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# $L^1$—decay of higher-order norms of solutions to the Navier–Stokes equations in the upper-half space

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**Abstract**
The aim of this article devotes to establishing the $L^1$-decay of cubic order spatial derivatives of solutions to the Navier–Stokes equations, which is a long-time challenging problem. To solve this problem, new tools have to be found to overcome these main difficulties: $L^1 - L^1$ estimate fails for the Stokes flow; the projection operator $P : L^1(\mathbb{R}^n_+) \to L^1_\sigma(\mathbb{R}^n_+)$ becomes unbounded; the steady Stokes's estimates does not work any more in $L^1(\mathbb{R}^n_+)$. We first give the asymptotic behavior with weights of negative exponent for the Stokes flow and Navier–Stokes equations in $L^1(\mathbb{R}^n_+)$, and these are also independent of interest by themselves. Secondly, we decompose the convection term into two parts, and translate the unboundedness of projection operator into studying an $L^1$-estimate for an elliptic problem with homogeneous Neumann boundary conditions, which is established by using the weighted estimates of the Gaussian kernel's convolution. Finally, a crucial new formula is given for the fundamental solution of the Laplace operator, which is employed for overcoming the strong singularity in studying the cubic order spatial derivatives in $L^1(\mathbb{R}^n_+)$.

**Keywords** Navier–Stokes flows · Solution formula · Asymptotic behavior · Strong solution

**Mathematics Subject Classification** 35Q35 · 35B40 · 75D05 · 76D07

## 1 Introduction and main results

In this article, we are concerned with the asymptotic behavior of solutions to the Navier–Stokes initial-value problem in the half space

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0, \infty), \\ u(x, t) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

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where $n \geq 2$, and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is the upper-half space of $\mathbb{R}^n$; $u = (u_1, u_2, \cdots, u_n)(x, t)$ and $p = p(x, t)$ denote unknown velocity vector and the pressure respectively, while initial datum $u_0(x)$ is assumed to satisfy a *compatibility condition*: $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$ and the normal component of $u_0$ equals to zero on $\partial\mathbb{R}^n_+$; and

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla = (\partial_1, \partial_2, \cdots, \partial_n), \quad \partial_j = \frac{\partial}{\partial x_j} \ (j = 1, 2, \cdots, n),$$
$$\Delta u = \sum_{j=1}^{n} \partial_j \partial_j u, \quad (u \cdot \nabla)u = \sum_{j=1}^{n} u_j \partial_j u, \quad \nabla \cdot u = \sum_{j=1}^{n} \partial_j u_j.$$

Throughout this paper, write

$$x = (x', x_n), \quad x' = (x_1, x_2, \cdots, x_{n-1}), \quad \nabla_{x'} = \nabla' = (\partial_1, \partial_2, \cdots, \partial_{n-1}),$$

$C_0^\infty(\mathbb{R}^n_+)$ denotes the set of all $C^\infty$ real functions with compact support in $\mathbb{R}^n_+$, and

$$C_{0,\sigma}^\infty(\mathbb{R}^n_+) = \{\phi = (\phi_1, \cdots, \phi_n) \in C_0^\infty(\mathbb{R}^n_+); \ \nabla \cdot \phi = 0 \ \text{ in } \ \mathbb{R}^n_+\};$$

$L^q_\sigma(\mathbb{R}^n_+)$ $(1 < q < \infty)$ is the closure of $C_{0,\sigma}^\infty(\mathbb{R}^n_+)$ with respect to $\|\cdot\|_{L^q(\mathbb{R}^n_+)}$, where $L^q(\mathbb{R}^n_+)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^q(\mathbb{R}^n_+)$ is denoted by $\|u\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |u(x)|^q dx)^{\frac{1}{q}}$ if $1 \leq q < \infty$; and $\|u\|_{L^\infty(\mathbb{R}^n_+)} = ess \sup_{x \in \mathbb{R}^n_+} |u(x)|$; $\|\omega(x)u(t)\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |\omega(x)u(x,t)|^q dx)^{\frac{1}{q}}$, $\|\omega(x) f(x, y)\|_{L^q_x(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |\omega(x) f(x, y)|^q dx)^{\frac{1}{q}}$, $1 \leq q < \infty$, $\omega(x) = x_n^\alpha$, $|x|^\alpha$, $\alpha \geq 0$. $O(f(x)) = g(x)$ means $|f(x)| \leq C|g(x)|$ for some constant $C$. By symbol $C$, it means a generic positive constant which may vary from line to line.

A vector function $u$ is called a weak solution of (1.1) if $u \in L^\infty(0, \infty; L^2_\sigma(\mathbb{R}^n_+)) \cap L^2_{loc}([0, \infty); H^1_0(\mathbb{R}^n_+))$ satisfies problem (1.1) in the sense of distributions. Moreover, the energy inequality holds for almost all $t \in [0, \infty)$ including $t = 0$:

$$\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} ds \leq \|u_0\|^2_{L^2(\mathbb{R}^n_+)}.$$

The weak solution $u$ of (1.1) is so far known to be unique only if $u$ belongs to a certain class of functions which, however, does not cover the whole space $L^\infty(0, \infty; L^2_\sigma(\mathbb{R}^n_+)) \cap L^2_{loc}([0, \infty); H^1_0(\mathbb{R}^n_+))$. Furthermore if the Serrin's condition holds: $u \in L^q(0, \infty; L^r(\mathbb{R}^n_+))$ with $\frac{2}{q} + \frac{n}{r} \leq 1$, $2 \leq q < \infty$, $n < r \leq \infty$, then $u$ is called a strong solution, which is smooth in $\mathbb{R}^n_+ \times (0, +\infty)$.

Before stating the main results, we first recall a small-data global existence of classical solution in the half-space, see, e.g., Theorem 3.2 in [12].

**Theorem 1.1** *Let $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$. Then there exists a number $\epsilon_0 > 0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)} \leq \epsilon_0$ (smallness condition is unnecessary if $n = 2$), problem (1.1) admits a unique strong solution $u$.*

Let $A$ denote the Stokes operator $-P\Delta$ in $\mathbb{R}^n_+$, where the Laplacian $\Delta$ in $\mathbb{R}^n_+$ is endowed with the homogeneous Dirichlet boundary condition, $P$ is the projection operator: $L^r(\mathbb{R}^n_+) \longrightarrow L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$. Then the function $e^{-tA}u_0$ solves the Stokes system, that is, problem (1.1) with $u \cdot \nabla u$ deleted, with the initial datum $u_0$. The (weak or strong) solution $u$ of problem (1.1) can be written as follows:

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)ds.$$

Note that for $t \geq s > 0$, $e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)$ is also a Stokes flow with the initial data $P(u(s)\cdot\nabla)u(s)$. That is, set $w(t) = e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)$, $t \geq s > 0$, then

$$\begin{cases} \partial_t w - \Delta w + \nabla\pi = 0 & \text{in } \mathbb{R}^n_+ \times (s,\infty),\\ \nabla\cdot w = 0 & \text{in } \mathbb{R}^n_+ \times (s,\infty),\\ w(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (s,\infty),\\ w(x,t)|_{t=s} = P(u(s)\cdot\nabla)u(s) & \text{in } \mathbb{R}^n_+. \end{cases}$$

Whence in order to establish the estimates of solutions to Navier–Stokes equations, it is necessary to investigate the linear Stokes problem.

If $u_0 \in L^1_\sigma(\mathbb{R}^n_+)$ satisfies some additional assumptions, Bae and Choe [3] proved the decay rate for $t > 0$: $\|\nabla e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-1}$ with $1 < q < \infty$. If the initial data $u_0$ lies in an appropriate weighted space, and satisfies the average condition: $\int_{\mathbb{R}^{n-1}} u_0(y', y_n)dy' = 0$ for $a.e.\ y_n > 0$, Bae showed $L^1$-decay of the Stokes flow in [1], and $L^1$–time estimate in [2] for the gradient of Stokes flow, respectively. It is natural to ask whether the Stokes flow $e^{-tA}a$ belongs to $L^1(\mathbb{R}^n_+)$, $t > 0$ for every $a \in L^1_\sigma(\mathbb{R}^n_+)$. The answer to this question is negative, see a specific counterexample given in the Appendix.

The first result focuses on the weighted $L^1$-time decay estimate of the Stokes flow, as far as we know, which is first considered for the negative exponent case. Of course, the following Theorem 1.2 is also very interesting by itself.

**Theorem 1.2** *Let $a = (a_1, a_2, \cdots, a_n) \in L^1(\mathbb{R}^n_+)$, $a_n|_{\partial\mathbb{R}^n_+} = 0$, $\nabla\cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$. Then the Stokes flow $e^{-tA}a$ satisfies for $t > 0$*

$$\||x'|^{-\alpha}e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\|a\|_{L^1(\mathbb{R}^n_+)}, \quad 0 < \alpha < n-1;$$

*and*

$$\|x_n^{-\alpha}e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\|a\|_{L^1(\mathbb{R}^n_+)}, \quad 0 < \alpha < 1;$$

*where $C$ depends only on $n, \alpha$.*

The decay problem for solutions to the Navier–Stokes equations was first proposed by Leray [17] for the Cauchy problem. Schonbek [18, 21] attacked this problem and succeeded for the first time in showing existence of weak solutions with explicit decay rate. In [22], Schonbek first developed a very effective new method, *Fourier splitting method*, called also *Schonbek's method*, which has been applied extensively for studying decay properties of solutions to various diffusive partial differential equations. This method does not depend on the linearized underlying equations. In a series of articles (see [18]–[23]), Schonbek completed systematic outstanding research work, made many innovative achievements on decay properties of Navier–Stokes flows in the whole space, which subsequently have been cited and generalized widely by many mathematical researchers.

All of these techniques employed in the whole space are not applicable directly to problem (1.1), because the projection operator $P : L^1(\mathbb{R}^n_+) \to L^1_\sigma(\mathbb{R}^n_+)$ is not bounded any more, and $P\Delta \neq \Delta P$, which causes many essential difficulties in treating the nonlinear term $(u\cdot\nabla)u$. Here we mention briefly some known results obtained on the half-space. Bae and Choe [3], Fujigaki and Miyakawa [12] studied asymptotic behavior for weak and strong solutions of (1.1) in $L^r(\mathbb{R}^n_+)$ with $1 < r < \infty$. Starting from the proof of existence, Farwig, Kozono and

David [10] get a weak solution satisfying $\|v(t)\|_{L^2} \longrightarrow 0$ as $t \longrightarrow \infty$, and determine an upper bound for the decay rate. Relevant topics are referred to see [4]– [9], [11, 15, 16] and the references therein.

Our second result is to establish $L^1$-decay rates for the Navier–Stokes flows with negative exponent weights. To our knowledge, few weighted $L^1$-decay estimates in such cases are available on solutions of problem (1.1).

**Theorem 1.3** *Assume $u_0 \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$. Then the strong solution $u$ of (1.1) obtained in Theorem 1.1 satisfies for $t > 0$*

$$\||x'|^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} \leq \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{if } n \geq 3, \\ Ct^{-\frac{\alpha}{2}}\log_e(1+t) & \text{if } n = 2, \end{cases} \qquad 0 < \alpha < \min\{2, n-1\};$$

*and*

$$\|x_n^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} \leq \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{if } n \geq 3, \\ Ct^{-\frac{\alpha}{2}}\log_e(1+t) & \text{if } n = 2, \end{cases} \qquad 0 < \alpha < 1.$$

*Furthermore if $x_n u_0 \in L^1(\mathbb{R}^n_+)$, there holds*

$$\||x'|^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}, \quad 0 < \alpha < \min\{2, n-1\};$$

*and*

$$\|x_n^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}, \quad 0 < \alpha < 1.$$

***Remark*** From the view point of mathematics, it is necessary to discuss how the classical weighted function $|x|^\alpha$ $(\alpha \in \mathbb{R}^1)$ affects the large time asymptotic behavior of the Navier–Stokes flows. Due to the fact that the Stokes flow $e^{tA}u_0$ is not in $L^1$ space (see Appendix), we do not expect the solution of the Navier–Stokes system to be in $L^1$ space with weighted function $|x|^\alpha$, $\alpha \geq 0$. Therefore, it is natural to consider the case of $\alpha < 0$, for which no conclusion has been found yet. The negative power $\alpha$ for the weighted function $|x|^\alpha$ implies the energy increases in the spatial direction, and simultaneously in the temporal direction the energy decreases as the power $\frac{\alpha}{2}$.

The $L^r$−asymptotic behavior of higher-order spatial derivatives of solutions of problem (1.1) is established for $1 < r \leq \infty$ in [14]. However, up to now, the case for $r = 1$ remains still open, because the *a priori* estimates on the steady Stokes system are not valid any more in $L^1(\mathbb{R}^n_+)$. Applying the nonstationary Stokes's estimates to the integral equation on the solution of (1.1), we inevitably encounter the strong singularity. Exactly speaking, under the assumptions of Theorem 1.4 below on the initial data $u_0$, together with the following Lemma 3.1, we conclude that for $t > 0$

$$\begin{aligned}\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} &\leq Ct^{-\frac{3}{2}}\int_{\mathbb{R}^n_+}|u_0(y)|dy + C\int_0^t (t-s)^{-\frac{3}{2}}\|P(u(s)\cdot\nabla)u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\ &\leq Ct^{-\frac{3}{2}}\int_{\mathbb{R}^n_+}|u_0(y)|dy + C\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)(t-s)^{-\frac{3}{2}}(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)})ds.\end{aligned}$$

In the above calculations, we made use of the Stokes's estimate (see [14]):

Let $a = (a_1, a_2, \cdots, a_n) \in L^1(\mathbb{R}^n_+)$, $a_n|_{\partial\mathbb{R}^n_+} = 0$, $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$. Then

$$\|\nabla^3[e^{-tA}a]\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{3}{2}}\int_{\mathbb{R}^n_+}|a(y)|dy, \quad \forall t > 0.$$

Since the strong solution $u$ of problem (1.1) satisfies for any $0 < s < t$

$$\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + \int_s^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^n_+)} d\tau = \|u(s)\|^2_{L^2(\mathbb{R}^n_+)}.$$

A strong singularity arises from the following term for $t > 0$:

$$\begin{aligned}
&\int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{2}} (\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}) ds \\
&\geq \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{2}} \|u(s)\|^2_{L^2(\mathbb{R}^n_+)} ds \\
&\geq \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{2}} ds \|u(t)\|^2_{L^2(\mathbb{R}^n_+)} = +\infty.
\end{aligned}$$

It is almost impossible to avoid this kind of difficulty by making use of such a direct method, new ideas and innovative approaches have to be found and employed. The following result (i.e. Theorem 1.4) is an attempt on such topics. Schonbek and Wiegner [23] established the decay estimates of arbitrary spatial derivatives of solutions in the whole space $\mathbb{R}^n$, and their method depends on the Fourier transform and on the commutativity between projection and differential operators, neither of which seems to be effective directly in dealing with problem (1.1). The following result is a long-time unsolved problem, which is inspired mainly by the remarkable research work by Schonbek and Wiegner in the whole space, see [23].

**Theorem 1.4** *Assume $u_0 \in L^1(\mathbb{R}^n_+) \bigcap L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$. Then the strong solution $u$ given in Theorem 1.1 satisfies for any $t > 1$*

$$\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} \leq \begin{cases} Ct^{-\frac{3}{2}} & \text{if } n \geq 3, \\ Ct^{-\frac{3}{2}} \log_e(1+t) & \text{if } n = 2. \end{cases}$$

*Suppose $x_n u_0 \in L^1(\mathbb{R}^n_+)$, there holds for every $t > 1$*

$$\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{3}{2}}. \tag{1.2}$$

*Furthermore if $x_n u_0, (1+x_n)\nabla u_0 \in L^2(\mathbb{R}^n_+)$, then for $0 < \beta < 1$*

$$\|x_n^\beta \nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{3}{2}+\frac{\beta}{2}}, \quad \forall\, t > 1. \tag{1.3}$$

***Remarks*** (1) As seen below, the proof's procedure of $\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)}$ is technical, complicated and lengthy. However, it is still possible to establish the $L^1$-time decay of higher spatial derivatives $\nabla^k u$ $(k \geq 4)$ by using these methods employed in this article.

(2) To our knowledge, it is the first time to show the (weighted) decay estimates of the cubic spatial derivatives in $L^1(\mathbb{R}^n_+)$. Let $x = (x', x_n)$, $y = (y', y_n) \in \mathbb{R}^n_+$, and $0 < \beta < 1$. It is not sure whether similar decay results are true for $\||x'|^\beta \nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)}$. Because in the proof procedure of Theorem 1.4, we make full use of the special structure of the half space, the weight $x_n^\beta$ can be treated by $(x_n + y_n)^\beta$ in Solonnikov's solution formula. On the other hand, since $|x'|^\beta \leq |x' - y'|^\beta + |y'|^\beta$, and the strong singularity arises from the term containing the weight $|y'|^\beta$ in Solonnikov's solution formula, we readily find that the methods employed in the proof of Theorem 1.4 does not work in the case of the weight $|x'|^\beta$. Up to now, the decay estimate of $\||x'|^\beta \nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)}$ is still unsolved.

This article is organized as follows: In Sect. 2, we collect some basic known results, and give the proof of Theorem 1.2. Section 3 devotes to establishing the weighted $L^1$-time decay of the strong solution of problem (1.1). By means of properties of the Gaussian kernel's convolution, we construct some crucial weighted estimates on a class of elliptic problem relating to the convection term in problem (1.1), which will be frequently applied in the proof of Theorem 1.4. In fact, such a study is of independent interest. Together with some known decay estimates of solutions of (1.1), we establish $L^1$-time (weighted) decays of solutions of (1.1), see Theorem 1.3. In Sect. 4, we focus on studying the (weighted) decays of cubic spatial derivatives of the strong solution of (1.1). To do this, we first find an important formula on the fundamental solution of elliptic operator $-\Delta$, which is important in overcoming the strong singularity. Combining Theorem 1.3 and Solonnikov's solution formula, we finally achieve the desired results, e.g., Theorem 1.4.

## 2 Weighted $L^1$-decay for the Stokes flow

In this section, we first introduce Solonnikov's solution formula and related basic estimates.

Let $a = (a_1, a_2, \cdots, a_n) \in L^1(\mathbb{R}^n_+)$, $a_n|_{\partial\mathbb{R}^n_+} = 0$, $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$. Then it holds for $t > 0$ (see [24])

$$[e^{-tA}a](x) = \int_{\mathbb{R}^n_+} \mathcal{M}(x, y, t)a(y)dy, \quad x \in \mathbb{R}^n_+, \tag{2.1}$$

where $\mathcal{M} = (M_{ij})_{i,j=1,2,\cdots,n}$ is defined as follows

$$\begin{aligned} M_{ij}(x, y, t) &= \delta_{ij}(G_t(x-y) - G_t(x-y^*)) \\ &\quad +4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_i} G_t(z-y^*)dz \\ &\equiv \delta_{ij}(G_t(x-y) - G_t(x-y^*)) + M^*_{ij}(x, y, t), \end{aligned} \tag{2.2}$$

$y^* = (y_1, y_2, \cdots, -y_n)$, $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel,

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{cases} \qquad E(x) = \begin{cases} \dfrac{1}{n(2-n)\omega_n |x|^{n-2}} & \text{if } n \geq 3, \\ -\dfrac{1}{2\pi}\log_e \dfrac{1}{|x|} & \text{if } n = 2, \end{cases}$$

$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ denotes the volume of the unit ball in $\mathbb{R}^n$, $E(x)$ is the fundamental solution of the Laplace operator $-\Delta$. Namely, $-\Delta E(x) = \delta(x)$ in the sense of distributions, where $\delta(x)$ denotes the Dirac function concentrating at $x = 0$.

