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Some results on a class of nonlinear Schrödinger equations*

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Abstract. By using a Liapunov-Schmidt reduction we prove an existence result for the nonlinear Schrödinger equation $-h^2\Delta u + V(x)u = f(x,u)$ in R^N where $f(x,u)$ satisfies suitable assumptions. We also provide a necessary condition for the existence of solutions.

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1 Introduction

In this paper we study standing wave solutions of the nonlinear Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\Delta\psi + V(x)\psi - g(x,|\psi|)\psi \tag{1.1}$$

i.e. solutions of the form

$$\psi(x,t) = e^{i\frac{Et}{h}}u(x), \quad u : \mathbb{R}^N \to \mathbb{R}^+. \tag{1.2}$$

Here h, m and E are real numbers and $V \in C^1(\mathbb{R}^N;\mathbb{R}^+) \cap L^\infty(\mathbb{R}^N;\mathbb{R})$. In [7], Floer and Weinstein considered the case $N = 1, g(x,|t|) = |t|^2$ and they proved that for small h there exists a positive standing wave solution which concentrates at each given nondegenerate critical point of the potential V. This result was generalized by Oh ([15]) to the case $g(x,|t|) = |t|^{p-1}$ with

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$1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$. The arguments in those papers are based on a Liapunov-Schmidt reduction.

Substituting (1.2) in (1.1) and assuming that $m = \frac{1}{2}$ one has

$$\begin{cases} -h^2 \Delta u + (V(x) - E)u = g(x, |u|)u \\ \\ u > 0 \end{cases} \tag{1.3}$$

A suitable choice of E makes V bounded from below by a positive constant. Hence, without loss of generality, it is possible to assume that $E = 0$ and $V \geq V_0 > 0$. Let us set $f(x,t) = g(x, |t|)t$. So (1.3) becomes

$$\begin{cases} -h^2 \Delta u + V(x)u = f(x, u) \quad \text{in} \mathbb{R}^N \\ \\ u > 0 \end{cases} \tag{1.4}$$

The existence of solutions of (1.4) in the possibly degenerate setting was studied by many authors. In this context the first results seem due to Rabinowitz (see [17]) and Ding-Ni (see [6]). In [17] it was shown that if $\inf_{\mathbb{R}^N} V < \lim_{|x| \to \infty} \inf V(x)$ then the mountain pass theorem provides a solution for small h. This solution concentrates around a global minimum of V as $h \to 0$, as shown later by X. Wang (see [18]). Moreover in [18] it was observed that concentration of any family of solutions with uniformly bounded energy may occur only at critical points of V.

Later Ambrosetti, Badiale and Cingolani (see [1]) obtained existence of standing wave solutions by assuming that the potential V has a local minimum or maximum with nondegenerate $m - th$ derivative, for some integer m.

This result was generalized by Li (see [13]), where a degeneracity of any order of the derivative is allowed. In [13] the author proves the existence of a solution for (1.4) by only assuming that the critical points of V are "stable" with respect to a small C^1-perturbation of V.
Here we remark that all the previous papers deal with the case $f(x,t) = t^p$.

In [5] Del Pino and Felmer consider a more general nonlinearity $f(t)$ and obtained a solution of (1.4) by considering a "topologically nontrivial" critical value of the energy functional associated.

When the nonlinearity f depends on x the first result seems to appear in [17] where $f(x,t) = K(x)|t|^{p-1} + Q(x)|t|^{q-1}, p > q, K, Q$ satisfy suitable assumptions and V is coercive. Such a result was improved by Bartsch and Z.Q. Wang in [2] where the assumption on V are weakened provided the functions V, K, Q are invariant under the action of some suitable group of rotations. Other results regarding this type of nonlinearity $f(x,t)$ are due to X. Wang and Zeng (see [19]) and Cingolani and Lazzo (see [3]).

In [19] the authors proved, among other results, a sufficient condition involving the functions V, K, Q in order to deduce the existence of the solution of (1.4). This condition is generalized in [3] where the number of the solutions of (1.4) is related with the topology of the set of the global minima of a suitable ground energy function.

In this paper we consider a more general class of nonlinearities depending both on x and t (see $(f_0) - (f_2)$ in Sect. 2).

The first result we get concerns solutions which concentrate at some point.

Definition 1.1 *We say that u_h concentrates at P_0 if there exist positive constants C, γ, R such that*

$$\begin{aligned} &For\ \ any\, \varepsilon > 0\, there\ exists\, h_0 > 0\, such\, that\, if\, h < h_0\, we\, have \\ &u_h(x) < \varepsilon\, for\, |x - P_h| \geq Rh \quad and \\ &u_h(P_h) \geq \gamma > 0 \end{aligned} \tag{1.5}$$

where $P_h \to P_0$ is the point where the maximum of u_h is achieved.

In this context the following vector field $G : \mathbb{R}^N \to \mathbb{R}^N$ seems to play a crucial role (see $(f_0) - (f_2)$ for the definition of w_P and F_{x_j}).

$$G_j(P) = -\frac{1}{2}\frac{\partial V}{\partial x_j}(P)\int_{\mathbb{R}^N} w_P^2 + \int_{\mathbb{R}^N} F_{x_j}(P, w_P). \tag{1.6}$$

Indeed, we have the following result

Theorem 1.1 *Assume (V_0)-(V_2) and $(f_0) - (f_2)$. Let us consider a positive solution u_h which concentrates at P_0. Then P_0 is a zero of the vector field G.*

In order to state our existence result we need the definition of stable zero ; let us set $B_{y,\rho} = \{x \in \mathbb{R}^N : |x - y| \leq \rho\}$. Then

Definition 1.2 *Let $G \in C(\mathbb{R}^N; \mathbb{R}^N)$ be a vector field. We say that Z is a ''set of stable zeroes" for G if $G(P) = 0$ for any $P \in Z$ and if G_n is a sequence of vector fields such that $||G_n - G||_{C(B_{P,\rho})} \to 0$ for some $\rho > 0$, then there exists P_n such that $G_n(P_n) = 0$ and $dist(P_n, Z) \to 0$*

If G is a conservative vector field this type of condition was considered by Li in [13].

