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# On the behavior of solutions for a semilinear parabolic equation with supercritical nonlinearity

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**Abstract.** This paper is concerned with a Cauchy problem

$$\text{(P)}\qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(x,0) = \lambda\varphi(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $p > p_* \equiv (N+2)/(N-2), \lambda > 0$ and $\varphi$ is a nonnegative radially symmetric function in $C^1(\mathbf{R}^N)$ with compact support. Denote the solution of (P) by $u_\lambda$. Let $p^* = \infty$ if $3 \le N \le 10$ and $p^* = 1 + 6/(N-10)$ if $N \ge 11$. We show that if $p_* < p < p^*$, then there is $\lambda_\varphi > 0$ such that:

(i) If $\lambda < \lambda_\varphi$, then $u_\lambda$ exists globally in time in the classical sense and $u_\lambda(t)$ converges to zero locally uniformly in $\mathbf{R}^N$ as $t \to \infty$.
(ii) If $\lambda = \lambda_\varphi$, then $u_\lambda$ blows up *incompletely* in finite time.
(iii) If $\lambda > \lambda_\varphi$, then $u_\lambda$ blows up *completely* in finite time .

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## 1 Introduction

In this paper, we are concerned with a Cauchy problem

$$\text{(1.1)}\qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $p > 1$ and $u_0$ is a nonnegative function in $L^\infty(\mathbf{R}^N)$. The behavior of solutions of (1.1) depends on the value of $p$. Fujita [2] showed that all nontrivial solutions of (1.1) necessarily blow up in finite time if $1 < p \leq 1 + 2/N$, while there exists a nontrivial global classical solution of (1.1) if $p > 1 + 2/N$. Here we say that a solution $u$ of (1.1) blows up in finite time if $|u(t)|_\infty \to \infty$ as $t \to T$ for some $T < +\infty$, where $|\cdot|_\infty$ denotes the supremum norm in $\mathbf{R}^N$.

Furthermore when a solution $u$ of (1.1) blows up in finite time, the blowup is called complete if the continuation of the solution is trivial, that is, $u(x,t) \equiv \infty$ for $t > T$ with some $T < +\infty$ and incomplete otherwise in Galaktionov and Vazquez [3], [4]. In [4] it was proved that if $p$ is critical or subcritical in the sense of Sobolev embedding, that is, $(N-2)p \leq N+2$, then any radially symmetric solution of (1.1) exhibits the complete blowup, but both the complete and the incomplete blowup occur in the supercritical case, that is, $p > p_* \equiv (N+2)/(N-2)$ with $N \geq 3$. Some sufficient conditions on initial data for the complete blowup were given there. They also obtained a radially symmetric selfsimilar solution of (1.1) which blows up in finite time and then decays to zero selfsimilarly as $t \to \infty$ for $p_* < p < p^*$. Here $p^* = \infty$ if $3 \leq N \leq 10$ and $p^* = 1 + 6/(N-10)$ if $N \geq 11$.

On the other hand, Ni, Sacks and Tavantzis [8] showed the following: Let $p \geq p_*$ and $\Omega$ be a bounded convex domain instead of the whole space. Then for any nonnegative function $\varphi \in L^\infty(\Omega)$ there is $\lambda > 0$ such that a solution of (1.1) in $\Omega$ under the Dirichlet boundary condition with initial data $\lambda\varphi$ is global in the sense of $L^1(\Omega)$ but unbounded in $L^\infty(\Omega)$.

The purpose of this paper is to obtain solutions blowing up *incompletely* for a large class of initial data and to show that such a class forms a separatrix between global classical solutions converging to zero locally uniformly in $\mathbf{R}^N$ as $t \to \infty$ and *completely* blowing-up solutions in finite time. We call a function $u$ a global solution in the sense of $L^1_{loc}$ if $u \in C([0,\infty); L^1_{loc}(\mathbf{R}^N))$ satisfies

$$\int_s^t \int_{\mathbf{R}^N} \{u\rho_\tau + u\Delta\rho + u^p\rho\}\, dx d\tau - \left[\int_{\mathbf{R}^N} u(\tau)\rho dx\right]_s^t = 0$$

for any $0 \leq s < t < \infty$ and $\rho \in C^2(\mathbf{R}^N \times [0,\infty))$ with compact support in $\mathbf{R}^N \times [0,\infty)$, where $L^1_{loc}(\mathbf{R}^N)$ denotes the space of locally integrable functions on $\mathbf{R}^N$. Denote by $\mathcal{D}$ the set of nonnegative radially symmetric functions $f(r)$ of class $C^1$ with compact support in $[0,\infty)$ such that the set of local minima of $f(r)$ is bounded away from zero. Here $f(r_0)$ is called a local minimum of $f(r)$ if $f(r_0) \leq f(r)$ in $U$ and $f(r_0) < f(r)$ on $\partial U$ for some bounded neighborhood of $r_0$ in $[0,\infty)$.

