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The N-stable category

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Abstract

A well-known theorem of Buchweitz provides equivalences between three categories: the stable category of Gorenstein projective modules over a Gorenstein algebra, the homotopy category of acyclic complexes of projectives, and the singularity category. To adapt this result to N-complexes, one must find an appropriate candidate for the N-analogue of the stable category. We identify this "N-stable category" via the monomorphism category and prove Buchweitz's theorem for N-complexes over a Grothendieck abelian category. We also compute the Serre functor on the N-stable category over a self-injective algebra and study the resultant fractional Calabi–Yau properties.

1 Introduction

The notion of N-complexes, which goes back to Mayer [22] and was first studied from a homological point of view by Kapranov [16] and Dubois-Violette [8], has received significant interest in recent years. As well as having applications in physics via spin gauge fields (see e.g. [9]), they are homologically interesting in their own right (see e.g. [23]. In addition, they provide the simplest examples of N-differential graded categories, which, for N a prime number, play an important role in categorification at roots of unity, see e.g. [10–12, 19, 20].

In the classical case of $N = 2$, which recovers the usual notion of homological algebra, there are numerous deep and important theorems connecting various categories obtained from complexes. One such example is a celebrated theorem by Buchweitz [4, Theorem 4.4.1], which, adapted to the setting of a Gorenstein abelian category $\mathcal{A}$, provides equivalences between a) $K^{ac}(\mathrm{Proj}(\mathcal{A}))$, the homotopy category of acyclic complexes of projective objects; b) $D^s(\mathcal{A})$, the singularity category of $\mathcal{A}$ (i.e., the Verdier quotient of the bounded derived category by the thick subcategory of perfect complexes); and c) $\mathrm{stab}(\mathrm{Gproj}(\mathcal{A}))$, the stable category of Gorenstein projective objects in $\mathcal{A}$. The equivalence between b) and c) was

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independently proved by Rickard [25, Theorem 2.1] in the special case of Frobenius exact abelian categories.

There are obvious N-complex analogues of categories a) and b), and an equivalence $K_N^{ac}(\mathrm{Proj}(\mathcal{A})) \cong D_N^s(\mathcal{A})$ generalizing Buchweitz was discovered by Bahiraei et al. [1]. This raises a question: is there an "N-stable" category which completes the missing link in Buchweitz's theorem? In this paper, we determine the correct object by investigating the monomorphism category, $\mathrm{MMor}_{N-2}(\mathcal{A})$, whose objects are diagrams of $N-2$ successive monomorphisms in $\mathcal{A}$. The monomorphism category has been intensively studied, particularly for $N = 3$ [26, 27], but also for general N [29]. Monomorphism categories associated to arbitrary species have also recently been studied by [13].

If $\mathcal{E}$ is an exact category, then $\mathrm{MMor}_{N-2}(\mathcal{E})$ can be given the structure of an exact category (Proposition 3.5). If $\mathcal{E}$ is Frobenius, then $\mathrm{MMor}_{N-2}(\mathcal{E})$ inherits this property (Theorem 3.12); in this case, we define the N-stable category, $\mathrm{stab}_N(\mathcal{E})$ to be the stable category of $\mathrm{MMor}_{N-2}(\mathcal{E})$. For a Gorenstein abelian category $\mathcal{A}$, we construct equivalences of triangulated categories $K_N^{ac}(\mathrm{Proj}(\mathcal{A})) \xrightarrow{\sim} \mathrm{stab}_N(\mathrm{Gproj}(\mathcal{A}))$ (Theorem 4.12) and $\mathrm{stab}_N(\mathrm{Gproj}(\mathcal{A})) \xrightarrow{\sim} D_N^s(\mathcal{A})$ (Theorem 5.3) generalizing Buchweitz, demonstrating that the N-stable category merits the name.

Classically, the stable category of a finite-dimensional self-injective algebra A provides a rich source of examples of negative or fractional Calabi–Yau categories, a topic of major interest in homological representation theory with connections to many areas of mathematics, see e.g. [6, 7, 17, 18]. One might hope the N-stable category enjoys similar properties, and in Corollary 6.11 we prove that if the Nakayama automorphism of A has finite order, then $\mathrm{stab}_N(A)$ is fractional Calabi–Yau with the denominator parametrized by N.

To prove result, we provide an explicit description of the Serre functor on $\mathrm{stab}_N(A)$ in Theorem 6.10. The effect of the Auslander-Reiten translation (from which the Serre functor can easily be derived) on the objects of the stable monomorphism category has already been computed by Ringel and Schmidmeier [26] for $N = 3$ and Xiong et al. [28] for general N. However, utilizing the connection with N-complexes, we are able to provide a simpler version of their construction which is also functorial.