A sufficient condition on G and Z which implies that Z is a ''set of stable zeroes" is the following one

There exists a sequence of compact sets $D_n \supset Z$ such that
i) $G \neq 0$ on ∂D_n for any $n \in N$,
ii) $dist(\partial D_n, Z) \to 0$ as $n \to \infty$
iii) the Brouwer degree satisfies $deg(G, D_n, 0) \neq 0$ for any $n \in N$.

If $Z = \{P\}$ where P is an isolated zero of G, the previous condition becomes

$$i(G, P, 0) \neq 0, \tag{1.7}$$

where the index of P at zero $i(G, P, 0)$ is given by

$$i(G, P, 0) = \lim_{\varepsilon \to 0} deg(G, B_{P,\varepsilon}, 0).$$

Now we are ready to state our main theorem.

Theorem 1.3 *Assume* (V_0)-(V_2) *and* $(f_0) - (f_2)$ *and let us suppose that* Z *is some stable bounded set of zeros of* G. *Then there exists* h_0 *such that for* $0 < h < h_0$ *the problem (1.4) admits a family of solutions* $u_h \in C^2(\mathbb{R}^N)$ *whose unique maximum point* Q_h *satisfies* $dist(Q_h, Z) \to 0$ *as* $h \to 0$.

Theorem 1.3 has the following corollary (see [13] or [5] for analogous results):

Corollary 1.4 *Assume* V_0-V_2 *and* $(f_0) - (f_2)$ *with* $f(x, t) = f(t)$. *If* Z *is some stable bounded set of zeros of* ∇V *then there exists* h_0 *such that for* $0 < h < h_0$ *the problem (1.4) admits a family of solutions* $u_h \in C^2(\mathbb{R}^N)$ *whose unique maximum point* Q_h *satisfies* $dist(Q_h, Z) \to 0$ *as* $h \to 0$

When $f(x, t) = K(x)t^p$ Theorem 1.3 provides the following result (which generalizes the previous one of [19] and [3]).

Corollary 1.5 *Let us suppose that* Z *is some stable bounded set of critical points of* $\frac{V^{2p+2+N-Np/(2p-2)}(x)}{K^{2/(p-1)}(x)}$. *Then there exists* h_0 *such that for* $0 < h < h_0$ *the problem (2.4) admits a family of solutions* $u_h \in C^2(\mathbb{R}^N)$ *whose unique maximum point* Q_h *satisfies* $dist(Q_h, Z) \to 0$ *as* $h \to 0$.

We would like to point out that the Proof of Theorem 1.3 is based on a Liapunov-Schmidt procedure as in the pioneering paper [7]. This approach was recently used to study (1.4) in bounded domains (see [10]).

The paper is organized as follows: in Sect. 2 and 3 we state some preliminaries and repeat the classical Liapunov-Schmidt procedure used in [7]. In Sect. 4 we prove Theorem 1.3 and Corollary 1.5. In Sect. 5 we prove Theorem 1.1.

2 Preliminaries

Let us consider the following problem

$$(P_h) \qquad \begin{cases} -h^2 \Delta u + V(x)u = f(x, u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

where $h > 0$, $N \geq 2$, the potential V satisfies the following assumptions

(V_0) $V \in C^1(\mathbb{R}^N)$,

(V_1) $0 < V_0 \leq V(x) \leq V_1$,

(V_2) $|\nabla V(x)| \leq Ce^{\delta|x|}$ $for\ |x|\ large\ and\ for\ some\ \delta > 0.$

and the nonlinearity f satisfies the following assumptions:

(f_0) $f \in C^1(\mathbb{R}^N \times \mathbb{R})\ and\ f(\cdot, u) \equiv 0\ \forall\, u \leq 0,$

$$(f_1)\ There\ exist\ \alpha \in]0,1],\ s \in]1, \frac{N}{N-4}[\ if\ N \geq 5\ and s > 1 if N < 5,$$

$$M > 0,\ \delta > 0,\ such\ that,\ if\ denote\ by\ F(x,t) = \int_0^t f(x,z)dz$$

$$(i)|f(x,t) - f(x,t')| \leq k|t-t'|^\alpha\ \forall x \in \mathbb{R}^N,\ t,t' \in \mathbb{R},$$

$$(ii)|f'_t(x,t)| \leq Cu^{s-1}\ \forall\, t > M,$$

$$(iii)|F_{x_i}(x,t)| \leq \begin{cases} Ct^{\alpha+1} & \text{if } t \leq M \\ Ce^{\delta|x|}t^s & \text{if } t > M \end{cases}$$

(f_2) $For\ any\ P \in \mathbb{R}^N\ the\ following\ problem$

$$\begin{cases} -\Delta w + V(P)w = f(P,w) & \text{in } \mathbb{R}^N \\ w > 0 & \text{in } \mathbb{R}^N \\ w(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

$has\ a\ unique\ solution\ w_P\ which\ is\ nondegenerate\ in\ the\ space\ of\ the\ radial\ function,\ i.e.\ the\ operator\ L_P = -\Delta + V(P) - f_u(P, w_P) is\ invertible\ in\ H^2_r(\mathbb{R}^N) = \{u \in H^2(\mathbb{R}^N) : u = u(|x|)\}.$

A class of nonlinearities which satisfy (f_0), (f_1) and (f_2) is the following one:
$f(x,t) = K(x)t^p - Q(x)t^q$ for $t \geq 0$, $f(x,t) \equiv 0$ for $t \leq 0$ with $K(x) \geq k_0 > 0$, $Q(x) \geq 0$ and $1 < q < p < \frac{N+2}{N-2}$ if $N \geq 3$ or $1 < q < p < +\infty$ if $N = 2$ (see [12] and [4]).