**Theorem 1.1** *Let* $p_* = (N+2)/(N-2)$, *and* $p^* = \infty$ *if* $3 \leq N \leq 10$ *and* $p^* = 1 + 6/(N-10)$ *if* $N \geq 11$. *If* $p_* < p < p^*$, *then for each* $\varphi \in \mathcal{D}$ *there exists* $\lambda_\varphi > 0$ *such that:*

(i) *If $\lambda < \lambda_\varphi$, then $u_\lambda$ exists globally in time in the classical sense and $u_\lambda(t)$ converges to zero locally uniformly in $\mathbf{R}^N$ as $t \to \infty$.*
(ii) *If $\lambda = \lambda_\varphi$, then $u_\lambda$ is a global solution in the sense of $L^1_{loc}$ and blows up incompletely in finite time.*
(iii) *If $\lambda > \lambda_\varphi$, then $u_\lambda$ blows up completely in finite time,*

*where $u_\lambda$ denotes the solution of (1.1) with initial data $\lambda\varphi$.*

*Furthermore in the case of $\lambda \le \lambda_\varphi$, there is $t_\lambda > 0$ such that for $t \ge t_\lambda$ the solution $u_\lambda(x,t)$ is nonincreasing in $|x|$ and satisfies*

$$u_\lambda(x,t) \le C|x|^{-2/(p-1)} \quad \textit{for } \; x \in \mathbf{R}^N \backslash \{0\},$$

*where $C$ is a positive constant depending only on $N$ if the first derivative $\varphi_r$ with respect to $r = |x|$ changes its sign at most finitely many times.*

This result is similar to the above one due to [8]. The most important step in the proof is to show that a set of parameter $\lambda > 0$ defined by

$$(1.2) \qquad \begin{aligned} \Lambda = \{\lambda > 0 : u_\lambda \text{ is a global classical solution of (1.1) and} \\ u_\lambda(t) \to 0 \text{ locally uniformly in } \mathbf{R}^N \text{ as } t \to \infty\} \end{aligned}$$

is open in $(0,\infty)$. In [8], the proof of openness of $\Lambda$ is based on the fact that zero is an isolated stationary solution of (1.1) on a bounded domain. However under the present circumstance, zero is not isolated (see e.g. [5], [6], [10]), which makes the situation different from that in [8]. In order to prove the openness of $\Lambda$, we make use of the nonincrease of intersection number between two solutions of (1.1) in time , the properties of stationary solutions and selfsimilar blowup solutions of (1.1) and an estimate of solutions of (1.1) at spatial infinity at each time.

Throughout the present paper, we use notations $p_*$ and $p^*$ to denote the special exponents in Theorem 1.1.

This paper is organized as follows: In Sect. 2, we get some preliminary results. Section 3 is devoted to the proof of openness of $\Lambda$. We complete the proof of main theorem in Sect. 4.

## 2 Preliminary results

In this section, we prepare some results which are used in the subsequent sections.

For $R > 0$, let $\gamma_R$ be the first eigenvalue of $-\Delta$ in $B_R(0)$ with the Dirichlet boundary condition and $\phi_R$ the corresponding eigenfunction normalized in $L^1(B_R(0))$, where $B_R(x)$ is the open ball with radius $R > 0$ centered at $x$ in $\mathbf{R}^N$. Then it is immediate that

$$(2.1) \qquad \gamma_R = \frac{\gamma_1}{R^2} \quad \text{and} \quad \phi_R(r) = \frac{1}{R^N}\phi_1\left(\frac{r}{R}\right) \quad \text{for } \; r \ge 0.$$

**Lemma 2.1** *If $u$ is a global classical solution of (1.1) with nonnegative initial data in $L^\infty(\mathbf{R}^N)$, then for any $R > 0$ it holds*

$$\int_{B_R(0)} u(x,t)\phi_R(x)dx \le \gamma_1^{1/(p-1)} R^{-2/(p-1)} \quad \text{for all } t \ge 0.$$

*Proof* Fix $R > 0$ arbitrarily. Multiplying (1.1) by $\phi_R$ and integrating by parts yields

$$\begin{aligned}&\frac{d}{dt}\int_{B_R(0)} u(x,t)\phi_R(x)dx \\ &\ge -\gamma_R \int_{B_R(0)} u(x,t)\phi_R(x)dx + \left(\int_{B_R(0)} u(x,t)\phi_R(x)dx\right)^p\end{aligned}$$

for all $t > 0$ from Jensen's inequality. Then the assertion follows from (2.1) since $u(t) \in L^1_{loc}(\mathbf{R}^N)$ for all $t \ge 0$. □

We next obtain a pointwise estimate of a global classical solution of (1.1). For a radially symmetric function $f$ with $f \not\equiv 0$, define $z(f)$ by the supremum over all $j$ such that there exist $0 \le r_1 < r_2 < \cdots < r_{j+1} < +\infty$ with

$$f(r_i) \cdot f(r_{i+1}) < 0 \quad \text{for } i = 1, 2, \cdots, j.$$

For two radially symmetric solutions $u_1$ and $u_2$ of (1.1) in $(0, t_0)$ with $t_0 > 0$, the following is shown in the same way as in [1];

(i) $z(u_1(t) - u_2(t)) < \infty$ for $0 < t < t_0$
(ii) $z(u_1(t) - u_2(t))$ is nonincreasing in $0 \le t < t_0$
(iii) if

$$u_1(r_1, t_1) - u_2(r_1, t_1) = 0 \text{ and } (u_1(r, t_1) - u_2(r, t_1))_r\,|_{r=r_1} = 0$$

for some $r_1 \ge 0$, and $0 < t_1 < t_0$, then

$$z(u_1(t) - u_2(t)) < z(u_1(s) - u_2(s)) \quad \text{for } 0 < s < t_1 < t < t_0.$$

