Math. Z. 235, 99–109 (2000)
Digital Object Identifier (DOI) 10.1007/s002090000125

Mathematische Zeitschrift

# Representations of the symmetric group are reducible over simply transitive subgroups

**A. S. Kleshchev, J. K. Sheth**

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
(e-mail: klesh@math.uoregon.edu, sheth@noether.uoregon.edu)

Received April 8, 1999 / Published online July 3, 2000 – © Springer-Verlag 2000

*Mathematics Subject Classification (1991):*20C30, 20B20

## Introduction

Let $F$ be an algebraically closed field of characteristic $p \geq 0$ and $\Sigma_n = Sym(\Omega)$ be the symmetric group on an $n$-element set $\Omega$. We are interested in the following

**Problem.** For any $n$ describe all pairs $(G, D)$ where $G \leq \Sigma_n$ is a subgroup and $D$ is a simple $F\Sigma_n$-module of dimension greater than 1 such that the restriction $D_G$ is irreducible.

If the characteristic of $F$ is zero, this problem has been solved by Saxl [16]. An important feature of Saxl's result is that a group $G$ as in the problem above is either 2-transitive or fixes a point, i.e. is contained in some $\Sigma_{n-1}$. The case $G \leq \Sigma_{n-1}$ can then be settled using the branching rule and induction. On the other hand, an explicit list of $k$-transitive groups (for $k \geq 2$) is available, which can be used to complete the proof in characteristic zero, see [16] for more details.

From now on we assume that $p > 0$. In this case the problem is important for determining maximal subgroups of finite classical groups [1],[13]. However, the situation is now more complicated. For example, to determine the pairs $(G, D)$ as above with $G = A_n$ one needs the Mullineux conjecture [7], [2].

If $G$ is intransitive then, up to a conjugation, it is contained in a standard Young subgroup of the form $\Sigma_{n-k} \times \Sigma_k$. The irreducible restrictions from $\Sigma_n$ to $\Sigma_{n-1}$ have been described in [14], see also [11], [6]. In [11] it is also

The first author partially supported by the NSF, grant DMS-9600124.

shown for $p > 2$ that the Young subgroups $\Sigma_{n-k} \times \Sigma_k$ with $1 < k < n-1$ never act irreducibly on a simple $F\Sigma_n$-module of dimension $> 1$ (this fact is also proved in this paper, see Theorem 3.10). This reduces the intransitive case to the subgroups of $\Sigma_{n-1}$. The main result here is

**Main Theorem.** Let $p > 2, n \geq 4$, $G \leq \Sigma_n$, and $D$ be a simple $F\Sigma_n$-module with $\dim D > 1$. If the restriction $D_G$ is irreducible then either $G \leq \Sigma_{n-1}$ or $G$ is 2-transitive on $\Omega$.

We refer the reader to Theorem 3.10 for a partial result in the case $p = 2$.

For $p > 2$ the problem above is thus reduced to 2-transitive groups. This turns out to be a very important step in a complete solution of the problem, which is obtained in [3] (for the case $p > 3$).

## 1. Preliminaries

Let $\Omega$ be a finite set with $|\Omega| = n$, and $k \leq n/2$. A group $G \leq Sym(\Omega)$ is called **k-homogeneous** (resp. **k-transitive** ) on $\Omega$ if $G$ acts transitively on the unordered (resp. ordered) $k$-element subsets of $\Omega$. Moreover, $G$ is called **k*-homogeneous** on $\Omega$ if $G$ is $k$-homogeneous but not $k$-transitive on $\Omega$.

Let $F$ be an algebraically closed field of characteristic $p > 0$, $G$ be a group, and $V$ be an $FG$-module. Denote by $V^G$ the space of $G$-fixed points in $V$. We write $V \cong V_1|\cdots|V_k$ if $V$ admits a *unique* filtration with sections isomorphic to $V_j, 1 \leq j \leq k$, counted from bottom to top. In particular, if the sections are irreducible this means that $V$ is uniserial. If $H \leq G$ is a subgroup, then $V_H$ or $V{\downarrow}_H$ denotes the restriction of $V$ to $H$. If $W$ is an $FH$-module then $W{\uparrow}^G$ denotes the induced module. If $V \cong V^*$ as $FG$-modules then $V_H \cong (V_H)^*$ as $FH$-modules. We let $\mathbf{1}$=$\mathbf{1}_G$ denote the trivial representation of $G$. If $V$ and $W$ are $FG$-modules we write $\mathrm{Hom}_G(V, W)$ (resp. $\mathrm{Hom}(V, W)$) for the space of all $FG$-homomorphisms (resp. $F$-linear maps) from $V$ to $W$. Note that $\mathrm{Hom}(V, W)$ is an $FG$-module with $\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}(V, W)^G$.