Remark 2.1 We recall that by [8] w_P is spherically symmetric with respect to some point of $\mathbb{R}^N$, say the origin, $\lim_{r\to\infty} w_P(r)e^r r^{\frac{N-1}{2}} = \gamma_P > 0$ and $\lim_{r\to\infty} \frac{w'(r)}{w(r)} = -1$

Moreover from (f_2) it follows that

$$Ker\ L_P = span\, \{\frac{\partial w_P}{\partial x_1}, \dots, \frac{\partial w_P}{\partial x_N}\} \tag{2.1}$$

(see Lemma 4.2 in [14] for example).

Remark 2.2 Assumptions $(f_0) - (f_2)$ imply that

$$|f(x,t)| \le \begin{cases} Ct^{\alpha+1} & \text{if } t < M \\ Ct^s & \text{if } t \ge M \end{cases} \tag{2.2}$$

We would like to point out that $f(x,t)$ may have a different behavior at the origin and at infinity. This allows us to treat nonlinearities of the type $f(x,t) = t^p + t^q$. We remark that we consider only the case $s - 1 > \alpha$. In fact if $s - 1 \le \alpha$ the function $f(x,t)$ satisfies a unique inequality which holds everywhere, namely

$$|f(x,t)| \le Ct^{s-1} \tag{2.3}$$

and obviously it can be treated in the same way.

Of course the problem (P_h) is equivalent to the following one

$$\begin{cases} -\Delta u + V(P+hx)u = f(P+hx,u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2.4}$$

Let us consider the operator $S_{h,P} \ :\ H^2(\mathbb{R}^N) \mapsto L^2(\mathbb{R}^N)$

$$S_{h,P}(v) = -\Delta v + V(P+hx)v - f(P+hx,v). \tag{2.5}$$

If $v = w_P + \Phi_P$ we have the following expansion to $S_{h,P}$

$$S_{h,P}(w_P + \Phi_P) = S_{h,P}(w_P) + S'_{h,P}(w_P)\Phi_P + R_{h,P}(\Phi_P) \tag{2.6}$$

where

$$R_{h,P}(\Phi_P) = f(P+hx, w_P) - f(P+hx, w_P+\Phi_P) + f_t'(P+hx, w_P)\Phi_P. \tag{2.7}$$

Finally let us denote by

$$L_{h,P} = \Pi_P^{\perp} \circ S'_{h,P}(w_P)\Big|_{K_P^{\perp} \cap H^2(\mathbb{R}^N)} \tag{2.8}$$

with

$$K_P^{\perp} = \left\{ \phi \in L^2(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} \phi \frac{\partial w_P}{\partial x_i} = 0 \quad \text{for } i = 1, \ldots, N \right\} \tag{2.9}$$

and

$$\Pi_P^{\perp} : L^2(\mathbb{R}^N) \mapsto K_P^{\perp} \tag{2.10}$$

is the projection operator. The following proposition is a classical result (see [7] or [15])

Proposition 2.3 *There are constants* $\gamma, h_1 > 0$ *such that if* $0 < h < h_1$ *and* $\phi \in K_P^{\perp} \cap H^2(\mathbb{R}^N)$ *then*

$$||L_{h,P}(\phi)||_{L^2(\mathbb{R}^N)} \geq \gamma ||\phi||_{H^2(\mathbb{R}^N)}. \tag{2.11}$$

3 Reduction to finite dimensions

In this section we prove that, for any $P \in \mathbb{R}^N$ and h small enough there exists a unique $\Phi_{h,P}$ such that

$$\Pi_P^{\perp} \circ S_{h,P}(w_P + \Phi_P) = 0. \tag{3.1}$$

By using (2.6) we see that (3.1) is equivalent to prove that $\Phi_{h,P}$ is a fixed point of the map $F_{h,P}$ on $H^2(\mathbb{R}^N)$ defined by

$$F_{h,P}(\Phi) = -L_{h,P}^{-1} \circ [\Pi_P^{\perp} \circ R_{h,P} + \Pi_P^{\perp} S_{h,P}(w_P)](\Phi). \tag{3.2}$$

Let us denote by $|| \ ||_H$ the standard norm in the Sobolev space $H^2(\mathbb{R}^N)$. We need the following lemma

Lemma 3.1 *There exists a positive constant* C *independent of* P *and* h *such that for all* ϕ *and* ϕ' *in* $H^2(\mathbb{R}^N)$ *we have*

$$||R_{h,P}(\Phi_P)||_{L^2(\mathbb{R}^N)} \leq C(||\Phi||_H^s + ||\Phi||_H^{1+\alpha}) \tag{3.3}$$

and

$$||R_{h,P}(\widetilde{\Phi}_P) - R_{h,P}(\Phi_P)||_{L^2(\mathbb{R}^N)} \le CM(\Phi_P, \widetilde{\Phi}_P)||\Phi_P - \widetilde{\Phi}_P||_H$$
$$\text{where } M(\Phi_P, \widetilde{\Phi}_P) \to 0 \text{ as } ||\Phi_P||_H, ||\widetilde{\Phi}_P||_H \to 0. \tag{3.4}$$

Proof. By using the mean value theorem we have

$$|R_{h,P}(\Phi_P)| = |f(P+hx, w_P+\Phi_P) - f(P+hx, w_P) + f'_t(P+hx, w_P)\Phi_P| \le |\Phi_P| \int_0^1 |f'_t(P+hx, w_P+t\Phi_P) - f'_t(P+hx, w_P)|dt, \tag{3.5}$$

and by $(f_0) - (f_1)$ and recalling that $2+2\alpha < 2s < \frac{2N}{N-4}$, by Sobolev embedding Theorem we get

$$\int_{\mathbb{R}^N} R_{h,P}(\Phi_P)^2 dx \tag{3.6}$$
$$\le \int_{\mathbb{R}^N} |\Phi_P|^2 \Big(\int_0^1 (f'_t(P+hx, w_P+t\Phi_P) - f'_t(P+hx, w_P)|dt \Big)^2 dx$$
$$\le C \int_{|\Phi_P| \le M} |\Phi_P|^{2+2\alpha} dx + C \int_{|\Phi_P| \ge M} |\Phi_P|^{2s} dx \le C(||\Phi_P||_H^{2+2\alpha} + ||\Phi_P||_H^{2s})$$

which proves (3.3).