**Lemma 2.2** *If $u$ is a radially symmetric global classical solution of (1.1) with positive initial data $u_0$ in $C^1(\mathbf{R}^N)$ satisfying $z((u_0)_r) < \infty$, then it holds $u_r(r,t) \le 0$ and*

$$u(r,t) \le \frac{N\gamma_1^{1/(p-1)}}{\omega_N} \cdot \inf_{0<k<1} \left\{ \frac{k^{2/(p-1)-N}}{\phi_1(k)} \right\} \cdot r^{-2/(p-1)}$$

*for all $r > 0$ and $t \ge t_0$ with some positive constant $t_0$ satisfying*

$$t_0 \ge \frac{1}{(p-1)m_0^{p-1}},$$

*where*

$$m_0 = \min\{u_0(r) : r \text{ is a local minimizer of } u_0\}.$$

*Proof* We sketch the proof to get a lower estimate of $t_0$ although our method is similar to that in [7]. Let $S$ be the maximal existence time of a curve $r(t)$ of local minimum of $u(r,t)$. The differentiability of $r(t)$ was studied in [7]. Then we see

$$r(t)^{N-1}u_{rr}(r(t),t) + (N-1)u_r(r(t),t) \geq 0 \quad \text{for all } t > 0.$$

Putting $m(t) = u(r(t),t)$, we have

$$\begin{aligned} m'(t) &= u_r(r(t),t)r'(t) + u_t(r(t),t) \\ &\geq m(t)^p. \end{aligned}$$

This implies

$$S \leq \frac{1}{(p-1)m(0)^{p-1}} \leq \frac{1}{(p-1)m_0^{p-1}}.$$

Thus $u_r \leq 0$ in $[0,\infty)$ for all $t \geq t_0$ with some positive constant $t_0$ with $t_0 \geq \dfrac{1}{(p-1)m_0^{p-1}}$ since $z((u_0)_r) < \infty$.

Then it follows from Lemma 2.1 and (2.1) that

$$\begin{aligned} \frac{\omega_N k^N \phi_1(k)}{N} u(kr,t) &\leq \int_{S^{N-1}} \int_0^{kr} u(\rho,t)\phi_r(\rho)\rho^{N-1} d\rho \\ &\leq \gamma_1^{1/(p-1)} r^{-2/(p-1)} \end{aligned}$$

for all $r > 0$ and $t \geq t_0$, where $k$ is an arbitrary constant with $0 < k < 1$ and $\omega_N$ denotes the area of the $(N-1)$-dimensional unit sphere. Therefore we obtain

$$u(r,t) \leq \frac{N\gamma_1^{1/(p-1)}}{\omega_N} \cdot \inf_{0<k<1} \left\{ \frac{k^{2/(p-1)-N}}{\phi_1(k)} \right\} \cdot r^{-2/(p-1)}$$

for all $r > 0$ and $t \geq t_0$. This completes the proof. □

Putting

(2.2) $$\psi(r) = \alpha r^{-2/(p-1)} \quad \text{for } r > 0$$

with

(2.3) $$\alpha = \left( \frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right) \right)^{1/(p-1)},$$

it is trivial that $\psi$ is a singular stationary solution of (1.1).

The following property of stationary solutions of (1.1) was investigated in [5], which is summarized in Lemma 9.3 in [4].

**Proposition 2.1** *For $p > p_*$, the set of positive radially symmetric stationary solutions of (1.1) consists of $\{\mu^{2/(p-1)}h(\mu r) : \mu > 0\}$ with a fixed $h$ and any positive radially symmetric stationary solution $w$ of (1.1) satisfies*

$$\frac{w(r)}{\psi(r)} \to 1 \quad \text{as } r \to \infty,$$

*where $\psi$ is defined in (2.2).*

For an arbitrary $T > 0$, put

(2.4) $$v_T(r,t) = (T-t)^{-1/(p-1)}V(\eta) \quad \text{and} \quad \eta = (T-t)^{-1/2}r,$$

where $V$ satisfies

(2.5) $$V_{\eta\eta} + \frac{N-1}{\eta}V_\eta - \frac{\eta}{2}V_\eta - \frac{1}{p-1}V + V^p = 0 \quad \text{for } \eta > 0.$$

Then $v_T$ is a backward selfsimilar solution of (1.1) which blows up at $t = T$.

The following result was shown in Theorems 12.1 and 12.2 of [4].

**Proposition 2.2** *If $p_* < p < p^*$, then there exists a solution $V$ of (2.5) satisfying*

$$\frac{V(\eta)}{\psi(\eta)} \to \beta \quad \text{as } \eta \to \infty$$

*for some positive constant $\beta < 1$.*

We get an estimate of solutions of (1.1) at spatial infinity enough to compare them with a stationary solution in Proposition 2.1 and a backward selfsimilar blowup solution in Proposition 2.2. Let $\mathrm{supp}(f)$ denote the support of a function $f$.

