written in a unique way as $\xi = \Phi(x^{\frac{1}{2}})\sigma$, with $x \in \Omega$ and $\sigma \in \Sigma$. The Euclidean structure induces a Riemannian structure on the submanifold Σ, and hence a measure $d\sigma$ on Σ which we normalize so that $\int_\Sigma d\sigma = 1$. Then there is an integration formula in (generalized) polar coordinates :

$$(17) \qquad \int_E f(\xi)d\xi = \frac{\pi^{N/2}}{\Gamma_\Omega(\frac{N}{2r})}\int_\Omega\int_\Sigma f(\Phi(x^{1/2})\sigma)\Delta(x)^{\frac{N}{2r}}\,d^*x\,d\sigma \quad .$$

The *zeta integral* is either of the following expressions

$$Z(f,\mathbf{s}) = \int_E P_{\mathbf{s}}(\xi)f(\xi)d\xi \quad .$$

$$Z^*(f,\mathbf{s}) = \int_E P^*_{\mathbf{s}}(\xi)f(\xi)d\xi \quad .$$

For f in the Schwartz class, the first integral converges for

$$\Re s_j > (j-1)\frac{d}{2} - \frac{N}{2r} \, ,$$

as it is easily deduced from (17). As observed earlier, the zeta integral can be continued meromorphically in $\mathbf{s}$ as a tempered distribution.

Theorem 2. *For any function f in the Schwartz space $\mathcal{S}(E)$,*

$$Z^*(\hat{f},-\mathbf{s}^*) = \gamma(\mathbf{s})Z(f,\mathbf{s}-\frac{N}{2r}) \quad ,$$

with

$$\gamma(\mathbf{s}) = \pi^{\frac{N}{2}}4^{-|\mathbf{s}|+\frac{N}{2}}\,\frac{\Gamma_\Omega(-\mathbf{s}^*+\frac{N}{2r})}{\Gamma_\Omega(\mathbf{s})} \quad .$$

We need an auxiliary result.

Proposition 3. *For* $a \in \Omega$, $\Re s_j > \frac{N}{2r} + \frac{d}{2}(r-j)$ *and* $\eta \in E$,

$$\int_E \Delta^*_{-\mathbf{s}}(a+Q(\xi))e^{-i<\xi,\eta>}d\xi = \frac{\pi^{\frac{N}{2}}}{\Gamma_\Omega(\mathbf{s}^*)}\, K_{\mathbf{s}^*-\frac{N}{2r}}(a, \frac{1}{4}Q(\eta)). \tag{18}$$

The proof can be read in [F-K], Prop. XVI.3.2., although the formula is erroneously stated for $\Delta_{-\mathbf{s}}$ instead of $\Delta^*_{-\mathbf{s}}$. One also needs the observation that the conditions on $\mathbf{s}$ imply that $K_{\mathbf{s}^*-\frac{N}{2r}}$ is indeed extendible to $\Omega \times \overline{\Omega}$, as $Q(\eta)$ may belong to the boundary of Ω.

Let us now prove Theorem 2. Assume first that $\Re s_j > \frac{N}{2r} + \frac{d}{2}(r-j)$. Let f be a function in the Schwartz class $\mathcal{S}(E)$, and let $\varepsilon > 0$. By the previous proposition,

(19)

$$\int_E \Delta^*_{-\mathbf{s}}(\varepsilon e + Q(\xi))\hat{f}(\xi)d\xi = \frac{\pi^{\frac{N}{2}}}{\Gamma_\Omega(\mathbf{s}^*)} \int_E K_{\mathbf{s}^*-\frac{N}{2r}}(\varepsilon e, \frac{1}{4}Q(\xi))f(\xi)d\xi \quad .$$

As $\Delta^*_j(a+b) \geq \Delta^*_j(a) > 0$ for any $a \in \Omega$ and $b \in \overline{\Omega}$, there are no singularities for $\xi \longmapsto \Delta^*_{\mathbf{s}}(\varepsilon e + Q(\xi))$ on E, and hence the left handside has a holomorphic extension to $\mathbb{C}^r$. Hence $\int_E K_{\mathbf{s}^*-\frac{N}{2r}}(\varepsilon e, \frac{1}{4}Q(\xi))f(\xi)d\xi$ has a meromorphic continuation as a function of $\mathbf{s}$, with singularities possibly at poles of $\Gamma_\Omega(\mathbf{s}^*)$.