Basic facts on the representation theory of the symmetric group can be found in [8]. For $\lambda = (\lambda_1, \dots, \lambda_r)$, a composition of $n$ with non-zero parts, we let $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_r}$ denote the corresponding Young subgroup of $\Sigma_n$. We may identify $\Sigma_{n-1}$ with $\Sigma_{(n-1,1)}$. We let $Y^\lambda, S^\lambda$ and $M^\lambda = (\mathbf{1}_{\Sigma_\lambda}){\uparrow}^{\Sigma_n}$ denote the Young, Specht, and permutation modules labelled by $\lambda$, respectively. If $\lambda$ is $p$-regular we write $D^\lambda$ for the unique irreducible quotient of $S^\lambda$. It is well known that $M^\lambda, Y^\lambda$, and $D^\lambda$ are all self-dual $F\Sigma_n$-modules. Let $\mathbf{sgn}$ denote the sign representation of $\Sigma_n$.

We will need the structure of the permutation modules $M^{(n-1,1)}$ and $M^{(n-2,2)}$. The proof of the next three lemmas is obtained by applying [8, 17.17, 24.15] and the 'Nakayama Conjecture' [10, 6.1.21, 2.7.41].

**Lemma 1.1.** *The module* $M^{(n-1,1)}$ *is isomorphic to* $D^{(n-1,1)} \oplus \mathbf{1}$ *if* $n \not\equiv 0 \pmod p$*, and* $\mathbf{1}|D^{(n-1,1)}|\mathbf{1} \cong S^{(n-1,1)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-1,1)})^*$ *otherwise.*

**Lemma 1.2.** *Let* $p > 2$ *and* $n \geq 4$.

(i) *If* $n \not\equiv 1, 2 \pmod p$ *then* $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ *where* $Y^{(n-2,2)} \cong S^{(n-2,2)} \cong D^{(n-2,2)}$.
(ii) *If* $n \equiv 1 \pmod p$ *then* $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ *where* $Y^{(n-2,2)} \cong \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-2,2)})^*$.
(iii) *If* $n \equiv 2 \pmod p$ *then* $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus \mathbf{1}$ *where*

$$Y^{(n-2,2)} \cong D^{(n-1,1)}|D^{(n-2,2)}|D^{(n-1,1)} \cong D^{(n-1,1)}|(S^{(n-2,2)})^*.$$

We will only need the case $p = 2$ when $n$ is odd:

**Lemma 1.3.** *Let* $p = 2$ *and* $n \geq 4$ *be odd.*

(i) *If* $n \equiv 1 \pmod 4$ *then* $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ *where* $Y^{(n-2,2)} \cong \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \cong \mathbf{1}|(S^{(n-2,2)})^*$.
(ii) *If* $n \equiv 3 \pmod 4$ *then* $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ *where* $Y^{(n-2,2)} \cong D^{(n-2,2)}$.

The following easy result is well known.

**Lemma 1.4.** *Let* $X$ *be a* $G$*-set, and* $M$ *be the corresponding permutation* $FG$*-module. Then* $\dim M^G$ *equals the number of* $G$*-orbits on* $X$.

*Proof.* It suffices to consider the case where $G$ is transitive on $\Omega$. Then $M = (1_H){\uparrow}^G$ where $H$ is a point stabilizer. But $\dim \mathrm{Hom}_G(\mathbf{1}_G, (\mathbf{1}_H){\uparrow}^G) = \dim \mathrm{Hom}_H(\mathbf{1}_H, \mathbf{1}_H) = 1$ by Frobenius reciprocity. □

For $1 \leq k \leq n$ denote by $d_k$ the number of $G$-orbits on the unordered $k$-element subsets of $\Omega$. The following lemma characterizes $d_k$ in terms of the corresponding permutation module.

**Lemma 1.5.** *Let* $1 \leq k \leq n$ *and* $G \leq \Sigma_n$ *be a subgroup. Then* $\dim (M^{(n-k,k)})^G = d_k$.

*Proof.* Note that $M_G^{(n-k,k)}$ is the permutation $FG$-module on the $k$-element subsets. Now we can use Lemma 1.4. □

**Lemma 1.6.** *If* $G \leq \Sigma_n$ *and* $n \not\equiv 0 \pmod p$ *then* $\dim(D^{(n-1,1)})^G = d_1 - 1$.

*Proof.* This follows from Lemmas 1.1 and 1.5. □

**Lemma 1.7.** *Let* $n \geq 4, G \leq \Sigma_n$ *and* $n$ *be odd if* $p = 2$*. Suppose* $d_1 < d_2$*. Then* $(Y^{(n-2,2)})^G$ *and* $((S^{(n-2,2)})^*)^G$ *are non-zero. Moreover if* $p > 2$ *and* $n \equiv 1 \pmod p$*, or* $p = 2$ *and* $n \equiv 1 \pmod 4$*, then* $\dim(Y^{(n-2,2)})^G \geq 2$.