The structure of the paper is as follows: In Sect. 2, we briefly summarize relevant background material while establishing our terminology and notational conventions. Section 3 develops the theory of the monomorphism category, culminating in the definition of the N-stable category. The two relevant equivalences of Buchweitz's theorem are generalized in Sects. 4 and 5. In Sect. 6, we describe the Serre functor of the N-stable category, discuss its Calabi–Yau properties, and provide a worked example.

2 Definitions and notation

2.1 Triangulated categories

We shall assume the reader is familiar with the basic theory of triangulated categories. In lieu of a detailed explanation, we give a quick overview of the relevant topics and terminology; for more details, the reader may consult Neeman [24] or Gelfand-Manin [14].

Let $\mathcal{T}$ be an additive category, and let $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be an additive automorphism of $\mathcal{T}$. We shall call Σ the **suspension functor** on $\mathcal{T}$. A **triangle** in $\mathcal{T}$ is any diagram of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. A **triangulated category** is the data of $\mathcal{T}$, Σ, and a collection of triangles (called the **distinguished triangles**), satisfying certain axioms.

If $(\mathcal{T}_1, \Sigma_1)$ and $(\mathcal{T}_2, \Sigma_2)$ are triangulated categories, a **triangulated functor** $F : \mathcal{T}_1 \to \mathcal{T}_2$ is the data of an additive functor F and an isomorphism $\phi : F\Sigma_1 \xrightarrow{\sim} \Sigma_2 F$, such that F (together with ϕ) maps distinguished triangles in $\mathcal{T}_1$ to distinguished triangles in $\mathcal{T}_2$.

Any morphism $f : X \to Y$ in a triangulated category $\mathcal{T}$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. We refer to Z as the **cone** of f; it is unique up to (non-canoncial) isomorphism. Similarly, we refer to X as the **cocone** of g.

A full, replete, additive subcategory $\mathcal{S} \subseteq \mathcal{T}$ is said to be a **triangulated subcategory** if $\mathcal{S}$ is closed under $\Sigma^{\pm 1}$ and the cone of any morphism in $\mathcal{S}$ lies in $\mathcal{S}$. A triangulated subcategory $\mathcal{S}$ is said to be **thick** if it is closed under direct summands. In this case, we can form a new triangulated category $\mathcal{T}/\mathcal{S}$, called the **Verdier quotient**, with the same objects and suspension functor as $\mathcal{T}$. There is a natural triangulated functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ which is the identity on objects and whose kernel is precisely $\mathcal{S}$. $\mathcal{T}/\mathcal{S}$ can also be viewed as the localization of $\mathcal{T}$ with respect to the multiplicative set of morphisms with cone in $\mathcal{S}$, hence morphisms in $\mathcal{T}/\mathcal{S}$ can be expressed in terms of a calculus of left and right fractions. A triangle in $\mathcal{T}/\mathcal{S}$ is distinguished if and only if it is isomorphic (in $\mathcal{T}/\mathcal{S}$) to a distinguished triangle in $\mathcal{T}$.

2.2 Serre duality and Calabi–Yau categories

Let F be a field and let $(\mathcal{T}, \Sigma)$ be an F-linear, Hom-finite triangulated category. A **Serre functor** on $\mathcal{T}$ is an equivalence of triangulated categories $S : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ together with isomorphisms $\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong D\,\mathrm{Hom}_{\mathcal{T}}(Y, SX)$ which are natural in X and Y. Here $D := \mathrm{Hom}_F(-, F)$ is the F-linear duality.

Let $m, l \in \mathbb{Z}$. We say that $\mathcal{T}$ is **(weakly)** (m, l)**-Calabi–Yau** if $\mathcal{T}$ has a Serre functor S and there is an isomorphism of functors $S^l \cong \Sigma^m$. (Elsewhere in the literature, this is often written using the "fraction" $\frac{m}{l}$.) Note that a triangulated category may be (m, l)-Calabi–Yau for many different integer pairs (m, l). If $l = 1$, then we shall simply say that $\mathcal{T}$ is (weakly) m-Calabi–Yau. There is a stronger notion of the Calabi–Yau property, due to Keller [17], which requires the isomorphism be compatible with the triangulated structure, but our focus will be on the weaker notion.

2.3 Exact categories

We recall some basic definitions and terminology regarding exact categories. For a more comprehensive overview, we refer to Bühler [5].