Assume now that $\Re s_j < \frac{N}{2r} - \frac{d}{2}(j-1)$, and that $\mathbf{s}$ is not a singularity of $\Gamma_\Omega(\mathbf{s}^*)$. The conditions on $\mathbf{s}$ guarantee that $\Re(\mathbf{s}^* - \frac{N}{2r})_j < -\frac{d}{2}(r-j)$ so that we may use inequalities (10) to get the estimate (uniformly with respect to ε)

$$|K_{\mathbf{s}^*-\frac{N}{2r}}(\varepsilon e, \frac{1}{4}Q(\xi))| \leq \Gamma(-\Re\mathbf{s} + \frac{N}{2r})P_{\Re(\mathbf{s}^*-\frac{N}{2r})}(\frac{\xi}{2}) \quad .$$

Choose $\mathbf{m} \in \mathbb{N}^r$ with $m_1 \gg m_2 \gg \cdots \gg m_r \gg 0$ so that the integral

$$\int_E P_{\Re\mathbf{s}^*-\frac{N}{2r}}(\frac{\eta}{2})P_{\mathbf{m}}(\eta)\,|f(\eta)|\,d\eta$$

converges. Now, as $P_{\mathbf{m}}$ is of degree $2\,|\mathbf{m}\,|$

$$\big(P_{\mathbf{m}}(\frac{\partial}{\partial\xi})\hat{f}\big)(\xi) = (-1)^{|\mathbf{m}|}\big(\widehat{P_{\mathbf{m}}f}\big)(\xi) \quad .$$

Substitue $P_{\mathbf{m}}f$ to f in (19), and let $\varepsilon \longrightarrow 0$, using Lebesgue dominated convergence theorem and the limit result (11), to get

$$(-1)^{|\mathbf{m}|}\int_E P^*_{-\mathbf{s}}(\xi)\big(P_{\mathbf{m}}(\frac{\partial}{\partial\xi})\hat{f}\big)(\xi)d\xi = \gamma(\mathbf{s}) \int_E P_{\mathbf{s}^*-\frac{N}{2r}}(\eta)P_{\mathbf{m}}(\eta)f(\eta)d\eta\ ,$$

where

$$\gamma(s) = 4^{-|\mathbf{s}|+\frac{N}{2}} \frac{\pi^{\frac{N}{2}} \mathit{\Gamma}_\Omega(\frac{N}{2r} - \mathbf{s})}{\mathit{\Gamma}_\Omega(\mathbf{s}^*)} \quad .$$

This may be rewritten as

$$(-1)^{|\mathbf{m}|} Z^* \big(P_\mathbf{m}(\frac{\partial}{\partial \xi})\hat{f}, -\mathbf{s}\big) = \gamma(\mathbf{s}) Z(f, \mathbf{s}^* + \mathbf{m} - \frac{N}{2r}) \quad .$$

But after $2|\,\mathbf{m}\,|$ integrations by parts and using the Bernstein-type identity (13), we get

$$Z^* \big(P_\mathbf{m}(\frac{\partial}{\partial \xi})\hat{f}, -\mathbf{s}\big) = B_\mathbf{m}(-\mathbf{s}) Z^*(\hat{f}, -\mathbf{s} - \mathbf{m}^*)$$

Using (8)

$$(-1)^{|\mathbf{m}|} B_\mathbf{m}(-\mathbf{s}) = 4^{|\,\mathbf{m}\,|}(-\mathbf{s} - \mathbf{m}^* + \frac{N}{2r})_{\mathbf{m}^*} (\mathbf{s}^*)_\mathbf{m}$$

$$= 4^{|\,\mathbf{m}\,|} \frac{\mathit{\Gamma}_\Omega(-\mathbf{s} + \frac{N}{2r})}{\mathit{\Gamma}_\Omega(-\mathbf{s} - \mathbf{m}^* + \frac{N}{2r})} \frac{\mathit{\Gamma}_\Omega(\mathbf{s}^* + \mathbf{m})}{\mathit{\Gamma}_\Omega(\mathbf{s}^*)}$$

Hence,

$$Z^*(\hat{f}, -\mathbf{s} - \mathbf{m}^*) = 4^{-|\mathbf{s}+\mathbf{m}|+\frac{N}{2}} \pi^{\frac{N}{2}} \frac{\mathit{\Gamma}_\Omega(-\mathbf{s} - \mathbf{m}^* + \frac{N}{2r})}{\mathit{\Gamma}_\Omega(\mathbf{s}^* + \mathbf{m})} Z(f, \mathbf{s}^* + \mathbf{m} - \frac{N}{2r}) \ .$$

The change of variable $\sigma = \mathbf{s}^* + \mathbf{m}$ gives the result, at least in some open set. The full result is obtained by analytic continuation.