*Proof.* By Lemmas 1.2 and 1.3 (or [8, 17.17]), there is an exact sequence

$$0 \to M^{(n-1,1)} \to M^{(n-2,2)} \to (S^{(n-2,2)})^* \to 0,$$

which implies that $((S^{(n-2,2)})^*)^G \neq 0$, by assumption and Lemma 1.5. By 1.2(i) and 1.3(ii), $Y^{(n-2,2)} = (S^{(n-2,2)})^*$, unless $p > 2$, $n \equiv 1, 2 \pmod p$ or $p = 2$, $n \equiv 1 \pmod 4$. If $p > 2$ and $n \equiv 2 \pmod p$ then by Lemma 1.2(iii), $\dim(Y^{(n-2,2)})^G = d_2 - 1 \geq d_1 > 0$. In the remaining cases, using Lemma 1.6, we get $\dim(Y^{(n-2,2)})^G = d_2 - (d_1 - 1) \geq 2$. □

**Lemma 1.8.** *Let $G \leq \Sigma_n$ be transitive and $n \not\equiv 0 \pmod p$. Then $G$ is 2-transitive if and only if* $\dim \mathrm{End}_G(D_G^{(n-1,1)}) = 1$.

*Proof.* Note that $G$ is 2-transitive on $\Omega$ if and only if $G$ has exactly two orbits on $\Omega \times \Omega$. However, the corresponding permutation module is isomorphic to $M_G^{(n-1,1)} \otimes M_G^{(n-1,1)}$. By Lemma 1.4, we now have that $G$ is 2-transitive if and only if $\dim(M_G^{(n-1,1)} \otimes M_G^{(n-1,1)})^G = 2$. By Lemmas 1.6 and 1.1, this is equivalent to $\dim(D^{(n-1,1)} \otimes D^{(n-1,1)})^G = 1$, which by self-duality of irreducible modules is equivalent to our claim. □

## 2. 2*-homogeneous groups

The main result of this section is Theorem 2.5 which shows that the restriction of a simple $F\Sigma_n$-module $D$ of dimension $> 1$ to a 2*-homogeneous subgroup is reducible. We will use a theorem of Kantor [12] (see also [4, 9.4B]) which describes 2*-homogeneous groups. Let $\Omega = \mathbb{F}_q$ be a finite field of order $q = r^e$ for a prime $r$. Let $\mathrm{Aut}_{\mathbb{F}_r}(\mathbb{F}_q)$ denote the Galois group, which is a cyclic group $C_e$ of order $e$. Let $A\Sigma L_1(q)$ (resp. $ASL_1(q)$) be the group of all transformations of $\mathbb{F}_q$ of the form $x \mapsto a^2x^\sigma + b$ (resp. $x \mapsto a^2x + b$) with $a \in \mathbb{F}_q^*, b \in \mathbb{F}_q$ and $\sigma \in \mathrm{Aut}_{\mathbb{F}_r}(\mathbb{F}_q)$. These are permutation groups on the elements of $\mathbb{F}_q$.

**Proposition 2.1.** [12] *Let $G$ be a 2*-homogeneous group. Then, up to permutation isomorphism, $ASL_1(q) \leq G \leq A\Sigma L_1(q)$ with $q \equiv 3 \pmod 4$ $q > 3$.*

Let $R_n(m)$ denote the class of simple $F\Sigma_n$-modules $D$ such that for some $p$-regular partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq n - m$, we have $D \cong D^\lambda$ or $D \cong D^\lambda \otimes \mathbf{sgn}$. We need the following results from [9] and the main results in [17],[18].

**Proposition 2.2.** [9, Theorem 7] *If $n \geq 15$ and $D$ is a simple $F\Sigma_n$-module, then either* $\dim D > \frac{1}{2}(n-1)(n-2)$ *or* $D \in R_n(2)$.

**Proposition 2.3.** [9, p.420] *If* $D \in R_n(2) \backslash R_n(1)$ *then* $\dim D \geq \frac{1}{2}n(n-5)$. *If* $D \in R_n(1)$ *then* $\dim D \geq n-2$.

**Proposition 2.4.** [17], [18], [9, Theorem 6(i)] *If* $n \geq 7$ *and* $D$ *is a simple* $F\Sigma_n$*-module, then either* $\dim D > n-1$ *or* $D \in R_n(1)$.