Let $\vec{\ell}$ denote $(\ell_1, \ell_2, \cdots, \ell_{n-1}, \ell_n) = (\ell', \ell_n)$. Then the estimate for $M^*_{ij}$ $(1 \leq i, j \leq n)$ in (2.2) holds for $x, y \in \mathbb{R}^n_+$, $t > 0$

$$|\partial_t^s \nabla_x^{\vec{k}} \nabla_y^{\vec{m}} M^*_{ij}(x, y, t)| \leq Ct^{-s-\frac{m_n}{2}}(t + x_n^2)^{-\frac{k_n}{2}}(|x-y^*|^2 + t)^{-\frac{n+|k'|+|m'|}{2}} e^{-\frac{cy_n^2}{t}}. \tag{2.3}$$

***Proof of Theorem 1.2.*** Let $y = (y', y_n) \in \mathbb{R}^n_+$. Then for $t > 0$

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}}\int_0^\infty |x'|^{-\alpha}(|x'-y'| + x_n + y_n + \sqrt{t})^{-n}dx_n dx' \\ &\leq \frac{1}{n-1}\int_{\mathbb{R}^{n-1}} |x'|^{-\alpha}(|x'-y'| + \sqrt{t})^{-n+1}dx' \end{aligned}$$

$$
\begin{aligned}
&= \frac{t^{-\frac{\alpha}{2}}}{n-1}\int_{\mathbb{R}^{n-1}} |z'|^{-\alpha}(|z'-t^{-\frac{1}{2}}y'|+1)^{-n+1}dz' \\
&\le \frac{t^{-\frac{\alpha}{2}}}{n-1}\int_{|z'|\le 1} |z'|^{-\alpha}dz' + \frac{t^{-\frac{\alpha}{2}}}{n-1}\left(\int_{|z'|>1}|z'|^{-n}dz'\right)^{\frac{\alpha}{n}} \\
&\quad \times\left(\int_{|z'|>1}(|z'-t^{-\frac{1}{2}}y'|+1)^{-\frac{n(n-1)}{n-\alpha}}dz'\right)^{1-\frac{\alpha}{n}} \\
&\le Ct^{-\frac{\alpha}{2}}\int_0^1 s^{-\alpha+n-2}ds + Ct^{-\frac{\alpha}{2}}\left(\int_1^\infty s^{-2}ds\right)^{\frac{\alpha}{n}} \\
&\quad \times\left(\int_0^\infty (s+1)^{-\frac{n(n-1)}{n-\alpha}+n-2}ds\right)^{1-\frac{\alpha}{n}} \\
&\le C_1(n,\alpha)t^{-\frac{\alpha}{2}}, \quad 0<\alpha<n-1;
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^{n-1}} x_n^{-\alpha}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n}dx'dx_n \\
&\le \int_0^\infty x_n^{-\alpha}\int_{\mathbb{R}^{n-1}}(|x'|+x_n+\sqrt{t})^{-n}dx'dx_n \\
&\le Ct^{-\frac{\alpha}{2}}\int_0^\infty x_n^{-\alpha}\int_{\mathbb{R}^{n-1}}(|x'|+x_n+1)^{-n}dx'dx_n \\
&\le Ct^{-\frac{\alpha}{2}}\int_0^\infty x_n^{-\alpha}\int_0^\infty (s+x_n+1)^{-n}s^{n-2}dsdx_n \\
&\le Ct^{-\frac{\alpha}{2}}\int_0^\infty x_n^{-\alpha}\int_0^\infty (s+x_n+1)^{-2}dsdx_n \\
&\le Ct^{-\frac{\alpha}{2}}\int_0^\infty x_n^{-\alpha}(x_n+1)^{-1}dx_n \\
&\le Ct^{-\frac{\alpha}{2}}\left(\int_0^1 x_n^{-\alpha}dx_n + \int_1^\infty x_n^{-\alpha-1}dx_n\right) \\
&\le C_2(n,\alpha)t^{-\frac{\alpha}{2}}, \quad 0<\alpha<1.
\end{aligned} \tag{2.5}
$$

Set $G_t^{(n-1)}(x') = (4\pi t)^{-\frac{n-1}{2}}e^{-\frac{|x'|^2}{4t}}, x'\in\mathbb{R}^{n-1}; G_t^{(1)}(x_n) = (4\pi t)^{-\frac{1}{2}}e^{-\frac{|x_n|^2}{4t}}, x_n\in\mathbb{R}^1$. Then $G_t(x) = G_t^{(n-1)}(x')G_t^{(1)}(x_n)$, $x=(x',x_n)$. Whence for $y=(y',y_n)\in\mathbb{R}_+^n$ and $t>0$

$$
\begin{aligned}
&\int_{\mathbb{R}_+^n}|x'|^{-\alpha}|G_t(x-y)-G_t(x+y^*)|dx \\
&= \int_0^\infty\int_{\mathbb{R}^{n-1}}|x'|^{-\alpha}G_t^{(n-1)}(x'-y')[G_t^{(1)}(x_n-y_n)-G_t^{(1)}(x_n+y_n)]dx'dx_n \\
&\le \int_0^\infty\int_{\mathbb{R}^{n-1}}|x'|^{-\alpha}G_t^{(n-1)}(x'-y')G_t^{(1)}(x_n-y_n)dx'dx_n \\
&\le \int_{\mathbb{R}^{n-1}}|x'|^{-\alpha}G_t^{(n-1)}(x'-y')dx' \\
&= t^{-\frac{\alpha}{2}}\int_{\mathbb{R}^{n-1}}|z'|^{-\alpha}G_1^{(n-1)}(z'-t^{-\frac{1}{2}}y')dz'
\end{aligned}
$$

$$
\begin{aligned}
&\le Ct^{-\frac{\alpha}{2}}\left(\int_{|z'|\le 1}|z'|^{-\alpha}dz'+\int_{|z'|>1}G_1^{(n-1)}(z'-t^{-\frac{1}{2}}y')dz'\right)\\
&\le Ct^{-\frac{\alpha}{2}}\left(\int_0^1 s^{-\alpha+n-2}ds+\int_{\mathbb{R}^{n-1}}G_1^{(n-1)}(z')dz'\right)\\
&\le C_3(n,\alpha)t^{-\frac{\alpha}{2}},\quad 0<\alpha<n-1; \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}x_n^{-\alpha}|G_t(x-y)-G_t(x+y^*)|dx\\
&=\int_0^\infty\int_{\mathbb{R}^{n-1}}x_n^{-\alpha}G_t^{(n-1)}(x'-y')[G_t^{(1)}(x_n-y_n)-G_t^{(1)}(x_n+y_n)]dx'dx_n\\
&\le\int_{\mathbb{R}^{n-1}}G_t^{(n-1)}(x'-y')dx'\int_0^\infty x_n^{-\alpha}G_t^{(1)}(x_n-y_n)dx_n\\
&=t^{-\frac{\alpha}{2}}\int_0^\infty x_n^{-\alpha}G_1^{(1)}(x_n-y_n)dx_n\\
&\le Ct^{-\frac{\alpha}{2}}\left(\int_0^1 x_n^{-\alpha}dx_n+\int_1^\infty G_1^{(1)}(x_n-y_n)dx_n\right)\\
&\le C_4(n,\alpha)t^{-\frac{\alpha}{2}},\quad 0<\alpha<1. \qquad (2.7)
\end{aligned}
$$

Using the representation (2.1) of the Stokes flow $e^{-tA}a$, and the estimates of (2.3)–(2.7), we find for $t>0$

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}|x'|^{-\alpha}|[e^{-tA}a](x)|dx\\
&\quad\le\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|x'|^{-\alpha}|G_t(x-y)-G_t(x+y^*)||a(y)|dxdy\\
&\qquad+C\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|x'|^{-\alpha}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n}|a(y)|dxdy\\
&\quad\le C(n,\alpha)t^{-\frac{\alpha}{2}}\|a\|_{L^1(\mathbb{R}^n_+)},\quad 0<\alpha<n-1;
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}x_n^{-\alpha}|[e^{-tA}a](x)|dx\\
&\quad\le\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}x_n^{-\alpha}|G_t(x-y)-G_t(x+y^*)||a(y)|dxdy\\
&\qquad+C\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}x_n^{-\alpha}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n}|a(y)|dxdy\\
&\quad\le C(n,\alpha)t^{-\frac{\alpha}{2}}\|a\|_{L^1(\mathbb{R}^n_+)},\quad 0<\alpha<1.
\end{aligned}
$$

□

## 3 Asymptotic behavior for the Navier–Stokes flows in $L^1(\mathbb{R}^n_+)$

Let $g = \mathcal{N} f$ denote the solution of the Neumann problem

$$\begin{cases} -\Delta g = f & \text{in } \mathbb{R}^n_+, \\ g(x) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty, \\ \partial_\nu g \,|_{\partial \mathbb{R}^n_+} = 0. \end{cases}$$

Then (see [13])

$$\mathcal{N} = \int_0^\infty F(\tau) d\tau, \tag{3.1}$$

where the operator $F(t)$ is defined by

$$[F(t) f](x) = \int_{\mathbb{R}^n_+} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y) dy,$$

and $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel in $\mathbb{R}^n$.

It follows from (3.1) that the solution $g = \mathcal{N} f$ can be written as follows

$$(\mathcal{N} f)(x) = \int_0^\infty \int_{\mathbb{R}^n_+} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y) dy dt. \tag{3.2}$$

In addition, there holds for any $u, v \in L^2_\sigma(\mathbb{R}^n_+) \bigcap H^1_0(\mathbb{R}^n_+)$ (see [13])

$$P(u \cdot \nabla) v = (u \cdot \nabla) v + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i v_j). \tag{3.3}$$

**Lemma 3.1** *Let $0 < \theta < 1$ and $0 \le \beta \le \alpha < 1$. Then for any $u, v \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$*

$$\begin{aligned} &\left\| \sum_{i,j=1}^n x_n^\alpha \nabla \mathcal{N} \partial_i \partial_j (u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} \\ &\quad \le C(\|u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} \\ &\qquad + \|y_n^{\alpha-\beta} u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta v\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)}); \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\left\| \sum_{i,j=1}^n x_n^\alpha \nabla^2 \mathcal{N} \partial_i \partial_j (u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} \\ &\quad \le C \Big( \|u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{L^2(\mathbb{R}^n_+)} \\ &\qquad + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} \Big) \\ &\qquad + \|y_n^{\alpha-\beta} u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta v\|_{L^2(\mathbb{R}^n_+)} \\ &\qquad + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{\alpha-\beta} \nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)}); \end{aligned} \tag{3.5}$$

*and*

$$\left\| \sum_{i,j=1}^n x_n^{-\theta} \nabla \mathcal{N} \partial_i \partial_j (u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)}$$

$$\leq C(\|u\|_{L^2(\mathbb{R}^n_+)}\|v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)}). \tag{3.6}$$

***Remark*** Replacing the weight $x_n^\alpha$ by $|x|^\alpha$ in (3.4) and (3.5) respectively, the following estimates hold (the proofs are similar to these of (3.4) and (3.5)) for any $u, v \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$

$$\begin{aligned}
&\left\|\sum_{i,j=1}^n |x|^\alpha \nabla\mathcal{N}\partial_i\partial_j(u_i v_j)\right\|_{L^1(\mathbb{R}^n_+)} \\
&\leq C(\|u\|_{L^2(\mathbb{R}^n_+)}\|v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^{\alpha-\beta}u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta v\|_{L^2(\mathbb{R}^n_+)} + \||y|^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)}); \\
&\left\|\sum_{i,j=1}^n |x|^\alpha \nabla^2\mathcal{N}\partial_i\partial_j(u_i u_j)\right\|_{L^1(\mathbb{R}^n_+)} \\
&\leq C\Big(\|u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^{\alpha-\beta}u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} + \||y|^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y|^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta \nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \||y|^{\alpha-\beta}\nabla^2 u\|_{L^2(\mathbb{R}^n_+)}\||y|^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)}\Big).
\end{aligned}$$

***Proof*** Denote the odd and even extensions of a function $f$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, respectively by

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ -f(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$f_*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ f(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Let $u, v \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$. Then from (3.2), one has for any $1 \leq k \leq n$

$$\begin{aligned}
&\left\|\sum_{i,j=1}^n x_n^\alpha \partial_k\mathcal{N}\partial_i\partial_j(u_i v_j)\right\|_{L^1(\mathbb{R}^n_+)} = \left\|\sum_{i,j=1}^n x_n^\alpha \partial_k \int_0^\infty F(\tau)\partial_i\partial_j(u_i v_j)d\tau\right\|_{L^1(\mathbb{R}^n_+)} \\
&= C\left\|\sum_{i,j=1}^n \theta(x_n)x_n^\alpha\partial_k\left(\int_0^1 + \int_1^\infty\right) G_\tau * [\partial_i\partial_j(u_i v_j)]_* d\tau\right\|_{L^1(\mathbb{R}^n)} \\
&\leq C\left\|\int_0^1\int_{\mathbb{R}^n} |x_n - y_n|^\alpha |\partial_k G_\tau(x-y)|\left|\left[\sum_{i,j=1}^n \partial_i\partial_j(u_i v_j)\right]_*(y)\right| dy d\tau\right\|_{L^1_x(\mathbb{R}^n)} \\
&\quad + C\left\|\int_0^1\int_{\mathbb{R}^n} |\partial_k G_\tau(x-y)||y_n|^\alpha\left|\left[\sum_{i,j=1}^n \partial_i\partial_j(u_i v_j)\right]_*(y)\right| dy d\tau\right\|_{L^1_x(\mathbb{R}^n)} \\
&\quad + C\sum_{i,j=1}^n\left\|\int_1^\infty\int_{\mathbb{R}^n} |x_n - y_n|^\alpha |\partial_k\partial_i\partial_j G_\tau(x-y)||w_{ij}(y)| dy d\tau\right\|_{L^1_x(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{i,j=1}^{n}\left\|\int_1^\infty\int_{\mathbb{R}^n}|\partial_k\partial_i\partial_j G_\tau(x-y)||y_n|^\alpha|w_{ij}(y)|dyd\tau\right\|_{L^1_x(\mathbb{R}^n)}\\
&=M_1+M_2+M_3+M_4,
\end{aligned}
\tag{3.7}
$$

where $\theta(x_n)=1$ if $x_n\geq 0$, $\theta(x_n)=0$ if $x_n<0$; $w_{ij}=(u_iv_j)_*$ if $1\leq i,j\leq n-1$ or $i=j=n$, $w_{in}=(u_iv_n)^*$ if $1\leq i\leq n-1$, $w_{nj}=(u_nv_j)^*$ if $1\leq j\leq n-1$.

Let $0\leq\beta\leq\alpha<1$. Then

$$
\begin{aligned}
M_1+M_2\leq\ & C\int_0^1\left\||x_n|^\alpha\partial_k G_\tau(x',x_n)\right\|_{L^1(\mathbb{R}^n)}d\tau\|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)}\\
&+C\int_0^1\|\partial_k G_\tau\|_{L^1(\mathbb{R}^n)}d\tau\|y_n^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\|y_n^{\beta}\nabla v\|_{L^2(\mathbb{R}^n_+)}\\
\leq\ & C\int_0^1\tau^{\frac{\alpha}{2}-\frac12}d\tau\|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)}\\
&+C\int_0^1\tau^{-\frac12}d\tau\|y_n^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\|y_n^{\beta}\nabla v\|_{L^2(\mathbb{R}^n_+)}\\
\leq\ & C\big(\|\nabla u\|_{L^2(\mathbb{R}^n_+)}\|\nabla v\|_{L^2(\mathbb{R}^n_+)}+\|y_n^{\alpha-\beta}\nabla u\|_{L^2(\mathbb{R}^n_+)}\|y_n^{\beta}\nabla v\|_{L^2(\mathbb{R}^n_+)}\big);
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
M_3+M_4\leq\ & C\sum_{i,j=1}^{n}\int_1^\infty\int_{\mathbb{R}^n}|x_n|^\alpha|\partial_k\partial_i\partial_j G_\tau(x',x_n)|dxd\tau\|w_{ij}\|_{L^1(\mathbb{R}^n_+)}\\
&+C\sum_{i,j=1}^{n}\int_1^\infty\int_{\mathbb{R}^n}|\partial_k\partial_i\partial_j G_\tau(x',x_n)|dxd\tau\|y_n^\alpha w_{ij}\|_{L^1(\mathbb{R}^n_+)}\\
\leq\ & C\int_1^\infty\tau^{\frac{\alpha}{2}-\frac32}d\tau\|w_{ij}\|_{L^1(\mathbb{R}^n_+)}+C\sum_{i,j=1}^{n}\int_1^\infty\tau^{-\frac32}d\tau\|y_n^\alpha w_{ij}\|_{L^1(\mathbb{R}^n_+)}\\
\leq\ & C\|u\|_{L^2(\mathbb{R}^n_+)}\|v\|_{L^2(\mathbb{R}^n_+)}+C\|y_n^{\alpha-\beta}u\|_{L^2(\mathbb{R}^n_+)}\|y_n^{\beta}v\|_{L^2(\mathbb{R}^n_+)}.
\end{aligned}
\tag{3.9}
$$

From (3.7)–(3.9), we infer that (3.4) holds.

