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The ring structure on the cohomology of coordinate subspace arrangements

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Abstract. Every simplicial complex $\Delta \subset 2^{[n]}$ on the vertex set $[n] = \{1, \dots, n\}$ defines a real resp. complex arrangement of coordinate subspaces in $\mathbb{R}^n$ resp. $\mathbb{C}^n$ via the correspondence $\Delta \ni \sigma \mapsto \mathrm{span}\{e_i : i \in \sigma\}$. The linear structure of the cohomology of the complement of such an arrangement is explicitly given in terms of the combinatorics of Δ and its links by the Goresky–MacPherson formula. Here we derive, by combinatorial means, the ring structure on the integral cohomology in terms of data of Δ. We provide a non-trivial example of different cohomology rings in the real and complex case. Furthermore, we give an example of a coordinate arrangement that yields non-trivial multiplication of torsion elements.

1 Introduction and results

This article is concerned with coordinate subspace arrangements, a family of (linear) subspace arrangements in real and complex space associated with simplicial complexes. For a detailed survey of subspace arrangements we refer to [Bj]; all we need here is given in Sect. 2. Associated with any subspace arrangement are its link and its complement. The homology of the link, the cohomology of the complement, and in particular its ring structure, have motivated a lot of research [Ar], [BZ], [Br], [CP], [FZ], [GM], [OS], [OT], [Zi].

The Goresky–MacPherson formula for the homology of the link is the starting point of our investigation. By analyzing Alexander duality combina-

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torially in the case of coordinate subspace arrangements, we give a complete combinatorial description of the ring structure of the integral cohomology. In this analysis the duality of the cross polytope and the cube plays a crucial role.

This work was motivated by a result of S. Yuzvinsky [Yu] on the rational cohomology ring structure of complex arrangements. We can give a partial positive answer to Conjecture 6.6 on the integral cohomology ring structure of complex arrangements of his article. Our modeling of the cohomology of the complement was inspired by the article [BC] of E. Babson and C. Chan.

We provide an example of a simplicial complex not containing faces of cardinality $n-1$, so that the complement of the associated real coordinate subspace arrangement is connected, that yields different ring structures for the cohomology of the complement of the associated real and complex arrangement. This answers a question by Gasharov, Peeva and Welker [GPW].

Finally, we give an example of a coordinate subspace arrangement that yields non trivial multiplication of torsion elements.

Results

Our main result – the description of the ring structure on the cohomology of the complement C_Δ of a coordinate subspace arrangement – is based on the Goresky–MacPherson formula for the link (cf. [GM]). After applying Alexander duality it is given in our situation by

$$\tilde{H}^i(C_\Delta;\mathbb{Z}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma;\mathbb{Z}).$$

To describe the multiplication in $\tilde{H}^*(C_\Delta;\mathbb{Z})$ it suffices to describe how to multiply classes $[u]$ and $[v]$ that correspond to $[c] \in H_r(\operatorname{link}_\Delta \sigma;\mathbb{Z})$ and $[c'] \in H_{r'}(\operatorname{link}_\Delta \sigma';\mathbb{Z})$ under the Goresky–MacPherson isomorphism. Note that there is a double grading of cohomology classes by assigning the grade (r,σ) to $[u]$.

Our main result is the following.

Theorem 1.1 *Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let C_Δ denote the complement of the associated real coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ is given by the homomorphisms*

$$\tilde{H}_r(\operatorname{link}_\Delta \sigma;\mathbb{Z}) \otimes \tilde{H}_{r'}(\operatorname{link}_\Delta \sigma';\mathbb{Z}) \longrightarrow \tilde{H}_{r+r'+2}(\operatorname{link}_\Delta \sigma\cap\sigma';\mathbb{Z})$$

$$[c]\otimes[c'] \longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \text{if } \sigma\cup\sigma' = [n],\\ 0 & \text{otherwise.}\end{cases}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon\in\{\pm1\}$ is a sign depending on n,σ,σ',r,r' computed in Sect. 3.4. If C_Δ is not connected there is additional non-trivial multiplication of cohomology classes in dimension zero.

This implies in particular that the multiplication respects the double grading of cohomology classes. The condition $\sigma \cup \sigma' = [n]$ is the "standard codimension condition" (cf., e.g., [Yu], [HRW, Proposition 6]). As a consequence we obtain the following Corollary, which answers Conjecture 6.6 in [Yu] in the case of coordinate subspace arrangements.

Corollary 1.1 *Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let $C_\Delta^{\mathbb{C}}$ denote the complement of the associated complex coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$ is given by the homomorphisms*

$$\tilde{H}_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}) \otimes \tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z}) \longrightarrow \tilde{H}_{r+r'+2}(\mathrm{link}_\Delta\,\sigma\cap\sigma';\mathbb{Z})$$

$$[c]\otimes[c'] \longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \text{if } \sigma\cup\sigma' = [n],\\ 0 & \text{otherwise.}\end{cases}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon \in \{\pm 1\}$ a sign depending on n, r, r' computed in Sect. 3.6.

The fact that the sign ε depends on σ and σ' in the real case, but not in the complex case, is the reason why in general there is no (dimension-shifting) isomorphism of graded rings between the cohomology rings of the real and complex arrangement associated with Δ (compare Corollary 2.1 and Sect. 4).

Example 1.1 There is a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices such that the following holds.

- ▷ The complement of the associated real arrangement is connected.
- ▷ The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ differs from $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$.

Example 1.2 There is a simplicial complex $\Delta \subset 2^{[10]}$ on ten vertices such that the cohomology ring of the complement of the associated real (or complex) arrangement yields non-trivial multiplication of torsion elements.

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2 Objects, tools and facts

In this section we recall basic facts on coordinate subspace arrangements, provide combinatorial models for their links and complements, and describe Lefschetz duality in the framework of cubical cohomology for the complement of a coordinate subspace arrangement.

2.1 Coordinate subspace arrangements

Simplicial complexes give rise to real and complex subspace arrangements. For that, let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$, resp. $\{e_1^{\mathbb{C}},\dots,e_n^{\mathbb{C}}\}$ the standard basis of $\mathbb{C}^n$. Let $\varDelta \subset 2^{[n]}$ be a simplicial complex on the vertex set $[n]=\{1,\dots,n\}$. We define that always $\emptyset \in \varDelta$ is a face. To avoid trivial cases we assume throughout the article that $\varDelta \neq 2^{[n]}$ and $n \geq 2$. The *(real) coordinate subspace arrangement* in $\mathbb{R}^n$ associated with $\varDelta$ is

$$\mathcal{A}_\varDelta = \{\operatorname{span}_{\mathbb{R}}\{e_{i_0},\dots,e_{i_k}\} : \{i_0,\dots,i_k\} \in \varDelta\},$$

the *(complex) coordinate subspace arrangement* in $\mathbb{C}^n$ associated with $\varDelta$ is

$$\mathcal{A}_\varDelta^{\mathbb{C}} = \left\{\operatorname{span}_{\mathbb{C}}\{e_{i_0}^{\mathbb{C}},\dots,e_{i_k}^{\mathbb{C}}\} : \{i_0,\dots,i_k\} \in \varDelta\right\}.$$

For every subspace arrangement we have the notion of the link and the complement, which in our case we denote by $L_\varDelta$ and $C_\varDelta$, resp. $L_\varDelta^{\mathbb{C}}$ and $C_\varDelta^{\mathbb{C}}$.

$$L_\varDelta = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}_\varDelta \qquad C_\varDelta = \mathbb{R}^n \setminus \bigcup \mathcal{A}_\varDelta$$
$$L_\varDelta^{\mathbb{C}} = \mathbb{S}^{2n-1} \cap \bigcup \mathcal{A}_\varDelta^{\mathbb{C}} \qquad C_\varDelta^{\mathbb{C}} = \mathbb{C}^n \setminus \bigcup \mathcal{A}_\varDelta^{\mathbb{C}}$$

2.2 Models for the real case

We introduce combinatorial models $\varLambda_\varDelta$ and $\varGamma_\varDelta$ for $L_\varDelta$ and $C_\varDelta$. Consider the n-dimensional cross polytope $Q^n = \operatorname{conv}\{\pm e_i : i=1,\dots,n\}$. Its proper faces form a simplicial complex, which we denote by ∂Q^n. Let $\varLambda_\varDelta$ be the subcomplex of ∂Q^n of all simplices that are contained in $\bigcup \mathcal{A}_\varDelta$.

$$\varLambda_\varDelta = \left\{\{\varepsilon_0 e_{i_0},\dots,\varepsilon_k e_{i_k}\} : \{i_0,\dots,i_k\} \in \varDelta, (\varepsilon_0,\dots,\varepsilon_k) \in \{\pm 1\}^{k+1}\right\}$$

Let $\varGamma_\varDelta$ be the "mirror complex" of $\mathcal{A}_\varDelta$ (cf. [BBC]), i.e., the faces of the n-cube $C^n = [-1,1]^n$ disjoint to $\bigcup \mathcal{A}_\varDelta$ considered as a polytopal subcomplex of the cube.

$$\varGamma_\varDelta = \{c : c \text{ a proper face of } C^n,\ [n] \setminus \{\text{varying coord. of } c\} \notin \varDelta\}$$

The underlying spaces $|\varLambda_\varDelta|$ and $|\varGamma_\varDelta|$ are homeomorphic, resp. homotopy equivalent, to the link $L_\varDelta$ and the complement $C_\varDelta$, see e.g. [Mu, p. 414].