On the other hand

$$|R_{h,P}(\widetilde{\Phi}_P) - R_{h,P}(\Phi_P)| = |f(P+hx, w_P+\Phi_P) - f(P+hx, w_P+\widetilde{\Phi}_P) + f'_t(P+hx, w_P)\widetilde{\Phi}_P - f'_t(P+hx, w_P)\Phi_P| \le |f(P+hx, w_P+\Phi_P) - f(P+hx, w_P+\widetilde{\Phi}_P) - f'_t(P+hx, w_P+\widetilde{\Phi}_P)(\Phi_P - \widetilde{\Phi}_P)| + |f'_t(P+hx, w_P+\widetilde{\Phi}_P)(\Phi_P - \widetilde{\Phi}_P) - f'_t(P+hx, w_P)(\Phi_P - \widetilde{\Phi}_P)|. \tag{3.7}$$

Integrating (3.7) we obtain

$$\int_{\mathbb{R}^N} |R_{h,P}(\widetilde{\Phi}_P) - R_{h,P}(\Phi_P)|^2 dx \le 2\int_{\mathbb{R}^N} |f(P+hx, w_P+\Phi_P) - f(P+hx, w_P+\widetilde{\Phi}_P) - f'_t(P+hx, w_P+\widetilde{\Phi}_P)(\Phi_P - \widetilde{\Phi}_P)|^2 dx + 2\int_{\mathbb{R}^N} |f'_t(P+hx, w_P+\widetilde{\Phi}_P) - f'_t(P+hx, w_P)|^2 |\Phi_P - \widetilde{\Phi}_P|^2 dx = I_1 + I_2. \tag{3.8}$$

Again by (f_1) we deduce that there exists a constant C independent of x such that

$$|f(x,s_1) - f(x,s_2) - f_t'(x,s_1)(s_1-s_2)| \leq C|s_1-s_2|^{\alpha+1}. \quad (3.9)$$

So, if we set $D = x \in \mathbb{R}^N \mid \{|\Phi_P(x)| < M$ and $|\widetilde{\Phi}_P(x)| < M\}$ we get

$$\begin{aligned}
&|I_1| = 2 \int\limits_{D\cup\{\mathbb{R}^N\setminus D\}} |f(P+hx, w_P+\Phi_P) - f(P+hx, w_P+\widetilde{\Phi}_P) \\
&-f_t'(P+hx, w_P+\widetilde{\Phi}_P)(\Phi_P-\widetilde{\Phi}_P)|^2 dx + \quad (3.10)\\
&\leq C\int\limits_D |\Phi_P-\widetilde{\Phi}_P|^{2\alpha+2}dx + C\int\limits_{\mathbb{R}^N} |\Phi_P+\widetilde{\Phi}_P|^{2s-2}|\Phi_P-\widetilde{\Phi}_P|^2 dx \\
&\leq C[(||\Phi_P||+||\widetilde{\Phi}_P||)^{2\alpha} + (||\Phi_P||+||\widetilde{\Phi}_P||)^{2s-2}]||\Phi_P-\widetilde{\Phi}_P||^2
\end{aligned}$$

On the other hand

$$|I_2| \leq C(||\widetilde{\Phi}_P||^{2\alpha} + ||\widetilde{\Phi}_P||^{2s-2})||\Phi_P-\widetilde{\Phi}_P||^2 \quad (3.11)$$

and so the claim follows

Lemma 3.2 *Let $A \subset \mathbb{R}^N$ be a compact set. Then*

$$\int_{\mathbb{R}^N} |S_{h,P}(w_P)|^2 dx \to 0 \textit{ as } h \to 0 \textit{ uniformly with respect to } P \in A\,. \quad (3.12)$$

Proof. We have

$$\begin{aligned}
&|S_{h,P}(w_P)|^2 = |-\Delta w_P + V(P+hx)w_P - f(P+hx, w_P)|^2 \\
&= |(V(P+hx)-V(P))w_P + f(P,w_P) - f(P+hx,w_P)|^2 \\
&\leq 2|V(P+hx)-V(P)|^2 w_P^2 + 2|f(P,w_P) - f(P+hx,w_P)|^2
\end{aligned} \quad (3.13)$$

and integrating on R^N we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |S_{h,P}(w_P)|^2 dx \\
&\leq 2\int\limits_{|x|\leq K} |V(P+hx)-V(P)|^2 w_P^2 dx \\
&+2\int\limits_{|x|\geq K} |V(P+hx)-V(P)|^2 w_P^2 dx
\end{aligned}$$

$$+2\int\limits_{|x|\le K}|f(P,w_P)-f(P+hx,w_P)|^2dx$$
$$+2\int\limits_{|x|\ge K}|f(P,w_P)-f(P+hx,w_P)|^2dx. \tag{3.14}$$

Now by (2.2) we get,

$$\int_{|x|\ge K}|f(P,w_P)-f(P+hx,w_P)|^2dx\le C\int_{|x|\ge K}w_P^{2\alpha+2}dx. \tag{3.15}$$

Then for any $\varepsilon>0$, let us set K_ε such that

$$\int_{|x|\ge K_\varepsilon}w_P^2dx<\varepsilon^2. \tag{3.16}$$

After we choose h small such that, for $|x|\le K_\varepsilon$

$$|f(P,w_P)-f(P+hx,w_P)|<\frac{\varepsilon}{(meas\ B_{O,k\varepsilon})^{1/2}}$$
$$|V(P+hx)-V(P)|<\frac{\varepsilon}{(\int_{\mathbb{R}^N}w_P^2)^{1/2}}. \tag{3.17}$$

Here we point out that the estimates are uniform with respect to P if P belongs to a compact set.