Now we give the proof of (3.5). Let $0\leq\beta\leq\alpha<1$, then for any $1\leq k\leq n$

$$
\begin{aligned}
&\left\|\sum_{i,j=1}^{n}x_n^\alpha\nabla\partial_k\mathcal{N}\partial_i\partial_j(u_iv_j)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\leq C\left\|\int_0^1\int_{\mathbb{R}^n}|x_n-y_n|^\alpha|\partial_k G_\tau(x-y)||\nabla\left[\sum_{i,j=1}^{n}\partial_i\partial_j(u_iv_j)\right]_*(y)|dyd\tau\right\|_{L^1_x(\mathbb{R}^n)}\\
&\quad+C\left\|\int_0^1\int_{\mathbb{R}^n}|\partial_k G_\tau(x-y)||y_n|^\alpha|\nabla\left[\sum_{i,j=1}^{n}\partial_i\partial_j(u_iv_j)\right]_*(y)|dyd\tau\right\|_{L^1_x(\mathbb{R}^n)}\\
&\quad+C\sum_{i,j=1}^{n}\left\|\int_1^\infty\int_{\mathbb{R}^n}|x_n-y_n|^\alpha|\partial_k\partial_i\partial_j G_\tau(x-y)||\nabla w_{ij}(y)|dyd\tau\right\|_{L^1_x(\mathbb{R}^n)}\\
&\quad+C\sum_{i,j=1}^{n}\left\|\int_1^\infty\int_{\mathbb{R}^n}|\partial_k\partial_i\partial_j G_\tau(x-y)||y_n|^\alpha|\nabla w_{ij}(y)|dyd\tau\right\|_{L^1_x(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&= N_1 + N_2 + N_3 + N_4; \qquad (3.10)\\
N_1 + N_2 &\le C \int_0^1 \| |x_n|^\alpha \partial_k G_\tau(x', x_n)\|_{L^1(\mathbb{R}^n)} d\tau \\
&\quad \times (\|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)}) \\
&\quad + C \int_0^1 \|\partial_k G_\tau\|_{L^1(\mathbb{R}^n)} d\tau (\|y_n^{\alpha-\beta} \nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla^2 v\|_{L^2(\mathbb{R}^n_+)}) \\
&\le C \int_0^1 \tau^{\frac{\alpha}{2}-\frac{1}{2}} d\tau (\|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)}) \\
&\quad + C \int_0^1 \tau^{-\frac{1}{2}} d\tau (\|y_n^{\alpha-\beta} \nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla^2 v\|_{L^2(\mathbb{R}^n_+)}) \\
&\le C\big(\|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla^2 v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|y_n^{\alpha-\beta} \nabla^2 u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla^2 v\|_{L^2(\mathbb{R}^n_+)}\big); \qquad (3.11)\\
N_3 + N_4 &\le C \sum_{i,j=1}^n \int_1^\infty \int_{\mathbb{R}^n} |x_n|^\alpha |\partial_k \partial_i \partial_j G_\tau(x', x_n)| dx d\tau \|\nabla w_{ij}\|_{L^1(\mathbb{R}^n_+)} \\
&\qquad + C \sum_{i,j=1}^n \int_1^\infty \int_{\mathbb{R}^n} |\partial_k \partial_i \partial_j G_\tau(x', x_n)| dx d\tau \|y_n^\alpha \nabla w_{ij}\|_{L^1(\mathbb{R}^n_+)} \\
&\le C \sum_{i,j=1}^n \left( \int_1^\infty \tau^{\frac{\alpha}{2}-\frac{3}{2}} d\tau \|\nabla w_{ij}\|_{L^1(\mathbb{R}^n_+)} + \int_1^\infty \tau^{-\frac{3}{2}} d\tau \|y_n^\alpha \nabla w_{ij}\|_{L^1(\mathbb{R}^n_+)} \right) \\
&\le C(\|u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + \|y_n^{\alpha-\beta} u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta \nabla v\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{\alpha-\beta} \nabla u\|_{L^2(\mathbb{R}^n_+)} \|y_n^\beta v\|_{L^2(\mathbb{R}^n_+)}). \qquad (3.12)
\end{aligned}
$$

From (3.10)–(3.12), we conclude that (3.5) holds. Next we show the validity of (3.6).

Let $u, v \in C_{0,\sigma}^\infty(\mathbb{R}^n_+)$. Then from (3.2), one has for any $1 \le k \le n$

$$
\begin{aligned}
&\left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \mathcal{N} \partial_i \partial_j (u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} \\
&\quad \le \left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \int_0^1 G_\tau * [\partial_i \partial_j (u_i v_j)]_* d\tau \right\|_{L^1(\mathbb{R}^{n-1} \times (0,1))} \\
&\qquad + \| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \int_1^\infty G_\tau * [\partial_i \partial_j (u_i v_j)]_* d\tau \|_{L^1(\mathbb{R}^{n-1} \times (0,1))} \\
&\qquad + \left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C \sup_{y\in\mathbb{R}^n} \left\| \int_0^1 x_n^{-\theta} \partial_k G_\tau(x-y) d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \left\| \sum_{i,j=1}^n \partial_i\partial_j(u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + C \sup_{y\in\mathbb{R}^n} \sum_{i,j=1}^n \left\| \int_1^\infty x_n^{-\theta} \partial_k\partial_i\partial_j G_\tau(x-y) d\tau \|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \right\| w_{ij} \|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i\partial_j(u_i u_j) \right\|_{L^1(\mathbb{R}^n_+)} . \qquad (3.13)
\end{aligned}
$$

Let $1 \le i, j, k \le n$. $0 < \theta < 1$. Then for any $y = (y', y_n) \in \mathbb{R}^n$

$$
\begin{aligned}
&\left\| \int_0^1 x_n^{-\theta} \partial_k G_\tau(x-y) d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \\
&\le \int_0^1 \int_0^1 \int_{\mathbb{R}^{n-1}} (4\pi\tau)^{-\frac{n}{2}} \tau^{-\frac{1}{2}} x_n^{-\theta} \frac{|x_k - y_k|}{2\sqrt{\tau}} e^{-\frac{|x-y|^2}{4\tau}} dx' dx_n d\tau \\
&\le C \int_0^1 \int_0^1 \tau^{-1} x_n^{-\theta} e^{-\frac{(x_n-y_n)^2}{8\tau}} dx_n d\tau \\
&\le C \int_0^1 \tau^{-1} \left( \int_0^1 x_n^{-\theta(1+\eta_0)} dx_n \right)^{\frac{1}{1+\eta_0}} \left( \int_0^1 e^{-\frac{(1+\eta_0)(x_n-y_n)^2}{8\eta_0\tau}} dx_n \right)^{\frac{\eta_0}{1+\eta_0}} d\tau \\
&\le C \int_0^1 \tau^{-1+\frac{\eta_0}{2(1+\eta_0)}} d\tau \le C, \quad \text{where } 0 < \eta_0 < \frac{1}{\theta} - 1; \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \int_1^\infty x_n^{-\theta} \partial_k\partial_i\partial_j G_\tau(x-y) d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \\
&\le \int_0^1 \int_1^\infty \tau^{-\frac{3}{2}} \int_{\mathbb{R}^{n-1}} x_n^{-\theta} (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2+|x_n-y_n|^2}{8\tau}} dx' dx_n d\tau \\
&\le C \int_1^\infty \int_0^1 \tau^{-2} x_n^{-\theta} e^{-\frac{(x_n-y_n)^2}{8\tau}} dx_n d\tau \\
&\le C \int_1^\infty \tau^{-2} d\tau \int_0^1 x_n^{-\theta} dx_n \le C. \qquad (3.15)
\end{aligned}
$$

In addition, it follows from (3.4) with $\alpha = \beta = 0$ that

$$
\begin{aligned}
&\left\| \sum_{i,j=1}^n \partial_i\partial_j(u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} + \sum_{i,j=1}^n \|w_{ij}\|_{L^1(\mathbb{R}^n_+)} \\
&\le C\big( \|u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} \big). \qquad (3.16)
\end{aligned}
$$

From (3.13)–(3.16), we conclude that for $0 < \theta < 1$, $1 \le k \le n$

$$
\left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \mathcal{N} \partial_i\partial_j(u_i v_j) \right\|_{L^1(\mathbb{R}^n_+)} \le C\big( \|u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} \big),
$$

which is (3.6). □

**Lemma 3.2** [12–14]. *Assume $u_0 \in L^2_\sigma(\mathbb{R}^n_+) \bigcap L^n(\mathbb{R}^n_+)$ $(n \geq 2)$. Then the strong solution $u$ of (1.1) given in Theorem 1.1 satisfies for $t > 1$*

$$\|\nabla^m u(t)\|_{L^2(\mathbb{R}^n_+)} \leq C(1+t)^{-\frac{m}{2}-\frac{n}{4}}, \quad m = 0, 1, 2, 3;$$
$$\|\partial_t \nabla^\ell u(t)\|_{L^2(\mathbb{R}^n_+)} \leq Ct^{-1-\frac{\ell}{2}-\frac{n}{4}}, \ \ell = 0, 1.$$

*Furthermore, if $u_0$ satisfies*

$$\|x_n u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+x_n)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+x_n)u_0\|_{L^1(\mathbb{R}^n_+)} < \infty.$$

*Then following estimates are true for the strong solution $u$ with $t > 1$*

$$\|\nabla^m u(t)\|_{L^2(\mathbb{R}^n_+)} \leq Ct^{-\frac{m+1}{2}-\frac{n}{4}}, \quad m = 0, 1, 2, 3;$$
$$\|\partial_t \nabla^\ell u(t)\|_{L^2(\mathbb{R}^n_+)} \leq Ct^{-\frac{3}{2}-\frac{\ell}{2}-\frac{n}{4}}, \ \ell = 0, 1;$$

*and for $0 < \gamma < 1$ and $t > 1$*

$$\|x_n^\gamma \nabla^\ell u(t)\|_{L^2(\mathbb{R}^n_+)} \leq Ct^{-\frac{\ell}{2}-\frac{n}{4}+\frac{\gamma}{2}}, \ \ell = 0, 1.$$

***Proof of Theorem 1.3.*** The strong solution $u$ of problem (1.1), which is given in Theorem 1.1, can be represented as follows

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} Pu(s) \cdot \nabla u(s) ds.$$

It follows from Theorem 1.2, Lemma 3.2 that for $0 < \alpha < \min\{2, n-1\}$ and $t > 0$

$$\begin{aligned}
\||x'|^{-\alpha} u(t)\|_{L^1(\mathbb{R}^n_+)} &\leq \||x'|^{-\alpha} e^{-tA} u_0\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \int_0^t \||x'|^{-\alpha} e^{-(t-s)A} P(u(s)\cdot\nabla)u(s)\|_{L^1(\mathbb{R}^n_+)} ds \\
&\leq Ct^{-\frac{\alpha}{2}} \|u_0\|_{L^1(\mathbb{R}^n_+)} + C\int_0^t (t-s)^{-\frac{\alpha}{2}} \|P(u(s)\cdot\nabla)u(s)\|_{L^1(\mathbb{R}^n_+)} ds \\
&\leq Ct^{-\frac{\alpha}{2}} \|u_0\|_{L^1(\mathbb{R}^n_+)} + C\int_0^t (t-s)^{-\frac{\alpha}{2}} \Big( \|(u(s)\cdot\nabla)u(s)\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \left\| \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^1(\mathbb{R}^n_+)} \Big) ds \\
&\leq Ct^{-\frac{\alpha}{2}} + C\left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (t-s)^{-\frac{\alpha}{2}} \left( \|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \right) ds \\
&\leq Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\alpha}{2}} \int_0^{\frac{t}{2}} (1+s)^{-\frac{n}{2}} ds + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha}{2}} (1+s)^{-\frac{n}{2}} ds \\
&\leq \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{if } n \geq 3, \\ Ct^{-\frac{\alpha}{2}} \log_e(1+t) & \text{if } n = 2. \end{cases}
\end{aligned}$$

Similarly, we have for $0 < \alpha < 1$ and $t > 0$

$$
\begin{aligned}
\|x_n^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} &\le \|x_n^{-\alpha}e^{-tA}u_0\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \int_0^t \|x_n^{-\alpha}e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\le Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\alpha}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2}}ds + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha}{2}}(1+s)^{-\frac{n}{2}}ds \\
&\le \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{if } n\ge 3,\\ Ct^{-\frac{\alpha}{2}}\log_e(1+t) & \text{if } n=2.\end{cases}
\end{aligned}
$$

Now suppose $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Using Theorem 1.2, Lemma 3.2, we derive for $t>0$

$$
\begin{aligned}
\||x'|^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} &\le Ct^{-\frac{\alpha}{2}} + C\int_0^t (t-s)^{-\frac{\alpha}{2}}\left(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}\right)ds \\
&\le Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\alpha}{2}}\int_0^{\frac{t}{2}}(1+s)^{-1-\frac{n}{2}}ds + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha}{2}}(1+s)^{-1-\frac{n}{2}}ds \\
&\le Ct^{-\frac{\alpha}{2}}, \quad 0<\alpha<\min\{2,n-1\};
\end{aligned}
$$

and similarly

$$
\|x_n^{-\alpha}u(t)\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-\frac{\alpha}{2}}, \quad 0<\alpha<1.
$$

□

## 4 $L^1$−behavior for cubic spatial derivatives of Navier–Stokes flows

The first preliminary result is an important identity, which plays a crucial role in avoiding the strong singularity in the proof of Theorem 1.4.

Set $\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \lim_{|x|\to\infty} |x^\eta \nabla^\gamma f(x)| = 0$ for any multi index $\eta,\gamma\}$, where $x^\eta = x_1^{\eta_1}x_2^{\eta_2}\cdots x_n^{\eta_n}$, $\nabla^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}\cdots\partial_n^{\gamma_n}$, $\eta=(\eta_1,\eta_2,\cdots,\eta_n)$, $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ are multi indexes.

**Lemma 4.1** *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$ $(n\ge 2)$. Then*

$$
\sum_{i=1}^n \frac{\partial}{\partial x_i}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} g(y)\frac{\partial E(x-y)}{\partial x_i}dy = \frac{1}{2}g(x), \qquad \forall g\in\mathscr{S}(\mathbb{R}^n),
$$

*where $E$ is the fundamental solution of the elliptic operator $-\Delta$ in $\mathbb{R}^n$, its specific expression is given in Sect. 2.*

***Proof*** Observe that for every $h\in\mathscr{S}(\mathbb{R}^n)$ and $x=(x',x_n)\in\mathbb{R}^n_+$,

$$
\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-y)}{\partial x_i}\frac{\partial h(y)}{\partial y_n}dy = \int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_i}\frac{\partial h(x-y)}{\partial x_n}dy, \quad 1\le i\le n; \tag{4.1}
$$

and $0<\epsilon<x_n$,

$$
\int_\epsilon^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_n}\frac{\partial h(x-y)}{\partial x_n}dy = -\int_\epsilon^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_n}\frac{\partial h(x-y)}{\partial y_n}dy
$$

$$
\begin{aligned}
&= \int_{\epsilon}^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2 E(y)}{\partial y_n^2} h(x-y)dy \\
&\quad - \int_{\mathbb{R}^{n-1}} \frac{\partial E(y', x_n)}{\partial x_n} h(x'-y', 0)dy' \\
&\quad + \int_{\mathbb{R}^{n-1}} \frac{\partial E(y', \epsilon)}{\partial y_n} h(x'-y', x_n-\epsilon)dy'. \qquad (4.2)
\end{aligned}
$$

In addition, it holds for $\epsilon > 0$ and $n \geq 2$,

$$
\frac{\partial E(y', \epsilon)}{\partial y_n} = \frac{\partial E(y', y_n)}{\partial y_n}\Big|_{y_n=\epsilon} = \frac{1}{n\omega_n} \frac{\epsilon}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}}. \qquad (4.3)
$$

Let $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})$ and $\eta > 0$ be small enough. Then

$$
\varphi(y') = \varphi(0') + y' \cdot \nabla' \varphi(0') + O(|y'|^2), \quad \forall |y'| < \eta.
$$

Whence for $\epsilon > 0$ and $n \geq 2$,

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{\epsilon\varphi(y')}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' &= \int_{|y'|\geq\eta} \frac{\epsilon\varphi(y')}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' + \int_{|y'|<\eta} \frac{\epsilon\varphi(0')}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' \\
&\quad + \int_{|y'|<\eta} \frac{\epsilon y' \cdot \nabla'\varphi(0')}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' + O(1) \int_{|y'|<\eta} \frac{\epsilon|y'|^2}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' \\
&= \sum_{j=1}^{4} I_j(\epsilon). \qquad (4.4)
\end{aligned}
$$

Now we calculate and estimate each term $I_j(\epsilon)$, $j = 1, 2, 3, 4$.

$$
\begin{aligned}
|I_1(\epsilon)| &\leq \eta^{-n}\epsilon \int_{\mathbb{R}^{n-1}} |\varphi(y')|dy' \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0; \qquad (4.5) \\
|I_3(\epsilon)| &\leq \epsilon|\nabla'\varphi(0')| \int_{|y'|<\eta} \frac{|y'|dy'}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} \\
&\leq C\epsilon|\nabla'\varphi(0')| \int_0^{\eta} \frac{s^{n-1}ds}{(s^2+\epsilon^2)^{\frac{n}{2}}} \\
&\leq C\epsilon|\nabla'\varphi(0')| \int_0^{\eta} (s^2+\epsilon^2)^{-\frac{1}{2}} ds \\
&\leq C|\nabla'\varphi(0')|\epsilon \log_e(1+\epsilon^{-1}\eta) \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0; \qquad (4.6) \\
|I_4(\epsilon)| &\leq C\epsilon| \int_{|y'|<\eta} \frac{|y'|^2dy'}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} \\
&\leq C\epsilon \int_0^{\eta} \frac{s^n ds}{(s^2+\epsilon^2)^{\frac{n}{2}}} \\
&\leq C\eta\epsilon \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0; \qquad (4.7) \\
I_2(\epsilon) &= \varphi(0') \int_{|y'|<\eta} \frac{\epsilon dy'}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} \\
&= \varphi(0') \int_{|y'|<\eta} \frac{\epsilon^{1-n}dy'}{(|\epsilon^{-1}y'|^2+1)^{\frac{n}{2}}} \\
&= \varphi(0') \int_{|z'|<\frac{\eta}{\epsilon}} \frac{dz'}{(|z'|^2+1)^{\frac{n}{2}}}
\end{aligned}
$$

$$\longrightarrow \varphi(0') \int_{\mathbb{R}^{n-1}} \frac{dz'}{(|z'|^2+1)^{\frac{n}{2}}} \quad \text{as} \quad \epsilon \longrightarrow 0. \tag{4.8}$$

Note that the volume $\omega_m$ of the unit ball in $\mathbb{R}^m$ $(m \geq 1)$ is expressed by $\omega_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(1+\frac{m}{2})}$. Set $m = n-1$, then

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{dz'}{(|z'|^2+1)^{\frac{n}{2}}} &= (n-1)\omega_{n-1} \int_0^\infty \frac{s^{n-2}ds}{(s^2+1)^{\frac{n}{2}}} \\ &= \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\left(1+\frac{n-1}{2}\right)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} d\theta \\ &= \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned} \tag{4.9}$$

In the proof of (4.9), we used the known result: let $m \geq 0$ be an integer, then

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^m d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta)^m d\theta = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)}.$$

From (4.4)-(4.9), we find

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^{n-1}} \frac{\epsilon\varphi(y')}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} dy' = \left\langle \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\delta(y'), \varphi(y') \right\rangle,$$

which, together with (4.3) and $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ implies

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{\partial E(y',\epsilon)}{\partial y_n} &= \lim_{\epsilon\to 0} \frac{1}{n\omega_n} \frac{\epsilon}{(|y'|^2+\epsilon^2)^{\frac{n}{2}}} \\ &= \frac{\Gamma\left(1+\frac{n}{2}\right)}{n\pi^{\frac{n}{2}}} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \delta(y') \\ &= \frac{1}{2}\delta(y') \quad \text{in the sense of the distribution.} \end{aligned} \tag{4.10}$$

In addition, there holds that for any $x = (x', x_n) \in \mathbb{R}^n_+$, $x_n > \epsilon$,

$$\int_\epsilon^{x_n} \int_{\mathbb{R}^{n-1}} h(x-y) \frac{\partial^2 E(y)}{\partial y_n^2} dy = -\int_\epsilon^{x_n} \int_{\mathbb{R}^{n-1}} h(x-y) \sum_{j=1}^{n-1} \frac{\partial^2 E(y)}{\partial y_j^2} dy. \tag{4.11}$$

Combining (4.2), (4.10) and (4.11) yields

$$\begin{aligned} &\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_n} \frac{\partial h(x-y)}{\partial x_n} dy \\ &= \lim_{\epsilon\to 0} \int_\epsilon^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_n} \frac{\partial h(x-y)}{\partial x_n} dy \\ &= -\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} E(y)h(x-y)dy \end{aligned}$$

$$
-\int_{\mathbb{R}^{n-1}} \frac{\partial E(y', x_n)}{\partial x_n} h(x'-y', 0)dy' + \frac{1}{2}h(x', x_n). \tag{4.12}
$$