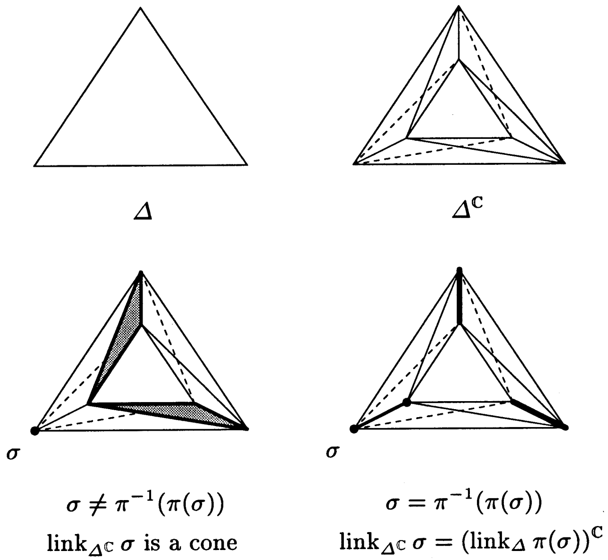

Fig. 1. Example for the "complexification" of a complex Δ

2.3 From complex to real arrangements

As far as the topology is concerned any complex coordinate arrangement can be modeled as a real subspace arrangement. Let $\Delta \subset 2^{[n]}$ be a simplicial complex on the vertex set $\{1, \ldots, n\}$. Let $\pi : [2n] \longrightarrow [n]$ the map defined by $2i-1, 2i \mapsto i$ for $i \in [n]$. Define the *"complexification" of* Δ by

$$\Delta^{\mathbb{C}} = \{\sigma \subset [2n] : \pi(\sigma) \in \Delta\}.$$

For an example of a "complexification" and the following Lemma see Fig. 1.

Lemma 2.1

- *Under the standard identification* $\mathbb{C}^n \cong \mathbb{R}^{2n}$ *the spaces* $\bigcup \mathcal{A}_\Delta^{\mathbb{C}}$ *and* $\bigcup \mathcal{A}_{\Delta^{\mathbb{C}}}$ *correspond to each other.*
- *For* $\sigma \in \Delta^{\mathbb{C}}$ *the following homotopy equivalence holds*

$$\operatorname{link}_{\Delta^{\mathbb{C}}} \sigma \simeq \begin{cases} * & \textit{if } \pi^{-1}(\pi(\sigma)) \neq \sigma, \\ \operatorname{link}_\Delta \pi(\sigma) & \textit{if } \pi^{-1}(\pi(\sigma)) = \sigma. \end{cases}$$

□

2.4 The Goresky–MacPherson theorem

Let $\mathcal{A}$ be a (linear) subspace arrangement in $\mathbb{R}^n$ with link $L = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}$ and complement $C = \mathbb{R}^n \setminus \bigcup \mathcal{A}$. Denote by P the intersection poset of $\mathcal{A}$ ordered by reversed inclusion, and by $d{:}P \longrightarrow \mathbb{N}$ the dimension function. For $v \in P$ let $P_{<v}$ be the subposet of all elements in P that are smaller than v. For any finite poset Q denote by $\Delta(Q)$ the order complex of Q.

Theorem (Goresky–MacPherson [GM, Part III]) *The homology of the link $L_{\mathcal{A}}$, and the cohomology of the complement $C_{\mathcal{A}}$, of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$ can be computed from the data (P, d) and n:*

$$\tilde{H}_i(L_{\mathcal{A}};\mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{i-d(v)}(\Delta(P_{<v});\mathbb{Z}),$$
$$\tilde{H}^i(C_{\mathcal{A}};\mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{n-i-d(v)-2}(\Delta(P_{<v});\mathbb{Z}).$$

This theorem, originally proven by means of stratified Morse theory in [GM], was given an elementary proof by Ziegler and Živaljević in [ZZ].

2.5 The Goresky–MacPherson theorem for coordinate subspace arrangements

In the situation of a real coordinate subspace arrangement $\mathcal{A}_\Delta$ the order complexes $\Delta(P_{<v})$ can be described more explicitly. The poset P is given by the face poset of the simplicial complex Δ ordered by inverse inclusion. The poset $P_{<\sigma}$ then is isomorphic to the opposite face lattice of $\operatorname{link}_\Delta \sigma = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}$. Thus we obtain the following formulation of the Goresky–MacPherson theorem.

Theorem *Let $\Delta \subset 2^{[n]}$ be a simplicial complex with vertex set $\{1, \dots, n\}$. Then*

$$\tilde{H}_i(L_\Delta;\mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{i-|\sigma|}(\operatorname{link}_\Delta \sigma;\mathbb{Z}),$$
$$\tilde{H}^i(C_\Delta;\mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma;\mathbb{Z}).$$

Here $|\sigma|$ denotes the cardinality of σ, i.e., $|\sigma| = \dim \sigma + 1$.

In view of section 2.2 this yields the following result for the associated complex coordinate subspace arrangement.

Corollary 2.1 *For simplicial complexes $\Delta \subset 2^{[n]}$ we have*

$$\tilde{H}_i \left(L_\Delta^{\mathbb{C}}; \mathbb{Z} \right) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{i-2|\sigma|} \left(\operatorname{link}_\Delta \sigma; \mathbb{Z} \right)$$
$$\tilde{H}^i \left(C_\Delta^{\mathbb{C}}; \mathbb{Z} \right) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{2n-i-2|\sigma|-2} \left(\operatorname{link}_\Delta \sigma; \mathbb{Z} \right),$$

and hence there is a dimension-shifting group isomorphism between the (co)homologies of the real and complex coordinate subspace arrangements. Every homology class

$$[c] \in \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z}) = \tilde{H}_{2n-(n+|\sigma|+i)-2|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z})$$

corresponds to

$$[u] \in \tilde{H}^i(C_\Delta; \mathbb{Z})$$

and to

$$[u^{\mathbb{C}}] \in \tilde{H}^{n+|\sigma|+i} \left(C_\Delta^{\mathbb{C}}; \mathbb{Z} \right).$$

The correspondence $[u] \longmapsto [u^{\mathbb{C}}]$ sets up the isomorphism.

2.6 A homology model and a map into the link

We establish a simplicial version of the Ziegler–Živaljević [ZZ] proof for the Goresky–MacPherson theorem. Let $\Delta \subset 2^{[n]}$ be a simplicial complex. We construct a simplicial complex $\mathfrak{L}_\Delta$ together with a simplicial map $\Phi : \mathfrak{L}_\Delta \longrightarrow \Lambda_\Delta$ to the link that induces an isomorphism in homology. Let $\mathfrak{L}_\Delta$ be the following one-point union of spaces.

$$\mathfrak{L}_\Delta = \left(\dot{\bigcup_{\sigma \in \Delta}} \partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma \right) /\!\sim\, = \left(\Delta \,\dot\cup \dot{\bigcup_{\sigma \in \Delta \setminus \{\emptyset\}}} \partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma \right) /\!\sim$$

The one-point union is given by the following identifications $\sim$. For each $\sigma = \{i_0 < \ldots < i_k\} \in \Delta$, $\sigma \neq \emptyset$, identify $e_1 \in \partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma$ with the vertex $i_0 \in \Delta = \partial Q^{|\emptyset|} * \operatorname{link}_\Delta \emptyset$. Compare Fig. 2.