Finally we get

$$\int_{\mathbb{R}^N}|S_{h,P}(w_P)|^2dx<2\varepsilon^2+8V_1^2\int_{|x|\ge K_\varepsilon}w_P^2dx+$$
$$+2\varepsilon^2+C\int_{|x|\ge K_\varepsilon}w_P^{2\alpha+2}dx<(4+8V_1^2+C)\varepsilon^2 \tag{3.18}$$

and so the claim follows.

Proposition 3.3 *For any $P\in\mathbb{R}^N$ there exists h_0 such that for any $h<h_0$ there exists a unique $\Phi_{h,P}$ in $H^2(\mathbb{R}^N)\cap K_P^\perp$ such that*

$$\Pi^\perp S_{h,P}(w_P+\Phi_{h,P})=0 \tag{3.19}$$

and

$$||\Phi_{h,P}||_H\le C||S_{h,P}(w_P)||_{L^2(\mathbb{R}^N)}. \tag{3.20}$$

Proof. First let us choose $\varepsilon > 0$ such that $\varepsilon^{s-1} + \varepsilon^{\alpha} < \frac{\gamma}{2C}$ where γ and C are the constants appearing in Lemma 3.1 and Proposition 2.3. Now we choose h small enough such that $||\Pi^{\perp} S_{h,P}||_{L^2(\mathbb{R}^N)} < \varepsilon\frac{\gamma}{2}$. We will prove that $F_{h,P}$ is a contraction from $\{\Phi \in H^2(\mathbb{R}^N) : ||\Phi||_H < \varepsilon\} \cap K_P^{\perp}$ into itself. We have that if $||\Phi||_H < \varepsilon$ then $F_{h,P}(\Phi)$ is in $K_P^{\perp}$ and

$$\begin{aligned} &||F_{h,P}(\Phi)||_H \leq \frac{1}{\gamma}||\Pi_P^{\perp} R_{h,P}(\Phi_P) + \Pi_P^{\perp} S_{h,P}||_{L^2(\mathbb{R}^N)} \\ &\leq \frac{C}{\gamma}(||\Phi||^s + ||\Phi||_H^{1+\alpha} + ||\Pi_P^{\perp} S_{h,P}||_{L^2(\mathbb{R}^N)}) \leq \\ &\leq C\frac{\varepsilon^s + \varepsilon^{1+\alpha}}{\gamma} + \frac{1}{\gamma}\left(\varepsilon\frac{\gamma}{2}\right) \leq \varepsilon. \end{aligned} \tag{3.21}$$

This proves that $||F_{h,P}||_H < \varepsilon$.

Moreover $F_{h,P}$ is contracting since, if we choose ε small enough such that $M(\Phi, \Phi') \leq \frac{\gamma}{2C}$ in (3.4) we get

$$\begin{aligned} &||F_{h,P}(\Phi) - F_{h,P}(\Phi')||_H = ||L_{h,P} \circ [\Pi_P^{\perp} R_{h,P}(\Phi_P) - \Pi_P^{\perp} R_{h,P}()]||_H \leq \\ &\leq \frac{C}{\gamma}||\Phi - \Phi'||_H \leq \frac{1}{2}||\Phi - \Phi'||_H. \end{aligned} \tag{3.22}$$

So by the contracting map Theorem we deduce (3.19) and (3.20).

Remark 3.4 Note that, from Lemma 3.2, h_0 does not depend on P for P belonging to a compact set.

4 The existence result

Let us consider the vector field $G : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$G_j(P) = -\frac{1}{2}\frac{\partial V}{\partial x_j}(P)\int_{\mathbb{R}^N} w_P^2 dx + \int_{\mathbb{R}^N} F_{x_j}(P, w_P)dx.$$

Note that by the exponential decay of w_P and the assumptions on V and F_{x_j} we get that G is well defined. Now we prove a technical lemma which will be useful in the following.

Lemma 4.1 *The vector field G is a continuous map for any $P \in \mathbb{R}^N$.*

Proof. Let us consider a sequence $P_n \to P$. If we prove that

$$\int_{\mathbb{R}^N} w_{P_n}^2 dx \to \int_{\mathbb{R}^N} w_P^2 dx \tag{4.1}$$

and

$$\int_{\mathbb{R}^N} F_{x_j}(P_n, w_{P_n})dx \to \int_{\mathbb{R}^N} F_{x_j}(P, w_P)dx \tag{4.2}$$

then the claim follows from the smoothness of the potential V.

Let us show that (4.1) holds. For this let us consider the operator $L : H_r^2(\mathbb{R}^N) \times \mathbb{R}^N \mapsto L^2(\mathbb{R}^N)$ defined by

$$L(u, Q) = \Delta u + V(Q)u - f(Q, u). \tag{4.3}$$

For any $P \in \mathbb{R}^N$ we have that

$$L(w_P, P) = 0 \text{ and } \frac{\partial L}{\partial u}(w_P, P) \text{ is invertible.} \tag{4.4}$$

So by implicit function theorem, for any $P \in \mathbb{R}^N$ there exists only one $Q \in B_{P,\rho_0}$ and exactly one function $\widetilde{w}_Q$ such that

$$L(\widetilde{w}_Q, Q) = 0 \text{ and } \lim_{Q \to P} ||\widetilde{w}_Q - w_P||_H = 0. \tag{4.5}$$

By the uniqueness of the solution of the problem (2.1) we deduce that $\widetilde{w}_{P_n} = w_{P_n}$ and from (4.5) we deduce (4.1). Now let us prove (4.2). By Remark 2.2 and (4.5) we get

$$\int\limits_{\mathbb{R}^N} |f(P_n, w_{P_n}|^2 dx \le C_1 \int\limits_{w_{P_n} \le M} |w_{P_n}|^{2\alpha+2} dx + C_2 \int\limits_{w_{P_n} > M} |w_{P_n}|^{2s} dx$$
$$\le C||w_{P_n}||_H \le C||w_P||_H. \tag{4.6}$$