Differentiating with respect to $x_n > 0$ in (4.12), we get

$$
\begin{aligned}
&\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial y_n}\frac{\partial h(x-y)}{\partial x_n}dy \\
&= -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} E(y)\frac{\partial h(x-y)}{\partial x_n}dy + \frac{1}{2}\frac{\partial h(x', x_n)}{\partial x_n} \\
&\quad -\int_{\mathbb{R}^{n-1}} h(x'-y', 0)\left(\sum_{j=1}^{n-1}\frac{\partial^2}{\partial y_j^2}E(y', x_n) + \frac{\partial^2 E(y', x_n)}{\partial x_n^2}\right)dy' \\
&= -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} E(y)\frac{\partial h(x-y)}{\partial x_n}dy + \frac{1}{2}\frac{\partial h(x', x_n)}{\partial x_n}.
\end{aligned} \tag{4.13}
$$

Combining (4.1) and (4.13) yields

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-y)}{\partial x_i}\frac{\partial h(y)}{\partial y_n}dy \\
&= \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} E(y)\frac{\partial h(x-y)}{\partial x_n}dy + \frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(y)}{\partial x_n}\frac{\partial h(x-y)}{\partial x_n}dy \\
&= \frac{1}{2}\frac{\partial h(x', x_n)}{\partial x_n}.
\end{aligned} \tag{4.14}
$$

Let $g \in \mathscr{S}(\mathbb{R}^n)$, take $h(x) = \int_0^{x_n} g(x', t)dt$ in (4.14), we complete the proof of Lemma 4.1. □

**Lemma 4.2** *Let $1 \le j, \ell \le n-1$, $1 \le k \le n$, $n \ge 2$. Then for any $t > 0$,*

$$
\begin{aligned}
&\left|\frac{\partial^2}{\partial x_j \partial x_k}\int_{\mathbb{R}^{n-1}} G_t(x'-z', x_n)\nabla E(z', z_n)dz'\right| \\
&\le Ct^{-\frac{1+\delta_{kn}}{2}}\left(|x'| + x_n + z_n + \sqrt{t}\right)^{-n-1+\delta_{kn}} e^{-\frac{x_n^2}{64}}, \quad \forall x' \in \mathbb{R}^{n-1}, \quad x_n, \ z_n > 0;
\end{aligned} \tag{4.15}
$$

*and*

$$
\begin{aligned}
&\left|\frac{\partial^3}{\partial x_j \partial x_\ell \partial x_n}\int_{\mathbb{R}^{n-1}} G_t(x'-z', x_n)\nabla E(z', z_n)dz'\right| \\
&\le Ct^{-1}\left(|x'| + x_n + z_n + \sqrt{t}\right)^{-n-1} e^{-\frac{x_n^2}{64}}, \quad \forall x' \in \mathbb{R}^{n-1}, \quad x_n, \ z_n > 0.
\end{aligned} \tag{4.16}
$$

***Proof*** Let $x = (x', x_n) \in \mathbb{R}^n_+$ $(n \ge 2)$, $z_n > 0$ and set

$$
J_i(x, z_n, t) = \int_{\mathbb{R}^{n-1}} G_t(x'-z', x_n)\frac{\partial E(z', z_n)}{\partial z_i}dz', \quad 1 \le i \le n.
$$

Observe that for any $\lambda, t > 0$

$$
J_i(\lambda x, \lambda z_n, \lambda^2 t) = \int_{\mathbb{R}^{n-1}} G_{\lambda^2 t}(\lambda x' - z', \lambda x_n)\frac{\partial E(z', \lambda z_n)}{\partial z_i}dz'
$$

$$= \lambda^{-n} \int_{\mathbb{R}^{n-1}} G_t(x'-z', x_n) \frac{\partial E(z', z_n)}{\partial z_i} dz'$$
$$= \lambda^{-n} J_i(x, z_n, t), \quad 1 \le i \le n. \tag{4.17}$$

A direct calculation shows that for $z_n > 0$,

$$\int_{|z'|<1} \big|E(z', z_n)\big| dz' \le C, \quad \text{where } C \text{ is independent of } z_n.$$

Whence for $1 \le i, j \le n-1$, $1 \le k \le n$,

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial x_j \partial x_k} J_i(x, z_n, 1)\Big| &= \Big|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \int_{\mathbb{R}^{n-1}} G_1(x'-z', x_n) E(z) dz'\Big| \\
&\le \int_{\mathbb{R}^{n-1}} \Big|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} G_1(x'-z', x_n)\Big| \big|E(z', z_n)\big| dz' \\
&\le Ce^{-\frac{x_n^2}{8}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-z'|^2}{8}} \big|E(z', z_n)\big| dz' \\
&\le Ce^{-\frac{x_n^2}{8}} \Big(\int_{|z'|<1} \big|E(z', z_n)\big| dz' + \int_{|z'|\ge 1} e^{-\frac{|x'-z'|^2}{8}} dz'\Big) \\
&\le Ce^{-\frac{x_n^2}{8}}, \quad \forall x = (x', x_n) \in \mathbb{R}^n_+, \quad z_n > 0;
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial x_j \partial x_k} J_n(x, z_n, 1)\Big| &\le \int_{\mathbb{R}^{n-1}} \Big|\frac{\partial^2}{\partial x_j \partial x_k} G_1(x'-z', x_n)\Big| \Big|\frac{\partial E(z', z_n)}{\partial z_n}\Big| dz' \\
&\le Ce^{-\frac{x_n^2}{8}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-z'|^2}{8}} \Big|\frac{\partial E(z', z_n)}{\partial z_n}\Big| dz' \\
&\le Ce^{-\frac{x_n^2}{8}} \Big(\int_{|z'|<1} \Big|\frac{\partial E(z', z_n)}{\partial z_n}\Big| dz' + \int_{|z'|\ge 1} e^{-\frac{|x'-z'|^2}{8}} dz'\Big) \\
&\le Ce^{-\frac{x_n^2}{8}} \Big(z_n \int_0^1 (s+z_n)^{-n} s^{n-2} ds + \int_{\mathbb{R}^{n-1}} e^{-\frac{|z'|^2}{8}} dz'\Big) \\
&\le Ce^{-\frac{x_n^2}{8}}.
\end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) yields for $1 \le j \le n-1$, $1 \le i, k \le n$

$$\Big|\frac{\partial^2}{\partial x_j \partial x_k} J_i(x, z_n, 1)\Big| \le Ce^{-\frac{x_n^2}{8}}, \quad \forall x = (x', x_n) \in \mathbb{R}^n_+, \quad z_n > 0. \tag{4.20}$$

Similar to the proofs of (4.18) and (4.19), we find for $1 \le j, \ell \le n-1$, $1 \le i \le n$

$$\Big|\frac{\partial^3}{\partial x_j \partial x_\ell \partial x_n} J_i(x, z_n, 1)\Big| \le Ce^{-\frac{x_n^2}{16}}, \quad \forall x = (x', x_n) \in \mathbb{R}^n_+, \quad z_n > 0. \tag{4.21}$$

Let $x = (x', x_n) \in \mathbb{R}^n_+$, $z_n > 0$, and set

$$\Pi_\rho(x) = \big\{z' \in \mathbb{R}^{n-1} : \ |x'-z'|^2 + x_n^2 \le \frac{\rho^2}{4}\big\}, \quad \text{where } \rho = \sqrt{|x'|^2 + (x_n+z_n)^2}.$$

Using the triangle inequality yields for any $z' \in \Pi_\rho(x)$, $x \in \mathbb{R}^n_+$, $z_n > 0$,

$$\rho = \sqrt{|x'|^2 + (x_n+z_n)^2} \le \sqrt{|x'-z'|^2 + x_n^2} + \sqrt{|z'|^2 + z_n^2} \le \frac{\rho}{2} + |z| \Longrightarrow |z| \ge \frac{\rho}{2}.$$

Then for $1 \le j, k \le n-1$ and $x = (x', x_n) \in \mathbb{R}^n_+, \quad z_n > 0$,

$$
\begin{aligned}
&\Big|\frac{\partial^2}{\partial x_j \partial x_k} J_n(x, z_n, 1)\Big| \\
&= \Big|\int_{\mathbb{R}^{n-1}} G_1(x'-z', x_n)\frac{\partial^3 E(z)}{\partial z_j \partial z_k \partial z_n} dz'\Big| \\
&= \Big|\int_{\Pi_\rho(x)} G_1(x'-z', x_n)\frac{\partial^3 E(z)}{\partial z_j \partial z_k \partial z_n} dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} G_1(x'-z', x_n)\frac{\partial^3 E(z)}{\partial z_j \partial z_k \partial z_n}\Big| \\
&\le \int_{\Pi_\rho(x)} G_1(x'-z', x_n)\Big|\frac{\partial^3 E(z)}{\partial z_j \partial z_k \partial z_n}\Big| dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} \Big|\frac{\partial^2 G_1(x'-z', x_n)}{\partial x_j \partial x_k}\Big|\Big|\frac{\partial E(z)}{\partial z_n}\Big| dz' \\
&\quad + \int_{\partial\Pi_\rho(x)} G_1(x'-z', x_n)\Big|\frac{\partial^2 E(z)}{\partial z_k \partial z_n}\Big| dS_{z'} \\
&\quad + \int_{\partial\Pi_\rho(x)} \Big|\frac{\partial G_1(x'-z', x_n)}{\partial x_j}\Big|\Big|\frac{\partial E(z)}{\partial z_n}\Big| dS_{z'} \\
&\le C\left(\int_{\Pi_\rho(x)} |z|^{-n-1} e^{-\frac{|x'-z'|^2+x_n^2}{4}} dz'\right. \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}} e^{-\frac{|x'-z'|^2+x_n^2}{8}} dz' \\
&\quad \left. + e^{-\frac{\rho^2}{16}} \int_{\partial\Pi_\rho(x)} |z|^{-n} dS_{z'} + e^{-\frac{\rho^2}{32}} \int_{\partial\Pi_\rho(x)} |z|^{-n+1} dS_{z'}\right) \\
&\le C\Big(\rho^{-n-1} e^{-\frac{x_n^2}{4}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-z'|^2}{4}} dz' \\
&\quad + e^{-\frac{\rho^2}{64}} \int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}} e^{-\frac{|x'-z'|^2+x_n^2}{16}} dz' \\
&\quad + e^{-\frac{\rho^2}{32}} \rho^{n-2}(\rho^{-n}+\rho^{-n+1})\Big) \\
&\le C\big(\rho^{-n-1} e^{-\frac{x_n^2}{4}} + e^{-\frac{\rho^2}{32}}(\rho^{-2}+\rho^{-1})\big) \\
&\quad + Ce^{-\frac{\rho^2}{64}}\left(\int_{\int_{|z'|<1}} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}} dz' + \int_{|z'|\ge 1} e^{-\frac{|x'-z'|^2}{16}} dz'\right) \\
&\le C\left(\rho^{-n-1} e^{-\frac{x_n^2}{4}} + e^{-\frac{\rho^2}{32}}(\rho^{-2}+\rho^{-1}) + e^{-\frac{\rho^2}{64}}\right) \\
&\le Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1});
\end{aligned} \tag{4.22}
$$

and for $1 \le j \le n-1$

$$
\begin{aligned}
\Big|\frac{\partial^2}{\partial x_j \partial x_n} J_n(x, z_n, 1)\Big| &= \Big| \int_{\mathbb{R}^{n-1}} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^2 E(z)}{\partial z_j \partial z_n} dz' \Big| \\
&= \Big| \int_{\Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^2 E(z)}{\partial z_j \partial z_n} dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus \Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^2 E(z)}{\partial z_j \partial z_n} \Big| \\
&\le \int_{\Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \Big| \frac{\partial^2 E(z)}{\partial z_j \partial z_n} \Big| dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus \Pi_\rho(x)} \Big| \frac{\partial^2 G_1(x'-z', x_n)}{\partial x_j \partial x_n} \Big| \Big| \frac{\partial E(z)}{\partial z_n} \Big| dz' \\
&\quad + \int_{\partial \Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \Big| \frac{\partial E(z)}{\partial z_n} \Big| dS_{z'} \\
&\le C \Bigg( \int_{\Pi_\rho(x)} |z|^{-n} e^{-\frac{|x'-z'|^2 + x_n^2}{8}} dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus \Pi_\rho(x)} z_n (|z'|^2 + z_n^2)^{-\frac{n}{2}} e^{-\frac{|x'-z'|^2 + x_n^2}{8}} dz' \\
&\quad + e^{-\frac{\rho^2}{32}} \int_{\partial \Pi_\rho(x)} |z|^{-n+1} dS_{z'} \Bigg) \\
&\le C \Big( \rho^{-n} e^{-\frac{x_n^2}{8}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-z'|^2}{8}} dz' \\
&\quad + e^{-\frac{\rho^2}{64}} \int_{\mathbb{R}^{n-1}\setminus \Pi_\rho(x)} z_n (|z'|^2 + z_n^2)^{-\frac{n}{2}} e^{-\frac{|x'-z'|^2 + x_n^2}{16}} dz' \\
&\quad + e^{-\frac{\rho^2}{32}} \rho^{n-2} \rho^{-n+1} \Big) \\
&\le C \left( \rho^{-n} e^{-\frac{x_n^2}{8}} + e^{-\frac{\rho^2}{32}} \rho^{-1} + e^{-\frac{\rho^2}{64}} \right) \\
&\le C e^{-\frac{x_n^2}{64}} (1 + \rho^{-n}).
\end{aligned}
\tag{4.23}
$$

In addition, for $1 \le j, \ell \le n-1$

$$
\begin{aligned}
&\Big| \frac{\partial^3}{\partial x_j \partial x_\ell \partial x_n} J_n(x, z_n, 1) \Big| \\
&= \Big| \int_{\mathbb{R}^{n-1}} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^3 E(z)}{\partial z_j \partial z_\ell \partial z_n} dz' \Big| \\
&= \Big| \int_{\Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^3 E(z)}{\partial z_j \partial z_\ell \partial z_n} dz' \\
&\quad + \int_{\mathbb{R}^{n-1}\setminus \Pi_\rho(x)} \frac{\partial G_1(x'-z', x_n)}{\partial x_n} \frac{\partial^3 E(z)}{\partial z_j \partial z_\ell \partial z_n} \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{\Pi_\rho(x)} \left|\frac{\partial G_1(x'-z',x_n)}{\partial x_n}\right|\left|\frac{\partial^3 E(z)}{\partial z_j\partial z_\ell\partial z_n}\right|dz' \\
&\quad+\int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} \left|\frac{\partial^3 G_1(x'-z',x_n)}{\partial x_j\partial x_\ell\partial x_n}\right|\left|\frac{\partial E(z)}{\partial z_n}\right|dz' \\
&\quad+\int_{\partial\Pi_\rho(x)} \left|\frac{\partial G_1(x'-z',x_n)}{\partial x_n}\right|\left|\frac{\partial^2 E(z)}{\partial z_\ell\partial z_n}\right|dS_{z'} \\
&\quad+\int_{\partial\Pi_\rho(x)} \left|\frac{\partial^2 G_1(x'-z',x_n)}{\partial x_j\partial x_n}\right|\left|\frac{\partial E(z)}{\partial z_n}\right|dS_{z'} \\
&\le C\Bigg(\int_{\Pi_\rho(x)} |z|^{-n-1}e^{-\frac{|x'-z'|^2+x_n^2}{8}}dz' \\
&\quad+\int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}}e^{-\frac{|x'-z'|^2+x_n^2}{8}}dz' \\
&\quad+e^{-\frac{\rho^2}{32}}\int_{\partial\Pi_\rho(x)}|z|^{-n}dS_{z'}+e^{-\frac{\rho^2}{32}}\int_{\partial\Pi_\rho(x)}|z|^{-n+1}dS_{z'}\Bigg) \\
&\le C\Big(\rho^{-n-1}e^{-\frac{x_n^2}{8}}\int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-z'|^2}{8}}dz' \\
&\quad+e^{-\frac{\rho^2}{64}}\int_{\mathbb{R}^{n-1}\setminus\Pi_\rho(x)} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}}e^{-\frac{|x'-z'|^2+x_n^2}{16}}dz' \\
&\quad+e^{-\frac{\rho^2}{32}}\rho^{n-2}(\rho^{-n}+\rho^{-n+1})\Big) \\
&\le C\big(\rho^{-n-1}e^{-\frac{x_n^2}{8}}+e^{-\frac{\rho^2}{32}}(\rho^{-2}+\rho^{-1})\big) \\
&\quad+Ce^{-\frac{\rho^2}{64}}\left(\int_{\int_{|z'|<1}} z_n(|z'|^2+z_n^2)^{-\frac{n}{2}}dz'+\int_{|z'|\ge 1} e^{-\frac{|x'-z'|^2}{16}}dz'\right) \\
&\le C\left(\rho^{-n-1}e^{-\frac{x_n^2}{8}}+e^{-\frac{\rho^2}{32}}(\rho^{-2}+\rho^{-1})+e^{-\frac{\rho^2}{64}}\right) \\
&\le Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1});
\end{aligned}
\tag{4.24}
$$

From (4.20), (4.22) and (4.23), we have for $1 \le j \le n-1$, $1 \le k \le n$

$$
\left|\frac{\partial^2}{\partial x_j\partial x_k}J_n(x,z_n,1)\right| \le Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1+\delta_{kn}}), \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0.
\tag{4.25}
$$

Combining (4.20) and (4.25) yields for $1 \le j \le n-1$, $1 \le i, k \le n$

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x_j\partial x_k}J_i(x,z_n,1)\right| &\le Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1+\delta_{kn}}) \\
&\le Ce^{-\frac{x_n^2}{64}}(1+|x'|+x_n+z_n)^{-n-1+\delta_{kn}}, \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0.
\end{aligned}
\tag{4.26}
$$

From (4.17) and (4.26), we find for $1 \le j \le n-1$, $1 \le i, k \le n$ and $t > 0$

$$
\left|\frac{\partial^2}{\partial x_j\partial x_k}J_i(x,z_n,t)\right| = t^{-\frac{n}{2}}\left|\frac{\partial^2}{\partial x_j\partial x_k}J_i(t^{-\frac{1}{2}}x,t^{-\frac{1}{2}}z_n,1)\right|
$$

$$\leq Ce^{-\frac{x_n^2}{64}} t^{-\frac{n}{2}-1}\left(1+t^{-\frac{1}{2}}\left(|x'|+x_n+z_n\right)\right)^{-n-1+\delta_{kn}}$$
$$= Ce^{-\frac{x_n^2}{64}} t^{-\frac{1+\delta_{kn}}{2}}\left(|x'|+x_n+z_n+\sqrt{t}\right)^{-n-1+\delta_{kn}}, \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0,$$

which implies that (4.15) holds.