Let $\mathcal{E}$ be an additive category. A **kernel-cokernel pair** in $\mathcal{E}$ is a diagram $X \xrightarrow{i} Y \xrightarrow{p} Z$ such that i is the kernel of p and p is the cokernel of i. Let $\mathcal{S}$ be a collection of kernel-cokernel pairs which is closed under isomorphisms; its elements will be called the **admissible short exact sequences**. The kernels in $\mathcal{S}$ are called **admissible monomorphisms** and the cokernels are called **admissible epimorphisms**. If the class of admissible monomorphisms (resp., admissible epimorphisms) contains all identity morphisms, is closed under composition, and is stable under pushouts (resp., pullbacks), we say that the pair $(\mathcal{E}, \mathcal{S})$ is an **exact category**. For a more precise statement of the axioms, see [5, Definition 2.1]. Note that $(\mathcal{E}, \mathcal{S})$ is exact if and only if $(\mathcal{E}^{op}, \mathcal{S}^{op})$ is exact. If $(\mathcal{E}, \mathcal{S})$ and $(\mathcal{E}', \mathcal{S}')$ are exact categories, we say an additive functor $F : \mathcal{E} \to \mathcal{E}'$ is **exact** if $F(\mathcal{S}) \subseteq \mathcal{S}'$.

If $\mathcal{E}$ is an exact category, we say that a subcategory $\mathcal{E}'$ of $\mathcal{E}$ is **closed under extensions** if whenever $X \rightarrowtail Y \twoheadrightarrow Z$ is an admissible short exact sequence in $\mathcal{E}$ with $X, Z \in \mathcal{E}'$, then Y is isomorphic to an object in $\mathcal{E}'$. If $\mathcal{E}'$ is a full, additive subcategory of $\mathcal{E}$ which is closed under extensions, then $\mathcal{E}'$ inherits the structure of an exact category: a kernel-cokernel pair in $\mathcal{E}'$ is admissible if and only if it is admissible in $\mathcal{E}$. (See [5, Lemma 10.20].) With this inherited structure, we say $\mathcal{E}'$ is a **fully exact** subcategory of $\mathcal{E}$.

Any additive category can be given the structure of an exact category by defining the split exact sequences to be admissible. Any abelian category can be given the structure of an exact category by defining every short exact sequence to be admissible. A small exact category $\mathcal{E}$ can be embedded as a fully exact subcategory of an abelian category [5, Theorem A.1].

An object P in an exact category $\mathcal{E}$ is **projective** if, for every admissible epimorphism $p : Y \twoheadrightarrow Z$ and every morphism $f : P \to Z$, there exists a lift $g : P \to Y$ satisfying $f = pg$. Injective objects are defined dually. We let $\mathrm{Proj}(\mathcal{E})$ (resp., $\mathrm{Inj}(\mathcal{E})$) denote the full subcategory of $\mathcal{E}$ consisting of the projective (resp., injective) objects. We say $\mathcal{E}$ has **enough projectives** if for every object $X \in \mathcal{E}$ there exists an admissible epimorphism $P \twoheadrightarrow X$ with P projective; likewise $\mathcal{E}$ has **enough injectives** if for every object X there is an admissible monomorphism $X \rightarrowtail I$ with I injective.

We define the **projectively stable category** of $\mathcal{E}$ to be the category $\underline{\mathcal{E}}$ whose objects are those of $\mathcal{E}$ and whose morphisms are given by $\mathrm{Hom}_{\underline{\mathcal{E}}}(X, Y) := \mathrm{Hom}_{\mathcal{E}}(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ is the additive subgroup of morphisms which factor through a projective object. Dually, we can quotient out by morphisms factoring through injective objects to form the **injectively stable category** $\overline{\mathcal{E}}$. If $\mathrm{Proj}(\mathcal{E}) = \mathrm{Inj}(\mathcal{E})$ and $\mathcal{E}$ has enough projectives and injectives, we say $\mathcal{E}$ is a **Frobenius exact category**. In this case, both stable categories coincide and can be given the structure of a triangulated category, which we shall denote by $(\mathrm{stab}(\mathcal{E}), \Omega^{-1})$. The suspension functor Ω^{-1} is defined by choosing for each object X an admissible monomorphism $X \rightarrowtail I_X$ into an injective object; $\Omega^{-1}X$ is then defined to be the cokernel of this map. An admissible short exact sequence $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ in $\mathcal{E}$ induces a natural map $h : Z \to \Omega^{-1}X$ in $\mathrm{stab}(\mathcal{E})$, which gives rise to a triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} \Omega^{-1}X$. The distinguished triangles in $\mathrm{stab}(E)$ are those isomorphic to triangles arising in this way.

2.4 *N*-complexes

For a comprehensive introduction to N-complexes, we refer the reader to the work of Iyama, Kato, and Miyachi [15]. Let $\mathcal{A}$ be an additive category, and let $N \geq 2$ be an integer.