Since w_{P_n} solves (2.1) with $P = P_n$ by the standard regularity theory we deduce that (up to a subsequence) $w_{P_n} \to w_P$ in $C^2_{loc}\mathbb{R}^N$. So

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F_{x_j}(P_n, w_{P_n}) - F_{x_j}(P, w_P)|dx \\
&\le \int_{|x| \ge M} (|F_{x_j}(P_n, w_{P_n}) - F_{x_j}(P, w_P)|)dx + o(1) \\
&\le \int_{|x| \ge M} |F_{x_j}(P_n, w_{P_n})| + |F_{x_j}(P, w_P)| + o(1) \\
&\le C \int_{|x| \ge M} (e^{\delta|P_n|}|w_{P_n}|^s + e^{\delta|P|}|w_P|^s)dx + o(1) \\
&\le Ce^{2\delta|P|} \int_{|x| \ge M} |w_P|^s dx + o(1)
\end{aligned} \tag{4.7}$$

and the claim follows by choosing M large enough and pointing out that (4.2) holds for any subsequence of P_n.

Now we are able to prove Theorem 1.3

Proof of Theorem 1.3. By the previous section we have that, for any $P \in \mathbb{R}$ there exists $h = h(P)$ such that the function $u_{h,P} = w_P + \Phi_{h,P}$ solves

$$-\Delta u_{h,P} + V(P+hx)u_{h,P} - f(P+hx, u_{h,P}) = \sum_{i=1}^{N} \alpha_{i,h} \frac{\partial w_P}{\partial x_i}. \quad (4.8)$$

Let us point out that (see Remark 3.4) h does not depend on P for any point in $B_{P,1}$.

So let us multiply (4.8) by $\frac{\partial u_{h,P}}{\partial x_j}$ and integrate on $\mathbb{R}^N$. We get

$$\begin{aligned} &-\int_{\mathbb{R}^N} \Delta u_{h,P} \frac{\partial u_{h,P}}{\partial x_j} dx + \int_{\mathbb{R}^N} V(P+hx) u_{h,P} \frac{\partial u_{h,P}}{\partial x_j} dx \\ &-\int_{\mathbb{R}^N} f(P+hx, u_{h,P}) \frac{\partial u_{h,P}}{\partial x_j} dx = \sum_{i=1}^{N} \alpha_{i,h} \int_{\mathbb{R}^N} \frac{\partial w_P}{\partial x_i} \frac{\partial u_{h,P}}{\partial x_j} dx. \quad (4.9) \end{aligned}$$

Let us remark that

$$\begin{aligned} &-\int_{B_R} \Delta u_{h,P} \frac{\partial u_{h,P}}{\partial x_j} dx \\ &= -\int_{B_R} div \left(\nabla u_{h,P} \frac{\partial u_{h,P}}{\partial x_j} \right) dx + \frac{1}{2} \int_{B_R} \frac{\partial}{\partial x_j} \left(|\nabla u_{h,P}|^2 \right) dx \\ &= \int_{\partial B_R} \left(-\frac{\partial u_{h,P}}{\partial \nu} \frac{\partial u_{h,P}}{\partial x_j} + \frac{1}{2} |\nabla u_{h,P}|^2 \nu_j \right) d\sigma. \quad (4.10) \end{aligned}$$

Moreover

$$\begin{aligned} &\int_{B_R} V(P+hx) u_{h,P} \frac{\partial u_{h,P}}{\partial x_j} dx \quad (4.11) \\ &= \int_{B_R} \frac{\partial}{\partial x_j} (V(P+hx) \frac{u_{h,P}^2}{2}) dx - \frac{h}{2} \int_{B_R} \frac{\partial V(P+hx)}{\partial x_j} u_{h,P}^2 dx \\ &= \frac{1}{2} \int_{\partial B_R} V(P+hx) u_{h,P}^2 \nu_j d\sigma - \frac{h}{2} \int_{B_R} \frac{\partial V(P+hx)}{\partial x_j} u_{h,P}^2 dx \end{aligned}$$

and

$$\int_{B_R} f(P+hx,u_{h,P})\frac{\partial u_{h,P}}{\partial x_j}dx$$
$$= \int_{B_R}\Big(\frac{\partial}{\partial x_j}F(P+hx,u_{h,P}) - hF_{x_j}(P+hx,u_{h,P})\Big)dx$$
$$= \int_{\partial B_R} F(P+hx,u_{h,P})\nu_j d\sigma - h\int_{B_R} F_{x_j}(P+hx,u_{h,P})dx. \quad (4.12)$$

Finally

$$\sum_{i=1}^N \alpha_{i,h}\int_{B_R}\frac{\partial w_P}{\partial x_i}\frac{\partial u_{h,P}}{\partial x_j}dx = h\int_{B_R}(F_{x_j}(P+hx,u_{h,P})$$
$$-\frac{1}{2}\frac{\partial V(P+hx)}{\partial x_j}u_{h,P}^2)dx \quad (4.13)$$
$$+\int_{\partial B_R}\Big(\frac{\partial u_{h,P}}{\partial \nu}\frac{\partial u_{h,P}}{\partial x_j}+\frac{1}{2}|\nabla u_{h,P}|^2\nu_j$$
$$+\frac{1}{2}V(P+hx)u_{h,P}^2\nu_j - F(P+hx,u_{h,P})\nu_j\Big)d\sigma. \quad (4.14)$$

Now we proceed as in [18] and set

$$I_R = \quad (4.15)$$
$$\int_{\partial B_R}\Big(\frac{\partial u_{h,P}}{\partial \nu}\frac{\partial u_{h,P}}{\partial x_j}+\frac{1}{2}|\nabla u_{h,P}|^2\nu_j+\frac{1}{2}V(P+hx)u_{h,P}^2\nu_j$$
$$-F(P+hx,u_{h,P})\nu_j\Big)d\sigma. \quad (4.16)$$