We get the map Φ by defining it on the pieces $\partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma$. Let

$$\phi_\sigma : \partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma \longrightarrow \Lambda_\Delta$$

be defined by the simplicial homeomorphism

$$\partial Q^{|\sigma|} \longrightarrow \operatorname{span}_{\mathbb{R}}\{e_{i_0}, \ldots, e_{i_k}\} \cap \partial Q^n,$$

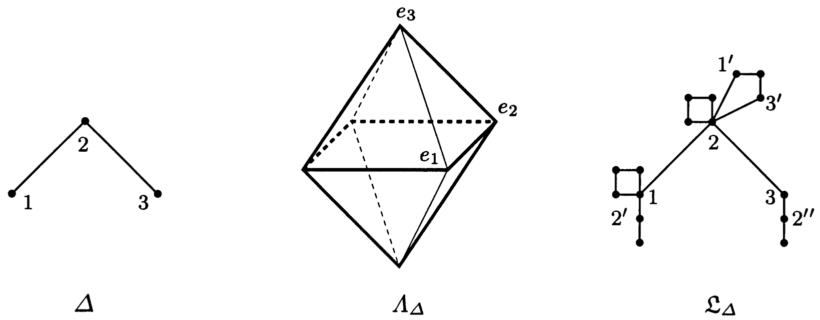

Fig. 2. An easy example for the model space $\mathfrak{L}_\Delta$

$\sigma = \{i_0 < \cdots < i_k\}$, such that $\phi_\sigma(e_{j+1}) = e_{i_j}$, in particular $\phi_\sigma(e_1) = e_{i_0}$. On $\mathrm{link}_\Delta\, \sigma$ the map ϕ_σ is defined by

$$\{j_0, \ldots, j_l\} \longmapsto \{e_{j_0}, \ldots, e_{j_l}\} \in \Lambda_\Delta$$

for $\{j_0, \ldots, j_l\} \in \mathrm{link}_\Delta\, \sigma$. By construction all these maps fit together and yield a simplicial map Φ.

Proposition 2.1 *The map Φ induces an isomorphism in homology. (In fact, it is a homotopy equivalence.)*

Proof. (Sketch of proof) The proof works as in [ZZ] by induction on the cardinality of Δ. In the induction step one removes a maximal simplex of Δ and uses the Mayer-Vietoris sequence along with the induction hypotheses (resp. the Glueing Lemma, to obtain the homotopy equivalence). □

2.7 Cubical cohomology

The homotopy model Γ_Δ of the complement C_Δ is a subcomplex of the boundary of the cube. We compute its cohomology by using "cubical cohomology." We give a short overview of the most important notation and the formula for the cup product (see also [Ma]).

Let Γ be a subcomplex of the n-cube C^n, and let $T \in \Gamma$ be a t-dimensional cube. We use two descriptions of T:
Denote the projection to the i-th coordinate by π_i. On the one hand, we can identify T with a vector in $\{+, -, *\}^n$, where the i-th coordinate is $+$, $-$ or $*$ iff $\pi_i(T) = \{+1\}$, $\{-1\}$, resp. $[-1, +1]$. On the other hand, there are three sets $T_+, T_-, T_* \subseteq \{1, \ldots, n\}$ that uniquely define the cube,

$$T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, T_*),$$

where $|T_*| = t$ and the following holds for the coordinate projections.

$$\begin{aligned} \pi_i(T) &= \{+1\} && \text{for } i \in T_+, \\ \pi_j(T) &= \{-1\} && \text{for } j \in T_-, \\ \pi_k(T) &= [-1,+1] && \text{for } k \in T_*. \end{aligned}$$

Let $C_t(\Gamma)$ be the free abelian group generated by the t-cubes in Γ. In order to get a boundary map we begin by defining face operators. Let $T \in \Gamma$ be a t-dimensional cube $T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, T_*)$ with $T_* = \{k_1 < \cdots < k_t\}$. For $A = \{a_1, \ldots, a_p\} \subseteq \{1, \ldots, t\}$ and $\varepsilon = \pm 1$ define the $(t-p)$-cube

$$D_A^\varepsilon T = \begin{cases} (T_+ \cup \{k_{a_1}, \ldots, k_{a_p}\}, T_-, T_* \setminus \{k_{a_1}, \ldots, k_{a_p}\}) & \text{if } \varepsilon = +1, \\ (T_+, T_- \cup \{k_{a_1}, \ldots, k_{a_p}\}, T_* \setminus \{k_{a_1}, \ldots, k_{a_p}\}) & \text{if } \varepsilon = -1. \end{cases}$$

$D_A^\varepsilon T$ is the face of T obtained by fixing the varying coordinates $\{k_{a_1}, \ldots, k_{a_p}\}$ to ε. A boundary operator is now defined by

$$\begin{aligned} \partial_t : C_t(\Gamma) &\longrightarrow C_{t-1}(\Gamma), \\ T &\longmapsto \sum_{a=1}^{t} (-1)^a \left(D_{\{a\}}^{+1} T - D_{\{a\}}^{-1} T \right). \end{aligned}$$

The homology of the resulting *cubical chain complex* $(C_*(\Gamma), \partial_*)$ is canonically isomorphic to singular homology. The cup product formula in this situation is given on the chain level by the following. Let $u \in \mathrm{Hom}(C_p(\Gamma), \mathbb{Z})$ and $v \in \mathrm{Hom}(C_q(\Gamma), \mathbb{Z})$, then for a $(p+q)$-cube T we obtain

$$(u \cup v)(T) = \sum \rho_{H,K} \cdot u\left(D_H^{+1} T\right) v\left(D_K^{-1} T\right),$$

where the sum is taken over all q-subsets H of $\{1, \ldots, p+q\}$, K is the complement of H, and $\rho_{H,K}$ is the sign of the permutation HK of $\{1, \ldots, p+q\}$, i.e., the signature of the shuffle (H, K).

2.8 Lefschetz duality for the cross polytope

As a crucial part of Alexander duality, we describe Lefschetz duality explicitly for simplicial homology of the cross polytope and cubical cohomology of the cube (cf. [Mu]).

Theorem [Lefschetz Duality] *Let (X, A) be a compact, orientable, triangulated relative homology n-manifold. Then there is an isomorphism*

$$H_k(X, A) \cong H^{n-k}(|X| \setminus |A|).$$

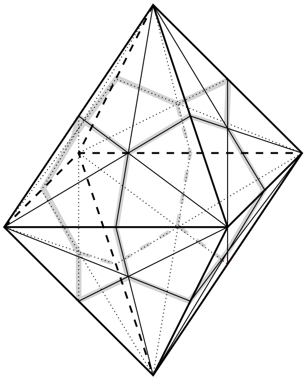

Fig. 3. The 3-dimensional cross polytope with the 1-skeleton of the 3-dimensional cube in the barycentric subdivision

Proof. (Outline of the proof) Let X^- be the simplicial complex consisting of all simplices of the barycentric subdivision $\mathrm{sd}\, X$ that are disjoint from $|A|$. Then

- ▷ $|X^-|$ is a deformation retract of $|X| \setminus |A|$.
- ▷ $|X^-|$ equals the union of all blocks $D(\sigma)$ dual to simplices $\sigma \in X$ that are not in A.

Now there is a chain isomorphism

$$C^k(X, A) \xrightarrow{\cong} D_{n-k}(X^-),$$

where $D_*(X^-)$ denotes the dual chain complex of X^-. Dualization yields

$$C_k(X, A) \cong \mathrm{Hom}(C^k(X, A), \mathbb{Z}) \xleftarrow{\cong} \mathrm{Hom}(D_{n-k}(X^-), \mathbb{Z}).$$

The inverse map $C_k(X, A) \longrightarrow \mathrm{Hom}(D_{n-k}(X^-), \mathbb{Z})$ is given by $\sigma \mapsto D(\sigma)^*$, where σ is a k-simplex of X not in A. This induces the desired isomorphism. □

Lefschetz duality is dealing with the complex X^-, whose underlying space is the union of the dual blocks $D(\sigma)$, $\sigma \in X \setminus A$. In case X is the boundary of the cross polytope Q^n, the dual blocks $|D(\sigma)|$, $\sigma \in X$, correspond to the faces of the boundary of the n-dimensional cube C^n. See Fig. 3.