An N-**complex** over $\mathcal{A}$ is a sequence of objects of $X^n \in \mathcal{A}$, together with a sequence of morphisms (called **differentials**) $d_X^n : X^n \to X^{n+1}$ such that the composition of any N successive differentials is zero. A morphism $f^\bullet : X^\bullet \to Y^\bullet$ of N-complexes is a sequence of morphisms $f^n : X^n \to Y^n$ which commute with the differentials. We denote the category of N-complexes over $\mathcal{A}$ by $C_N(\mathcal{A})$. As with complexes, we say an N-complex $X^\bullet$ is bounded (resp., bounded above, bounded below) if $X^n = 0$ for $|n| \gg 0$ (resp., $n \gg 0$, $n \ll 0$). We write $C_N^b(\mathcal{A})$ (resp., $C_N^-(\mathcal{A})$, $C_N^+(\mathcal{A})$) for the full subcategory of $C_N(\mathcal{A})$ consisting of the bounded (resp., bounded above, and bounded below) N-complexes. In the classical case of $N = 2$, we shall always omit the subscript.

As an abbreviation, we shall write $d_X^{n,r}$ for the composition $d_X^{n+r-1} \dots d_X^n$ of r successive differentials, beginning with d_X^n. We shall interpret $d_X^{n,0}$ as the identity map on X. To improve readability, in complex formulae we shall sometimes write $d_X^{\circ,r}$ when the value of n is clear from context.

For $\natural \in \{\text{nothing}, b, +, -\}$, $C_N^\natural(\mathcal{A})$ carries the structure of a Frobenius exact category, in which the admissible exact sequences are precisely the chainwise split exact sequences of complexes. For $i \in \mathbb{Z}$, $1 \leq k \leq N$ and $X \in \mathcal{A}$, let $\mu_k^i(X)$ be the N-complex

$$\cdots \to 0 \to X \xrightarrow{id_X} \cdots \xrightarrow{id_X} X \to 0 \to \cdots$$

with k terms equal to X, in positions $i-k+1$ through i. For any $i \in \mathbb{Z}$ and any $X \in \mathcal{A}$, $\mu_N^i(X)$ is projective-injective in $C_N^\natural(\mathcal{A})$, and every projective-injective object is a direct sum of complexes of this form. [15, Theorem 2.1] The stable category of $C_N^\natural(\mathcal{A})$ is denoted $K_N^\natural(\mathcal{A})$ and is called the **homotopy category** of N-complexes over $\mathcal{A}$.

A morphism $f : X^\bullet \to Y^\bullet$ in $C_N^\natural(\mathcal{A})$ is **null-homotopic** if there exists a sequence of morphisms $h^i : X^i \to Y^{i-N+1}$ satisfying

$$f^i = \sum_{j=1}^{N} d_Y^{i+j-N,N-j} \circ h^{i+j-1} \circ d_X^{i,j-1}$$

The null-homotopic morphisms are precisely those which factor through a projective-injective object [15, Theorem 2.3], hence two morphisms of complexes are equal in $K_N^\natural(\mathcal{A})$ if and only if their difference is null-homotopic.

The suspension functor for the triangulated structure on $K_N^\natural(\mathcal{A})$ will be denoted by Σ. While Σ is induced by the Frobenius structure on $C_N^\natural(\mathcal{A})$, there is a useful explicit description. Given any N-complex $X^\bullet$, for each $n \in \mathbb{Z}$, there are natural morphisms $X^\bullet \to \mu_N^n(X^n)$ and $\mu_N^{n+N-1}(X^n) \to X^\bullet$. By taking direct sums of these morphisms, we obtain chainwise split exact sequences

$$0 \longrightarrow X^\bullet \rightarrowtail \bigoplus_{n\in\mathbb{Z}} \mu_N^n(X^n) \twoheadrightarrow \Sigma X^\bullet \longrightarrow 0$$

$$0 \longrightarrow \Sigma^{-1} X^\bullet \rightarrowtail \bigoplus_{n\in\mathbb{Z}} \mu_N^{n+N-1}(X^n) \twoheadrightarrow X^\bullet \longrightarrow 0$$

whose middle terms are projective-injective. These sequences are functorial in $X^\bullet$ and define Σ and Σ^{-1} on $C_N^\natural(\mathcal{A})$. (Despite the notation, these functors only become mutually inverse on $K_N^\natural(\mathcal{A})$.)

Let $[n] : C_N^\natural(\mathcal{A}) \to C_N^\natural(\mathcal{A})$ denote the standard shift of complexes, with $(X[n])^i = X^{n+i}$. For $N > 2$, Σ does not agree with [1]; however, we have the relation $\Sigma^2 \cong [N]$ in $K_N^\natural(\mathcal{A})$ [15, Theorem 2.4].

2.5 Derived category of *N*-complexes

In this section, let $\mathcal{A}$ be an abelian (not merely additive) category. Let $N \geq 2$ be an integer.