**Lemma 2.3** *Suppose that $u_0 \in L^\infty(\mathbf{R}^N)$ is nonnegative, not identically equal to zero and has compact support, that is, supp$(u_0) \subset B_R(0)$ for some $R > 0$. Let $u$ be a global classical solution of (1.1) with initial data $u_0$. Then for each $t > 0$ there is a positive constant $C_t$ such that*

$$u(x,t) \le C_t \exp\left(-\frac{|x|^2}{32t}\right) \quad \text{for } x \in \mathbf{R}^N \text{ with } |x| \ge 2R.$$

*Proof* Setting $M(t) = \sup\{|u(s)|_\infty : 0 \le s \le t\}$ for $t \ge 0$, it holds

$$u_t \le \Delta u + M(t)^{p-1}u \quad \text{in } \mathbf{R}^N \times (0,t).$$

According to the comparison theorem, we see

(2.6) $$u(x,t) \le K(t)U(x,t) \quad \text{in } \mathbf{R}^N \times (0,t),$$

where $U$ is a solution of the heat equation in $\mathbf{R}^N$ with initial data $u_0$ and

$$K(t) = \exp\left(\int_0^t M(s)^{p-1} ds\right) \quad \text{for } t > 0.$$

For $x \in \mathbf{R}^N$ with $|x| \geq 2R$ and $t > 0$, we have

$$\begin{aligned} U(x,t) &= \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy \\ &\leq \frac{M(0)}{(4\pi t)^{N/2}} \int_{B_R(0)} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \\ &\leq \frac{M(0)}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{32t}\right) \int_{B_R(0)} \exp\left(-\frac{|x-y|^2}{8t}\right) dy \\ &= 2^{N/2} M(0) \exp\left(-\frac{|x|^2}{32t}\right). \end{aligned}$$

This inequality together with (2.6) yields

$$u(x,t) \leq 2^{N/2} M(0) K(t) \exp\left(-\frac{|x|^2}{32t}\right)$$

for $x \in \mathbf{R}^N$ with $|x| \geq 2R$ and $t > 0$. This completes the proof. □

Define the energy functional $E$ by

$$E(w) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla w|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |w|^{p+1} dx$$

for $w \in L^{p+1}(\mathbf{R}^N)$ with $\nabla w \in \left(L^2(\mathbf{R}^N)\right)^N$. Let $u$ be a global classical solution of (1.1) with $u_0 \in L^\infty(\mathbf{R}^N) \cap L^{p+1}(\mathbf{R}^N)$ and $\nabla u_0 \in \left(L^2(\mathbf{R}^N)\right)^N$. Multiplying (1.1) by $u_t$ and integrating by parts, we have

$$\int_{\mathbf{R}^N} u_t(t)^2 dx = -\frac{d}{dt} E(u(t)) \quad \text{for } t > 0 \tag{2.7}$$

and hence $E(u(t))$ is nonincreasing in $t$.

**Lemma 2.4** *Let $u_0 \in L^\infty(\mathbf{R}^N) \cap L^{p+1}(\mathbf{R}^N)$ with $\nabla u_0 \in \left(L^2(\mathbf{R}^N)\right)^N$. If $u$ is a global classical solution of (1.1) with nonnegative initial data $u_0$, then $E(u(t)) \geq 0$ for all $t \geq 0$.*

*Proof* Assume that $E(u(t_0)) < 0$ for some $t_0 \geq 0$. Then there are $R > 0$ and $\tilde{u}_0 \in L^{p+1}(B_R(0))$ with $\nabla \tilde{u}_0 \in \left(L^2(B_R(0))\right)^N$ such that $\operatorname{supp}(\tilde{u}_0) \subset B_R(0)$, $0 \leq \tilde{u}_0 \leq u(t_0)$ in $B_R(0)$ and $E(\tilde{u}_0) < 0$. Let $\tilde{u}$ be a solution of

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + \tilde{u}^p & \text{in } B_R(0) \times (0, \infty), \\ \tilde{u}(x,t) = 0 & \text{on } \partial B_R(0) \times (0, \infty), \\ \tilde{u}(x,0) = \tilde{u}_0(x) & \text{in } B_R(0). \end{cases} \tag{2.8}$$

Let $\tilde{E}$ be the energy functional in $B_R(0)$ defined by

$$\tilde{E}(w) = \frac{1}{2} \int_{B_R(0)} |\nabla w|^2 dx - \frac{1}{p+1} \int_{B_R(0)} |w|^{p+1} dx$$

for $w \in H_0^1(B_R(0))$. Multiplying (2.8) by $\tilde{u}$ and integrating by parts yields

$$\begin{aligned} \frac{d}{dt} \int_{B_R(0)} \tilde{u}(t)^2 dx &= -2\tilde{E}(\tilde{u}(t)) + \frac{p-1}{p+1} \int_{B_R(0)} \tilde{u}(t)^{p+1} dx \\ &\geq C \left( \int_{B_R(0)} \tilde{u}(t)^2 dx \right)^{(p+1)/2} \end{aligned}$$

for some $C > 0$ since $\tilde{E}(\tilde{u}(t))$ is nonincreasing in $t$. Here we used Jensen's inequality to get the above inequality. This implies that $\tilde{u}$ blows up at some $\tilde{T} < +\infty$. On the other hand, we get

$$u(x, t + t_0) \geq \tilde{u}(x,t) \quad \text{in } B_R(0) \times (0, \tilde{T})$$

by the comparison theorem and hence $u$ blows up in finite time. This contradiction completes the proof. □