4. Zeta functions associated to $\mathit{\Phi}$-homogeneous harmonic polynomials

A polynomial h on E is said to be *$\mathit{\Phi}$-homogeneous* of degree k if

$$h(\mathit{\Phi}(x)\xi) = (\det x)^k h(\xi), \ \forall x \in V, \ \forall \xi \in E \quad . \tag{20}$$

Notice that h is then necessarily homogeneous of degree kr. The polynomial h is said to be harmonic if $\mathit{\Delta} h = 0$, where, as usal $\mathit{\Delta}$ denotes the Laplace operator associated to the inner product on E.

In preparation for the main theorem, we give two lemmas on Fourier transforms, which generalize classical formulae.

Lemma 1. *Let h be a polynomial on E which is $\mathit{\Phi}$-homogeneous of degree k and harmonic. Then, for all $x \in \Omega$ and for all $\eta \in E$,*

$$\int_E h(\xi) e^{-<\mathit{\Phi}(x)\xi, \xi>} e^{-i<\xi, \eta>} d\xi \ldots \tag{21}$$

$$= \pi^{\frac{N}{2}}(-i)^{kr}2^{-k}(\det x)^{-\frac{N}{2r}-k}h(\eta)e^{-\frac{1}{4}<\Phi(x^{-1})\eta,\eta>} \quad .$$

The lemma is a consequence of the classical Hecke formula for the Fourier transform of a homogeneous harmonic polynomial times a Gaussian function, after the change of variable $\xi \longmapsto \Phi(x^{\frac{1}{2}})\xi$.

Lemma 2. *Let h be a polynomial on E, which is Φ-homogeneous of degree k and harmonic. Then,*

$$\partial(h)P_{\mathbf{s}}(\xi) = (-1)^{kr}2^k \frac{\Gamma_\Omega(-\mathbf{s}^* + k)}{\Gamma_\Omega(-\mathbf{s}^*)} h(\xi)\, P_{\mathbf{s}-k}(\xi) \quad . \tag{22}$$

Needless to say, the equality has to be interpreted in the sense of tempered distributions. For convenience, we will prove the corresponding result for $P^*_{\mathbf{s}^*}$. First reinterpret (21) by using the classical result on Fourier transform that $\widehat{(pf)} = i^m\partial(p)\widehat{f}$ for p any homogeneous polynomial of degree m on E, to get

$$\partial(h)(e^{-<\Phi(x)\eta,\eta>}) = (-1)^{kr}2^k(\det x)^k h(\eta)e^{-<\Phi(x)\eta,\eta>} \quad . \tag{23}$$

Now assume that $\Re s_j > (j-1)\frac{d}{2}$, for $1 \le j \le r$. From (5) and (6), for any $\xi \in E'$,

$$P^*_{-\mathbf{s}^*}(\xi) = \frac{1}{\Gamma_\Omega(\mathbf{s})}\int_\Omega e^{-<\Phi(x)\xi,\xi>}\Delta_{\mathbf{s}}(x)d^*x \quad .$$

Thanks to (23),

$$\partial(h)P^*_{-\mathbf{s}^*}(\xi) = (-1)^{kr}2^k \frac{1}{\Gamma_\Omega(\mathbf{s})}h(\xi)\int_\Omega e^{-<\Phi(x)\xi,\xi>}\Delta_{\mathbf{s}+k}(x)d^*x \quad ,$$

from which follows

$$\partial(h)P^*_{-\mathbf{s}^*}(\xi) = (-1)^{kr}2^k \frac{\Gamma_\Omega(\mathbf{s}+k)}{\Gamma_\Omega(\mathbf{s})} h(\xi)P^*_{-\mathbf{s}^*-k}(\xi) \quad .$$

So, by analytic continuation both sides coincide on E' for all $\mathbf{s}$. Now for $\Re s_r \ll \Re s_{r-1} \ll \cdots \ll \Re s_1 \ll 0$, $P^*_{-\mathbf{s}^*}$ is continuously differentiable as many times as wanted, and hence the formula is valid on E by continuity. In particular the associated tempered distributions coincide. But now, $P^*_{-\mathbf{s}^*}$ viewed as a tempered distribution is meromorphic in s, and hence the result is valid everywhere.