Let now $A = \Lambda_\Delta$ be the subcomplex of $X = \partial Q^n$ given by the arrangement associated with a simplicial complex Δ (Section 2.2). Then there is a chain isomorphism from the dual block complex of $(\partial Q^n)^-$ to the cubical chain complex of Γ_Δ

$$D_j((\partial Q^n)^-) \longrightarrow C_j(\Gamma_\Delta),$$

which yields a chain isomorphism

$$\begin{aligned}\Psi : C_k(\partial Q^n, \Lambda_\Delta) &\longrightarrow \mathrm{Hom}(D_{n-1-k}((\partial Q^n)^-), \mathbb{Z}) \\ &\longrightarrow \mathrm{Hom}(C_{n-1-k}(\Gamma_\Delta), \mathbb{Z})\end{aligned}$$

where

$$\Psi(\sigma) = (-1)^{i_0+\cdots+i_k}(-1)^{|T_-(\sigma)|}(T_+(\sigma), T_-(\sigma), T_*(\sigma))^*,$$

for $\sigma = \langle \varepsilon_0 e_{i_0}, \ldots, \varepsilon_k e_{i_k}\rangle \in \partial Q^n \setminus \Lambda_\Delta$, $i_0 < \cdots < i_k$, with

$$\begin{aligned}T_+(\sigma) &= \{i_j \in [n] : \varepsilon_j = +1\}, \\ T_-(\sigma) &= \{i_j \in [n] : \varepsilon_j = -1\}, \\ T_*(\sigma) &= [n] \setminus (T_+(\sigma) \cup T_-(\sigma)).\end{aligned}$$

The signs in $\Psi(\sigma)$ result from the condition that Ψ must commute with the respective boundary maps.

3 Proofs of results

In this section we prove Theorem 1.1. We begin by introducing joins of chains, and then exhibit explicit cohomology classes in $\tilde{H}^*(\Gamma_\Delta)$ with respect to the Goresky–MacPherson theorem. We derive an explicit formula for the cup product of two such classes. In most of the cases the product vanishes as stated in Theorem 1.1. Then we treat the case in which the product does not vanish. The considerations of the complex case follow then.

3.1 Joins of chains

Definition 3.1 *The* join $c * c'$ of two simplicial chains $c = \sum_j \alpha_j \tau_j$ and $c' = \sum_k \alpha'_k \tau'_k$ *in a simplicial complex $\Delta \subset 2^{[n]}$ is defined by*

$$\sum_{\substack{j,k \\ \tau_j \cap \tau'_k = \emptyset}} \alpha_j \alpha'_k \, \tau_j * \tau'_k,$$

where the join of two disjoint oriented simplices is defined by

$$\langle v_0, \ldots, v_r\rangle * \langle w_0, \ldots, w_s\rangle = \langle v_0, \ldots, v_r, w_0, \ldots, w_s\rangle.$$

Lemma 3.1 *Let $R = \{r_0, \ldots, r_s\}$ be a subset of the vertex set, $c = \sum_j \alpha_j \tau_j$ a cycle. For $R \subset \tau_j$ define the (oriented) simplex $\bar{\tau}_j$ by the equation $\tau_j = \bar{\tau}_j * \langle r_0, \ldots, r_s\rangle$. Then $\sum_{j : R \subset \tau_j} \alpha_j \bar{\tau}_j$ is a cycle.*

Proof. We write c as

$$c = \sum_{j:R\not\subset\tau_j} \alpha_j\tau_j + \sum_{j:R\subset\tau_j} \alpha_j\bar{\tau}_j * \langle r_0, \dots, r_s\rangle,$$

and obtain for the boundary

$$\partial\left(\sum_{j:R\not\subset\tau_j} \alpha_j\tau_j\right) + \partial\left(\sum_{j:R\subset\tau_j} \alpha_j\bar{\tau}_j\right) * \langle r_0, \dots, r_s\rangle \\ \pm \sum_{j:R\subset\tau_j} \alpha_j\bar{\tau}_j * \partial(\langle r_0, \dots, r_s\rangle) = 0.$$

The only simplices that contain R appear in the second summand, and hence this summand must be zero on its own. □

Lemma 3.2 *Let i be a vertex and let $c = \sum_j \alpha_j\tau_j$ and $c' = \sum_k \alpha'_k\tau'_k$ be two cycles that share at most the vertex i. Then*

$$\partial(\langle i\rangle * c * c') = c * c'.$$

Proof.

$$\begin{aligned}
\partial(\langle i\rangle * c * c') &= \partial\left(\langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right) \\
&= \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k - \langle i\rangle * \partial\left(\sum_{j:i\notin\tau_j} \alpha_j\tau_j\right) * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k \\
&\quad \pm \langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right) \\
&= \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k + \langle i\rangle * \partial\left(\sum_{j:i\in\tau_j} \alpha_j\tau_j\right) * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k \\
&\quad \pm \langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right) \\
&= c * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k - \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \langle i\rangle * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right)
\end{aligned}$$

$$= c * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k + \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \langle i\rangle * \partial\left(\sum_{k:i\in\tau'_k} \alpha'_k\tau'_k\right)$$
$$= c * c',$$

where possible empty sums are considered to be zero. □

3.2 Explicit cocycles

Using the Goresky–MacPherson theorem and the explicit description of Alexander duality we now derive explicit cohomology cocycles for the complement of a coordinate subspace arrangement. For that, we use the following sequence of homomorphisms.

$$\begin{aligned}\tilde{H}_r(\operatorname{link}_\Delta \sigma) &\xrightarrow[\text{suspension}]{\cong} \tilde{H}_{r+|\sigma|}\left(\partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma\right)\\ &\xrightarrow[(\phi_\sigma)_*]{\hookrightarrow} \tilde{H}_{r+|\sigma|}(\Lambda_\Delta) \xrightarrow[\text{pair sequence}]{} \tilde{H}_{r+|\sigma|+1}\left(\partial Q^n, \Lambda_\Delta\right)\\ &\xrightarrow[\text{Lefschetz duality}]{\cong} \tilde{H}^{n-r-|\sigma|-2}(\Gamma_\Delta)\end{aligned} \tag{1}$$

Before describing the maps explicitly, we introduce some notation.

Notation 3.1

▷ *For each subset* $\{j_1, \dots, j_s\} \subset [n]$ *we define*

$$\operatorname{sign}(j_1 j_2 \cdots j_s) = \operatorname{sign}\pi,$$

where π *is the permutation of* $(1, \dots, s)$ *such that* $j_{\pi(1)} < \cdots < j_{\pi(s)}$. *For every family of subsets* $A_1, \dots, A_k \subset [n]$, *where* $A_i = \{j^i_1 < \cdots < j^i_{m_i}\}$, *we define*

$$\operatorname{sign}(A_1 \cdots A_k) = \operatorname{sign}(j^1_1, \dots, j^1_{m_1}, j^2_1, \dots, j^2_{m_2}, \dots, j^k_1, \dots, j^k_{m_k}).$$

Furthermore, for every set $A = \{a_1, \dots, a_k\} \subset [n]$ *we abbreviate* $(-1)^{a_1+\cdots+a_n}$ *by* $(-1)^{\Sigma A}$.

▷ *For each* $\sigma \in \Delta$ *let*

$$s_\sigma = \sum_{\boldsymbol{\varepsilon}=(\varepsilon_0,\dots,\varepsilon_k)\in\{\pm1\}^{k+1}} \varepsilon_0\cdots\varepsilon_k \cdot \langle \varepsilon_0 e_0, \dots, \varepsilon_k e_k\rangle$$

be a generating simplicial cycle of $\tilde{H}_{|\sigma|-1}(\partial Q^{|\sigma|})$.