**Lemma 2.5** *If $u$ is a radially symmetric global classical solution of (1.1) with nonnegative initial data $u_0 \in \mathcal{D}$ such that $\limsup_{t \to \infty} |u(t)|_\infty < +\infty$, then $u(t)$ converges to a nonnegative radially symmetric stationary solution of (1.1) locally uniformly in $\mathbf{R}^N$ as $t \to \infty$.*

*Proof* Integrating (2.7) in $(0, s)$ for any $s > 0$, we get

$$\begin{aligned} \int_0^s \int_{\mathbf{R}^N} u_t(t)^2 dx dt &= E(u_0) - E(u(s)) \\ &\leq E(u_0) \end{aligned}$$

for any $s > 0$ since $E(u(s)) \geq 0$ from Lemma 2.4 and hence

$$\int_0^\infty \int_{\mathbf{R}^N} u_t(t)^2 dx dt < \infty.$$

Thus there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $u_t(t_n)$ converges to 0 in $L^2(\mathbf{R}^N)$ as $n \to \infty$. Then we can take a subsequence, which is written by $\{t_n\}$ again, such that $u(t_n)$ converges to a stationary solution $w$ of (1.1) locally uniformly in $\mathbf{R}^N$ as $n \to \infty$ by the parabolic regularity theory. It is immediate that $w$ is nonnegative and radially symmetric.

We assume that there are $\{t_n\}, \{s_n\}$ with $t_n \to \infty$ and $s_n \to \infty$ as $n \to \infty$ and nonnegative radially symmetric stationary solutions $w_1, w_2$ with $w_1 \not\equiv w_2$ such that

(2.9) $u(t_n) \to w_1, \quad u(s_n) \to w_2$ locally uniformly in $\mathbf{R}^N$ as $n \to \infty$.

We may suppose without loss of generality that $w_1(0) < w_2(0)$. Choose a positive radially symmetric stationary solution $w_3$ such that $z(u_0 - w_3) < +\infty$ and $w_1(0) < w_3(0) < w_2(0)$. On the other hand, there is $\{\tau_n\}$ with $\tau_n \to \infty$ as $n \to \infty$ such that $u(0, \tau_n) = w_3(0)$ for all $n$ from (2.9). Since $(u(t) - w_3)_r\,|_{r=0} = 0$ for $t > 0$, this cannot occur by the finiteness of $z(u_0 - w_3)$. This contradiction implies that $u(t)$ converges to $w$ locally uniformly in $\mathbf{R}^N$ as $t \to \infty$. □

## 3 Proof of openness of $\Lambda$

This section is devoted to the proof of openness of $\Lambda$ defined by (1.2). To do that, we need the following result.

**Lemma 3.1** *Let $\varphi \in \mathcal{D}$ and $u_\lambda$ be a solution of (1.1) with initial data $\lambda\varphi$ for $\lambda > 0$. If $u_\lambda(t)$ converges to a nonnegative radially symmetric stationary solution $w$ of (1.1) locally uniformly in $\mathbf{R}^N$ as $t \to \infty$, then $w$ is identically equal to zero.*

*Proof* Now we assume that $w \not\equiv 0$. Take a positive constant $\delta$ with $\delta < (\alpha - \beta)/(\alpha + \beta)$, where $\alpha$ and $\beta$ are positive constants in (2.3) and Proposition 2.2, respectively. It follows from Proposition 2.1 that

$$w(r) \geq \alpha(1-\delta)r^{-2/(p-1)} \quad \text{for } r \geq r_2 \tag{3.1}$$

with some $r_2 > 0$. Fix a positive constant $\varepsilon$ with $\varepsilon < \{\alpha - \beta - \delta(\alpha + \beta)\}r_2^{-2/(p-1)}$. Since $u(t) \to w$ locally uniformly in $\mathbf{R}^N$ as $t \to \infty$, there is $t_1 > 0$ such that

$$|u_\lambda(r,t) - w(r)| < \varepsilon \quad \text{for } 0 \leq r \leq r_2 \text{ and } t \geq t_1. \tag{3.2}$$

Thus we get

$$u_\lambda(r_2, t) \geq \alpha(1-\delta)r_2^{-2/(p-1)} - \varepsilon \quad \text{for } t \geq t_1. \tag{3.3}$$

from (3.1) and (3.2).

On the other hand, let $v_T$ be defined by (2.4) using $V$ obtained in Proposition 2.2. According to Proposition 2.2, there exists $r_3 > 0$ such that

$$V(\eta) \leq \beta(1+\delta)\eta^{-2/(p-1)} \quad \text{for } \eta \geq r_3 \tag{3.4}$$

and hence

$$v_T(r,t) \leq \beta(1+\delta)r^{-2/(p-1)} \quad \text{for } r \geq (T-t)^{1/2} r_3. \tag{3.5}$$

Since $\varphi \in \mathcal{D}$ , we have $z(\lambda\varphi - v_T(0)) \leq 1$ if $T > \max\{t_1, (r_2/r_3)^2\}$ is sufficiently large. Since $v_T$ blows up at the origin at $t = T$ and $u_\lambda(t)$ is uniformly bounded in $\mathbf{R}^N$ from (3.8), we see

$$u_\lambda(0, s_T) < v_T(0, s_T) \quad \text{for some positive } s_T < T.$$

By Proposition 2.2 and Lemma 2.3, for each $0 < t < T$

$$u_\lambda(r,t) < v_T(r,t) \quad \text{for } r \geq R_t$$

with some $R_t > 0$. Therefore it follows from the nonincrease of intersection number between $u_\lambda$ and $v_T$ that

$$u_\lambda(r, s_T) \leq v_T(r, s_T) \quad \text{for } r \geq 0$$

and hence

$$u_\lambda(r,t) \leq v_T(r,t) \quad \text{for } r \geq 0 \text{ and } s_T < t < T. \tag{3.6}$$