Introduce the *zeta integral with coefficient h* to be either of the following expressions :

$$Z(f,\mathbf{s}\,;h) = \int_E P_{\mathbf{s}}(\xi)h(\xi)f(\xi)d\xi$$

$$Z^*(f,\mathbf{s}\,;\,h) = \int_E P^*_{\mathbf{s}}(\xi)h(\xi)f(\xi)d\xi$$

where f is any function int the Schwartz space $\mathcal{S}(E)$. As before, both expressions converge for $\Re s_j > (j-1)\frac{d}{2} - \frac{N}{2r}$, define tempered distributions and can be continued meromorphically in $\mathbb{C}^r$.

Theorem 3. *Let h be a harmonic polynomial, Φ- homogeneous of degree k. For any function f in the Schwartz space $\mathcal{S}(E)$,*

$$Z^*(\hat{f}, -\mathbf{s}^*; h) = \gamma_k(\mathbf{s}) Z(f, \mathbf{s} - \frac{N}{2r} - k\,; h)$$

where

$$\gamma_k(\mathbf{s}) = \pi^{\frac{N}{2}} 4^{-|\mathbf{s}|+\frac{N}{2}+\frac{k}{2}} (-i)^{kr} \frac{\Gamma_\Omega(-\mathbf{s}^* + \frac{N}{2r} + k)}{\Gamma_\Omega(\mathbf{s})} \quad .$$

On one hand, $Z^*(\hat{f}, -\mathbf{s}^*\,; h) = Z^*(h\hat{f}, -\mathbf{s}^*)$. Now for any polynomial p on E which is homogeneous of degree m, $p\hat{f} = (-i)^m\, \widehat{\partial(p) f}$. Hence

$$Z^*(\hat{f}, -\mathbf{s}^*\,; h) = (-i)^{kr} Z^*(\widehat{\partial(h) f}, -\mathbf{s}^*) \quad . \tag{24}$$

On the other hand,

$$\begin{aligned} Z(\partial(h) f, \mathbf{s}) &= (\partial(h) f, P_{\mathbf{s}}) = (-1)^{kr}(f, \partial(h) P_{\mathbf{s}}) \\ &= 2^k \frac{\Gamma_\Omega(-\mathbf{s}^* + k)}{\Gamma_\Omega(-\mathbf{s}^*)}(f, hP_{\mathbf{s}-k}) \end{aligned}$$

using successively integration by parts and (22), so that

$$Z(\partial(h) f, \mathbf{s}) = 2^k \frac{\Gamma_\Omega(-\mathbf{s}^* + k)}{\Gamma_\Omega(-\mathbf{s}^*)} Z(f, \mathbf{s} - k\,;\, h) \quad . \tag{25}$$

The result now follows from Theorem 2, (24) and (25).

5. Examples

Consider $V = Sym(r, \mathbb{R})$ the space of $m \times m$ symmetric matrices with real entries with its Jordan product $x.y = \frac{1}{2}(xy + yx)$, the inner product being $(x, y) = \operatorname{tr} xy$. The dimension of V is $n = \frac{1}{2} r(r+1)$, the rank is r and the integer d is 1. The set Ω in V is the cone of positive-definite matrices, the functions Δ_j are the usual principal minors d_j $(1 \leq j \leq r)$.

Let k be any integer, and let $E = M_{rk}$ be the vector space of $r \times k$ matrices with real entries, equipped with the inner product $< \xi, \eta >= \operatorname{tr} \xi\, \eta^t$. To each $x \in V$, define

$$\Phi(x)\xi = x\xi \quad .$$

Then $\Phi(x)$ is a symmetric operator on E and the mapping Φ defines a representation of V on E, which is regular if (and only if) we assume the condition $k \geq r$. The corresponding quadratic map $Q : E \longmapsto V$ is given by

$$Q(\xi) = \xi\,\xi^t \quad .$$

Hence, for $\mathbf{s} = (s_1, s_2, \dots, s_r)$

$$P_{\mathbf{s}}(\xi) = d_1(\xi\,\xi^t)^{s_1 - s_2} d_2(\xi\,\xi^t)^{s_2 - s_3} \dots d_r(\xi\,\xi^t)^{s_r} \quad .$$

The Φ-homogeneous polynomials are the so-called *determinantally homogeneous* polynomials. If $k > 2r$, the space of polynomials which are harmonic and Φ-homogeneous of degree m is spanned by the following polynomials

$$p(\xi) = \big(\mathrm{Det}(\xi\,\eta^t)\big)^m$$

where $\eta \in E^{\mathbb{C}}$ and $\eta\,\eta^t = 0$ (see [T] Corollary 3.7).