▷ *For each* $\sigma \in \Delta$ *choose* $i_\sigma \in [n] \setminus \sigma$ *arbitrarily.*

Now, let $\sigma \in \Delta$ and $[c] \in \tilde{H}_r(\mathrm{link}_\Delta \sigma)$, $c = \sum_j \alpha_j \tau_j$. Consider $\Phi{:}\mathfrak{L}_\Delta \longrightarrow \Lambda_\Delta$ as defined in Section2.6 and the induced chain map $\Phi_\sharp : C_*(\mathfrak{L}_\Delta) \to C_*(\Lambda_\Delta)$. The first two steps in the sequence (1) of homomorphisms are given by

$$[c] \longmapsto [s_\sigma * c] \longmapsto [\Phi_\sharp(s_\sigma * c)].$$

Now we construct the pair sequence map. Consider the following "cone" over the chain $\Phi_\sharp(s_\sigma * c)$:

$$\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c).$$

Observation 3.1

- ▷ $\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c) \in C_{r+|\sigma|+1}(\partial Q^n, \Lambda_\Delta)$ *by the definition of* Φ *and* i_σ,
- ▷ $\partial(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)) = \Phi_\sharp(s_\sigma * c)$ *as a special case of Lemma 3.2, and*
- ▷ *for any* $i'_\sigma \in [n] \setminus \sigma$, *the cycles* $\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)$ *and* $\langle e_{i'_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)$ *in* $C_{r+|\sigma|+1}(\partial Q^n, \Lambda_\Delta)$ *are homologous.*

Hence an element $[c] \in \tilde{H}_r(\mathrm{link}_\Delta \sigma)$ is mapped under (1) as follows.

$$\begin{aligned}[c] \longmapsto [s_\sigma * c] \longmapsto [\Phi_\sharp(s_\sigma * c)] \longmapsto \\ \longmapsto [\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)] \longmapsto [\Psi(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c))]\end{aligned}$$

The cocycle $\Psi(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c))$ is explicitly given by

$$\sum_{j: i_\sigma \notin \tau_j} \sum_{\boldsymbol{\varepsilon} \in \{\pm 1\}^{k+1}} \mathrm{sign}(i_\sigma \sigma \tau_j) \cdot (-1)^{i_\sigma + \sum \sigma + \sum \tau_j} \cdot \alpha_j \cdot \\ (T_+(j, \boldsymbol{\varepsilon}), T_-(j, \boldsymbol{\varepsilon}), T_*(j, \boldsymbol{\varepsilon}))^*,$$

where

$$\begin{aligned} T_+(j, \boldsymbol{\varepsilon}) &= \tau_j \cup \{i_\sigma\} \cup \{i_l : \varepsilon_l = +1\}, \\ T_-(j, \boldsymbol{\varepsilon}) &= \{i_l : \varepsilon_l = -1\}, \\ T_*(j, \boldsymbol{\varepsilon}) &= [n] \setminus (T_+(j, \boldsymbol{\varepsilon}) \cup T_-(j, \boldsymbol{\varepsilon})) \\ &= [n] \setminus (\sigma \cup \tau_j \cup \{i_\sigma\}). \end{aligned}$$

Here we made use of the equality $\varepsilon_0 \cdots \varepsilon_k \cdot (-1)^{|T_-(j, \boldsymbol{\varepsilon})|} = +1$. In the other representation the cubes $(T_+(j, \boldsymbol{\varepsilon}), T_-(j, \boldsymbol{\varepsilon}), T_*(j, \boldsymbol{\varepsilon}))$ look as in Fig. 4 (up to a permutation of coordinates), where the $\pm$-signs correspond to the sign vector $\boldsymbol{\varepsilon}$.

Throughout the rest of the article we will use this correspondence between homology cycles of the links of Δ and cocycles of the complement of the arrangement.

$$(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm}_{\sigma}|*\ *\ *\ *|\underbrace{\ +\ }_{\{i_\sigma\}}|*\ *\ *\ *\ *|\underbrace{++++}_{\tau_j}|*),$$

Fig. 4. Schematic description of $(T_+(j,\varepsilon), T_-(j,\varepsilon), T_*(j,\varepsilon))$

3.3 The cup product

Now consider two cohomology classes $[u]$ and $[v]$ of Γ_Δ corresponding to two homology classes $[c] \in \tilde{H}_r(\mathrm{link}_\Delta\,\sigma)$ and $[c'] \in \tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma')$ for simplices $\sigma, \sigma' \in \Delta$, $c = \sum_j \alpha_j \tau_j$ and $c' = \sum_k \alpha'_k \tau'_k$. Let $p = n-r-|\sigma|-2$ and $q = n - r' - |\sigma'| - 2$ and let $T \in \Gamma_\Delta$ be a $(p+q)$-cube. For the cup product of $[u]$ and $[v]$ evaluated at T we obtain

$$\sum_{H,K} \sum_{\substack{j:i_\sigma\notin\tau_j \\ k:i_{\sigma'}\notin\tau'_k}} \sum_{\varepsilon,\varepsilon'} \rho_{H,K}\,\mathrm{sign}(i_\sigma\sigma\tau_j)\,\mathrm{sign}(i_{\sigma'}\sigma'\tau'_k)\cdot$$
$$(-1)^{i_\sigma+i_{\sigma'}+\sum\sigma+\sum\tau_j+\sum\sigma'+\sum\tau'_k}\cdot\alpha_j\alpha'_k\cdot$$
$$(T_+(j,\varepsilon),T_-(j,\varepsilon),T_*(j,\varepsilon))^*\left(D_H^{+1}T\right)\cdot$$
$$(T'_+(k,\varepsilon'),T'_-(k,\varepsilon'),T'_*(k,\varepsilon'))^*\left(D_K^{-1}T\right),$$

where the first summation is over all (q,p)-shuffles (H,K). Let us first consider only the last term

$$(T_+(j,\varepsilon),T_-(j,\varepsilon),T_*(j,\varepsilon))^*\left(D_H^{+1}T\right)\cdot$$
$$(T'_+(k,\varepsilon'),T'_-(k,\varepsilon'),T'_*(k,\varepsilon'))^*\left(D_K^{-1}T\right). \quad (*)$$

Observation 3.2 *The term $(*)$ vanishes for all H, K and $\varepsilon, \varepsilon'$ unless*

$$\sigma\cup\sigma'\cup\{i_\sigma\}\cup\{i_{\sigma'}\}\cup\tau_j\cup\tau'_k = [n].$$

Proof. The sets of varying coordinates in $D_H^{+1}T$ and $D_K^{-1}T$ are disjoint. This gives

$$\emptyset = T_*(j,\varepsilon)\cap T'_*(k,\varepsilon')$$
$$= ([n]\setminus(\sigma\cup\tau_j\cup\{i_\sigma\}))\cap\left([n]\setminus(\sigma'\cup\tau'_k\cup\{i_{\sigma'}\})\right),$$

which yields the result. □

Now we turn to the general computation of the cup product.

Case I: $\sigma \neq \sigma'$ and $\sigma\cup\sigma' \neq [n]$.

In this case we will show that the cup product vanishes as demanded in the statement of the main Theorem. By anti-commutativity of the cup

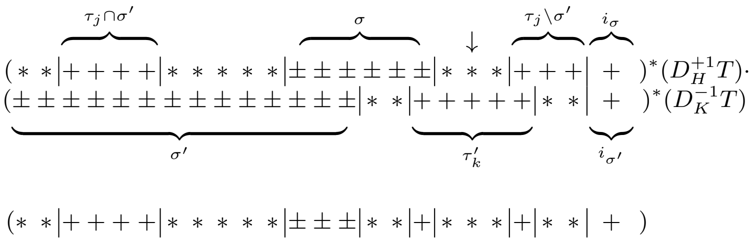

Fig. 5. The term $(*)$ schematically and all cubes T for which $(*)$ has a chance not to vanish

product we may assume $\sigma' \not\subset \sigma$. By Observation 3.1 we can assume $i_\sigma = i_{\sigma'} \notin [n] \setminus (\sigma \cup \sigma')$. Our situation is represented in Fig. 5.