Now by Remark 2.2 and by the Sobolev embedding Theorem we get

$$\int_0^\infty |I_R|dx \quad (4.17)$$
$$\le \int_0^\infty\int_{\partial B_R}(\frac{3}{2}|\nabla u_{h,P}|^2+\frac{1}{2}V(P+hx)u_{h,P}^2+|F(P+hx,u_{h,P})|)d\sigma$$
$$\le C\int_{\mathbb{R}^N}|\nabla u_{h,P}|^2+V(P+hx)u_{h,P}^2+|F(P+hx,u_{h,P})|dx<\infty$$

and then there exists a sequence $R_n \to \infty$ such that $I_{R_n} \to 0$. Passing to the limit we deduce

$$\sum_{i=1}^{N} \alpha_{i,h} \int_{\mathbb{R}^N} \frac{\partial w_P}{\partial x_i} \frac{\partial u_{h,P}}{\partial x_j} dx$$
$$= h \int_{\mathbb{R}^N} (F_{x_j}(P + hx, u_{h,P}) - \frac{1}{2} \frac{\partial V(P + hx)}{\partial x_j} u_{h,P}^2) dx. \quad (4.18)$$

Let us prove that

$$\int_{\mathbb{R}^N} (F_{x_j}(P + hx, u_{h,P}) - \frac{1}{2} \frac{\partial V(P + hx)}{\partial x_j} u_{h,P}^2) dx \to G_j(P) \quad \text{as } h \to 0 \quad (4.19)$$

uniformly on the compact set of $\mathbb{R}^N$.
By the assumption $|F_{x_j}(x, u)| \leq e^{\delta|x|} u^r$ for $u > M$ and since $u_{h,P} \to w_P$ in $H^2(\mathbb{R}^N) \cap C^2_{loc}(\mathbb{R}^N)$ we have that

$$| \int_{\mathbb{R}^N} F_{x_j}(P + hx, u_{h,P}) - F_{x_j}(P, w_P)| dx$$
$$\leq \int_{|x|>\rho} (|F_{x_j}(P + hx, u_{h,P})| + |F_{x_j}(P, w_P)|) dx + o(1)$$
$$\leq C \int_{|x|>\rho} (e^{\delta(h|x|+|P|)} u_{h,P}^{\alpha+1} + e^{\delta|P|} w_P^{\alpha+1}) dx + o(1)$$
$$\leq Ce^P \int_{|x|>\rho} e^{\frac{\alpha+1}{2}} u_{h,P}^{\alpha+1} dx + o(1) = Ce^P \int_{|x|>\rho} e^{\frac{\alpha+1}{2}} w_P^{\alpha+1} + o(1). \quad (4.20)$$

By Remark 2.1 and since ρ is arbitrary we have that $\int_{\mathbb{R}^N} (F_{x_j}(P + hx, u_{h,P}) - F_{x_j}(P, w_P)) \to 0$ as $h \to 0$. The same proof applies to show that

$$\int_{\mathbb{R}^N} \frac{\partial V(P + hx)}{\partial x_j} u_{h,P}^2 dx \to \frac{\partial V}{\partial x_j}(P) \int_{\mathbb{R}^N} w_P^2 dx \quad (4.21)$$

and this gives (4.19).
By assumption $G_j(P)$ has a stable zero at P and so there exists $P_h \to P$ such that

$$\int_{\mathbb{R}^N} \left(F_{x_j}(P_h + hx, u_{h,P_h}) - \frac{1}{2} \frac{\partial V(P_h + hx)}{\partial x_j} u_{h,P_h}^2\right) dx = 0. \quad (4.22)$$

Hence (4.18) becomes

$$\sum_{i=1}^{N} \alpha_{i,h} \int_{\mathbb{R}^N} \frac{\partial w_{P_h}}{\partial x_i} \frac{\partial u_{h,P_h}}{\partial x_j} dx = 0 \tag{4.23}$$

Since the matrix $\int_{\mathbb{R}^N} \frac{\partial w_{P_h}}{\partial x_i} \frac{\partial u_{h,P_h}}{\partial x_j} dx \to \delta_i^j \int_{\mathbb{R}^N} (\frac{\partial w_P}{\partial x_i})^2$ this implies that the linear system (4.23) admits only the trivial solution $\alpha_{i,h} = 0$.

So we have proved that u_{h,P_h} satisfies $-\Delta u_{h,P_h} + V(P+hx)u_{h,P_h} = f(P+hx, u_{h,P_h})$. Since $f(x,u) \le 0$ for $u \le 0$ we get that $u_{h,P_h} \ge 0$ and the strong maximum principle implies $u_{h,P_h} > 0$.

Now let us prove that $u_{h,P_h} \to 0$ as $|x| \to \infty$. First of all we remark that by the standard regularity theory from (2.4) we get that $||u_{h,P}||_{H^{2,p}(\mathbb{R}^N)} \le C$ for any $p \ge 2$. So $||u_{h,P}||_{L^\infty(\mathbb{R}^N)} \le C$. Moreover, since $||u_{h,P}||_H$ is uniformly bounded, we have

$$\int_{|x|>R} u_{h,P_h}^{\frac{2N}{N-2}} \to 0 \text{ as } R \to \infty \text{ uniformly with respect to } h \tag{4.24}$$

Then we remark that u_{h,P_h} is a subsolution of $\Delta u + c(x)u = 0$ with $c(x) = \frac{f(x,u_{h,P_h})}{u_{h,P_h}} \le C(u_{h,P_h}^{s-1}(x) + u_{h,P_h}^{\alpha}(x)) \le C$. So by the Harnack inequality (see [9]) we have

$$\max_{B_{y,1}} u_{h,P_h} \le C \left(\int_{B_2(y)} u_{h,P_h}^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \tag{4.25}$$

where y is an arbitrary point of $\mathbb{R}^N$. So by (4.23) we obtain that $u_{h,P_h} \to 0$ as $|x| \to \infty$.