**Proposition 2.5.** *Let* $G$ *be a 2*-homogeneous subgroup of* $\Sigma_n$ *and* $D$ *be a simple* $F\Sigma_n$*-module with* $\dim D > 1$. *Then the restriction* $D_G$ *is reducible.*

*Proof.* In view of Proposition 2.1 we may assume $G = A\Sigma L_1(q) < \Sigma_q$, $q \equiv 3 \pmod 4$, and $q > 3$. In particular, $q \geq 7$ and $r > 2$. Let $D_G$ be irreducible. Then

$$\dim D \leq \sqrt{|G|} = \sqrt{q(q-1)e/2}. \tag{1}$$

Moreover, $e = \log_r q < q-5$ as $q \geq 7$ and $r > 2$. So

$$\dim D < q\sqrt{(q-5)/2} \leq \frac{1}{2}q(q-5). \tag{2}$$

If $q = 7$ or $q = 11$ then $e = 1$ and we get a contradiction by (1) and Propositions 2.4, 2.3. Let $q \geq 15$. Then from (2) and Propositions 2.2, 2.3 we conclude that $D \in R_n(1)$, i.e. $D \cong D^{(n-1,1)}$ or $D \cong D^{(n-1,1)} \otimes \mathbf{sgn}$. If $D_G$ is irreducible, we have by Schur's lemma

$$1 = \dim \mathrm{End}_G(D_G) = \dim \mathrm{End}_G(D_G^{(n-1,1)}).$$

Now, if $p$ does not divide $q$ then $G$ is 2-transitive on $\mathbb{F}_q$ by Lemma 1.8, giving a contradiction. Otherwise the restriction of $D$ to $\mathbb{F}_q \lhd G$ is semisimple by Clifford's theorem. But the trivial module is the only simple module over the group $\mathbb{F}_q$, as $\mathbb{F}_q$ is a $p$-group. Hence $\mathbb{F}_q$ acts trivially on $D$. This contradicts the fact that $D$ is faithful over $\Sigma_q$. □

## 3. Main Results

Denote by $\sigma$ the transposition $(n-1, n) \in \Sigma_n$. We will write $\Sigma_{n-2,2}$ for the Young subgroup $\Sigma_{(n-2,2)} \cong \Sigma_{n-2} \times \Sigma_2$. For an $F\Sigma_n$-module $V$ and a homomorphism $\theta \in \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow_{\Sigma_{n-1}}})$ define a map $\hat{\theta} : V \to V$ by setting

$$\hat{\theta}(v) := \theta(v) + \sigma\theta(\sigma v) \quad (v \in V).$$

**Lemma 3.1.** *Let* $p > 2$ *and* $n \geq 4$. *Then the map* $\theta \mapsto \hat{\theta}$ *is an injective map from* $\mathrm{End}_{\Sigma_{n-1}}(V{\downarrow_{\Sigma_{n-1}}})$ *to* $\mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow_{\Sigma_{n-2,2}}})$.

*Proof.* It is routine that $\hat{\theta} \in \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow_{\Sigma_{n-2,2}}})$. Assume $\hat{\theta} = 0$. Then $\theta(\sigma v) = -\sigma\theta(v)$ for all $v \in V$. But $\theta$ is also an $F\Sigma_{n-1}$-homomorphism. So

$$\theta(v) = \theta((\sigma(1, n-1))^3 v) = -(\sigma(1, n-1))^3\theta(v) = -\theta(v),$$

whence $\theta(v) = 0$ for any $v \in V$. □

**Lemma 3.2.** *Let $p > 2$, and $W$ be a $F\Sigma_4$-module with $((1,4)+(2,3)-(1,3)-(2,4))W = 0$. Then $A_4$ acts trivially on $W$.*

*Proof.* By assumption we have $((1,3)+(2,4))v = ((1,4)+(2,3))v$ for all $v \in W$. Conjugating by $(2,3)$ we get

(3) $$((1,2)+(3,4))v = ((1,3)+(2,4))v = ((1,4)+(2,3))v.$$

The group algebra $F[\Sigma_2 \times \Sigma_2]$ is semisimple and has four irreducible modules: $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$, $\mathbf{1}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$, $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$, and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$. We claim that only $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$ may appear in the restriction $W{\downarrow}_{\Sigma_2 \times \Sigma_2}$. Indeed, assume for example that $\mathbf{1}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$ appears in this restriction. Pick a non-zero vector $v$ in the corresponding isotypic component. Then $(1,2)v = v$ and $(3,4)v = -v$. So $(1,2)(3,4)v = -v$ and $((1,2)+(3,4))v = 0$. By (3), we have $(1,3)v = -(2,4)v$ and $(1,4)v = -(2,3)v$. Hence

(4) $$(1,3)(2,4)v = -v \quad \text{and} \quad (1,4)(2,3)v = -v.$$

But $(1,2)(3,4) = (1,3)(2,4)(1,4)(2,3)$. So (4) implies $(1,2)(3,4)v = v$, and we have arrived to a contradiction. The case of $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ is considered similarly.

