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Laufer's vanishing theorem for embedded CR manifolds

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Abstract. Let D be a sufficiently small open subset of a generic CR manifold in $\mathbb{C}^n$. We show that the cohomology groups $H^{p,q}(D)$ are either zero or infinite dimensional.

Using Fredholm-operator theory or some L^2-estimates, it is often easier to obtain finiteness theorems for the natural cohomology groups associated to a complex manifold than to actually show the solvability of the $\overline{\partial}$-equation. Therefore one would like to have a criterion which permits to pass from finiteness theorems to vanishing theorems.

It was proved by Laufer that for any open subset D of a Stein manifold, the Dolbeault cohomology groups $H^{p,q}(D)$ are either zero or infinite dimensional ([La]). The purpose of this note is to give a version of this theorem for embedded CR manifolds.

Let us denote by M a smooth generic CR manifold embedded into $\mathbb{C}^n$, of real codimension k, and by $H^{p,q}(M)$, $0 \leq p \leq n$, $0 \leq q \leq n-k$, the cohomology groups of the Cauchy-Riemann complexes obtained from the $\overline{\partial}_M$-operator (see [Bo] for the definitions). We then obtain the following theorem.

Theorem. *Let z be an arbitrary point of M. Then there exists an open neighborhood U_z of z in M such that for every open subset $D \subset U_z$ with $\dim H^{p,q}(D) < +\infty$, we have $H^{p,q}(D) = 0$, $0 \leq p \leq n$, $1 \leq q \leq n-k$.*

We point out that this theorem holds without any hypothesis on the Levi form of M.

Our result holds only for sufficiently small open subsets of CR manifolds. We also provide an example of a CR manifold, globally embedded into some

$\mathbb{C}^n$, for which the corresponding global result is false. However, our example being a compact CR manifold, it remains an open question whether one can find a noncompact CR manifold, embedded into some Stein manifold, which does not satisfy the above theorem.

For later use, let us take a closer look at some local representation of M. We fix $z \in M$. We may, see [Bo] for the details, assume $z = 0$ and that M is (in a neighborhood of 0) of the form

$$M = \left\{ (z, w = u + iv) \in \mathbb{C}^{n-k} \times \mathbb{C}^k \mid v = h(z, u) \right\} \tag{1}$$

where

$$h = (h_1, \ldots, h_k) : \mathbb{C}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

is a $\mathcal{C}^\infty$-mapping with $h(0,0) = 0$, $Dh(0,0) = 0$. Moreover, there is a basis of $T^{0,1}M$ on a neighborhood of 0 of the form

$$\overline{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\ell=1}^{k} a_{j\ell} \frac{\partial}{\partial \bar{w}_\ell}, \quad j = 1, \ldots, n-k \tag{2}$$

We note (see [Ba-Tr]) that

$$[\overline{L}_i, \overline{L}_j] = 0, \quad i, j = 1, \ldots, n-k\ . \tag{3}$$

Let us define a vector field $M_1 \in \mathbb{C} \otimes TM$ by

$$M_1 = \frac{\partial}{\partial z_1} + \sum_{\ell=1}^{k} b_\ell \frac{\partial}{\partial \bar{w}_\ell} \tag{4}$$

M_1 will be tangent to M if and only if

$$\frac{\partial h_j}{\partial z_1} + \sum_{\ell=1}^{k} \frac{b_\ell}{2} \left(\frac{\partial h_j}{\partial u_\ell} - i\delta_{j\ell} \right) = 0, \quad j = 1, \ldots, k\ .$$

As $Dh(0,0) = 0$, we can certainly find $(b_1, \ldots, b_k)$ satisfying the above system on a neighborhood of 0.

We claim that we have

$$[M_1, \overline{L}_j] = 0, \quad j = 1, \ldots, n-k\ . \tag{5}$$

This can be seen as follows (see also [Ba-Tr]).

We have

$$[M_1, \overline{L}_j] z_\ell = [M_1, \overline{L}_j] w_m = 0, \quad \ell = 1, \ldots, n-k, \quad m = 1, \ldots, k\ .$$

As the annihilator of the forms $dz_1, \ldots, dz_{n-k}, dw_1, \ldots, dw_k$ in $\mathbb{C} \otimes TM$ is spanned by $\overline{L}_1, \ldots, \overline{L}_{n-k}$, this implies that $[M_1, \overline{L}_j]$ has to be a linear combination of the $\overline{L}_\alpha$'s. But $[M_1, \overline{L}_j]$ does not involve differentiation in the direction of $\bar{z}_\alpha$, hence $[M_1, \overline{L}_j] = 0$.