Observation 3.3 *The term $(*)$ vanishes for all H, K, $\varepsilon, \varepsilon'$, and k unless the following holds*

$$\sigma \cup \sigma' \cup \{i_\sigma\} \cup \tau_j = [n].$$

Proof. As the down arrow $\downarrow$ points out in Fig. 5, if there is a coordinate only covered by τ'_k it will be a fixed -1-coordinate in $D_K^{-1}T$. $\square$

Hence all terms that have a chance to contribute to a non trivial product are as shown in Fig. 6. Gathering all contributing terms with the right sign and

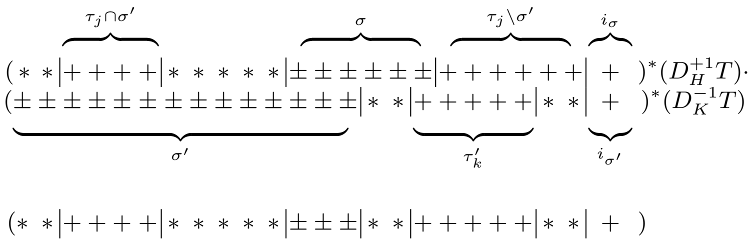

Fig. 6. The term $(*)$ schematically and all cubes T contributing non zero summands

coefficient, we obtain that the cup product is represented by the following cocycle (up to a global sign)

$$\Psi\left(\langle e_{i_\sigma}\rangle * \Phi_\sharp\left(s_{\sigma\cap\sigma'} * \sum_{\substack{j:i_\sigma\notin\tau_j\\ R\subset\tau_j}} \alpha_j\bar{\tau}_j * \sum_{k:i_\sigma\notin\tau'_k} \alpha'_k\tau'_k\right)\right),$$

where $R = \{r_0, \ldots, r_s\} = [n] \setminus (\sigma \cup \sigma' \cup \{i_\sigma\})$ and $\tau_j = \bar{\tau}_j * \langle r_0, \ldots, r_s \rangle$. Tracing this element back through the sequence of homomorphisms in (1) and using Lemma 3.2 we arrive (up to a global sign) at

$$\partial \left(\langle i_\sigma \rangle * \sum_{\substack{j: i_\sigma \notin \tau_j \\ R \subset \tau_j}} \alpha_j \bar{\tau}_j * \sum_{k: i_\sigma \notin \tau'_k} \alpha'_k \tau'_k \right) = \sum_{\substack{j \\ R \subset \tau_j}} \alpha_j \bar{\tau}_j * c'$$

as a representing cycle in $\tilde{H}_*(\mathrm{link}_\Delta(\sigma \cap \sigma'))$, which we denote by $\bar{c} * c'$. This is a chain in $C_k(\mathrm{link}_\Delta(\sigma \cap \sigma'))$ for the following reason. Consider an arbitrary pair of simplices $\bar{\tau}_j$ and τ'_k. Since $\tau'_k \in \mathrm{link}_\Delta \sigma'$ we have $\sigma' \cup \tau'_k \in \Delta$ and since $\bar{\tau}_j \subset \sigma'$ we obtain $\bar{\tau}_j \cup \tau'_k \in \Delta$.

We claim that the cycle $\bar{c} * c'$ is a boundary in $C_k(\mathrm{link}_\Delta(\sigma \cap \sigma'))$. Since $\sigma' \not\subset \sigma$ there is a $p \in \sigma' \setminus \sigma$ and as before all simplices $\{p\} \cup \bar{\tau}_j \cup \tau'_k \in \Delta$. Hence $\langle p \rangle * \bar{c} * c' \in C_{k+1}(\mathrm{link}_\Delta(\sigma \cap \sigma'))$ with boundary $\bar{c} * c'$ as follows by Lemma 3.2.

Case II: $\sigma = \sigma'$ (of course $\sigma \neq [n]$). We will show that the cup product vanishes unless the complement C_Δ is not connected. In this case we get non-trivial self multiplication of elements in cohomological dimension 0. Again we can assume $i_\sigma = i_{\sigma'}$. Our situation is shown in Fig. 7.

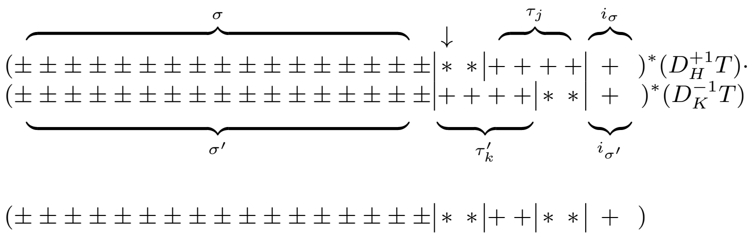

Fig. 7. The term $(*)$ schematically and all possible cubes T on which it does not vanish

As in the last case $(\downarrow)$ the only interesting terms are the ones with $\sigma \cup \tau_j \cup \{i_\sigma\} = [n]$. If such a term exists c must be a multiple of a generating cycle of the sphere on the vertices $[n] \setminus \sigma$. But for c' not to be trivial the same holds for c', since the reduced homology of the sphere is non trivial only in the dimension of the sphere. Now $n - |\sigma| - r - 2 = n - |\sigma'| - r' - 2 = 0$. Therefore the corresponding cohomology classes are not zero only if C_Δ is not connected, which means that there are simplices of dimension $n-2$ in Δ.

Case III: $\sigma \cup \sigma' = [n]$. Consider Fig. 8. Gathering the cocubes corresponding to the non vanishing summands together with signs and coefficients gives the following representing cocycle for the cup product (up to a global sign, see Sect. 3.4)

$$\Psi\left(\langle e_{i_\sigma}\rangle * \langle e_{i_{\sigma'}}\rangle * \Phi_\sharp\left(s_{\sigma\cap\sigma'} * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right)\right).$$

$$\begin{array}{l}(*\ *\overbrace{|+\ +\ +\ +|}^{\tau_j}*\ *\overbrace{|\ +\ |}^{i_\sigma}*\ *\overbrace{|\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\ \ \pm}^{\sigma}\ )^*(D_H^{+1}T)\cdot \\ (\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\ \ \pm\ \ \pm\pm\pm\pm\pm}_{\sigma'}|*\ *\underbrace{|+\ +\ +\ +\ +|}_{\tau'_k}*\ *\underbrace{|\ +\ }_{i_{\sigma'}})^*(D_K^{-1}T)\end{array}$$

$$(*\ *|+\ +\ +\ +|*\ *|\ +\ |*\ *|\pm\pm\pm|*\ *|+\ +\ +\ +\ +|*\ *|\ +\ )$$

Fig. 8. The term $(*)$ schematically and all cubes T for which $(*)$ does not vanish

Tracing this element back to a cycle in $C_{r+r'+2}(\mathrm{link}_\Delta(\sigma\cap\sigma'))$ leads up to a factor of $(-1)^{|\sigma\cap\sigma'|}$ to

$$\begin{aligned}
&\partial\left(\langle i_\sigma\rangle * \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right) = \\
&= \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad - \langle i_{\sigma'}\rangle * \langle i_\sigma\rangle * \partial\left(\sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j\right) * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad + \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \langle i_{\sigma'}\rangle * \partial\left(\sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right) \\
&= \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad + \langle i_{\sigma'}\rangle * \langle i_\sigma\rangle * \partial\left(\sum_{j:i_\sigma\in\tau_j}\alpha_j\tau_j\right) * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k
\end{aligned}$$

$$
\begin{aligned}
&- \langle i_\sigma \rangle * \sum_{j: i_\sigma \notin \tau_j} \alpha_j \tau_j * \langle i_{\sigma'} \rangle * \partial \left(\sum_{k: i_{\sigma'} \in \tau'_k} \alpha'_k \tau'_k \right) \\
&= \langle i_{\sigma'} \rangle * c * \sum_{k: i_{\sigma'} \notin \tau'_k} \alpha'_k \tau'_k - \langle i_\sigma \rangle * \sum_{j: i_\sigma \notin \tau_j} \alpha_j \tau_j * c' \\
&= \langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'.
\end{aligned}
$$

This finishes the proof of Theorem 1.1.

3.4 The global sign

We show how to compute the global sign. In the cup product formula we have the sign

$$
\rho_{H,K} \operatorname{sign}(i_\sigma \sigma \tau_j) \operatorname{sign}(i_{\sigma'} \sigma' \tau'_k) \cdot (-1)^{i_\sigma + i_{\sigma'} + \sum \sigma + \sum \tau_j + \sum \sigma' + \sum \tau'_k}. \quad (*)
$$

For the image of $\langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'$ under the sequence of homomorphisms (1) we obtain

$$
(-1)^{|\sigma \cap \sigma'|} \cdot \Psi \left(\langle e_{i_\sigma} \rangle * \langle e_{i_{\sigma'}} \rangle * \Phi_\sharp \left(s_{\sigma \cap \sigma'} * \sum_{j: i_\sigma \notin \tau_j} \alpha_j \tau_j * \sum_{k: i_{\sigma'} \notin \tau'_k} \alpha'_k \tau'_k \right) \right).
$$

In this sum the sign of the cube in question is

$$
(-1)^{|\sigma \cap \sigma'|} \cdot \operatorname{sign}\left(i_\sigma i_{\sigma'} (\sigma \cap \sigma') \tau_j \tau'_k \right) \cdot (-1)^{i_\sigma + i_{\sigma'} + \sum (\sigma \cap \sigma') + \sum \tau_j + \sum \tau'_k}. \quad (**)
$$