As the element $(1,2)(3,4)$ acts trvially on both $\mathbf{1}_{\Sigma_2} \otimes \mathbf{1}_{\Sigma_2}$ and $\mathbf{sgn}_{\Sigma_2} \otimes \mathbf{sgn}_{\Sigma_2}$, it must act trivially on $W$. By conjugation, we conclude that the whole Klein group

$$H := \{1,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$$

acts trivially on $W$. As $(1,2) \equiv (3,4) \pmod H$ and $(2,3) \equiv (1,4) \pmod H$, the equation (3) implies $(1,2)v = (2,3)v$. Hence $(1,2,3)v = v$, and so $A_4$ acts trivially on $W$. □

**Theorem 3.3.** *Let $p > 2$ and $n \geq 4$. Assume that $V$ is an $F\Sigma_n$-module such that the alternating group $A_n < \Sigma_n$ does not act trivially on $V$. Then*

$$\dim \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow}_{\Sigma_{n-1}}) < \dim \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow}_{\Sigma_{n-2,2}}).$$

*Proof.* By Lemma 3.1 it is enough to demonstrate an endomorphism $\psi \in \mathrm{End}_{\Sigma_{n-2,2}}(V{\downarrow}_{\Sigma_{n-2,2}})$ which is not in the image of $\theta \mapsto \hat{\theta}$. Set $\psi(v) := \sigma v$, $v \in V$. Assume that $\psi = \hat{\theta}$ for some $\theta \in \mathrm{End}_{\Sigma_{n-1}}(V{\downarrow}_{\Sigma_{n-1}})$. Then

(5) $$\theta(\sigma v) = v - \sigma\theta(v) \quad (v \in V).$$

For $i = 1, 2, \dots, n-2$ denote $\pi_i := (i, n-1)$ and $\tau_i := (i, n)$. Then, using (5), we get

$$\theta(\pi_i\sigma\pi_i v) = \pi_i\theta(\sigma\pi_i v) = \pi_i(\pi_i v - \sigma\theta(\pi_i v)) = v - \pi_i\sigma\pi_i\theta(v).$$

Note that $\pi_i\sigma\pi_i = \tau_i$ so we have proved that

$$\theta(\tau_i v) = v - \tau_i\theta(v), \quad i = 1, 2, \ldots, n-2 \quad (v \in V). \tag{6}$$

Using (5), (6) and the equality $\pi_i = \tau_i\sigma\tau_i$, we get for any $v \in V$ and $1 \le i \le n-2$

$$\begin{aligned}\pi_i\theta(v) &= \theta(\pi_i v) = \theta(\tau_i\sigma\tau_i v) = \sigma\tau_i v - \tau_i\theta(\sigma\tau_i v)\\ &= \sigma\tau_i v - \tau_i(\tau_i v - \sigma\theta(\tau_i v)) = \sigma\tau_i v - v + \tau_i\sigma(v - \tau_i\theta(v))\\ &= \sigma\tau_i v - v + \tau_i\sigma v - \tau_i\sigma\tau_i\theta(v) = \sigma\tau_i v - v + \tau_i\sigma v - \pi_i\theta(v).\end{aligned}$$

Solving for $\pi_i\theta(v)$ and multiplying by $\pi_i$, we find

$$\theta(v) = (1/2)(\sigma v - \tau_i\sigma\tau_i v + \tau_i v), \quad i = 1, 2, \ldots, n-2 \quad (v \in V). \tag{7}$$

Taking $i = 1$ and $i = 2$, we have

$$(1/2)(\sigma v - \tau_1\sigma\tau_1 v + \tau_1 v) = (1/2)(\sigma v - \tau_2\sigma\tau_2 v + \tau_2 v) \quad (v \in V).$$

Conjugating by $(3, n-1)(4, n)$, we get

$$((1,4) + (2,3) - (1,3) - (2,4))V = 0.$$

By Lemma 3.2, the natural subgroup $A_4 < \Sigma_n$ acts trivially on $V$, which implies by conjugation that $A_n$ acts trivially on $V$, giving a contradiction. □

Now we study the case $p = 2$. If $n = 2l$ is even we write $S$ for the irreducible module $D^{(l+1,l-1)}$ and if $n = 2l+1$ is odd we write $S$ for $D^{(l+1,l)}$. We call $S$ the *spinor* representation of $\Sigma_n$. The following result is proved in [15], see also [5] for a generalization.

**Lemma 3.4.** *Let $p = 2$, $n \ge 3$, and $D$ be an irreducible $F\Sigma_n$-module different from the spinor representation $S$. Then for any 3-cycle $\gamma \in \Sigma_n$ there exists a non-zero vector $v \in D$ such that $\gamma v = v$.*

*Proof.* It is proved in [15] that the only irreducible representations of the alternating group $A_n$ for which 3-cycles act fixed-point-freely come from the restriction of the spinor module to $A_n$. This implies the result as these irreducible $A_n$-modules do not appear in the restriction of $D^\lambda$ to $A_n$, unless $D^\lambda \cong S$. □

**Lemma 3.5.** *Let $p = 2$, $n \ge 3$, $b, c \in F$, and $D$ be an irreducible $F\Sigma_n$-module. If $b \ne 0$ and*

$$(b(1,2) + c(1,2,3) + c(1,3,2) + c)v = 0$$

*for all $v \in D$ then $D$ is the trivial module $\mathbf{1}_{\Sigma_n}$.*

*Proof.* By conjugating with $(2,3)$ we also get

$$(b(1,3)+c(1,2,3)+c(1,3,2)+c)D=0,$$

so $(1,2)v=(1,3)v$, hence $(1,2,3)v=v$ for all $v\in D$. By conjugating we also get $(1,3,2)v=v$ for any $v$. Now it follows from the assumption that $(1,2)$ acts trivially on $D$, whence any transposition acts trivially. □