Let $n \in \mathbb{Z}$, $1 \leq r < N$, and $X^\bullet \in C_N(\mathcal{A})$. Define the r**-th cycle** (resp., **boundary**, **homology**) **group at** n to be

$$\begin{aligned} Z_r^n(X^\bullet) &:= ker(d_X^{n,r}) \\ B_r^n(X^\bullet) &:= im(d_X^{n-N+r,N-r}) \\ H_r^n(X^\bullet) &:= Z_r^n(X^\bullet)/B_r^n(X^\bullet) \end{aligned}$$

It is clear that $B_r^n(X^\bullet)$ is a subobject of $Z_r^n(X^\bullet)$. Note that our notation convention for $B_r^n(X^\bullet)$ differs from that of [15].

For $\natural \in \{\text{nothing}, b, +, -\}$, $C_N^\natural(\mathcal{A})$ is an abelian category, with all limits and colimits computed component-wise. Given any short exact sequence $X^\bullet \xhookrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet$ of N-

complexes, there are long exact sequences in homology

$$\cdots \to H_r^n(X^\bullet) \xrightarrow{f_*} H_r^n(Y^\bullet) \xrightarrow{g_*} H_r^n(Z^\bullet) \xrightarrow{\delta} H_{N-r}^{n+r}(X^\bullet) \to \cdots$$

for all $1 \le r < N$. [8, Section 3]

We say that $X^\bullet \in C_N(\mathcal{A})$ is **acyclic** if $H_r^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$ and $1 \le r < N$. For $\natural \in \{\text{nothing}, b, +, -\}$, we let $C_N^{\natural,ac}(\mathcal{A}) \subseteq C_N^\natural(\mathcal{A})$ and $K_N^{\natural,ac}(\mathcal{A}) \subseteq K_N^\natural(\mathcal{A})$ denote the full subcategories of acyclic N-complexes. $K_N^{\natural,ac}(\mathcal{A})$ is a thick subcategory of $K_N^\natural(\mathcal{A})$ [15, Proposition 3.2]. We define the **derived category of N-complexes** to be the Verdier quotient $D_N^\natural(\mathcal{A}) := K_N^\natural(\mathcal{A})/K_N^{\natural,ac}(\mathcal{A})$. As with ordinary complexes, a short exact sequence in $C_N(\mathcal{A})$ induces a triangle in $D_N(\mathcal{A})$ [15, Proposition 3.7].

A morphism $s^\bullet$ in $K_N^\natural(\mathcal{A})$ is a **quasi-isomorphism** if its cone is acyclic. This occurs if and only if $H_r^n(s^\bullet)$ is an isomorphism for every $n \in \mathbb{Z}$ and all $1 \le r < N$.

Given an N-complex $X^\bullet$ and $n \in N$, define the **homological truncation of $X^\bullet$ at** n to be the complex $\sigma_{\le n} X^\bullet$ given by

$$\sigma_{\le n} X^i = \begin{cases} 0 & i > n \\ Z_{n+1-i}^i(X^\bullet) & n - N + 2 \le i \le n \\ X^i & i < n - N + 2 \end{cases}$$

with the differential induced by $d_X^\bullet$. Clearly $H_r^i(\sigma_{\le n} X^\bullet) = 0$ for all $i > n$. There is a natural inclusion of complexes $\sigma_{\le n} X^\bullet \hookrightarrow X^\bullet$ which induces an isomorphism $H_r^i(\sigma_{\le n} X^\bullet) \cong H_r^i(X^\bullet)$ for all r and all $i \le n$ [15, Lemma 3.9]. We define $\sigma_{>n} X^\bullet$ to be the cokernel of this morphism.

We also define the **sharp truncation of $X^\bullet$ at** n to be the complex $\tau_{\le n} X^\bullet$ which is zero in degrees greater than n and agrees with $X^\bullet$ in degrees less than or equal to n. We define $\tau_{\ge n} X^\bullet$ analogously.

We say $X^\bullet \in D_N^b(\mathcal{A})$ is **perfect** if it is isomorphic to a bounded complex of projective objects; let $D_N^{perf}(\mathcal{A})$ denote the full subcategory of such objects. In other words, $D_N^{perf}(\mathcal{A})$ is the essential image of $K_N^b(\mathrm{Proj}(\mathcal{A}))$ in $D_N^b(\mathcal{A})$. It is easily verified that $D_N^{perf}(\mathcal{A})$ is a thick subcategory of $D_N^b(\mathcal{A})$; we define the N-**singularity category** to be the Verdier quotient $D_N^s(\mathcal{A}) := D_N^b(\mathcal{A})/D_N^{perf}(\mathcal{A})$.