From a different point of view, E can be looked at as tensor product $\mathbb{R}^r \otimes \mathbb{R}^k$, or in other words, we may consider also actions on the right side. The group $GL_r(\mathbb{R})$ acts on the left by $\xi \longmapsto g\xi$, and we may consider the action of the group of isometries $O(k)$ on the *right* side by $\xi \longmapsto \xi\,u$. Finally, let introduce the Borel subgroup B_r of $GL_r(\mathbb{R})$ of invertible lower triangular $r \times r$ matrices. Then the space E under the action of $B_r \times O(k)$ is a prehomogeneous vector space. In fact, if ξ is an element of E' which we can think of as a set of r linearly independent vectors in $\mathbb{R}^k$, it can be transformed into an orthonormal r-frame by the Gram-Schmidt process, which is tantamount to a multiplication on the left by an element of B_r. Then, using the right action of $O(k)$, this can be transformed to the base point

$$I_r = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & \dots & 0 \end{pmatrix} \quad ,$$

thus showing that E' is a unique orbit under the action of $B_r \times O(k)$. The zeta distributions we have introduced and the corresponding functional equations could be obtained from the general theory developped by F. Sato (see [Sa] Theorem 3.1).

Similar examples come from the Jordan algebra of hermitian matrices (resp. quaternionic-hermitian matrices), and these examples can also be studied from the point of view of prehomogeneous vector spaces.

The situation for rank 2 Euclidean Jordan algebras exhibits new features. Let W be a Euclidean vector space of dimension q with inner product denoted by $\langle\ ,\ \rangle$, and define on $V = \mathbb{R} \oplus W$ the Jordan product

$$(\lambda, v)(\mu, w) = (\lambda + \langle v, w\rangle, \lambda w + \mu v) \quad .$$

Then V is a Euclidean Jordan algebra, of rank 2 and this construction exhausts all possibilities in rank 2 case (see [F-K]). The inner product is given by

$$\big((\lambda, v)\,,\,(\mu, w)\big) = \lambda\mu + \langle\, v, w\rangle$$

and the set Ω is then the *Lorentzian cone*

$$\Omega = \Big\{(\lambda, v) \mid \lambda^2 - < v, v > > 0, \lambda > 0\Big\} \quad .$$

The element $1 = (1, 0)$ is the neutral element of V. Choose an orthonormal basis of W, say $\{v_1, v_2, \ldots, v_q\}$. Then

$$1 = (\frac{1}{2}, \frac{1}{2}v_1) + (\frac{1}{2}, -\frac{1}{2}v_1)$$

is a Peirce decomposition. Moreover,

$$\Delta_2(\lambda, v) = \det(\lambda, v) = \lambda^2 - \langle v, v\rangle \text{ and } \Delta_1(\lambda, v) = \lambda + \langle v_1, v\rangle \quad .$$

Observe that for any $v, w \in W$, $(0, v)(0, w) =< v, w > 1$. Hence a representation of V on a Euclidean vector space $(E, < \ , \ >)$ is nothing but a *Clifford module* for the Clifford algebra $Cliff(W)$ associated to W with relations $v.w + w.v = 2 < v, w > 1$ (see [C2]). Let v be an element of W, which we can view as an element of $Cliff(W)$, and denote by $v.\xi = \Phi(0, v)\,\xi$ the corresponding action on the Clifford module E. Then an elementary calculation shows that the quadratic map Q is given by

$$Q(\xi) = \big(\|\xi\|^2, \sum_{i=1}^{q} < v_i.\xi, \xi > v_i\big)$$

In particular, for $\mathbf{s} = (s_1, s_2)$

$$P_{\mathbf{s}}(\xi) = \big(\|\xi\|^2 + < v_1\xi, \xi > \big)^{s_1 - s_2}\big(\|\xi\|^4 - \sum_{i=1}^{q} < v_i\xi, \xi >^2\big)^{s_2} \quad .$$

This situation is *not* related to a prehomogeneous vector space (except in low dimension), and hence the functional equations for the corresponding zeta distributions seem to be new.

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