The global sign is given by the quotient of the two signs $(*)$ and $(**)$.

$$
\begin{aligned}
&(-1)^{|\sigma \cap \sigma'|} \cdot \rho_{H,K} \operatorname{sign}(i_\sigma \sigma \tau_j) \cdot \operatorname{sign}\left(i_{\sigma'} \sigma' \tau'_k \right) \cdot \operatorname{sign}\left(i_\sigma i_{\sigma'} (\sigma \cap \sigma') \tau_j \tau'_k \right) \cdot \\
&\quad (-1)^{i_\sigma + i_{\sigma'} + \sum \sigma + \sum \tau_j + \sum \sigma' + \sum \tau'_k} \cdot (-1)^{i_\sigma + i_{\sigma'} + \sum (\sigma \cap \sigma') + \sum \tau_j + \sum \tau'_k} \\
&= (-1)^{|\sigma \cap \sigma'| + \sum (\sigma \cup \sigma')} \cdot \rho_{H,K} \operatorname{sign}(i_\sigma \sigma \tau_j) \cdot \operatorname{sign}\left(i_{\sigma'} \sigma' \tau'_k \right) \cdot \\
&\qquad \operatorname{sign}\left(i_\sigma i_{\sigma'} \left(\sigma \cap \sigma' \right) \tau_j \tau'_k \right),
\end{aligned}
$$

where

$$
\begin{aligned}
H &= [n] \setminus \left(\sigma' \cup \tau'_k \cup \{ i_{\sigma'} \} \right) \\
K &= [n] \setminus \left(\sigma \cup \tau_j \cup \{ i_\sigma \} \right).
\end{aligned}
$$

We will derive a formula that is easier to handle and, in particular, shows the independence of j, k and $i_\sigma, i_{\sigma'}$.

Lemma 3.3 *Let $\sigma, \sigma' \subset [n]$ such that $\sigma \cup \sigma' = [n]$, and $\iota = \{i\} \subset [n] \setminus \sigma$, $\iota' = \{i'\} \subset [n] \setminus \sigma'$, and $r, r' \geq 0$. Then for $\tau \subset [n] \setminus (\sigma \cup \iota)$ and $\tau' \subset [n] \setminus (\sigma' \cup \iota')$ of cardinality r, resp. r' we have*

$$\begin{aligned}&\operatorname{sign}(([n] \setminus (\sigma' \cup \tau' \cup \iota'))([n] \setminus (\sigma \cup \tau \cup \iota))) \cdot \\ &\qquad \operatorname{sign}(\iota\sigma\tau) \operatorname{sign}(\iota'\sigma'\tau') \operatorname{sign}(\iota\iota'(\sigma \cap \sigma')\tau\tau') \\ &\qquad\qquad = (-1)^{rr'+r'(n-|\sigma|-1)+1} \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)).\end{aligned}$$

Note that for simplicity we have used r, r' for the cardinalities of τ, τ' instead of the dimensions.

Proof. We proceed in two steps. First we show, what happens if we reduce (r, r') in the lexicographic order.

For $(r, r') = (0, 0)$, we just have

$$\begin{aligned}&\operatorname{sign}(([n] \setminus (\sigma' \cup \iota'))([n] \setminus (\sigma \cup \iota))) \\ &\quad \cdot \operatorname{sign}(\iota\sigma)\operatorname{sign}(\iota'\sigma') \operatorname{sign}(\iota\iota'(\sigma \cap \sigma')).\end{aligned} \tag{2}$$

Now assume $r = 0$ **and** $r' > 0$**.** Choose two r'-sets $\tau'_1, \tau'_2 \subset [n] \setminus (\sigma' \cup \iota')$, and choose elements $v_1 \in \tau'_1, v_2 \in \tau'_2$. Let $\bar{\tau}'_1 = \tau_1 \setminus \{v_1\}$ and $\bar{\tau}'_2 = \tau'_2 \setminus \{v_2\}$. Then

$$\begin{aligned}&\operatorname{sign}(([n] \setminus (\sigma' \cup \tau'_{1/2} \cup \iota'))([n] \setminus (\sigma \cup \iota))) \\ &\quad = \operatorname{sign}([n] \setminus (\sigma' \cup \bar{\tau}'_{1/2} \cup \iota')[n] \setminus (\sigma \cup \iota))(-1)^{|\{a \in [n] \setminus (\sigma \cup \iota) : a < v_{1/2}\}|}, \\ &\operatorname{sign}(\iota'\sigma'\tau'_{1/2}) = \operatorname{sign}(\iota'\sigma'\bar{\tau}'_{1/2})(-1)^{|\{a \in \iota' \cup \sigma' : a > v_{1/2}\}|}, \\ &\operatorname{sign}(\iota\iota'(\sigma \cap \sigma')\tau'_{1/2}) = \operatorname{sign}(\iota\iota'(\sigma \cap \sigma')\bar{\tau}'_{1/2})(-1)^{|\{a \in \iota \cup \iota' \cup (\sigma \cap \sigma') : a > v_{1/2}\}|}.\end{aligned}$$

Consider the sum of the (-1)-exponents.

$$\begin{aligned}&|\{a \in [n] \setminus (\sigma \cup \iota) : a < v_{1/2}\}| + |\{a \in \iota' \cup \sigma' : a > v_{1/2}\}| + \\ &|\{a \in \iota \cup \iota' \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\ &\quad \equiv |[n] \setminus (\iota \cup \sigma)| - |\{a \in [n] \setminus (\sigma \cup \iota) : a > v_{1/2}\}| + \\ &\qquad |\{a \in \sigma' : a > v_{1/2}\}| + |\{a \in \iota \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\ &\quad \equiv |[n] \setminus (\iota \cup \sigma)| + |\{a \in \sigma' : a > v_{1/2}\}| + \\ &\qquad |\{a \in [n] \setminus (\sigma \cup \iota) : a > v_{1/2}\}| + |\{a \in \iota \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\ &\quad \equiv |[n] \setminus (\iota \cup \sigma)| + 2|\{a \in \sigma' : a > v_{1/2}\}| \\ &\quad \equiv |[n] \setminus (\iota \cup \sigma)| \pmod 2.\end{aligned}$$

Hence, for reducing r' by one we obtain a factor of $(-1)^{n-|\sigma|-1}$ and thus in total a factor $(-1)^{r'(n-|\sigma|-1)}$.

Assume $r > 0$. This case works analogously, reducing two choices of r-sets $\tau_{1/2}$. In each step one gets a factor $(-1)^{r'}$. Hence, after r steps, we obtain a factor $(-1)^{rr'}$.
Treating the expression (2) similarly yields

$$(-1) \cdot \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)),$$

which gives the result. □

Thus, we derived the following global sign

$$\begin{aligned}(-1)^{\frac{n(n+1)}{2}+|\sigma\cap\sigma'|+(r+1)(r'+1)+(r'+1)(n-|\sigma|-1)+1} \cdot \operatorname{sign}(([n]\setminus\sigma')([n]\setminus\sigma)) \\ = (-1)^{\frac{n(n+1)}{2}+|\sigma\cap\sigma'|+(r'+1)(n+|\sigma|+r)+1} \cdot \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)). \quad (3)\end{aligned}$$

3.5 The complex case

We will explicitly compute the multiplication in $\tilde{H}^*(C^{\mathbb{C}}_{\Delta}; \mathbb{Z})$ using the results and notation of Sect. 2.3 and the previous Section. Let $[u], [v] \in \tilde{H}^*(C^{\mathbb{C}}_{\Delta}; \mathbb{Z})$ correspond to

$$[c] \in \tilde{H}_r(\operatorname{link}_{\Delta} \sigma) \cong \tilde{H}_r \left(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma)\right)$$

and

$$[c'] \in \tilde{H}_{r'}(\operatorname{link}_{\Delta} \sigma') \cong \tilde{H}_{r'} \left(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma')\right)$$

for simplices $\sigma, \sigma' \in \Delta$.

Case I: If $\sigma \cup \sigma' \neq [n]$ then $\pi^{-1}(\sigma) \cup \pi^{-1}(\sigma') \neq [2n]$ and hence the cup product of $[u]$ and $[v]$ is zero.

Case II: If $\sigma = \sigma' \neq [n]$ the cup product vanishes since the complement of a complex coordinate subspace arrangement is connected.