2.6 Gorenstein algebras

For a self-contained treatment of the theory of Gorenstein algebras, we refer to the upcoming book by Krause [21, Chapter 6]. Let A be a finite-dimensional associative algebra over a field F. We shall assume that A is a **Gorenstein** algebra; that is, A has finite injective dimension as both a left and right A-module. In this case, both the left and right injective dimension of A coincide [21, Lemma 6.2.1]. If this number is zero, i.e. A is injective as a right and left A-module, then we say that A is **self-injective**; in this case the projective and injective A-modules coincide.

We shall write mod-A and A-mod for the category of finitely-generated right and left A-modules, respectively; when we speak of an "A-module", we shall always mean an object of mod-A unless otherwise specified. We shall identify A-mod with mod-(A^{op}) when convenient. Given $X \in$ mod-A and $a \in A$, define $r_a : X \to X$ to be the F-linear map given by right multiplication by a; for $X \in A$-mod, we similarly define $l_a : X \to X$ to be left multiplication by a. If $\phi : A \xrightarrow{\sim} A$ is an F-algebra automorphism and $X \in \text{mod-}A$, define $X_\phi \in \text{mod-}A$ by $x \cdot a := x\phi(a)$, where the right-hand multiplication is done in X.

We shall abbreviate Proj(mod-A) by proj-A, and Inj(mod-A) by inj-A; for left modules we use the abbreviations A-proj and A-inj. We say that $X \in \text{mod-}A$ is **Gorenstein projective** (resp., **Gorenstein injective**) if $\text{Ext}^i_A(X, A) = 0$ (resp., $\text{Ext}^i_A(DA, X) = 0$) for all $i > 0$, where $D = \text{Hom}_F(-, F)$ is the F-linear duality. We denote the full subcategory of all Gorenstein projective (resp., Gorenstein injective) modules by Gproj(A) (resp., Ginj(A)).

Gproj(A) forms a fully exact subcategory of the abelian category mod-A. In fact, Gproj(A) is a Frobenius category whose projective-injective objects are precisely proj-A [21, Theorem 6.2.5]. D restricts to an equivalence $\text{Gproj}(A)^{op} \xrightarrow{\sim} \text{Ginj}(A^{op})$, hence Ginj($A$) is also Frobenius exact and its projective-injective objects are precisely inj-A. When A is self-injective, note that $\text{Gproj}(A) = \text{mod-}A = \text{Ginj}(A)$.

The **Nakayama functor** $\nu_A : \text{mod-}A \to \text{mod-}A$ is the composition $\nu_A := D\,\text{Hom}_A(-, A) \cong - \otimes_A DA$. The functor $\text{Hom}_A(-, A)$ restricts to an exact duality $\text{Gproj}(A) \xrightarrow{\sim} \text{Gproj}(A^{op})$ [21, Lemma 6.2.2], hence ν_A defines an exact equivalence $\text{Gproj}(A) \xrightarrow{\sim} \text{Ginj}(A)$ which descends to a triangulated equivalence of the respective stable categories.

If A is self-injective, then ν_A is an exact autoequivalence of both mod-A and A-mod and preserves projective-injectives; in this case, ν_A lifts to $D^b_N(A)$ and descends to $D^s_N(A)$. There is an F-algebra automorphism ϕ_A, called the **Nakayama automorphism**, such that $\nu_A(X) = X_{\phi_A}$. The Nakayama automorphism is unique up to a choice of inner automorphism.

2.7 Gorenstein Abelian categories

Just as a Frobenius exact abelian category serves as a useful categorical model for the module category of a self-injective algebra, a Gorenstein abelian category generalizes the module category of a Gorenstein algebra. For a detailed introduction to such categories, the interested reader may consult Beligiannis and Reiten [2]; we shall summarize the needed facts and definitions below.

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. We say that $\mathcal{A}$ is **Gorenstein** if the projective objects have bounded injective dimension and the injective objects have bounded projective dimension. An object $X \in \mathcal{A}$ is said to be **Gorenstein projective** if $\text{Ext}^i_{\mathcal{A}}(X, P) = 0$ for all $i > 0$ and every $P \in \text{Proj}(\mathcal{A})$. We define $\text{Gproj}(\mathcal{A})$ to be the full subcategory of $\mathcal{A}$ consisting of the Gorenstein projective objects. (Beligiannis and Reiten refer to this as the subcategory $\text{CM}(\mathcal{P})$ of **Cohen-Macaulay** objects using an equivalent definition.) It is easy to verify that $\text{Gproj}(\mathcal{A})$ is a fully exact subcategory of $\mathcal{A}$ containing $\text{Proj}(\mathcal{A})$. We also define $\mathcal{P}^{<\infty}(\mathcal{A})$ to be the full subcategory of $\mathcal{A}$ consisting of the objects with finite projective dimension.