**Theorem 3.6.** *Let $p=2$, $n\geq 4$, and $D$ be a non-trivial irreducible $F\Sigma_n$-module. Then*

$$\dim \mathrm{End}_{\Sigma_{n-1}}(D{\downarrow_{\Sigma_{n-1}}}) < \dim \mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}}),$$

*unless $n$ is odd and $D$ is the spinor module $S$.*

*Proof.* We first note that the kernel of the linear map $\theta\mapsto\hat\theta$ defined in the beginning of this section is 1-dimensional and is spanned by the identity map $\mathrm{id}_D$. Indeed, $\hat\theta(v)=0$ is equivalent to $\theta(\sigma v)=\sigma\theta(v)$. But $\theta$ is also an $F\Sigma_{n-1}$-homomorphism. Hence $\theta$ is an $F\Sigma_n$-endomorphism of $D$ so it must be proportional to $\mathrm{id}_D$, by Schur's lemma.

Let $\mathrm{id}_D,\theta_1,\dots,\theta_k$ be a basis of the vector space $\mathrm{End}_{\Sigma_{n-1}}(D{\downarrow_{\Sigma_{n-1}}})$. Define a map $\psi: D\to D$ by setting $\psi(v)=\sigma v$, $v\in D$. Then $\psi$ is an element of $\mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}})$. We claim that

$$\hat\theta_1,\ \hat\theta_2,\ \dots,\ \hat\theta_k,\ \mathrm{id}_D,\ \psi$$

are linearly independent elements of $\mathrm{End}_{\Sigma_{n-2,2}}(D{\downarrow_{\Sigma_{n-2,2}}})$. This of course implies the theorem.

A linear dependence between our endomorphisms looks like

$$a_1\hat\theta_1+a_2\hat\theta_2+\dots+a_k\hat\theta_k+b\,\mathrm{id}_D+c\psi\equiv 0 \tag{8}$$

with $a_i,b,c\in F$. We may assume that $a_i\neq 0$ for some $i$, since otherwise we would get a linear dependence between $\mathrm{id}_D$ and $\psi$, which is only possible if $D$ is trivial. Let $\theta:=a_1\theta_1+\dots+a_k\theta_k$. Then $a_1\hat\theta_1+\dots+a_k\hat\theta_k=\hat\theta$, and we may rewrite (8) as

$$\hat\theta+b\,\mathrm{id}_D+c\psi\equiv 0.$$

By the first paragraph of the proof, we have $(b,c)\neq(0,0)$. The last equality is equivalent to

$$\theta(\sigma v)=\sigma\theta(v)+b\sigma v+cv. \qquad (v\in D)$$

Let $\pi_i=(i,n-1),\ \tau_i=(i,n)\in\Sigma_n$. Then $\tau_i=\pi_i\sigma\pi_i$. So for any $v\in D$ we have

$$\begin{aligned}\theta(\tau_i v)&=\theta(\pi_i\sigma\pi_i v)=\pi_i\theta(\sigma\pi_i v)=\pi_i(\sigma\theta(\pi_i v)+b\sigma\pi_i v+c\pi_i v)\\ &=\pi_i\sigma\theta(\pi_i v)+b\pi_i\sigma\pi_i v+cv=\tau_i\theta(v)+b\tau_i v+cv. \qquad (v\in D)\end{aligned}$$

As $\pi_i = \tau_i\sigma\tau_i$ we also have for any $v \in D$

$$\begin{aligned}\pi_i\theta(v) &= \theta(\pi_i v) = \theta(\tau_i\sigma\tau_i v) = \tau_i\theta(\sigma\tau_i v) + b\tau_i\sigma\tau_i v + c\sigma\tau_i v\\ &= \tau_i(\sigma\theta(\tau_i v) + b\sigma\tau_i v + c\tau_i v) + b\tau_i\sigma\tau_i v + c\sigma\tau_i v\\ &= \tau_i\sigma\theta(\tau_i v) + b\tau_i\sigma\tau_i v + cv + b\tau_i\sigma\tau_i v + c\sigma\tau_i v\\ &= \tau_i\sigma(\tau_i\theta(v) + b\tau_i v + cv) + cv + c\sigma\tau_i v\\ &= \pi_i\theta(v) + b\pi_i v + c\tau_i\sigma v + cv + c\sigma\tau_i v,\end{aligned}$$

whence $b\pi_i v + c(\tau_i\sigma + \sigma\tau_i + 1)v = 0$. In particular, $c \neq 0$. If $b \neq 0$ we have