Finally let us prove that u_{h,P_h} has only one maximum point. First we show that if Q_h is a local maximum point of u_{h,P_h} then

$$|Q_h - P_h| \to 0. \tag{4.26}$$

Indeed since Q_h is a local maximum point of u_{h,P_h} we have that $\Delta u_{h,P_h} \le 0$. Therefore, since u_{h,P_h} is a solution of (1.4)

$$\frac{f(Q_h, u_{h,P_h}(Q_h))}{u_{h,P_h}(Q_h)} \ge V(Q_h) \ge V_0 > 0. \tag{4.27}$$

If $|Q_h| \to \infty$ we get that $u_{h,P_h} \to 0$ and by (f_0) we reach a contradiction. Then Q_h is bounded and we can assume that, up to a subsequence, $Q_h \to Q_0$. Since $u_{h,P_h} \to w_P$ in $C^2_{loc}(\mathbb{R}^N)$ we obtain $\nabla w(Q_0) = 0$ and so $Q_0 = P$. Hence (4.26) holds.

Now if $Q_{1,h}$ and $Q_{2,h}$ are two different local minima points then $Q_{1,h}$ and $Q_{2,h}$ tend to P as $h \to 0$. However, since $u_{h,P_h} \to w_P$ in $C^2_{loc}(\mathbb{R}^N)$ and w_P is strictly concave in a neighborhood of P we reach a contradiction.

Now we prove Corollary 1.5.

Proof of Corollary 1.5. Let w_P be the unique positive solution of

$$-\Delta w_P + V(P)w_P = K(P)w_P^s. \tag{4.28}$$

Then the function $\overline{w}(x) = \left(\frac{K(P)}{V(P)}\right)^{\frac{1}{s-1}} w_P(\frac{x}{\sqrt{V(P)}})$ satisfies $-\Delta\overline{w} + \overline{w} = \overline{w}^s$. So the vector field G defined at the beginning of the section becomes

$$G_j(P) = -\frac{1}{2}\frac{\partial V}{\partial x_j}(P)(\frac{V(P)}{K(P)})^{\frac{2}{s-1}}\frac{1}{V(P)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\overline{w}(y)^2dy + \\ +\frac{1}{s+1}\frac{\partial V}{\partial x_j}(P)(\frac{V(P)}{K(P)})^{\frac{s+1}{s-1}}\frac{1}{V(P)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\overline{w}(y)^{s+1}dy. \tag{4.29}$$

Now by the Pohozaev identity (see [16] or also [11]) we have that $\overline{w}$ satisfies

$$\left(\frac{N}{s+1} - \frac{N-2}{2}\right)\int_{B_R}\overline{w}^{s+1}dy - \int_{B_R}\overline{w}^2dy \\ = \int_{\partial B_R}[(x\cdot\nabla\overline{w})\frac{\partial\overline{w}}{\partial\nu} - (x\cdot\nu)|\nabla\overline{w}|^2 + (x\cdot\nu)F(u) + \frac{N-2}{2}\overline{w}\frac{\partial\overline{w}}{\partial\nu}]d\sigma \tag{4.30}$$

and by exponential decay of $\overline{w}$ we get, as $R\to\infty$

$$(\frac{N}{s+1} - \frac{N-2}{2})\int_{\mathbb{R}^N}\overline{w}^{s+1}dy = \int_{\mathbb{R}^N}\overline{w}^2dy. \tag{4.31}$$

Hence (4.29) becomes

$$G_j(P) \\ = \frac{(s-1)V(P)^{2-2N-2s}}{2(s+1)}\frac{\partial}{\partial x_j}(\frac{V^{2p+2+N-Np/(2p-2)}(x)}{K^{2/(p-1)}(x)})(P). \tag{4.32}$$

So the stable zeros of the vector field G are stable critical points of the function $\frac{V^{2p+2+N-Np/(2p-2)}(x)}{K^{2/(p-1)}(x)}$ and this proves the claim of the Corollary 1.5

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1
We follow the line of [18]. Let v_h a solution of (1.4) uniformly bounded in $H^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $v_h(x) = u_h(P_0 + hx)$. It easily seen that v_h satisfies

$$-\Delta v_h + V(P_0 + hx)v_h = f(P_0 + hx, v_h) \tag{5.1}$$

and we have that $v_h \to w_{P_0}$ in $C^2_{loc}\mathbb{R}^N$. Next multiplying (5.1) by $\frac{\partial v_h}{\partial x_j}$ and integrating on B_R we get

$$\int_{B_R} (\Delta v_h \frac{\partial v_h}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}(V(P_0 + hx)v_h^2 + \frac{\partial}{\partial x_j}F(P_0 + hx, v_h))dx$$
$$= h \int_{B_R} (F_{x_j}(P + hx, v_h) - \frac{1}{2}\frac{\partial V(P + hx)}{\partial x_j} v_h^2)dx. \tag{5.2}$$

Proceeding as in the Proof of Theorem 1.3 we get that the LHS tends to zero as $R \to \infty$. On the other hand, since $v_h \in H^2(\mathbb{R}^N)$ and the assumption on F we have

$$\lim_{R\to\infty} \int_{B_R} (F_{x_j}(P + hx, v_h) - \frac{1}{2}\frac{\partial V(P + hx)}{\partial x_j} v_h^2)dx$$
$$= \int_{\mathbb{R}^N} (F_{x_j}(P + hx, v_h) - \frac{1}{2}\frac{\partial V(P + hx)}{\partial x_j} v_h^2)dx. \tag{5.3}$$

So (5.2) becomes

$$\int_{\mathbb{R}^N} (F_{x_j}(P + hx, v_h) - \frac{1}{2}\frac{\partial V(P + hx)}{\partial x_j} v_h^2)dx = 0 \tag{5.4}$$

and the claim follows as $h \to 0$.

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