However it follows from (3.3) and (3.5) that if

$$\max\left\{s_T, t_1, T - (r_2/r_3)^2\right\} < t < T,$$

then

$$u_\lambda(r_2, t) > v_T(r_2, t)$$

by the choice of $\varepsilon > 0$, which contradicts (3.6). Therefore we obtain $w \equiv 0$. □

**Lemma 3.2** *If $\varphi \in \mathcal{D}$, then the set $\Lambda$ defined by (1.2) is open in* $(0, \infty)$.

*Proof* Suppose that $\lambda_0 \in \Lambda$. Then $u_{\lambda_0}$ is a global classical solution of (1.1) and $u_{\lambda_0}(t) \to 0$ locally uniformly in $\mathbf{R}^N$ as $t \to \infty$. From $\varphi \in \mathcal{D}$, we can take a positive radially symmetric stationary solution $w$ satisfying $z(\lambda\varphi - w) \leq 1$ for $\lambda > 0$ with $|\lambda - \lambda_0|$ sufficiently small by Proposition 2.1. Since $u_{\lambda_0}(t) \to 0$ locally uniformly in $\mathbf{R}^N$ as $t \to \infty$, there are $r_0, t_0 > 0$ such that

$$u_{\lambda_0}(r, t_0) < w(r) \quad \text{for } 0 \leq r \leq r_0.$$

Then we get

$$u_\lambda(r, t_0) < w(r) \quad \text{for } 0 \leq r \leq r_0$$

if $|\lambda-\lambda_0|$ is sufficiently small. On the other hand, it follows from Proposition 2.1 and Lemma 2.3 that

$$u_\lambda(r,t_0) < w(r) \quad \text{for } r \geq r_1(\lambda)$$

for some $r_1(\lambda) > r_0$ for $\lambda$ with $|\lambda-\lambda_0|$ sufficiently small. Therefore if $|\lambda-\lambda_0|$ is sufficiently small, then

$$u_\lambda(r,t_0) \leq w(r) \quad \text{for } r \geq 0. \tag{3.7}$$

Indeed, assuming that this does not hold, we see $z(u_\lambda(t_0)-w) \geq 2$, which is a contradiction since

$$z(u_\lambda(t_0)-w) \leq z(\lambda\varphi - w) \leq 1.$$

This implies (3.7). Then it holds

$$u_\lambda(r,t) \leq w(r) \quad \text{for } r \geq 0 \text{ and } t \geq t_0 \tag{3.8}$$

and hence $u_\lambda$ exists globally in time and $\sup_{t>0} |u_\lambda(t)|_\infty < +\infty$ if $|\lambda-\lambda_0|$ is sufficiently small. According to Lemmas 2.5 and 3.1, we see that $u_\lambda(t)$ converges to zero locally uniformly in $\mathbf{R}^N$ as $t\to\infty$ if $|\lambda-\lambda_0|$ is sufficiently small. This completes the proof. □

## 4 Proof of Theorem 1.1

In this section, the proof of main theorem is completed. The following result is similar to Theorem 15.1 in [4], which treated a global $L^1$-solution obtained by [8]. However when one considers (1.1) in the whole space, the comparison between two solutions must be done more carefully than the case of the Dirichlet boundary condition on a bounded domain.

**Lemma 4.1** *Suppose that $p_* < p < p^*$. If $u_0 \in \mathcal{D}$, then a global classical solution $u$ of (1.1) with initial data $u_0$ does not grow up, that is, $u$ is not a global classical solution of (1.1) for which $\limsup_{t\to\infty} |u(t)|_\infty = \infty$.*

*Proof* On the contrary, we assume that the solution $u$ grows up. Letting $v_T$ be a solution of (1.1) defined by (2.4), we see $z(u_0 - v_T(0)) \leq 1$ if $T>0$ is sufficiently large.

For $\mu > 0$, let $w_\mu$ be a positive radially symmetric stationary solution of (1.1) with $w_\mu(0)=\mu$ and $(w_\mu)_r(0)=0$. Take a positive constant $\delta$ with $\delta < (\alpha-\beta)/(\alpha+\beta)$. From Proposition 2.1, it holds

$$w_\mu(r) \geq \alpha(1-\delta)r^{-2/(p-1)} \quad \text{for } r \geq \frac{r_\delta}{\mu} \tag{4.1}$$

with some $r_\delta > 0$ independent of $\mu > 0$. Since $|u(t)|_\infty$ is not uniformly bounded in $t \geq 0$, for each $t > 0$ there is $r_\mu(t) \geq 0$ such that

$$u\left(r_\mu(t), t\right) > w_\mu\left(r_\mu(t)\right). \tag{4.2}$$

Fix $T > 0$ sufficiently large so that $z(u_0 - v_T(0)) \leq 1$. Put

$$M_T = \sup\{|u(t)|_\infty : 0 \leq t \leq T\}$$

and choose $r_T$ with

$$0 < r_T < \min\left\{\left(\frac{\alpha(1-\delta)}{2M_T}\right)^{(p-1)/2}, T^{1/2} r_3\right\},$$