From (4.21) and (4.24), we derive for $1\leq j,\ell\leq n-1$

$$\left|\frac{\partial^3}{\partial x_j\partial x_\ell\partial x_n}J_n(x,z_n,1)\right|\leq Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1}), \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0. \tag{4.27}$$

Combining (4.21) and (4.27), we conclude for $1\leq j,\ell\leq n-1$ and $1\leq i\leq n$

$$\left|\frac{\partial^3}{\partial x_j\partial x_\ell\partial x_n}J_i(x,z_n,1)\right|\leq Ce^{-\frac{x_n^2}{64}}(1+\rho^{-n-1}), \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0. \tag{4.28}$$

From (4.17) and (4.28), we find for $1\leq j,\ell\leq n-1$, $1\leq i\leq n$ and $t>0$

$$\left|\frac{\partial^3}{\partial x_j\partial x_\ell\partial x_n}J_i(x,z_n,t)\right| = t^{-\frac{n}{2}}\left|\frac{\partial^3}{\partial x_j\partial x_\ell\partial x_n}J_i(t^{-\frac{1}{2}}x,t^{-\frac{1}{2}}z_n,1)\right|$$
$$\leq Ce^{-\frac{x_n^2}{64}} t^{-\frac{n}{2}-\frac{3}{2}}\left(1+t^{-\frac{1}{2}}\left(|x'|+x_n+z_n\right)\right)^{-n-1}$$
$$= Ce^{-\frac{x_n^2}{64}} t^{-1}\left(|x'|+x_n+z_n+\sqrt{t}\right)^{-n-1}, \quad \forall x=(x',x_n)\in\mathbb{R}^n_+, \quad z_n>0,$$

which implies that (4.16) holds. □

***Proof of Theorem 1.4.*** Let $u$ be the strong solution of problem (1.1) given in Theorem 1.1. Then $u$ can be represented as follows for $t>0$ (see [24])

$$u(x,t)=\int_{\mathbb{R}^n_+}\mathcal{M}\left(x,y,\frac{t}{2}\right)u\left(y,\frac{t}{2}\right)dy-\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\mathcal{M}(x,y,t-s)P(u\cdot\nabla)u(y,s)dyds, \tag{4.29}$$

where the definition of $\mathcal{M}=(M_{ij})_{i,j=1,2,\cdots,n}$ is given in Sect. 2.

Note that $M^*_{kn}=0,\ \forall 1\leq k\leq n$. Then for $1\leq k\leq n$

$$\widetilde{w}_k(x,t):=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{j=1}^{n}M_{kj}(x,y,t-s)\big(P(u\cdot\nabla)u(y,s)\big)_j dyds$$
$$=\sum_{j=1}^{n}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\left(G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)\right)$$
$$\times\,\delta_{kj}\big(P(u\cdot\nabla)u(y,s)\big)_j dyds$$
$$+\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{j=1}^{n-1}M^*_{kj}(x,y,t-s)\big(P(u\cdot\nabla)u(y,s)\big)_j dyds$$
$$=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\left(G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)\right)$$
$$\times\big(P(u\cdot\nabla)u(y,s)\big)_k dyds$$

$$
\begin{aligned}
&+\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{j=1}^{n-1} M^*_{kj}(x,y,t-s)\big((u\cdot\nabla)u_j(y,s)\\
&+\sum_{i,\ell=1}^{n}\partial_{y_j}\mathcal{N}\partial_i\partial_\ell(u_iu_\ell)(y,s)\big)dyds\\
&=\widetilde{I}_k(x,t)+\widetilde{J}_k(x,t).
\end{aligned}\tag{4.30}
$$

Using the heat equation yields for any $(x',x_n)\in\mathbb{R}^n$ and $t>0$,

$$
\partial^2_{x_n}G_t(x',x_n)=\left(\partial_t-\sum_{j=1}^{n-1}\partial^2_{x_j}\right)G_t(x',x_n),
$$

and

$$
\lim_{t\to 0^+}G_t(x',x_n)=\delta(x',x_n)\ \text{ in the sense of the distribution.}
$$

Whence we have for $x=(x',x_n)\in\mathbb{R}^n_+$ and $t>0$,

$$
\begin{aligned}
\nabla_{x'}\partial_{x_n}\widetilde{I}_k(x,t)&=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\nabla_{x'}\partial_{x_n}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad -G_{t-s}(x'-y',x_n+y_n)]\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_n}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad -G_{t-s}(x'-y',x_n+y_n)]\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds;
\end{aligned}\tag{4.31}
$$

and

$$
\begin{aligned}
\partial^2_{x_n}\widetilde{I}_k(x,t)&=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\{\partial^2_{x_n}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad -G_{t-s}(x'-y',x_n+y_n)]\}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&=\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}(-\partial_s)[G_{t-s}(x'-y',x_n-y_n)\\
&\quad -G_{t-s}(x'-y',x_n+y_n)]\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&\quad -\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{j=1}^{n-1}\partial^2_{x_j}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad -G_{t-s}(x'-y',x_n+y_n)]\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&=-\big(P(u\cdot\nabla u)(x,t)\big)_k+\int_{\mathbb{R}^n_+}[G_{\frac{t}{2}}(x'-y',x_n-y_n)\\
&\quad -G_{\frac{t}{2}}(x'-y',x_n+y_n)]\left(P(u\cdot\nabla)u\left(y,\frac{t}{2}\right)\right)_k dy\\
&\quad +\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}[G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)]\\
&\quad \times\partial_s\big(P(u\cdot\nabla)u(y,s)\big)_k dyds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}[G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)]\\
&\times\sum_{j=1}^{n-1}\partial_{y_j}^2\big(P(u\cdot\nabla)u(y,s)\big)_k dyds. \qquad (4.32)
\end{aligned}
$$

Whence, using (4.31), (4.32) and Lemma 3.1, we conclude that for $1 \le k \le n$ and $t > 1$

$$
\begin{aligned}
&\|\partial_{x_n}^3\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla_{x'}\partial_{x_n}^2\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla_{x'}^2\partial_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}\\
&\le\|\nabla\big(P(u\cdot\nabla)u(\cdot,t)\big)_k\|_{L^1(\mathbb{R}^n_+)}+\int_{\mathbb{R}^n_+}\|\nabla\Big[G_{\frac{t}{2}}(x'-y',x_n-y_n)\\
&-G_{\frac{t}{2}}(x'-y',x_n+y_n)\Big]\|_{L^1_x(\mathbb{R}^n_+)}\left|\left(P(u\cdot\nabla)u\left(y,\frac{t}{2}\right)\right)_k\right|dy\\
&+\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|\nabla[G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)]\|_{L^1_x(\mathbb{R}^n_+)}\\
&\big(|\partial_s\big(P(u\cdot\nabla)u(y,s)\big)_k|+|\nabla_{y'}^2\big(P(u\cdot\nabla)u(y,s)\big)_k|\big)dyds\\
&+\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|\nabla[G_{t-s}(x'-y',x_n-y_n)\\
&-G_{t-s}(x'-y',x_n+y_n)]\|_{L^1_x(\mathbb{R}^n_+)}\sum_{j=1}^{n-1}|\partial_{y_j}^2\big(P(u\cdot\nabla)u(y,s)\big)_k|dyds\\
&\le\|\nabla\big(u\cdot\nabla\big)u(\cdot,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla\sum_{i,j=1}^{n}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,t)\|_{L^1(\mathbb{R}^n_+)}\\
&+Ct^{-\frac{1}{2}}\left(\|\big(u\cdot\nabla\big)u_k\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}+\|\sum_{i,j=1}^{n}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}\right)\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(\|\partial_s\big(u\cdot\nabla\big)u_k(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|\sum_{i,j=1}^{n}\partial_s\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\big)ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(\|\sum_{\ell=1}^{n-1}\partial_\ell^2\big(u\cdot\nabla\big)u_k(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|\sum_{\ell=1}^{n-1}\sum_{i,j=1}^{n}\partial_\ell^2\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\big)ds\\
&\le C\big(\|u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^2(\mathbb{R}^n_+)}+\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n_+)}\\
&+\|u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}+\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^2(\mathbb{R}^n_+)}\big)\\
&+Ct^{-\frac{1}{2}}\left(\|u(t)\|^2_{L^2(\mathbb{R}^n_+)}+\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n_+)}\right)\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\partial_s u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\Big)ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^3 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}+\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}+\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^3 u(s)\|_{L^2(\mathbb{R}^n_+)}\Big)ds.
\end{aligned}
$$

Using Lemma 3.2 yields for $1 \le k \le n$ and $t > 1$

$$
\begin{aligned}
&\|\partial^3_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla_{x'}\partial^2_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla^2_{x'}\partial_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}\\
&\le Ct^{-\frac{1}{2}-\frac{n}{2}}+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-1-\frac{n}{2}}ds\\
&\le \widetilde{C}t^{-\frac{1}{2}-\frac{n}{2}}.
\end{aligned}
\tag{4.33}
$$

Furthermore suppose $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, there holds for $1 \le k \le n$ and $t > 1$

$$
\begin{aligned}
&\|\partial^3_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla_{x'}\partial^2_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla^2_{x'}\partial_{x_n}\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)}\\
&\le Ct^{-\frac{3}{2}-\frac{n}{2}}+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-2-\frac{n}{2}}ds\\
&\le \widetilde{C}t^{-\frac{3}{2}-\frac{n}{2}}.
\end{aligned}
\tag{4.33'}
$$

Set

$$
N_{ij}(x,y,t)=\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_i}G_t(z-y^*)dz,\quad 1\le i,j\le n,
$$

and

$$
b_j(y,s)=(u\cdot\nabla)u_j(y,s)+\sum_{i,\ell=1}^{n}\partial_{y_j}\mathcal{N}\partial_i\partial_\ell(u_iu_\ell)(y,s).
$$

Then

$$
M^*_{ij}(x,y,t)=(1-\delta_{jn})N_{ij}(x,y,t),
$$

and

$$
\widetilde{J}_k(x,t)=4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{j=1}^{n-1}N_{kj}(x,y,t-s)b_j(y,s)dyds. \tag{4.34}
$$

Using Lemma 4.1, we get for $x, y \in \mathbb{R}^n_+$ and $t > 0$,

$$
\begin{aligned}
N_{nn}(x,y,t) &= \frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_n} G_t(z-y^*)dz'dz_n \\
&= -\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_j} G_t(z-y^*)dz'dz_n + \frac{1}{2}G_t(x-y^*) \\
&= -\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(z)}{\partial x_j} G_t(x-y^*-z)dz'dz_n + \frac{1}{2}G_t(x-y^*) \\
&= -\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_0^{x_n} J_j(x'-y', x_n+y_n-z_n, z_n, t)dz_n + \frac{1}{2}G_t(x-y^*),
\end{aligned} \tag{4.35}
$$

where

$$
J_i(x', x_n, z_n, t) = \int_{\mathbb{R}^{n-1}} G_t(x'-z', x_n)\frac{\partial E(z', z_n)}{\partial z_i}dz', \quad 1 \le i \le n.
$$

Let $1 \le i \le n-1$. Then for $x=(x',x_n)$, $y=(y',y_n)\in\mathbb{R}^n_+$ and $t>0$,

$$
\begin{aligned}
N_{in}(x,y,t) &= \frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_i} G_t(z-y^*)dz \\
&= \frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} G_t(x-y^*-z)\frac{\partial E(z)}{\partial z_i}dz \\
&= \int_0^{x_n}\frac{\partial}{\partial x_n}\int_{\mathbb{R}^{n-1}} G_t(x-y^*-z)\frac{\partial E(z)}{\partial z_i}dz \\
&\quad + \int_{\mathbb{R}^{n-1}} G_t(x'-y'-z', y_n)\partial_{z_i}E(z', x_n)dz' \\
&= \int_0^{x_n}\frac{\partial}{\partial x_n} J_i(x'-y', x_n+y_n-z_n, z_n, t)dz_n + J_i(x'-y', y_n, x_n, t);
\end{aligned} \tag{4.36}
$$

and

$$
\begin{aligned}
N_{ni}(x,y,t) &= \frac{\partial}{\partial x_i}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_n} G_t(z-y^*)dz \\
&= \int_0^{x_n}\frac{\partial}{\partial x_n}\int_{\mathbb{R}^{n-1}} G_t(x-y^*-z)\frac{\partial E(z)}{\partial z_i}dz \\
&= \int_0^{x_n}\frac{\partial}{\partial x_n} J_i(x'-y', x_n+y_n-z_n, z_n, t)dz_n.
\end{aligned} \tag{4.37}
$$

Combining (4.36) and (4.37), together with Lemma 4.2 yields for $1 \le i \le n-1$, $x=(x',x_n)$, $y=(y',y_n)\in\mathbb{R}^n_+$ and $t>0$,

$$
\begin{aligned}
&|\nabla^2_{x'}N_{ni}(x,y,t)| + |\nabla^2_{x'}N_{in}(x,y,t)| \\
&\le 2\int_0^{x_n} |\frac{\partial}{\partial x_n}\nabla^2_{x'}J_i(x'-y', x_n+y_n-z_n, z_n, t)|dz_n + |\nabla^2_{x'}J_i(x'-y', y_n, x_n, t)| \\
&\le Ct^{-1}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-1}\int_0^{x_n} e^{-\frac{(x_n+y_n-z_n)^2}{64t}}dz_n \\
&\quad + Ct^{-\frac{1}{2}}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-1}e^{-\frac{y_n^2}{64t}}
\end{aligned}
$$

$$\leq Ct^{-\frac{1}{2}}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-1}. \tag{4.38}$$

Using Lemma 4.1, we find for $1 \leq k \leq n-1$, $x=(x',x_n)$, $y=(y',y_n) \in \mathbb{R}^n_+$ and $t>0$,

$$\begin{aligned}
\frac{\partial^3}{\partial x_n^3}N_{nk}(x,y,t) &= \frac{\partial^3}{\partial x_k \partial x_n^2}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_n}G_t(z-y^*)dz \\
&= -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial^2}{\partial x_n^2}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_j}G_t(z-y^*)dz \\
&\quad +\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_n^2}G_t(x-y^*) \\
&= -\sum_{j=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_j \partial x_k}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_n}G_t(z-y^*)dz \\
&\quad -\sum_{j=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_j \partial x_k}\frac{\partial}{\partial x_n}\int_{\mathbb{R}^{n-1}}E(x'-z',0)G_t(z'-y',x_n+y_n)dz' \\
&\quad +\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_n^2}G_t(x-y^*) \\
&= \sum_{j,\ell=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_j \partial x_k}\frac{\partial}{\partial x_\ell}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_\ell}G_t(z-y^*)dz \\
&\quad -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial}{\partial x_n}J_j(x'-y',x_n+y_n,0,t) \\
&\quad -\frac{1}{2}\sum_{j=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_j \partial x_k}G_t(x-y^*)+\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_n^2}G_t(x-y^*);
\end{aligned} \tag{4.39}$$

and for $1 \leq k,m,q \leq n-1$

$$\begin{aligned}
\frac{\partial^3}{\partial x_m \partial x_n^2}N_{nk}(x,y,t) &= \frac{\partial^3}{\partial x_k \partial x_m \partial x_n}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_n}G_t(z-y^*)dz \\
&= -\sum_{j=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_k \partial x_m}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_j}G_t(z-y^*)dz \\
&\quad +\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_m \partial x_n}G_t(x-y^*) \\
&= -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial}{\partial x_m}N_{jn}(x,y,t)+\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_m \partial x_n}G_t(x-y^*);
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
&\frac{\partial^3}{\partial x_m \partial x_q \partial x_n}N_{nk}(x,y,t) \\
&= \frac{\partial^3}{\partial x_k \partial x_m \partial x_q}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_n}G_t(z-y^*)dz
\end{aligned}$$

$$= -\sum_{j=1}^{n-1} \frac{\partial^3}{\partial x_j \partial x_k \partial x_m} \frac{\partial}{\partial x_q} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_j} G_t(z-y^*) dz$$
$$+ \frac{1}{2} \frac{\partial^3}{\partial x_k \partial x_m \partial x_q} G_t(x-y^*)$$
$$= -\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j \partial x_k} \frac{\partial}{\partial x_m} N_{jq}(x,y,t) + \frac{1}{2} \frac{\partial^3}{\partial x_k \partial x_m \partial x_q} G_t(x-y^*); \tag{4.41}$$

Using (4.36) and Lemma 4.1 yields for $1 \le k, m \le n-1$, $x, y \in \mathbb{R}^n_+$ and $t > 0$

$$\frac{\partial^3}{\partial x_n^3} N_{mk}(x,y,t) = \frac{\partial^3}{\partial x_k \partial x_n^2} \frac{\partial}{\partial x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_m} G_t(z-y^*) dz$$
$$= \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_n} N_{mn}(x,y,t)$$
$$= \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_n} \int_0^{x_n} \frac{\partial}{\partial x_n} J_m(x'-y', x_n+y_n-z_n, z_n, t) dz_n$$
$$+ \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_n} J_m(x'-y', y_n, x_n, t)$$
$$= \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_n} \int_0^{x_n} \frac{\partial}{\partial x_m} J_n(x'-y', x_n+y_n-z_n, z_n, t) dz_n$$
$$+ \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_m} J_n(x'-y', y_n, x_n, t)$$
$$= -\sum_{\ell=1}^{n-1} \frac{\partial^3}{\partial x_k \partial x_m \partial x_n} \frac{\partial}{\partial x_\ell} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_\ell} G_t(z-y^*) dz$$
$$+ \frac{1}{2} \frac{\partial^3}{\partial x_k \partial x_m \partial x_n} G_t(x-y^*)$$
$$+ \frac{\partial^2}{\partial x_k \partial x_n} \frac{\partial}{\partial x_m} J_n(x'-y', y_n, x_n, t); \tag{4.42}$$

and for $1 \le j, q \le n-1$

$$\frac{\partial^3}{\partial x_j \partial x_n^2} N_{mk}(x,y,t) = \frac{\partial^3}{\partial x_k \partial x_j \partial x_n} \frac{\partial}{\partial x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_m} G_t(z-y^*) dz$$
$$= \frac{\partial^2}{\partial x_k \partial x_j} \frac{\partial}{\partial x_n} N_{mn}(x,y,t)$$
$$= \frac{\partial^2}{\partial x_k \partial x_j} \frac{\partial}{\partial x_n} \int_0^{x_n} \frac{\partial}{\partial x_n} J_m(x'-y', x_n+y_n-z_n, z_n, t) dz_n$$
$$+ \frac{\partial^2}{\partial x_k \partial x_j} \frac{\partial}{\partial x_n} J_m(x'-y', y_n, x_n, t)$$
$$= \frac{\partial^2}{\partial x_k \partial x_j} \frac{\partial}{\partial x_n} \int_0^{x_n} \frac{\partial}{\partial x_m} J_n(x'-y', x_n+y_n-z_n, z_n, t) dz_n$$

$$
\begin{aligned}
&+\frac{\partial^2}{\partial x_k \partial x_j}\frac{\partial}{\partial x_m} J_n(x'-y', y_n, x_n, t)\\
&= -\sum_{\ell=1}^{n-1} \frac{\partial^3}{\partial x_k \partial x_m \partial x_j}\frac{\partial}{\partial x_\ell}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_\ell} G_t(z-y^*)dz\\
&+\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_m \partial x_j} G_t(x-y^*) + \frac{\partial^2}{\partial x_k \partial x_j}\frac{\partial}{\partial x_m} J_n(x'-y', y_n, x_n, t); \qquad (4.43)\\
&\frac{\partial^3}{\partial x_j \partial x_q \partial x_n} N_{mk}(x, y, t) = \frac{\partial^3}{\partial x_k \partial x_j \partial x_q}\frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_m} G_t(z-y^*)dz\\
&= \frac{\partial^2}{\partial x_k \partial x_j}\frac{\partial}{\partial x_q} N_{mn}(x, y, t). \qquad (4.44)
\end{aligned}
$$