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$ be full subcategories, closed under isomorphisms and direct summands. Define the Ext-orthogonal subcategories

$$\mathcal{X}^{\perp} := \{M \in \mathcal{A} \mid \forall X \in \mathcal{X}, \text{Ext}^1_{\mathcal{A}}(X, M) = 0\}$$
$$^{\perp}\mathcal{X} := \{M \in \mathcal{A} \mid \forall X \in \mathcal{X}, \text{Ext}^1_{\mathcal{A}}(M, X) = 0\}$$

We say $(\mathcal{X}, \mathcal{Y})$ is a **cotorsion pair** if:

i) $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$.
ii) For all $M \in \mathcal{A}$, there exists a short exact sequence $Y \hookrightarrow X \twoheadrightarrow M$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.
iii) For all $M \in \mathcal{A}$, there exists a short exact sequence $M \hookrightarrow Y \twoheadrightarrow X$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.

We shall need the following three facts about Gorenstein abelian categories.

Theorem 2.1 *(Beligiannis and Reiten, [2], Chapter 7.2, Theorem 2.2; Chapter 7.1, Theorem 1.4; and Chapter 5.3, Lemma 3.3) Let $\mathcal{A}$ be a Gorenstein abelian category. Then:*

1) $(\mathrm{Gproj}(\mathcal{A}), \mathcal{P}^{<\infty}(\mathcal{A}))$ *is a cotorsion pair.*
2) $\mathrm{Gproj}(\mathcal{A})^{\perp} = \mathcal{P}^{<\infty}(\mathcal{A})$ *and* ${}^{\perp}\mathcal{P}^{<\infty}(\mathcal{A}) = \mathrm{Gproj}(\mathcal{A})$.
3) $\mathrm{Gproj}(\mathcal{A}) \cap \mathcal{P}^{<\infty}(\mathcal{A}) = \mathrm{Proj}(\mathcal{A})$.

Though Beligiannis and Reiten describe Gorenstein abelian categories using the language of cotorsion pairs, we shall not. The following corollary translates the above results into our preferred language of Frobenius exact categories.

Corollary 2.2 *Let $\mathcal{A}$ be a Gorenstein abelian category. Then* $\mathrm{Gproj}(\mathcal{A})$ *is a Frobenius exact category.*

Proof Note that $\mathrm{Proj}(\mathcal{A}) \subseteq \mathrm{Gproj}(\mathcal{A})$. It follows immediately that $\mathrm{Proj}(\mathcal{A}) \subseteq \mathrm{Proj}(\mathrm{Gproj}(\mathcal{A}))$. Also, if $P \in \mathrm{Proj}(\mathcal{A})$, then $\mathrm{Ext}^1_{\mathcal{A}}(X, P) = 0$ for all $X \in \mathrm{Gproj}(\mathcal{A})$. Therefore P is an injective object in $\mathrm{Gproj}(\mathcal{A})$ and so $\mathrm{Proj}(\mathcal{A}) \subseteq \mathrm{Inj}(\mathrm{Gproj}(\mathcal{A}))$.

If $I \in \mathrm{Inj}(\mathrm{Gproj}(\mathcal{A}))$, then $\mathrm{Ext}^1_{\mathcal{A}}(M, I) = 0$ for all $M \in \mathrm{Gproj}(\mathcal{A})$, so $I \in \mathrm{Gproj}(\mathcal{A})^{\perp} = \mathcal{P}^{<\infty}(\mathcal{A})$. Thus $I \in \mathrm{Gproj}(\mathcal{A}) \cap \mathcal{P}^{<\infty}(\mathcal{A}) = \mathrm{Proj}(\mathcal{A})$, and so $\mathrm{Inj}(\mathrm{Gproj}(\mathcal{A})) = \mathrm{Proj}(\mathcal{A})$.

Let $P \in \mathrm{Proj}(\mathrm{Gproj}(\mathcal{A}))$ and let $M \in \mathcal{A}$; it is enough to show that $\mathrm{Ext}^1_{\mathcal{A}}(P, M) = 0$. There is a short exact sequence $Y \hookrightarrow X \twoheadrightarrow M$ with $X \in \mathrm{Gproj}(\mathcal{A})$ and $Y \in \mathcal{P}^{<\infty}(\mathcal{A})$. Note that $\mathrm{Ext}^n_{\mathcal{A}}(P, X) = 0$ for all $n \geq 1$; it follows from the long exact sequence in Ext that $\mathrm{Ext}^1_{\mathcal{A}}(P, M) \cong \mathrm{Ext}^2_{\mathcal{A}}(P, Y)$. Y has finite projective dimension and therefore finite injective dimension, so let $I^{\bullet}$ be a finite injective resolution for Y. Define $Y' := Z^1(I^{\bullet})$. Clearly $Y' \in \mathcal{P}^{<\infty}(\mathcal{A}) = \mathrm{Gproj}(\mathcal{A})^{\perp}$, hence $\mathrm{Ext}^2_{\mathcal{A}}(P, Y) = \mathrm{Ext}^1_{\mathcal{A}}(P, Y') = 0$. Thus $P \in \mathrm{Proj}(\mathcal{A})$ and so $\mathrm{Proj}(\mathcal{A}) = \mathrm{Proj}(\mathrm{Gproj}(\mathcal{A}))$.