where $r_3$ is the positive constant in (3.4) from which (3.5) follows. Since $w_\mu(r)$ is decreasing in $r \geq 0$, we get

$$w_\mu(r) \geq 2M_T \quad \text{for } 0 \leq r \leq r_T$$

if $\mu \geq r_\delta/r_T$ by (4.1). Let $\mu \geq r_\delta/r_T$. Then we obtain $r_\mu(t) \geq r_T \geq r_\delta/\mu$ for $0 \leq t \leq T$. It follows from (4.1) and (4.2) that

$$u(r_\mu(t), t) \geq \alpha(1-\delta)\left(r_\mu(t)\right)^{-2/(p-1)} \quad \text{for } 0 \leq t \leq T. \tag{4.3}$$

On the other hand, since $v_T$ satisfies (3.5), we have

$$v_T(r_\mu(t), t) \leq \beta(1+\delta)\left(r_\mu(t)\right)^{-2/(p-1)} \tag{4.4}$$

for $T - (r_T/r_3)^2 < t < T$. It follows from (4.3) and (4.4) that

$$u(r_\mu(t), t) > v_T(r_\mu(t), t) \tag{4.5}$$

for $T - (r_T/r_3)^2 < t < T$ by the choice of $\delta > 0$. It is also trivial that

$$u(0, t) < v_T(0, t) \tag{4.6}$$

for $0 < t < T$ sufficiently close to $T$. By Proposition 2.2 and Lemma 2.3, for each $0 < t < T$ there is $R_t > 0$ such that

$$u(r, t) < v_T(r, t) \quad \text{for } r \geq R_t. \tag{4.7}$$

It follows from (4.5)-(4.7) that $z(u(t) - v_T(t)) \geq 2$ if $0 < t < T$ is sufficiently close to $T$, which contradicts $z(u(t) - v_T(t)) \leq z(u_0 - v_T(0)) \leq 1$ for $0 < t < T$. This completes the proof. □

The following result can be shown in the same way as in the proof of Theorem 5.1 in [4].

**Proposition 4.1** *Let $u$ be a positive radially symmetric solution of (1.1). If $B[u](T) \equiv \{r \geq 0 : u(r,T) = \infty\} \neq \{0\}$ for some $T > 0$, then $u$ exhibits the complete blowup.*

We are now in a position to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We first put $\lambda_\varphi = \sup \Lambda$. According to Fujita's result ([2]), $u_\lambda$ exists globally in time in the classical sense and $|u_\lambda(t)|_\infty \to 0$ as $t \to \infty$ if $\lambda > 0$ is sufficiently small. It is also immediate that $u_\lambda$ blows up in finite time for sufficiently large $\lambda > 0$. Hence we see $0 < \lambda_\varphi < \infty$. Define

$$u_{\lambda_\varphi}(x,t) = \lim_{\lambda \to \lambda_\varphi} u_\lambda(x,t) \quad \text{for } (x,t) \in \mathbf{R}^N \times [0,\infty)$$

since $\{u_\lambda : 0 < \lambda < \lambda_\varphi\}$ is nondecreasing in $\lambda$ by the comparison theorem.

Let $R > 0$ and $0 \leq s < t < \infty$ be arbitrary. Multiplying (1.1) with $u = u_\lambda$ by $\phi_R$ and integrating by parts yields

$$\int_s^t \int_{\mathbf{R}^N} u_\lambda(\tau)^p \phi_R dx d\tau \leq \int_{\mathbf{R}^N} u_\lambda(t)\phi_R dx + \gamma_R \int_s^t \int_{\mathbf{R}^N} u_\lambda(\tau)\phi_R dx d\tau \tag{4.8}$$

for $0 < \lambda < \lambda_\varphi$, where $\gamma_R$ and $\phi_R$ denote the first eigenvalue of $-\Delta$ in $B_R(0)$ with the Dirichlet boundary condition and the corresponding eigenfunction normalized in $L^1(B_R(0))$, respectively. From Lemma 2.1, we have

$$\int_{\mathbf{R}^N} u_\lambda(t)\phi_R dx \leq \gamma_1^{1/(p-1)} R^{-2/(p-1)} \tag{4.9}$$

and hence

$$\int_s^t \int_{\mathbf{R}^N} u_\lambda(\tau)\phi_R dx d\tau \leq \gamma_1^{1/(p-1)} R^{-2/(p-1)} t \quad \text{for } 0 < \lambda < \lambda_\varphi. \tag{4.10}$$

Thus it follows from (4.8) that

$$\int_s^t \int_{\mathbf{R}^N} u_\lambda(\tau)^p \phi_R dx d\tau \leq \gamma_1^{1/(p-1)} R^{-2/(p-1)} (1 + \gamma_R t) \tag{4.11}$$

for $0 < \lambda < \lambda_\varphi$. Using the above estimates (4.9)-(4.11) with $2R$ instead of $R$, there is $C > 0$ such that

$$\int_{B_R(0)} u_\lambda(t) dx \leq C R^{N-2/(p-1)}$$

$$\int_s^t \int_{B_R(0)} u_\lambda(\tau) dx d\tau \leq C R^{N-2/(p-1)} t$$

$$\int_s^t \int_{B_R(0)} u_\lambda(\tau)^p dx d\tau \le C R^{N-2/(p-1)}(1+\gamma_R t)$$

since $\phi_R(r) = R^{-N}\phi_1(R^{-1}r)$ for $r \ge 0$. Therefore it follows from Fatou's lemma that $u_{\lambda_\varphi}(t) \in L^1(B_R(0))$ and $u_{\lambda_\varphi} \in L^1(B_R(0)\times(s,t)) \cap L^p(B_R(0)\times(s,t))$.