It follows from (4.34) and (4.39) that for $x \in \mathbb{R}^n_+$ and $t > 0$

$$
\begin{aligned}
\frac{\partial^3}{\partial x_n^3}\widetilde{J}_n(x,t) &= 4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k=1}^{n-1} \frac{\partial^3}{\partial x_n^3} N_{nk}(x, y, t-s) b_k(y, s)dyds\\
&= 4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k=1}^{n-1} \Bigg( \sum_{j,\ell=1}^{n-1} \frac{\partial^3}{\partial x_j \partial x_j \partial x_k} N_{\ell\ell}(x, y, t-s)\\
&\quad - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial}{\partial x_n} J_j(x'-y', x_n+y_n, 0, t-s)\\
&\quad - \frac{1}{2}\sum_{j=1}^{n-1} \frac{\partial^3}{\partial x_j \partial x_j \partial x_k} G_{t-s}(x-y^*) + \frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_n \partial x_n} G_{t-s}(x-y^*) \Bigg) b_k(y,s)dyds\\
&= 4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k,j,\ell=1}^{n-1} \frac{\partial^2}{\partial x_k \partial x_j} N_{\ell\ell}(x, y, t-s)\frac{\partial}{\partial y_j} b_k(y, s)dyds\\
&\quad -4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k,j=1}^{n-1} \frac{\partial^2}{\partial x_j \partial x_j} J_n(x'-y', x_n+y_n, 0, t-s)\frac{\partial}{\partial y_k} b_k(y, s)dyds\\
&\quad -2\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k,j=1}^{n-1} \frac{\partial^2}{\partial x_k \partial x_j} G_{t-s}(x-y^*)\frac{\partial}{\partial y_j} b_k(y, s)dyds\\
&\quad +2\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_n \partial x_n} G_{t-s}(x-y^*)\frac{\partial}{\partial y_k} b_k(y, s)dyds; \qquad (4.45)
\end{aligned}
$$

From (4.34), (4.40) and (4.41), we have for $1 \le m, q \le n-1$, $x \in \mathbb{R}^n_+$ and $t > 0$,

$$
\begin{aligned}
\frac{\partial^3}{\partial x_m \partial x_n^2}\widetilde{J}_n(x,t) &= 4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k=1}^{n-1} \frac{\partial^3}{\partial x_m \partial x_n^2} N_{nk}(x, y, t-s) b_k(y, s)dyds\\
&= -4\int_{\frac{t}{2}}^t \int_{\mathbb{R}^n_+} \sum_{k=1}^{n-1} \Big( \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial}{\partial x_m} N_{jn}(x, y, t-s)\\
&\quad -\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_m \partial x_n} G_{t-s}(x-y^*) \Big) b_k(y,s)dyds
\end{aligned}
$$

$$= -4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\frac{\partial^2}{\partial x_k\partial x_j}N_{jn}(x,y,t-s)\frac{\partial}{\partial y_m}b_k(y,s)dyds$$
$$+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k\partial x_n}G_{t-s}(x-y^*)\frac{\partial}{\partial y_m}b_k(y,s)dyds; \quad (4.46)$$

and

$$\frac{\partial^3}{\partial x_m\partial x_q\partial x_n}\widetilde{J}_n(x,t) = 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^3}{\partial x_m\partial x_q\partial x_n}N_{nk}(x,y,t-s)b_k(y,s)dyds$$
$$= -4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j\partial x_k}\frac{\partial}{\partial x_m}N_{jq}(x,y,t)$$
$$-\frac{1}{2}\frac{\partial^3}{\partial x_k\partial x_m\partial x_q}G_t(x-y^*)\Big)b_k(y,s)dyds$$
$$= -4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\frac{\partial^2}{\partial x_k\partial x_j}N_{jq}(x,y,t-s)\frac{\partial}{\partial y_m}b_k(y,s)dyds$$
$$+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k\partial x_q}G_{t-s}(x-y^*)$$
$$\times\frac{\partial}{\partial y_m}b_k(y,s)dyds. \quad (4.47)$$

Let $1 \le j, m, q \le n-1$. Using (4.34) and (4.42)–(4.44) yields for $x \in \mathbb{R}^n_+$ and $t > 0$,

$$\frac{\partial^3}{\partial x_n^3}\widetilde{J}_m(x,t) = 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^3}{\partial x_n^3}N_{mk}(x,y,t-s)b_k(y,s)dyds$$
$$= 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big(-\sum_{\ell=1}^{n-1}\frac{\partial^3}{\partial x_k\partial x_m\partial x_\ell}N_{\ell n}(x,y,t-s)$$
$$+\frac{1}{2}\frac{\partial^3}{\partial x_k\partial x_m\partial x_n}G_{t-s}(x-y^*)$$
$$+\frac{\partial^2}{\partial x_k\partial x_n}\frac{\partial}{\partial x_m}J_n(x'-y',y_n,x_n,t-s)\Big)b_k(y,s)dyds$$
$$= -4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,\ell=1}^{n-1}\frac{\partial^2}{\partial x_\ell\partial x_k}N_{\ell n}(x,y,t-s)\frac{\partial}{\partial y_m}b_k(y,s)dyds$$
$$+4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_m\partial x_n}J_n(x'-y',y_n,x_n,t-s)\frac{\partial}{\partial y_k}b_k(y,s)dyds$$
$$+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_n\partial x_m}G_{t-s}(x-y^*)\frac{\partial}{\partial y_k}b_k(y,s)dyds; \quad (4.48)$$

$$\frac{\partial^3}{\partial x_j \partial x_n^2}\widetilde{J}_m(x,t) = 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_n^2}N_{mk}(x,y,t-s)b_k(y,s)dyds$$

$$= 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big(-\sum_{\ell=1}^{n-1}\frac{\partial^3}{\partial x_k \partial x_m \partial x_j}\frac{\partial}{\partial x_\ell}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_\ell}G_t(z-y^*)dz$$

$$+\frac{1}{2}\frac{\partial^3}{\partial x_k \partial x_m \partial x_j}G_{t-s}(x-y^*)$$

$$+\frac{\partial^2}{\partial x_k \partial x_j}\frac{\partial}{\partial x_m}J_n(x'-y', y_n, x_n, t-s)\Big)b_k(y,s)dyds$$

$$= -4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,\ell=1}^{n-1}\frac{\partial^2}{\partial x_k \partial x_j}N_{\ell\ell}(x,y,t-s)\frac{\partial}{\partial y_m}b_k(y,s)dyds$$

$$+4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k \partial x_j}J_n(x'-y', y_n, x_n, t-s)\frac{\partial}{\partial y_m}b_k(y,s)dyds$$

$$+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k \partial x_j}G_{t-s}(x-y^*)\frac{\partial}{\partial y_m}b_k(y,s)dyds; \quad (4.49)$$

and

$$\frac{\partial^3}{\partial x_j \partial x_q \partial x_n}\widetilde{J}_m(x,t) = 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^3}{\partial x_j \partial x_q \partial x_n}N_{mk}(x,y,t-s)b_k(y,s)dyds$$

$$= 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k \partial x_j}N_{mn}(x,y,t-s)\frac{\partial}{\partial y_q}b_k(y,s)dyds. \quad (4.50)$$

Using (4.38) yields for every $1 \le i, k, \ell \le n-1$, $y=(y', y_n) \in \mathbb{R}^n_+$ and $t>0$,

$$\left\|(x_n+y_n)^\epsilon \frac{\partial^2}{\partial x_k \partial x_\ell}N_{in}(x,y,t)\right\|_{L^1_x(\mathbb{R}^n_+)}$$

$$\le Ct^{-\frac{1}{2}}\int_{\mathbb{R}^n_+}(x_n+y_n)^\epsilon(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-1}dx'dx_n$$

$$\le Ct^{-1+\frac{\epsilon}{2}}, \quad \epsilon \in (0,1). \quad (4.51)$$

By Lemma 4.2, we get for every $1 \le k, \ell \le n-1$, $y=(y', y_n) \in \mathbb{R}^n_+$ and $t>0$,

$$\left\|(x_n+y_n)^\epsilon \frac{\partial^2}{\partial x_k \partial x_\ell}J_n(x'-y', y_n, x_n, t)\right\|_{L^1_x(\mathbb{R}^n_+)}$$

$$\le Ct^{-\frac{1}{2}}\int_{\mathbb{R}^n_+}(x_n+y_n)^\epsilon(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-1}dx'dx_n$$

$$\le Ct^{-1+\frac{\epsilon}{2}}, \quad \epsilon \in (0,1); \quad (4.52)$$

and

$$\left\|(x_n+y_n)^\epsilon \frac{\partial^2}{\partial x_\ell \partial x_n}J_n(x'-y', y_n, x_n, t)\right\|_{L^1_x(\mathbb{R}^n_+)}$$

$$
\begin{aligned}
&\le Ct^{-1}\int_{\mathbb{R}^n_+}(x_n+y_n)^{\epsilon}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n}e^{-\frac{(x_n+y_n)^2}{64t}}dx'dx_n\\
&\le Ct^{-1}\int_0^{\infty}\int_0^{\infty}(s+x_n+y_n+\sqrt{t})^{-n+\epsilon+n-2}e^{-\frac{(x_n+y_n)^2}{64t}}dsdx_n\\
&\le Ct^{-1}\int_0^{\infty}(x_n+y_n+\sqrt{t})^{-1+\epsilon}e^{-\frac{(x_n+y_n)^2}{64t}}dx_n\\
&\le Ct^{-1+\frac{\epsilon}{2}}\int_0^{\infty}(\tau+1)^{-1+\epsilon}e^{-\frac{\tau^2}{64}}d\tau\\
&\le Ct^{-1+\frac{\epsilon}{2}},\quad \epsilon\in(0,1).
\end{aligned}
\tag{4.53}
$$

From (4.45)–(4.47) and (4.51)–(4.53), we get for $t>0$

$$
\begin{aligned}
&\left\|\frac{\partial^3}{\partial x_n^3}\widetilde{J}_n(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\le 4\sum_{k,j,\ell=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}N_{\ell\ell}(x,y,t-s)\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times|y_n^{-\epsilon}\frac{\partial}{\partial y_j}b_k(y,s)|dyds\\
&\quad+4\sum_{k,j=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_j^2}J_n(x'-y',x_n+y_n,0,t-s)\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times|y_n^{-\epsilon}\frac{\partial}{\partial y_k}b_k(y,s)|dyds\\
&\quad+2\sum_{k,j=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}G_{t-s}(x-y^*)\|_{L^1_x(\mathbb{R}^n_+)}|y_n^{-\epsilon}\frac{\partial}{\partial y_j}b_k(y,s)|dyds\\
&\quad+2\sum_{k=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_n^2}G_{t-s}(x-y^*)\|_{L^1_x(\mathbb{R}^n_+)}|y_n^{-\epsilon}\frac{\partial}{\partial y_k}b_k(y,s)|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla' b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds,\quad \epsilon\in(0,1);
\end{aligned}
\tag{4.54}
$$

moreover, for $1\le m,q\le n-1$,

$$
\begin{aligned}
&\left\|\frac{\partial^3}{\partial x_m\partial x_n^2}\widetilde{J}_n(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\le 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}N_{jn}(x,y,t-s)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds\\
&\quad+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_n}G_{t-s}(x-y^*)\right\|_{L^1(\mathbb{R}^n_+)}|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)|dyds
\end{aligned}
$$

$$\leq C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla' b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds, \quad \epsilon\in(0,1); \tag{4.55}$$

and

$$\begin{aligned}
&\left\|\frac{\partial^3}{\partial x_m\partial x_q\partial x_n}\widetilde{J}_n(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\leq 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}N_{jq}(x,y,t-s)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds\\
&\quad+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_q}G_{t-s}(x-y^*)\right\|_{L^1(\mathbb{R}^n_+)}|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)|dyds\\
&\leq C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla' b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds, \quad \epsilon\in(0,1).
\end{aligned} \tag{4.56}$$

From (4.48)–(4.53), we derive that for $1\leq j,m,q\leq n-1$ and $t>0$

$$\begin{aligned}
&\left\|\frac{\partial^3}{\partial x_n^3}\widetilde{J}_m(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\leq 4\sum_{k,\ell=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_\ell\partial x_k}N_{\ell n}(x,y,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds\\
&\quad+4\sum_{k=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_m\partial x_n}J_n(x'-y',y_n,x_n,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_k}b_k(y,s)\right|dyds\\
&\quad+2\sum_{k=1}^{n-1}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_n\partial x_m}G_{t-s}(x-y^*)\right\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_k}b_k(y,s)\right|dyds\\
&\leq C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla' b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds, \quad \epsilon\in(0,1);
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
&\left\|\frac{\partial^3}{\partial x_j\partial x_n^2}\widetilde{J}_m(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\leq 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,\ell=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}N_{\ell\ell}(x,y,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds
\end{aligned}$$

$$+4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}J_n(x'-y',y_n,x_n,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}$$
$$\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds$$
$$+2\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}G_{t-s}(x-y^*)\right\|_{L^1_x(\mathbb{R}^n_+)}\left|y_n^{-\epsilon}\frac{\partial}{\partial y_m}b_k(y,s)\right|dyds$$
$$\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla'b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds,\quad \epsilon\in(0,1); \tag{4.58}$$

and

$$\left\|\frac{\partial^3}{\partial x_j\partial x_q\partial x_n}\widetilde{J}_m(x,t)\right\|_{L^1(\mathbb{R}^n_+)}$$
$$\le 4\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\left\|(x_n+y_n)^{\epsilon}\frac{\partial^2}{\partial x_k\partial x_j}N_{mn}(x,y,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}$$
$$\times\left|y_n^{-\epsilon}\frac{\partial}{\partial y_q}b_k(y,s)\right|dyds$$
$$\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla'b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds,\quad \epsilon\in(0,1). \tag{4.59}$$

From (4.54)–(4.59), we obtain for $t>0$

$$\|\nabla^3\widetilde{J}(\cdot,t)\|_{L^1(\mathbb{R}^n_+)}\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\epsilon}{2}}\|y_n^{-\epsilon}\nabla'b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}ds,\quad \epsilon\in(0,1). \tag{4.60}$$

Recall that

$$b_j(y,s)=(u\cdot\nabla)u_j(y,s)+\sum_{i,\ell=1}^{n}\partial_{y_j}\mathcal{N}\partial_i\partial_\ell(u_iu_\ell)(y,s),\quad 1\le j\le n.$$

Whence we conclude for $\epsilon\in(0,1)$,

$$\|y_n^{-\epsilon}\nabla'b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\le\|y_n^{-\epsilon}\nabla'((u\cdot\nabla)u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^{-\epsilon}\nabla'\left(\sum_{i,\ell=1}^{n}\nabla\mathcal{N}\partial_i\partial_\ell(u_iu_\ell)(\cdot,s)\right)\|_{L^1(\mathbb{R}^n_+)}$$
$$\le\|y_n^{-\epsilon}(\nabla'u\cdot\nabla)u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^{-\epsilon}(u\cdot\nabla'\nabla)u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^{-\epsilon}\sum_{i,\ell=1}^{n}\nabla\mathcal{N}\partial_i\partial_\ell\nabla'(u_iu_\ell)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}. \tag{4.61}$$

To proceed, let $0<\epsilon<1$. Then for $s>0$

$$\|y_n^{-\epsilon}(\nabla'u\cdot\nabla)u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}+\|y_n^{-\epsilon}(u\cdot\nabla'\nabla)u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}$$

$$\leq \int_0^1 \int_{\mathbb{R}^{n-1}} y_n^{-\epsilon} |\nabla' u(y,s)||\nabla u(y,s)| dy' dy_n + \int_0^1 \int_{\mathbb{R}^{n-1}} y_n^{-\epsilon} |u(y,s)||\nabla' \nabla u(y,s)| dy' dy_n$$
$$+ \|\nabla'(u \cdot \nabla) u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)} + \|(u \cdot \nabla' \nabla) u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)}$$
$$\leq \|y_n^{-\epsilon} \nabla' u(s)\|_{L^2(\mathbb{R}^{n-1}\times(0,1))} \|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{-\epsilon} u(s)\|_{L^2(\mathbb{R}^{n-1}\times(0,1))} \|\nabla' \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}$$
$$+ \|\nabla' u(s)\|_{L^2(\mathbb{R}^n_+)} \|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)} + \|u(s)\|_{L^2(\mathbb{R}^n_+)} \|\nabla' \nabla u(s)\|_{L^2(\mathbb{R}^n_+)}.$$

One-dimensional Hardy inequality yields for $s > 0$ and $\epsilon \in (0, 1)$,

$$\|y_n^{-\epsilon} \nabla' u(s)\|^2_{L^2(\mathbb{R}^{n-1}\times(0,1))} + \|y_n^{-\epsilon} u(s)\|^2_{L^2(\mathbb{R}^{n-1}\times(0,1))}$$
$$\leq \int_{\mathbb{R}^{n-1}} \left( \int_0^1 y_n^{2-2\epsilon} \frac{|\nabla' u(y', y_n, s)|^2}{y_n^2} dy_n + \int_0^1 y_n^{2-2\epsilon} \frac{|u(y', y_n, s)|^2}{y_n^2} dy_n \right) dy'$$
$$\leq \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty \frac{|\nabla' u(y', y_n, s)|^2}{y_n^2} dy_n + \int_0^\infty \frac{|u(y', y_n, s)|^2}{y_n^2} dy_n \right) dy'$$
$$\leq C \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty \left|\partial_n \nabla' u(y', y_n, s)\right|^2 dy_n + \int_0^\infty \left|\partial_n u(y', y_n, s)\right|^2 dy_n \right) dy'$$
$$\leq C(\|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}).$$

Whence we obtain for $0 < \epsilon < 1$ and $s > 0$

$$\begin{aligned} &\|y_n^{-\epsilon} \nabla'(u \cdot \nabla) u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)} + \|y_n^{-\epsilon} (u \cdot \nabla' \nabla) u(\cdot,s))\|_{L^1(\mathbb{R}^n_+)} \\ &\leq C(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}). \end{aligned} \tag{4.62}$$

In addition, it follows from Lemma 3.1 that for $0 < \epsilon < 1$ and $s > 0$

$$\begin{aligned} &\|y_n^{-\epsilon} \sum_{i,\ell=1}^n \nabla \mathcal{N} \partial_i \partial_\ell \nabla'(u_i u_\ell)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)} \\ &\leq C(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}). \end{aligned} \tag{4.63}$$

Inserting (4.62) and (4.63) into (4.61), we find for $0 < \epsilon < 1$ and $s > 0$

$$\|y_n^{-\epsilon} \nabla' b(\cdot,s)\|_{L^1(\mathbb{R}^n_+)} \leq C(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}). \tag{4.64}$$

Combining (4.60) and (4.64), we obtain for $0 < \epsilon < 1$ and $t > 1$

$$\begin{aligned} \left\|\nabla^3 \widetilde{J}(\cdot,t)\right\|_{L^1(\mathbb{R}^n_+)} &\leq C \int_{\frac{t}{2}}^t (t-s)^{-1+\frac{\epsilon}{2}} (\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}) ds \\ &\leq C \int_{\frac{t}{2}}^t (t-s)^{-1+\frac{\epsilon}{2}} s^{-1-\frac{n}{2}} ds \\ &\leq C t^{-1-\frac{n}{2}+\frac{\epsilon}{2}}. \end{aligned} \tag{4.65}$$

From (4.29), (4.30), (4.33) and (4.65), together with Theorem 1.3, we conclude for $t > 1$

$$\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} \leq \int_{\mathbb{R}^n_+} \left\| \nabla_x^3 \mathcal{M}\left(x, y, \frac{t}{2}\right) y_n^{\frac{1}{2}} \right\|_{L^1_x(\mathbb{R}^n_+)} \left| y_n^{-\frac{1}{2}} u\left(y, \frac{t}{2}\right) \right| dy$$
$$+ \sum_{k=1}^n (\|\nabla^3 \widetilde{I}_k(\cdot,t)\|_{L^1(\mathbb{R}^n_+)} + \|\nabla^3 \widetilde{J}_k(\cdot,t)\|_{L^1(\mathbb{R}^n_+)})$$

$$
\leq Ct^{-\frac{3}{2}+\frac{1}{4}}\|y_n^{-\frac{1}{2}}u\left(\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}+C\left(t^{-\frac{1}{2}-\frac{n}{2}}+t^{-1-\frac{n}{2}+\frac{\epsilon}{2}}\right)\quad \text{where } \ 0<\epsilon<1
$$

$$
\leq \begin{cases} Ct^{-\frac{3}{2}} & \text{if } n\geq 3,\\ Ct^{-\frac{3}{2}}\log_e(1+t) & \text{if } n=2.\end{cases}
$$

In addition, suppose $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)}<\infty$, together with (4.33′), there holds for any $t>1$

$$
\begin{aligned}
\|\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} &\leq \int_{\mathbb{R}^n_+}\left\|\nabla_x^3\mathcal{M}\left(x,y,\frac{t}{2}\right)y_n^{\frac{1}{2}}\right\|_{L^1_x(\mathbb{R}^n_+)}\left|y_n^{-\frac{1}{2}}u\left(y,\frac{t}{2}\right)\right|dy\\
&\quad+\sum_{k=1}^n(\|\nabla^3\widetilde{I}_k(\cdot,t)\|_{L^1(\mathbb{R}^n_+)}+\|\nabla^3\widetilde{J}_k(\cdot,t)\|_{L^1(\mathbb{R}^n_+)})\\
&\leq Ct^{-\frac{3}{2}+\frac{1}{4}}\|y_n^{-\frac{1}{2}}u\left(\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}+C\left(t^{-\frac{3}{2}-\frac{n}{2}}+t^{-1-\frac{n}{2}+\frac{\epsilon}{2}}\right)\\
&\leq C\left(t^{-\frac{3}{2}}+t^{-1-\frac{n}{2}+\frac{\epsilon}{2}}\right)\quad \text{where } \ 0<\epsilon<1\\
&\leq \widetilde{C}t^{-\frac{3}{2}}.
\end{aligned}
$$

which is (1.2).