Case III: Now let $\sigma \cup \sigma' = [n]$. Consider the isomorphism

$$\begin{aligned}\tilde{H}_r(\operatorname{link}_{\Delta} \sigma) &\longrightarrow \tilde{H}_r \left(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma)\right) \\ [c] &\longmapsto [c^{\mathbb{C}}]\end{aligned}$$

induced by the vertex map $i \mapsto 2i - 1$. It corresponds to the isomorphism induced by the homotopy equivalence. Using this isomorphism for the cup product computation we are in the well known situation as shown in Fig. 9.

$$
\begin{array}{l}
(\ast\ \ast|\overbrace{+\,+\,+\,+}^{\tau_j^{\mathbb{C}}}|\ast\ \ast|\overbrace{\ +\ }^{i_\sigma^{\mathbb{C}}}|\ast\ \ast\ \ast|\overbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\ \ \pm}^{\pi^{-1}(\sigma)}\ )^*(D_H^{+1}T)\cdot \\
(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\ \ \pm\ \ \pm\pm\pm\pm\pm\pm}_{\pi^{-1}(\sigma')}|\ast\ \ast|\underbrace{+\,+\,+\,+}_{\tau_k'^{\mathbb{C}}}|\ast\ \ast\ \ast|\underbrace{\ +\ }_{i_{\sigma'}^{\mathbb{C}}}\ )^*(D_K^{-1}T)
\end{array}
$$

$$
(\ast\ \ast|+\,+\,+\,+|\ast\ \ast|\ +\ |\ast\ \ast\ \ast|\pm\pm|\ast\ \ast|+\,+\,+\,+|\ast\ \ast\ \ast|\ +\ )
$$

Fig. 9. A typical summand of the cup product evaluated at T schematically and the cubes T for which it does not vanish

Collecting all summands yields the cocycle

$$\Psi\left(\langle e_{i_\sigma}\rangle * \langle e_{i_{\sigma'}}\rangle * \Phi_\sharp\left(s_{\pi^{-1}(\sigma)\cap\pi^{-1}(\sigma')} * \sum_{j:i_\sigma\notin\tau_j^{\mathbb{C}}}\alpha_j\tau_j^{\mathbb{C}} * \sum_{k:i_{\sigma'}\notin\tau_k'^{\mathbb{C}}}\alpha_k'\tau_k'^{\mathbb{C}}\right)\right)$$

for vertices $i_\sigma \in [2n]\setminus\pi^{-1}(\sigma)$ and $i_{\sigma'} \in [2n]\setminus\pi^{-1}(\sigma')$. As above this leads (up to the global sign) to

$$\begin{aligned}
[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] &\in \tilde{H}_{r+r'+2}\left(\operatorname{link}_\Delta \sigma\cap\sigma'\right)\\
&\cong \tilde{H}_{r+r'+2}\left(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma\cap\sigma')\right)\\
&= \tilde{H}_{r+r'+2}\left(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma)\cap\pi^{-1}(\sigma')\right).
\end{aligned}$$

3.6 The global sign in the complex case

First of all, from the computation in the real case, we obtain the sign

$$(-1)^{n(2n+1)+|\pi^{-1}(\sigma)\cap\pi^{-1}(\sigma')|+(r+1)(r'+1)+(r'+1)(2n-|\pi^{-1}(\sigma)|-1)+1}\cdot$$
$$\operatorname{sign}(([2n]\setminus\pi^{-1}(\sigma'))([2n]\setminus\pi^{-1}(\sigma))).$$

Now in $\pi^{-1}(\sigma)$, $\pi^{-1}(\sigma')$ resp., all elements appear in pairs. This simplifies the sign to

$$(-1)^{n+r(r'+1)+1}. \tag{4}$$

4 Example of a simplicial complex yielding different ring structures

Let $[u], [v], [w]$ be cohomology classes of the complement of a real coordinate subspace arrangement corresponding to homology classes of links of Δ, such that $[u] \cup [v] = [w]$. Then our results imply that for the corresponding cohomology classes of the complement of the associated complex arrangement we have (see Corollary 2.1)

$$\left[u^{\mathbb{C}}\right] \cup \left[v^{\mathbb{C}}\right] = \pm \left[w^{\mathbb{C}}\right].$$

Hence it arises the question if we can choose signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$ consistently such that it becomes a (dimension-shifting) ring isomorphism. An example of different ring structures containing hyperplanes was given in [GPW]: the existence of hyperplanes lead to additional multiplication in the real case. Our example shows that this is not the only case where non-isomorphic rings occur.

Remark 4.1 There is a (dimension shifting) ring isomorphism of $\tilde{H}^*(C_\Delta; \mathbb{Z}_2)$ and $\tilde{H}^*(C^{\mathbb{C}}_\Delta; \mathbb{Z}_2)$.

4.1 The example: different sign patterns

We construct a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices given by four facets $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$, and investigate the multiplication of cohomology classes stemming from the links of these facets in the case of the associated real and complex arrangement. For the real and complex case the resulting sign pattern implies that there is no ring isomorphism between $\tilde{H}^*(C_\Delta)$ and $\tilde{H}^*(C^{\mathbb{C}}_\Delta)$. The facets are given by the following scheme which also helps for computing the signs appearing in the multiplication. A black box in position (ρ, j) indicates that $j \in \rho$.

	1	2	3	4	5	6	7	8
σ_1	■	■	■	■	■	■	□	□
σ_2	■	□	■	■	■	□	□	■
σ_1'	■	■	□	□	■	■	■	■
σ_2'	□	■	■	■	□	■	■	■

Fig. 10. The facets of Δ

The sign patterns arising in the real and in the complex case according to (3) and (4) are given by the following table. Clearly, there is no consistent way of assigning signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$.

5 Example of non trivial multiplication of torsion elements

We construct a simplicial complex $\Delta \subset 2^{[10]}$. Let $\sigma := \{1, 2, 3, 4, 5, 6\}$ and $P \subset 2^{\{1,2,3,4,5,6\}}$ be a six-vertex triangulation of the projective plane.

		Sign (3)	Sign (4)
σ_1	σ'_1	-1	-1
σ_1	σ'_2	-1	-1
σ_2	σ'_1	+1	-1
σ_2	σ'_2	-1	-1

Let $\sigma' = \{7, 8, 9, 10\}$, and let S be a simplicial 1-sphere on four vertices as a subcomplex of $2^{\{7,8,9,10\}}$. Now define $\Delta = P * 2^{\sigma'} \cup 2^{\sigma} * S$. Then the homotopy type of Δ is $\Sigma(P * 2^{\sigma'} \cap 2^{\sigma} * S) = \Sigma(P * S)$. Hence Δ has the homotopy type of a threefold suspended projective plane. Now $\mathrm{link}_\Delta(\sigma * \emptyset) = \emptyset * S$ and $\mathrm{link}_\Delta(\emptyset * \sigma') = P * \emptyset$. Let $[c] \in \tilde{H}_1(\mathrm{link}_\Delta(\sigma * \emptyset)) \cong \mathbb{Z}$ and $[c'] \in \tilde{H}_1(\mathrm{link}_\Delta(\emptyset * \sigma')) \cong \mathbb{Z}_2$ be generating homology classes. They correspond to elements $[u] \in \tilde{H}^{10-1-6-2}(\Gamma_\Delta)$ and $[v] \in \tilde{H}^{10-1-4-2}(\Gamma_\Delta)$. Their cup product corresponds to a generating class

$$[\langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'] \in \tilde{H}_{10-4-0-2}(\mathrm{link}_\Delta \emptyset) \cong \mathbb{Z}_2$$

for $i_\sigma \in \{7, 8, 9, 10\}$ and $i_{\sigma'} \in \{1, 2, 3, 4, 5, 6\}$.

Note that this example works for the real as well as for the complex case.

6 Questions and remarks

- ▷ A very natural question is as to what extent our methods can be used to treat more general subspace arrangements.
- ▷ It is easy to see that if $\Delta \subset 2^{[n]}$ is a simplicial complex such that
  - ▷ $\dim \Delta \leq n-3$, i.e., the associated real arrangement does not contain hyperplanes, and
  - ▷ Δ is Cohen-Macaulay over $\mathbb{Z}$,

  then the ring structure of $\tilde{H}^*(C_\Delta; \mathbb{Z})$ is trivial. Using the specific description of the multiplication it would be nice to derive a better characterization of simplicial complexes yielding trivial multiplication. Confer also [HRW].

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