$$(b\pi_i + c\tau_i\sigma + c\sigma\tau_i + c)v = 0. \qquad (v \in D)$$

Conjugating we get

$$(b(1,2) + c(1,2,3) + c(1,3,2) + c)v = 0. \qquad (v \in D)$$

Lemma 3.5 now leads to a contradiction as $D \ncong \mathbf{1}_{\Sigma_n}$. So we may assume that $b = 0$. Then we get the identity

$$(\tau_i\sigma + \sigma\tau_i + 1)v = 0. \qquad (v \in D)$$

But $\tau_i\sigma =: \gamma$ is a 3-cycle and $\sigma\tau_i = \gamma^2$. So it suffices to show that there is a non-zero $v \in D$ such that $\gamma v = v$. But this follows from Lemma 3.4, provided $D \ncong S$. Finally if $D \cong S$ and $n = 2l$ is even, then $S = D^{(l+1,l-1)}$. By [14], $D^{(l+1,l-1)}{\downarrow_{\Sigma_{n-1}}} \cong D^{(l,l-1)}$ and $D^{(l,l-1)}{\downarrow_{\Sigma_{n-2,2}}}$ is reducible (and self-dual). So the Hom-spaces under consideration have dimensions 1 and 2, and the theorem is true. □

**Corollary 3.7.** *Let $n \geq 4$ and $D$ be an irreducible $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$ and $n$ is odd, assume that $D \ncong S$. Then*

$$\dim \operatorname{Hom}_{\Sigma_n}(M^{(n-2,2)}, \operatorname{End}(D)) > \dim \operatorname{Hom}_{\Sigma_n}(M^{(n-1,1)}, \operatorname{End}(D)).$$

*Proof.* Follows from Theorems 3.3, 3.6 and the isomorphisms

$$\operatorname{Hom}_{\Sigma_n}(M^\nu, \operatorname{End}(D)) \cong \operatorname{Hom}_{\Sigma_\nu}(\mathbf{1}_{\Sigma_\lambda}, \operatorname{End}(D)) \cong \operatorname{End}_{\Sigma_\lambda}(D{\downarrow_{\Sigma_\lambda}}).$$

□

**Lemma 3.8.** *Let $n \geq 4$ and $D$ be an irreducible $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that $n$ is odd and $D \ncong S$. Then $\operatorname{End}(D)$ contains either $Y^{(n-2,2)}$ or $(S^{(n-2,2)})^*$ or both as submodules.*

*Proof.* Assume first that $p > 2$, $n \not\equiv 1, 2 \pmod p$ or $p = 2$, $n \equiv 3 \pmod 4$. Then the result follows from Lemmas 1.2(i), 1.3(ii) and Corollary 3.7. Now let $p > 2$ and $n \equiv 1 \pmod p$. Then $M^{(n-1,1)}$ splits as $\mathbf{1} \oplus D^{(n-1,1)}$. By Lemma 1.2(ii), there is a surjection $M^{(n-2,2)} \to M^{(n-1,1)}$, and by Corollary 3.7, there must exist a homomorphism $\theta : M^{(n-2,2)} \to \mathrm{End}(D)$ which does not factor through this surjection. If the restriction $\theta|Y^{(n-2,2)}$ is injective then $Y^{(n-2,2)}$ is a submodule of $\mathrm{End}(D)$. Otherwise, in view of Lemma 1.2(ii), the kernel of $\theta|Y^{(n-2,2)}$ is $\mathbf{1}$. But $Y^{(n-2,2)}/\mathbf{1} \cong (S^{(n-2,2)})^*$, so $(S^{(n-2,2)})^*$ embeds into $\mathrm{End}(D)$. The cases $p > 2$, $n \equiv 2 \pmod p$ and $p = 2$, $n \equiv 1 \pmod 4$ are considered similarly to the case $n \equiv 1 \pmod p$ using 1.2(iii), 1.3(i) and 3.7. □

Recall that $d_1$ and $d_2$ denote the number of $G$-orbits on $\Omega$ and the 2-subsets of $\Omega$, respectively.

**Theorem 3.9.** *Let $n \geq 4$, $G \leq \Sigma_n$ be a subgroup, and $D$ be a simple $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that $n$ is odd and $D \not\cong S$. Then the restriction $D_G$ is reducible whenever $d_1 < d_2$.*

*Proof.* Suppose $D_G$ is irreducible and $d_1 < d_2$. Then $\mathrm{End}_G(D_G) \cong \mathrm{End}(D)^G$ is 1-dimensional by Schur's lemma. We consider the following two cases.

Case 1: $p > 2$, $n \not\equiv 1 \pmod p$ or $p = 2$, $n \equiv 3 \pmod 4$. Then by Lemmas 1.2(i),(iii), 1.3(ii) and 3.8, either $\mathbf{1}_{\Sigma_n} \oplus Y^{(n-2,2)} \subseteq \mathrm{End}(D)$ or $\mathbf{1}_{\Sigma_n} \oplus (S^{(n-2,2)})^* \subseteq \mathrm{End}(D)$ (or both). By Lemma 1.7, $\mathrm{End}(D)^G$ is at least 2-dimensional, giving a contradiction.