Let $\bar{\omega}_1, \ldots, \bar{\omega}_{n-k}$ be smooth 1-forms in a neighboorhood of 0 in M such that $\bar{\omega}_j(\overline{L}_\alpha) = \delta_{j\alpha}$. We observe that

$$d\bar{\omega}_j|_{T^{0,1}M} = 0, \;\; j = 1, \ldots, n-k \; . \tag{6}$$

Indeed, by the classical formula for d we have

$$\begin{aligned} d\bar{\omega}_j(\overline{L}_\alpha, \overline{L}_\beta) &= \overline{L}_\alpha(\bar{\omega}_j(\overline{L}_\beta)) - \overline{L}_\beta(\bar{\omega}_j(\overline{L}_\alpha)) - \bar{\omega}_j([\overline{L}_\alpha, \overline{L}_\beta]) \\ &= \overline{L}_\alpha(\delta_{j\beta}) - \overline{L}_\beta(\delta_{j\alpha}) - \bar{\omega}_j(0) = 0 \end{aligned}$$

because of (3).

Proof of the theorem. We fix a point $z \in M$ and choose a small neighborhood U_z of z such that all the computations (1)–(6) hold in U_z.

Let $D \subset U_z$ be an open subset with $\dim H^{p,q}(D) < +\infty$, $q \geq 1$. We assume $H^{p,q}(D) \neq 0$.

It is no loss of generality to take $p = n$ (see [Tr]). The (n,q)-forms in D can be uniquely written as

$$u = \sum_{|I|=q}{}' \, u_I dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I, \;\; u_I \in \mathcal{C}^\infty(D),$$

where the prime indicates summation over increasing multiindices, $I = (i_1, \ldots, i_q)$ and $\bar{\omega}^I = \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_q}$.

Moreover, the $\overline{\partial}_M$-operator is nothing else but the exterior differential d, i.e.

$$\begin{aligned} \overline{\partial}_M u = & \sum_{|I|=q}{}' d(u_I) \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I \\ & + (-1)^n \sum_{|I|=q}{}' u_I \; \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge d(\bar{\omega}^I) \\ = & \sum_{j=1}^{n-k} \sum_{|I|=q}{}' \overline{L}_j(u_I)\bar{\omega}_j \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I \end{aligned}$$

because of (6).

From the assumption $0 < H^{n,q}(D) < +\infty$, it follows that there exists a non constant polynomial P in one complex variable such that $P(z_1)u \in \mathrm{Im}(\overline{\partial}_M)$ for every $\overline{\partial}_M$-closed (n,q)-form u in D.

We take P to be of minimal degree N among all the polynomials having this property.

For each $\overline{\partial}_M$-closed (n,q)-form u in D we can thus find an (n,q)-form α such that

$$P(z_1)u = \overline{\partial}_M \alpha \; .$$

Applying the vector field M_1 defined by (4) coefficientwise to this equation

$$(\text{i.e. } M_1 u = \sum_{|I|=q}{}' \, (M_1 u_I) dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I)$$

yields

$$P'(z_1)u + P(z_1)M_1 u = M_1 \overline{\partial}_M \alpha$$

where P' is a polynomial of degree $< N$.

In view of (5), the operators M_1 and $\overline{\partial}_M$ commute, hence the above equations yield

$$P'(z_1)u \in \operatorname{Im}(\overline{\partial}_M)$$

for all $\overline{\partial}_M$-closed (n, q)-forms in D. This contradicts the minimality of N and completes the proof. □

Remark. The preceding theorem does not hold globally, not even for CR manifolds globally embedded into $\mathbb{C}^n$. Indeed, it was proved in [Ca-Le] that there exist compact strictly pseudoconvex CR manifolds of hypersurface type and of any dimension, embedded into some $\mathbb{C}^N$ that admit small deformations that are also embeddable but their embeddings cannot be chosen close to the original embedding. Moreover, it was shown by Tanaka [Ta] that if we are in a situation as above with $\dim_{\mathbb{R}} M \geq 5$ and $H^{0,1}(M) = 0$, then small deformations of M can be embedded by mappings close to the original embedding, i.e. the above phenomenon does not arise. In addition, if M is compact and strictly pseudoconvex of dimension greater than five, we always have $\dim H^{0,1}(M) < +\infty$ (see [Ta]). This implies the existence of compact strictly pseudoconvex hypersurfaces of dimension greater than five in $\mathbb{C}^n$ with

$$0 \neq \dim H^{0,1}(M) < +\infty \, .$$

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