Now suppose

$$
\|x_n u_0\|_{L^2(\mathbb{R}^n_+)}+\|(1+x_n)\nabla u_0\|_{L^2(\mathbb{R}^n_+)}+\|x_n u_0\|_{L^1(\mathbb{R}^n_+)}<\infty.
$$

We give the proof of $\|x_n^\beta\nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)}$, where $u$ is the strong solution of (1.1), given in Theorem 1.1.

From (4.31), we have for $x=(x',x_n)\in\mathbb{R}^n_+$ and $t>0$

$$
\begin{aligned}
\nabla_{x'}\partial_{x_n}\widetilde{I}_k(x,t) &= \int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{y_n}[-G_{t-s}(x'-y',x_n-y_n)\\
&\quad+G_{t-s}(x'-y',x_n+y_n)]\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&\quad-2\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}G_{t-s}(x'-y',x_n+y_n)\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&= \int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad-G_{t-s}(x'-y',x_n+y_n)]\nabla_{y'}\partial_{y_n}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&\quad-2\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}G_{t-s}(x'-y',x_n+y_n)\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\nabla_{x'}\partial_{x_n}\partial_{x_n}\widetilde{I}_k(x,t) &= \int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}[G_{t-s}(x'-y',x_n-y_n)\\
&\quad-G_{t-s}(x'-y',x_n+y_n)]\nabla_{y'}\partial_{y_n}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds\\
&\quad-2\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}\partial_{x_n}G_{t-s}(x'-y',x_n+y_n)\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds, \qquad (4.66)
\end{aligned}
$$

and

$$\nabla_{x'}\nabla_{x'}\partial_{x_n}\widetilde{I}_k(x,t)=\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\nabla_{x'}[G_{t-s}(x'-y',x_n-y_n)$$
$$-G_{t-s}(x'-y',x_n+y_n)]\nabla_{y'}\partial_{y_n}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds$$
$$-2\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}G_{t-s}(x'-y',x_n+y_n)\nabla_{y'}\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds. \quad (4.67)$$

From (4.32), we get for any $x=(x',x_n)\in\mathbb{R}^n_+$ and $t>0$

$$\partial^3_{x_n}\widetilde{I}_k(x,t)=-\partial_{x_n}\big(P(u\cdot\nabla u)(x,t)\big)_k+\int_{\mathbb{R}^n_+}\partial_{x_n}[G_{\frac{t}{2}}(x'-y',x_n-y_n)$$
$$-G_{\frac{t}{2}}(x'-y',x_n+y_n)]\left(P(u\cdot\nabla)u\left(y,\frac{t}{2}\right)\right)_k dy$$
$$+\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}[G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)]$$
$$\times\partial_s\big(P(u\cdot\nabla)u(y,s)\big)_k dyds$$
$$-\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}[G_{t-s}(x'-y',x_n-y_n)-G_{t-s}(x'-y',x_n+y_n)]$$
$$\times\sum_{j=1}^{n-1}\partial^2_{y_j}\big(P(u\cdot\nabla)u(y,s)\big)_k dyds. \quad (4.68)$$

In addition, applying Lemmas 3.1, 3.2 to the strong solution $u$ of problem (1.1), we find for any $1\le\ell,m\le n-1$ and $0\le\gamma<1$, $s>1$

$$\|y_n^\gamma\partial_s P(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}+\|y_n^\gamma\partial_{y_\ell}\partial_{y_m}P(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\partial_{y_\ell}\partial_{y_n}P(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$\le\|y_n^\gamma\sum_{i,j=1}^n\partial_s\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\sum_{i,j=1}^n\partial_{y_\ell}\partial_{y_m}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\sum_{i,j=1}^n\partial_{y_\ell}\partial_{y_n}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\partial_s(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}+\|y_n^\gamma\partial_{y_\ell}\partial_{y_m}(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\partial_{y_\ell}\partial_{y_n}(u\cdot\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$\le\|y_n^\gamma\sum_{i,j=1}^n\nabla\mathcal{N}\partial_i\partial_j(u_i\partial_su_j+u_j\partial_su_i)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$
$$+\|y_n^\gamma\sum_{i,j=1}^n\nabla\mathcal{N}\partial_i\partial_j(u_i\partial_{y_\ell}\partial_{y_m}u_j+u_j\partial_{y_\ell}\partial_{y_m}u_i$$
$$+\partial_{y_\ell}u_i\partial_{y_m}u_j+\partial_{y_\ell}u_j\partial_{y_m}u_i)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}$$

$$
\begin{aligned}
&+\|y_n^\gamma \sum_{i,j=1}^{n} \partial_{y_n}\nabla\mathcal{N}\partial_i\partial_j(u_i\partial_{y_\ell}u_j+u_j\partial_{y_\ell}u_i)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma(\partial_s u\cdot\nabla)u+(u\cdot\partial_s\nabla)u(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma(\partial_{y_\ell}\partial_{y_m}(u\cdot\nabla)u+(u\cdot\partial_{y_\ell}\partial_{y_m}\nabla)u\\
&+(\partial_{y_\ell}u\cdot\partial_{y_m}\nabla)u+(\partial_{y_m}u\cdot\partial_{y_\ell}\nabla)u)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma((\partial_{y_\ell}\partial_{y_n}u\cdot\nabla)u+(u\cdot\partial_{y_\ell}\partial_{y_n}\nabla)u\\
&+(\partial_{y_\ell}u\cdot\partial_{y_n}\nabla)u+(\partial_{y_n}u\cdot\partial_{y_\ell}\nabla)u)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\\
\le{}& C\big(\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s u(s)\|_{L^2(\mathbb{R}^n_+)}+\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s u(s)\|_{L^2(\mathbb{R}^n_+)}+\|y_n^\gamma\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}+\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^3 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}+\|y_n^\gamma\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^3 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}+\|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}+\|y_n^\gamma\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}+\|y_n^\gamma\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma u(s)\|_{L^2(\mathbb{R}^n_+)}\|\partial_s\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}+\|y_n^\gamma\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2 u(s)\|_{L^2(\mathbb{R}^n_+)}\\
&+\|y_n^\gamma u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^3 u(s)\|_{L^2(\mathbb{R}^n_+)}\big)\\
\le{}& Cs^{-\frac{3}{2}-\frac{n}{2}+\frac{\gamma}{2}}. \qquad (4.69)
\end{aligned}
$$

From (4.66)–(4.69), using Lemmas 3.1, 3.2, we obtain for $1\le k\le n$, $0<\beta<1$ and $t>1$

$$
\begin{aligned}
&\|x_n^\beta\partial_{x_n}^3\widetilde{I}_k(x,t)\|_{L^1_x(\mathbb{R}^n_+)}+\|x_n^\beta\nabla_{x'}\partial_{x_n}^2\widetilde{I}_k(x,t)\|_{L^1_x(\mathbb{R}^n_+)}+\|x_n^\beta\nabla_{x'}^2\partial_{x_n}\widetilde{I}_k(x,t)\|_{L^1_x(\mathbb{R}^n_+)}\\
&\le\|x_n^\beta\nabla\big(P(u\cdot\nabla)u(\cdot,t)\big)_k\|_{L^1(\mathbb{R}^n_+)}+\int_{\mathbb{R}^n_+}\big(\||x_n-y_n|^\beta\nabla_x G_{\frac{t}{2}}(x'-y',x_n-y_n)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|(x_n+y_n)^\beta\nabla_x G_{\frac{t}{2}}(x'-y',x_n+y_n)\|_{L^1(\mathbb{R}^n_+)}\big)|\big(P(u\cdot\nabla)u\left(y,\frac{t}{2}\right)\big)_k|dy\\
&+\int_{\mathbb{R}^n_+}\|\nabla_x G_{\frac{t}{2}}(x'-y',x_n-y_n)\|_{L^1(\mathbb{R}^n_+)}y_n^\beta\left|\left(P(u\cdot\nabla)u\left(y,\frac{t}{2}\right)\right)_k\right|dy\\
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\big(\||x_n-y_n|^\beta\nabla_x G_{t-s}(x'-y',x_n-y_n)\|_{L^1(\mathbb{R}^n_+)}\\
&+\|(x_n+y_n)^\beta\nabla_x G_{t-s}(x'-y',x_n+y_n)\|_{L^1(\mathbb{R}^n_+)}\big)\\
&\times\big(|\partial_s\big(P(u\cdot\nabla)u(y,s)\big)_k|+|(\Delta_{y'}+\nabla_{y'}\partial_{y_n})\big(P(u\cdot\nabla)u(y,s)\big)_k|\big)dyds\\
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|\nabla_x G_{t-s}(x'-y',x_n-y_n)\|_{L^1(\mathbb{R}^n_+)}\\
&\times\big(y_n^\beta|\partial_s\big(P(u\cdot\nabla)u(y,s)\big)_k|+y_n^\beta|(\Delta_{y'}+\nabla_{y'}\partial_{y_n}+\nabla^2_{y'})\big(P(u\cdot\nabla)u(y,s)\big)_k|\big)dyds\\
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\|(x_n+y_n)^\beta\nabla_x\partial_{x_n}G_{t-s}(x'-y',x_n+y_n)\|_{L^1(\mathbb{R}^n_+)}\\
&\times|\nabla_{y'}\big(P(u\cdot\nabla)u(y,s)\big)_k|dyds
\end{aligned}
$$

$$
\begin{aligned}
&\le \|x_n^{\beta}\nabla\big(u\cdot\nabla\big)u(\cdot,t)\|_{L^1(\mathbb{R}^n_+)} + \|x_n^{\beta}\nabla\sum_{i,j=1}^{n}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,t)\|_{L^1(\mathbb{R}^n_+)}\\
&\quad + Ct^{-\frac{1}{2}+\frac{\beta}{2}}\Big(\|\big(u\cdot\nabla\big)u_k\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)} + \|\sum_{i,j=1}^{n}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}\Big)\\
&\quad + Ct^{-\frac{1}{2}}\Big(\|y_n^{\beta}\big(u\cdot\nabla\big)u_k\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)} + \|y_n^{\beta}\sum_{i,j=1}^{n}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\left(\cdot,\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}\Big)\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}s^{-\frac{3}{2}-\frac{n}{2}}ds + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{3}{2}-\frac{n}{2}+\frac{\beta}{2}}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\Big(\|\nabla\big((u\cdot\nabla)u(\cdot,s)\big)\|_{L^1(\mathbb{R}^n_+)}\\
&\quad + \|\nabla\sum_{i,j=1}^{n}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(\cdot,s)\|_{L^1(\mathbb{R}^n_+)}\Big)\\
&\le Ct^{-1-\frac{n}{2}+\frac{\beta}{2}} + \|x_n^{\beta}u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^2(\mathbb{R}^n_+)} + \|x_n^{\beta}\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}\\
&\quad + \|u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^2(\mathbb{R}^n_+)}\\
&\quad + \|y_n^{\beta}u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(t)\|_{L^2(\mathbb{R}^n_+)} + \|y_n^{\beta}\nabla u(t)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(t)\|_{L^2(\mathbb{R}^n_+)}\\
&\quad + Ct^{-\frac{1}{2}+\frac{\beta}{2}}\left(\left\|u\left(\frac{t}{2}\right)\right\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u\left(\frac{t}{2}\right)\|^2_{L^2(\mathbb{R}^n_+)}\right)\\
&\quad + Ct^{-\frac{1}{2}}\left(\left\|y_n^{\beta}u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)}\left\|\nabla u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} + \left\|u\left(\frac{t}{2}\right)\right\|^2_{L^2(\mathbb{R}^n_+)} + \left\|\nabla u\left(\frac{t}{2}\right)\right\|^2_{L^2(\mathbb{R}^n_+)}\right.\\
&\quad \left. + \left\|u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)}\left\|y_n^{\beta}u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} + \left\|\nabla u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)}\left\|y_n^{\beta}\nabla u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)}\right)\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\Big(\|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(s)\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}\\
&\quad + \|u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|_{L^2(\mathbb{R}^n_+)}\|\nabla^2u(s)\|_{L^2(\mathbb{R}^n_+)}\Big)ds\\
&\le Ct^{-1-\frac{n}{2}+\frac{\beta}{2}} + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}s^{-\frac{3}{2}-\frac{n}{2}}ds\\
&\le \widetilde{C}t^{-1-\frac{n}{2}+\frac{\beta}{2}},
\end{aligned}
$$

which implies for $0<\beta<1$ and $t>1$

$$
\|x_n^{\beta}\nabla_x^3\widetilde{I}_k(x,t)\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-1-\frac{n}{2}+\frac{\beta}{2}}, \quad k=1,2\cdots,n. \tag{4.70}
$$

Let $0<\beta<1$, from (4.45)–(4.47), (4.51) and (4.52), we get for any $t>0$

$$
\begin{aligned}
&\left\|x_n^{\beta}\frac{\partial^3}{\partial x_n^3}\widetilde{J}_n(x,t)\right\|_{L^1(\mathbb{R}^n_+)}\\
&\quad \le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j,\ell=1}^{n-1}\left\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}N_{\ell\ell}(x,y,t-s)\right\|_{L^1_x(\mathbb{R}^n_+)}\left|\frac{\partial}{\partial y_j}b_k(y,s)\right|dyds
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_j\partial x_j}J_n(x'-y',x_n+y_n,0,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\\
&\times\Big|\frac{\partial}{\partial y_k}b_k(y,s)\Big|dyds\\
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_j}b_k(y,s)\Big|dyds\\
&+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_n\partial x_n}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_k}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds;
\end{aligned}
\tag{4.71}
$$

and there holds for $1\le m,q\le n-1$,

$$
\begin{aligned}
&\Big\|x_n^{\beta}\frac{\partial^3}{\partial x_m\partial x_n^2}\widetilde{J}_n(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}N_{jn}(x,y,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\quad+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_n}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds;
\end{aligned}
\tag{4.72}
$$

and

$$
\begin{aligned}
&\Big\|x_n^{\beta}\frac{\partial^3}{\partial x_m\partial x_q\partial x_n}\widetilde{J}_n(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,j=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}N_{jq}(x,y,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\quad+C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_q}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds.
\end{aligned}
\tag{4.73}
$$

Combining (4.71), (4.72) and (4.73) yields for $0<\beta<1$ and $t>0$

$$
\|x_n^{\beta}\nabla^3\widetilde{J}_n(x,t)\|_{L^1(\mathbb{R}^n_+)}\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds. \tag{4.74}
$$

Let $1\le j,m,q\le n-1$. Using (4.48)–(4.53) yields $0<\beta<1$ and $t>0$

$$
\Big\|x_n^{\beta}\frac{\partial^3}{\partial x_n^3}\widetilde{J}_m(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}
$$

$$
\begin{aligned}
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,\ell=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_\ell\partial x_k}N_{\ell n}(x,y,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\quad +C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_m\partial x_n}J_n(x'-y',y_n,x_n,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad \times\Big|\frac{\partial}{\partial y_k}b_k(y,s)\Big|dyds\\
&\quad +C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_n\partial x_m}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_k}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds; \qquad (4.75)
\end{aligned}
$$

$$
\begin{aligned}
&\Big\|x_n^{\beta}\frac{\partial^3}{\partial x_j\partial x_n^2}\widetilde{J}_m(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k,\ell=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}N_{\ell\ell}(x,y,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\quad +C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}J_n(x'-y',y_n,x_n,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\\
&\quad \times\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\quad +C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}G_{t-s}(x-y^*)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_m}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds; \qquad (4.76)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|x_n^{\beta}\frac{\partial^3}{\partial x_j\partial x_q\partial x_n}\widetilde{J}_m(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\sum_{k=1}^{n-1}\Big\|(x_n+y_n)^{\beta}\frac{\partial^2}{\partial x_k\partial x_j}N_{mn}(x,y,t-s)\Big\|_{L^1_x(\mathbb{R}^n_+)}\Big|\frac{\partial}{\partial y_q}b_k(y,s)\Big|dyds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds. \qquad (4.77)
\end{aligned}
$$

From (4.74)–(4.77), we derive for $0<\beta<1$ and $t>0$

$$
\|x_n^{\beta}\nabla^3\widetilde{J}_m(x,t)\Big\|_{L^1(\mathbb{R}^n_+)}\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|\nabla_{y'}b(y,s)\|_{L^1(\mathbb{R}^n_+)}ds,\quad m=1,2,\cdots,n. \qquad (4.78)
$$

Recall the definition of $b(y,s)=(b_1(y,s),b_2(y,s),\cdots,b_n(y,s))$:

$$
b_j(y,s)=(u\cdot\nabla)u_j(y,s)+\sum_{i,\ell=1}^{n}\partial_{y_j}\mathcal{N}\partial_i\partial_\ell(u_iu_\ell)(y,s),\quad s>0,\quad 1\le j\le n.
$$