For any $\rho \in C^2(\mathbf{R}^N \times [0,\infty))$ with bounded support in $\mathbf{R}^N \times [0,\infty)$, it holds

$$\int_s^t \int_{\mathbf{R}^N} \left\{u_\lambda \rho_\tau + u_\lambda \Delta\rho + u_\lambda^p \rho\right\} dx d\tau - \left[\int_{\mathbf{R}^N} u_\lambda(\tau)\rho dx\right]_s^t = 0$$

for $0 < \lambda < \lambda_\varphi$. Letting $\lambda \to \lambda_\varphi$, we get

$$\int_s^t \int_{\mathbf{R}^N} \left\{u_{\lambda_\varphi} \rho_\tau + u_{\lambda_\varphi} \Delta\rho + u_{\lambda_\varphi}^p \rho\right\} dx d\tau - \left[\int_{\mathbf{R}^N} u_{\lambda_\varphi}(\tau)\rho dx\right]_s^t = 0.$$

We also see

$$\begin{aligned}
&\int_{B_R(0)} |u_\lambda(t) - u_\lambda(s)| dx \\
&= \int_{B_R(0)} \left| \int_s^t u_{\lambda\tau}(\tau) d\tau \right| dx \\
&\le \left(\frac{\omega_N R^N}{N}(t-s)\right)^{1/2} \left(\int_s^t \int_{B_R(0)} u_{\lambda\tau}(\tau)^2 dx d\tau\right)^{1/2} \\
&= \left(\frac{\omega_N R^N}{N}(t-s)\right)^{1/2} \left(E(u_\lambda(s)) - E(u_\lambda(t))\right)^{1/2} \\
&\le \left(\frac{\omega_N R^N}{N} E(u_\lambda(0))\right)^{1/2} (t-s)^{1/2}
\end{aligned}$$

by Lemma 2.4. Passing to the limit as $\lambda \to \lambda_\varphi$ yields the continuity of $u_{\lambda_\varphi}(t)$ from $[0,\infty)$ to $L^1_{loc}(\mathbf{R}^N)$. Consequently $u_{\lambda_\varphi}$ is a global solution of (1.1) in the sense of $L^1_{loc}$.

Now we assume that $u_{\lambda_\varphi}$ is a global classical solution of (1.1). It follows from Lemma 4.1 that $u_{\lambda_\varphi}$ does not grow up, that is, $\sup_{t>0} |u_{\lambda_\varphi}(t)|_\infty < +\infty$. Then we see that $u_{\lambda_\varphi}(t) \to 0$ locally uniformly in $\mathbf{R}^N$ as $t \to \infty$ by Lemmas 2.5 and 3.2 and hence $\lambda_\varphi \in \Lambda$. This contradicts the definition of $\lambda_\varphi$ since $\Lambda$ is open in $(0,\infty)$ from Lemma 3.2. Therefore $u_{\lambda_\varphi}$ blows up in finite time.

If $\lambda < \lambda_\varphi$, then Lemma 2.2 is the same as the last statement of this theorem. In the case of $\lambda = \lambda_\varphi$, letting $t_\lambda$ be the positive constant $t_0$ in Lemma 2.2 with $u = u_\lambda$ and

$$m_\lambda = \lambda \min\{\varphi(r) : r \text{ is a local minimizer of } \varphi\}$$

for $\lambda < \lambda_\varphi$, it holds $m_\lambda \geq C$ for $\lambda$ close to $\lambda_\varphi$ with some $C > 0$. Therefore we can take $t_\lambda$ independently of $\lambda$ close to $\lambda_\varphi$. This implies the last assertion of Theorem 1.1 for $\lambda = \lambda_\varphi$ from the definition of $u_{\lambda_\varphi}$.

In order to show the statement (iii), fix an arbitrary $\lambda > \lambda_\varphi$. Putting

$$\varphi_\mu(x) = \lambda_\varphi \mu^{2/(p-1)} \varphi(\mu x) \quad \text{for } x \in \mathbf{R}^N$$

for $\mu > 1$, there are $\mu_0, \delta_0 > 0$ such that for $1 < \mu \leq \mu_0$ and $0 < \delta \leq \delta_0$

$$\varphi_\mu(x + \delta y) \leq \lambda \varphi(x) \quad \text{for } x \in \mathbf{R}^N \text{ and } y \in \mathbf{R}^N \text{ with } |y| = 1.$$

Since $\mu^{2/(p-1)} u_{\lambda_\varphi}(\mu x, \mu^2 t)$ is the solution of (1.1) with initial data $\varphi_\mu$, it holds $u_\lambda(r, S) = \infty$ for $0 \leq r \leq \delta_0$ with some $S > 0$ by the comparison theorem. Here we extend $u_\lambda$ as a proper solution introduced in [4] after the blowup time. Therefore we obtain the complete blowup of $u_\lambda$ from Proposition 4.1. □

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