Case 2: $p > 2$, $n \equiv 1 \pmod p$ or $p = 2$, $n \equiv 1 \pmod 4$. By Lemma 3.8 again, either $Y^{(n-2,2)} \subseteq \mathrm{End}(D)$ or $(S^{(n-2,2)})^* \subseteq \mathrm{End}(D)$. In the first case Lemma 1.7 implies that $\mathrm{End}(D)^G$ is at least 2-dimensional. In the second case $(S^{(n-2,2)})^* = D^{(n-2,2)}|\mathbf{1}$ implies that $(S^{(n-2,2)})^* \oplus \mathbf{1}_{\Sigma_n} \subseteq \mathrm{End}(D)$. Now apply Lemma 1.7 as in Case 1. □

**Theorem 3.10.** *Let $n \geq 4$, $G \leq \Sigma_n$ be a subgroup, and $D$ be a simple $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that $n$ is odd and $D \not\cong S$. If the restriction $D_G$ is irreducible then either $G \leq \Sigma_{n-1}$ or $G$ is 2-transitive.*

*Proof.* Let $D_G$ be irreducible. Assume first that $G$ is intransitive. Then, up to a conjugation, $G$ is contained in a standard Young subgroup $\Sigma_{n-k} \times \Sigma_k$ for some $1 \leq k \leq n/2$. If $k > 1$, then $d_1 = 2 < d_2 = 3$ with respect to the action of $\Sigma_{n-k} \times \Sigma_k$. Hence $D{\downarrow}_{\Sigma_{n-k}\times\Sigma_k}$ is reducible by Theorem 3.9 applied to the subgroup $\Sigma_{n-k} \times \Sigma_k$. This gives a contradiction. Therefore $k = 1$ and so $G \leq \Sigma_{n-1}$. We may now assume that $G$ is transitive. If $G$ is not 2-homogeneous, then $d_1 = 1 < d_2$ and we get a contradiction by Theorem 3.9. Thus, $G$ is 2-homogeneous. Now, if $G$ is not 2-transitive apply Theorem 2.5. □

Finally, observe that the Main Theorem is a special case of Theorem 3.10.

## References


1. M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math. **76** (1984), 469–514
2. C. Bessenrodt and J.B. Olsson, On residue symbols and the Mullineux conjecture, J. Algebraic Combin. **7** (1998), 227–251
3. J. Brundan and A.S. Kleshchev, Representations of the symmetric group which are irreducible over subgroups, preprint, University of Oregon, 1999
4. J. D. Dixon and B. Mortimer, Permutation groups. Graduate Texts in Mathematics, 163, Springer-Verlag, New York, 1996
5. P. Fleischmann, W. Lempken, A.E. Zalesskii, Linear groups over $GF(2^k)$ generated by a conjugacy class of a fixed point free element of order $3$, in preparation
6. B. Ford, Irreducible restrictions of representations of the symmetric groups, Bull. London Math. Soc. **27** (1995), 453–459
7. B. Ford and A. Kleshchev, A proof of the Mullineux Conjecture, Math. Z. **226** (1997), 267–308
8. G. D. James, The Representation Theory of the Symmetric Groups, volume 682 of Lecture Notes in Mathematics, Springer, New York/Heidelberg/Berlin, 1978
9. G. D. James, On the minimal dimensions of irreducible representations of symmetric groups, Math. Proc. Camb. Phil. Soc. **94** (1983), 417–424
10. G. James and A. Kerber, The representation theory of the symmetric group, Addison-Wesley, London, 1981
11. J. C. Jantzen and G. M. Seitz, On the representation theory of the symmetric groups, Proc. London Math. Soc. (3) **65** (1992), 475–504
12. W. M. Kantor, $k$-homogeneous groups, Math. Z. **124** (1972), 261–265
13. P. Kleidman and M. Liebeck, The subgroup structure of the finite classical groups, London Mathematical Society Lecture Note Series, 129, Cambridge, Cambridge University Press
14. A. S. Kleshchev, On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I Proc. London Math. Soc. (3) **69** (1994), 515–540
15. G. Mullineux, Representations of $A_n$ in characteristic $2$ with the $3$-cycles fixed-point-free, Quart. J. Math. Oxford (2) **29** (1978), 199–212
16. J. Saxl. The complex characters of the symmetric groups that remain irreducible in subgroups, J. Algebra **111** (1987), 210–219
17. A. Wagner, On the faithful linear representations of least degree of $S_n$ and $A_n$ over a field of characteristic 2, Math. Z. **151** (1976), 127–137
18. A. Wagner, The faithful linear representations of least degree of $S_n$ and $A_n$ over a field of odd characteristic, Math. Z. **154** (1977), 103–114