Whence using Lemma 3.1 yields for $s > 0$,

$$\begin{aligned}
\|\nabla' b(\cdot, s)\|_{L^1(\mathbb{R}^n_+)} &\leq \|\nabla'((u\cdot\nabla)u(\cdot, s))\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \|\nabla'\left(\sum_{i,\ell=1}^n \nabla\mathcal{N}\partial_i\partial_\ell(u_i u_\ell)(\cdot, s)\right)\|_{L^1(\mathbb{R}^n_+)} \\
&\leq \|(\nabla' u\cdot\nabla)u(\cdot, s))\|_{L^1(\mathbb{R}^n_+)} + \|(u\cdot\nabla'\nabla)u(\cdot, s))\|_{L^1(\mathbb{R}^n_+)} \\
&\quad + \|\sum_{i,\ell=1}^n \nabla\mathcal{N}\partial_i\partial_\ell\nabla'(u_i u_\ell)(\cdot, s)\|_{L^1(\mathbb{R}^n_+)} \\
&\leq C(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)}). \qquad (4.79)
\end{aligned}$$

Inserting (4.79) into (4.78), using Lemma 3.2, we find for $0 < \beta < 1$ and $t > 1$

$$\begin{aligned}
\|x_n^\beta\nabla^3\widetilde{J}_m(x, t)\|_{L^1(\mathbb{R}^n_+)} &\leq C\int_{\frac{t}{2}}^t (t-s)^{-1+\frac{\beta}{2}}(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^n_+)})ds \\
&\leq C\int_{\frac{t}{2}}^t (t-s)^{-1+\frac{\beta}{2}} s^{-1-\frac{n}{2}} ds \\
&\leq Ct^{-1-\frac{n}{2}+\frac{\beta}{2}}, \quad m = 1, 2, \cdots, n. \qquad (4.80)
\end{aligned}$$

Combining (4.30), (4.70) and (4.80), we conclude for $0 < \beta < 1$ and $t > 1$

$$\begin{aligned}
\|x_n^\beta\nabla^3\widetilde{w}_m(x, t)\|_{L^1(\mathbb{R}^n_+)} &\leq \|x_n^\beta\nabla^3\widetilde{I}_m(x, t)\|_{L^1(\mathbb{R}^n_+)} + \|x_n^\beta\nabla^3\widetilde{J}_m(x, t)\|_{L^1(\mathbb{R}^n_+)} \\
&\leq Ct^{-1-\frac{n}{2}+\frac{\beta}{2}}, \quad m = 1, 2, \cdots, n. \qquad (4.81)
\end{aligned}$$

Observe that for $t > 0$, $u(y, \frac{t}{2})|_{\partial\mathbb{R}^n_+} = 0$. So for $t > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \partial_{x_j} G_t(x-y)u\left(y, \frac{t}{2}\right)dy &= \int_{\mathbb{R}^n_+}(-\partial_{y_j})G_t(x-y)u\left(y, \frac{t}{2}\right)dy \\
&= \int_{\mathbb{R}^n_+} G_t(x-y)\partial_{y_j}u\left(y, \frac{t}{2}\right)dy, \quad 1 \leq j \leq n.
\end{aligned}$$

Combining the estimate (2.3), Theorem 1.3 and Lemma 3.2, we derive for $0 < \beta < 1$ and $t > 1$

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n_+} x_n^\beta\nabla_x^3\mathcal{M}\left(x, y, \frac{t}{2}\right)u\left(y, \frac{t}{2}\right)dy\right\|_{L^1_x(\mathbb{R}^n_+)} \\
&\leq \int_{\mathbb{R}^n_+}(|x_n - y_n|^\beta\nabla_x^2 G_t(x-y)\|_{L^1_x(\mathbb{R}^n_+)}|\nabla u\left(y, \frac{t}{2}\right)|dy \\
&\quad + \int_{\mathbb{R}^n_+}\|\nabla_x^2 G_t(x-y)\|_{L^1_x(\mathbb{R}^n_+)}|y_n^\beta\nabla u\left(y, \frac{t}{2}\right)|dy \\
&\quad + \int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\beta+\frac{1}{2}}\nabla_x^3 G_t(x-y^*)\|_{L^1_x(\mathbb{R}^n_+)}|y_n^{-\frac{1}{2}}u\left(y, \frac{t}{2}\right)|dy \\
&\quad + \int_{\mathbb{R}^n_+}\|(x_n+y_n)^{\beta+\frac{1}{2}}\nabla_x^3\mathcal{M}^*\left(x, y, \frac{t}{2}\right)\|_{L^1_x(\mathbb{R}^n_+)}|y_n^{-\frac{1}{2}}u\left(y, \frac{t}{2}\right)|dy \\
&\leq Ct^{-1+\frac{\beta}{2}}\|\nabla u\left(\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)} + Ct^{-1}\|y_n^\beta\nabla u\left(\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)} + Ct^{-1+\frac{\beta}{2}}\|y_n^{-\frac{1}{2}}u\left(\frac{t}{2}\right)\|_{L^1(\mathbb{R}^n_+)}
\end{aligned}$$

$$\le \widetilde{C}t^{-\frac{3}{2}+\frac{\beta}{2}}. \tag{4.82}$$

Let $0 < \beta < 1$. Combining (4.29), (4.81) and (4.82) yields for $t > 1$

$$\|x_n^\beta \nabla^3 u(t)\|_{L^1(\mathbb{R}^n_+)} \le C\left(t^{-\frac{3}{2}+\frac{\beta}{2}} + t^{-1-\frac{n}{2}+\frac{\beta}{2}}\right) \le \widetilde{C}t^{-\frac{3}{2}+\frac{\beta}{2}},$$

which is (1.3). □

## Appendix

This section devotes to finding a counterexample for the Stokes flow $e^{-tA}u_0$ with initial datum $u_0 \in L^1_\sigma(\mathbb{R}^n_+)$. That is,

**Proposition A** *There exists an initial vector function* $u_0 = (u_{01}, u_{02}, \cdots, u_{0n}) \in L^1(\mathbb{R}^n_+)$ $(n \ge 2)$*, which satisfies* $\nabla \cdot u_0 = 0$ *in* $\mathbb{R}^n_+$ *and* $u_{0n}|_{\partial\mathbb{R}^n_+} = 0$*, such that* $e^{-tA}u_0 \notin L^1(\mathbb{R}^n_+)$ *for each* $t > 0$.

We first give some notations and introduce some known results.

Let $\mathcal{F}_m$ ($\mathcal{F}_m^{-1}$) be the (inverse) Fourier transform in $\mathbb{R}^m$ given by

$$\mathcal{F}_m[f](\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} f(x)dx \ \left(\mathcal{F}_m^{-1}[f](\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{i\xi\cdot x} f(x)dx\right).$$

A direct calculation shows that for the function $f(x) = e^{-a|x|^2}$ in $\mathbb{R}^m$ with $a > 0$,

$$\mathcal{F}_m[f](y) = \left(\frac{\pi}{a}\right)^{\frac{m}{2}} e^{-\frac{|y|^2}{4a}}.$$

The Riesz operators $S_j$ $(j = 1, 2, \cdots, n-1)$ are defined by

$$\mathcal{F}_{n-1}[S_j f](\xi') = \frac{i\xi_j}{|\xi'|}\mathcal{F}_{n-1}[f](\xi'),$$

where $\xi' = (\xi_1, \xi_2, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$.

In the following arguments, for simplicity, we denote the Fourier transform $\mathcal{F}_{n-1}[g]$ in $\mathbb{R}^{n-1}$ by $\widehat{g}$. Recall $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$, $x = (x', x_n) \in \mathbb{R}^n$ satisfies $G_t(x) = G_t^{(n-1)}(x')G_t^{(1)}(x_n)$, where

$$G_t^{(n-1)}(x') = (4\pi t)^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{4t}}, \quad G_t^{(1)}(x_n) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x_n|^2}{4t}}.$$

Define the operator $E(t)$ by

$$\begin{aligned}(E(t)g)(x) &= \int_{\mathbb{R}^n_+} [G_t(x'-y', x_n - y_n) - G_t(x'-y', x_n + y_n)]g(y)dy \\ &= \int_{\mathbb{R}^n_+} G_t^{(n-1)}(x'-y')[G_t^{(1)}(x_n - y_n) - G_t^{(1)}(x_n + y_n)]g(y)dy.\end{aligned}$$

By the solution formula in [25], the Stokes flow $e^{-tA}u_0 = (u', u_n)$ can be represented as

$$\begin{cases} u_n = UE(t)V_1u_0, \\ u' = E(t)V_2u_0 - SUE(t)V_1u_0, \end{cases} \tag{A.1}$$

where

$$V_1u_0 = -S\cdot u_0' + u_{0n}, \quad V_2u_0 = u_0' + Su_{0n}, \quad S = (S_1, S_2, \cdots, S_{n-1}),$$
$$\widehat{Ug}(\xi', x_n) = |\xi'|\int_0^{x_n} e^{-|\xi'|(x_n-y_n)}\widehat{g}(\xi', y_n)dy_n.$$

***Proof of Proposition A.*** Set

$$f = E(t)V_1u_0 = E(t)[u_{0n} - S\cdot u_0'], \quad h(\xi', s) = \begin{cases} |\xi'|e^{-|\xi'|s} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Then $u_n = Uf$ and for $t > 0$

$$\begin{aligned}\widehat{u_n}(\xi', x_n, t) = \widehat{Uf}(\xi', x_n) &= |\xi'|\int_0^{x_n} e^{-|\xi'|(x_n-y_n)}\widehat{f}(\xi', y_n)dy_n \\ &= \int_0^\infty h(\xi', x_n - y_n)\widehat{f}(\xi', y_n)dy_n.\end{aligned}$$

Whence using the definition of $h$ yields for $t > 0$

$$\begin{aligned}\int_0^\infty \widehat{u_n}(\xi', x_n, t)dx_n &= \int_0^\infty \left(\int_0^\infty h(\xi', x_n - y_n)dx_n\right)\widehat{f}(\xi', y_n)dy_n \\ &= \int_0^\infty \left(\int_{-y_n}^\infty h(\xi', s)ds\right)\widehat{f}(\xi', y_n)dy_n \\ &= \int_0^\infty \left(\int_0^\infty h(\xi', s)ds\right)\widehat{f}(\xi', y_n)dy_n \\ &= \int_0^\infty \left(\int_0^\infty |\xi'|e^{-|\xi'|s}ds\right)\widehat{f}(\xi', y_n)dy_n \\ &= \int_0^\infty \widehat{f}(\xi', y_n)dy_n. \end{aligned} \tag{A.2}$$

Note that

$$\begin{aligned}\widehat{f}(\xi', y_n) &= E(t)[\widehat{u_{0n} - S\cdot u_0'}](\xi', y_n) \\ &= \widehat{G_t^{(n-1)}}(\xi')\int_0^\infty [G_t^{(1)}(y_n - z_n) - G_t^{(1)}(y_n + z_n)] \\ &\quad \times[\widehat{u_{0n}}(\xi', z_n) - \frac{i\xi'}{|\xi'|}\cdot\widehat{u_0'}(\xi', z_n)]dz_n.\end{aligned} \tag{A.3}$$

Inserting (A.3) into (A.2) yields for $t > 0$

$$\begin{aligned}\int_0^\infty \widehat{u_n}(\xi', x_n, t)dx_n &= \widehat{G_t^{(n-1)}}(\xi')\int_0^\infty \left(\int_0^\infty [G_t^{(1)}(y_n - z_n) - G_t^{(1)}(y_n + z_n)]dy_n\right) \\ &\quad \times[\widehat{u_{0n}}(\xi', z_n) - \frac{i\xi'}{|\xi'|}\cdot\widehat{u_0'}(\xi', z_n)]dz_n \\ &= \widehat{G_t^{(n-1)}}(\xi')\int_0^\infty \left(\int_{-z_n}^\infty G_t^{(1)}(s)ds - \int_{z_n}^\infty G_t^{(1)}(s)ds\right) \\ &\quad \times[\widehat{u_{0n}}(\xi', z_n) - \frac{i\xi'}{|\xi'|}\cdot\widehat{u_0'}(\xi', z_n)]dz_n \\ &= 2\widehat{G_t^{(n-1)}}(\xi')\int_0^\infty \left(\int_0^{z_n} G_t^{(1)}(s)ds\right)\end{aligned}$$

$$
\begin{aligned}
&\times[\widehat{u_{0n}}(\xi', z_n) - \frac{i\xi'}{|\xi'|}\cdot \widehat{u_0'}(\xi', z_n)]dz_n \\
&= 2\widehat{G_t^{(n-1)}}(\xi')\int_0^\infty \left(\int_0^{z_n} G_t^{(1)}(s)ds\right)\widehat{u_{0n}}(\xi', z_n)dz_n \\
&\quad -2\widehat{SG_t^{(n-1)}}(\xi')\cdot\int_0^\infty \left(\int_0^{z_n} G_t^{(1)}(s)ds\right)\widehat{u_0'}(\xi', z_n)dz_n \\
&= I_{1t}(\xi') + I_{2t}(\xi').
\end{aligned}
\tag{A.4}
$$

Set

$$
\begin{aligned}
u_{01}(x', x_n) &= G_1^{(n-1)}(x')(e^{-x_n} - 2e^{-2x_n}), \\
u_{0n}(x', x_n) &= -\partial_{x_1} G_1^{(n-1)}(x')\int_0^{x_n}(e^{-s} - 2e^{-2s})ds.
\end{aligned}
$$

Then $u_0 = (u_{01}, 0, \cdots, 0, u_{0n})$ satisfies $\|u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$ and

$$
\nabla\cdot u_0 = \partial_1 u_{01} + \partial_n u_{0n} = 0, \quad u_{0n}(x', 0) = 0, \quad \forall\, x' \in \mathbb{R}^{n-1}.
$$

For the given $u_0$, from (A.4), we have for $t > 0$

$$
\begin{aligned}
&\|\mathcal{F}_{n-1}^{-1}[I_{1t}]\|_{L^1(\mathbb{R}^{n-1})} \\
&= 2\left\|\int_{\mathbb{R}^{n-1}} G_t^{(n-1)}(x'-y')\int_0^\infty\left(\int_0^{z_n} G_t^{(1)}(s)ds\right)u_{0n}(y', z_n)dz_n dy'\right\|_{L^1(\mathbb{R}^{n-1})} \\
&\le 2\|G_t^{(n-1)}\|_{L^1(\mathbb{R}^{n-1})}\int_{\mathbb{R}^{n-1}}\int_0^\infty\left(\int_0^\infty G_t^{(1)}(s)ds\right)|u_{0n}(y', z_n)|dz_n dy' \\
&\le 2\|u_{0n}\|_{L^1(\mathbb{R}^n_+)}.
\end{aligned}
\tag{A.5}
$$

In addition,

$$
\begin{aligned}
I_{2t}(\xi') &= -2\widehat{S_1 G_t^{(n-1)}}(\xi')\int_0^\infty\left(\int_0^{z_n} G_t^{(1)}(s)ds\right)\widehat{u_{01}}(\xi', z_n)dz_n \\
&= -2\widehat{S_1 G_t^{(n-1)}}(\xi')\widehat{G_1^{(n-1)}}(\xi')\int_0^\infty\left(\int_0^{z_n} G_t^{(1)}(s)ds\right)(e^{-z_n} - 2e^{-2z_n})dz_n \\
&= 2\widehat{S_1 G_{t+1}^{(n-1)}}(\xi')\int_0^\infty\left(\int_0^{z_n} G_t^{(1)}(s)ds\right)(2e^{-2z_n} - e^{-z_n})dz_n,
\end{aligned}
$$

from which, we get for $t > 0$

$$
\begin{aligned}
|\mathcal{F}_{n-1}^{-1}[I_{2t}](x')| &= 2|[S_1 G_{t+1}^{(n-1)}](x')|\Bigg(\int_0^{\log_e 2}\Big(\int_0^{z_n} G_t^{(1)}(s)ds\Big)(2e^{-2z_n} - e^{-z_n})dz_n \\
&\qquad + \int_{\log_e 2}^\infty\Big(\int_0^{z_n} G_t^{(1)}(s)ds\Big)(e^{-z_n} - 2e^{-2z_n})dz_n\Bigg) \\
&\ge 2|[S_1 G_{t+1}^{(n-1)}](x')|\int_{\log_e 2}^\infty\left(\int_0^{z_n} G_t^{(1)}(s)ds\right)(e^{-z_n} - 2e^{-2z_n})dz_n \\
&\ge 2|[S_1 G_{t+1}^{(n-1)}](x')|\int_0^{\log_e 2} G_t^{(1)}(s)ds\int_{\log_e 2}^\infty (e^{-z_n} - 2e^{-2z_n})dz_n \\
&\ge C(t)|[S_1 G_{t+1}^{(n-1)}](x')|,
\end{aligned}
\tag{A.6}
$$

where

$$C(t)=\int_0^{\log_e 2}G_t^{(1)}(s)ds\int_{\log_e 2}^{\infty}(e^{-z_n}-2e^{-2z_n})dz_n=\frac{1}{4}\int_0^{\log_e 2}G_t^{(1)}(s)ds.$$

Observe that for $\xi'=(\xi_1,0,\cdots,0)$,

$$\lim_{\xi_1\to 0^+}\frac{\xi_1}{|\xi'|}e^{-(t+1)|\xi'|^2}=1,\quad \lim_{\xi_1\to 0^-}\frac{\xi_1}{|\xi'|}e^{-(t+1)|\xi'|^2}=-1.$$

This shows that for $t>0$, the following function

$$\widehat{S_1G_{t+1}^{(n-1)}}(\xi')=\frac{i\xi_1}{|\xi'|}e^{-(t+1)|\xi'|^2}$$

is not continuous at $\xi'=0$. Whence $S_1G_{t+1}^{(n-1)}\notin L^1(\mathbb{R}^{n-1})$. Together with (A.6), we conclude for $t>0$

$$\|\mathcal{F}_{n-1}^{-1}[I_{2t}]\|_{L^1(\mathbb{R}^{n-1})}\geq C(t)\|S_1G_{t+1}^{(n-1)}\|_{L^1(\mathbb{R}^{n-1})}=+\infty. \tag{A.7}$$

From (A.4), (A.5) and (A.7), we derive for $t>0$

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^{n-1}}|u_n(x',x_n,t)|dx'dx_n\\ &\geq\int_{\mathbb{R}^{n-1}}\left|\int_0^\infty u_n(x',x_n,t)dx_n\right|dx'\\ &\geq\int_{\mathbb{R}^{n-1}}|\mathcal{F}_{n-1}^{-1}[I_{2t}](x')|dx'-\int_{\mathbb{R}^{n-1}}|\mathcal{F}_{n-1}^{-1}[I_{1t}](x')|dx'\\ &\geq\|\mathcal{F}_{n-1}^{-1}[I_{2t}]\|_{L^1(\mathbb{R}^{n-1})}-2\|u_{0n}\|_{L^1(\mathbb{R}_+^n)}=+\infty,\end{aligned}$$

then by (A.1)

$$\begin{aligned}\|e^{-tA}u_0\|_{L^1(\mathbb{R}_+^n)}=\|u\|_{L^1(\mathbb{R}_+^n)}&=\int_0^\infty\int_{\mathbb{R}^{n-1}}\left(|u'(x',x_n,t)|^2+|u_n(x',x_n,t)|^2\right)^{\frac{1}{2}}dx'dx_n\\ &\geq\int_0^\infty\int_{\mathbb{R}^{n-1}}|u_n(x',x_n,t)|dx'dx_n=+\infty.\end{aligned}$$

□

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