Since $\mathcal{A}$ has enough projectives, so does $\mathrm{Gproj}(\mathcal{A})$. If $X \in \mathrm{Gproj}(\mathcal{A})$, we obtain a short exact sequence $X \hookrightarrow I \twoheadrightarrow X'$ for some $I \in \mathcal{P}^{<\infty}(\mathcal{A})$ and $X' \in \mathrm{Gproj}(\mathcal{A})$. Then I is an extension of Gorenstein projective objects, so $I \in \mathrm{Gproj}(\mathcal{A})$. Thus $I \in \mathrm{Gproj}(\mathcal{A}) \cap \mathcal{P}^{<\infty}(\mathcal{A}) = \mathrm{Proj}(\mathcal{A}) = \mathrm{Inj}(\mathrm{Gproj}(\mathcal{A}))$. Therefore $\mathrm{Gproj}(\mathcal{A})$ has enough injectives, and so is a Frobenius exact category. □

3 The N-stable category

3.1 The monomorphism category

Throughout this section, let $(\mathcal{E}, \mathcal{S})$ be an exact category.

For any integer $k \geq 1$, let $[[k]]$ denote the category corresponding to the poset $\{1 < \cdots < k\}$. For any $k \geq 0$, let $\mathrm{Mor}_k(\mathcal{E})$ denote the category $\mathcal{E}^{[[k+1]]}$ of functors from $[[k+1]]$ to $\mathcal{E}$. Namely, the objects of $\mathrm{Mor}_k(\mathcal{E})$ are diagrams $(X_{\bullet}, f_{\bullet}) = X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_k} X_{k+1}$ of k composable morphisms in $\mathcal{E}$. $\mathrm{Mor}_k(\mathcal{E})$ carries a natural structure of an exact category, in which the class of admissible exact sequences is $\mathcal{S}^{[[k+1]]}$. That is, $X_{\bullet} \rightarrowtail Y_{\bullet} \twoheadrightarrow Z_{\bullet}$ is admissible if and only if $X_i \rightarrowtail Y_i \twoheadrightarrow Z_i$ is admissible in $\mathcal{E}$ for each $1 \leq i \leq k+1$. (See Bühler, [5, Example 13.11].) As in all diagram categories, small limits and colimits in $\mathrm{Mor}_k(\mathcal{E})$ are computed component-wise and exist if and only if the component-wise limits and colimits exist (see, for instance, [3, Proposition 2.15.1]). Note that $\mathrm{Mor}_0(\mathcal{E})$ recovers $\mathcal{E}$ as an exact category.

Mimicking our notation for N-complexes, given $(X_\bullet, f_\bullet) \in \mathrm{Mor}_k(\mathcal{E})$ we will write $f_i^j := f_{i+j-1} \cdots f_i$ for the composition of j successive maps in $f_\bullet$, beginning with f_i. We shall let f_i^0 denote the identity map on X_i.

Definition 3.1 Let $(\mathcal{E}, \mathcal{S})$ be an exact category. Let $k \geq 0$. Let the **monomorphism subcategory** $\mathrm{MMor}_k(\mathcal{E})$ be the full subcategory of $\mathrm{Mor}_k(\mathcal{E})$ consisting of objects of the form

$$X_1 \overset{\iota_1}{\rightarrowtail} X_2 \overset{\iota_2}{\rightarrowtail} \cdots \overset{\iota_k}{\rightarrowtail} X_{k+1}$$

where each ι_j is an admissible monomorphism in $\mathcal{E}$.

An **admissible short exact sequence** in $\mathrm{MMor}_k(\mathcal{E})$ is any short exact sequence $X_\bullet \rightarrowtail Y_\bullet \twoheadrightarrow Z_\bullet$ which is admissible in $\mathrm{Mor}_k(\mathcal{E})$. Write $\mathrm{MMor}_k(\mathcal{S})$ for the class of admissible short exact sequences in $\mathrm{MMor}_k(\mathcal{E})$.

Remark We could also define the **epimorphism subcategory** $\mathrm{EMor}_k(\mathcal{E})$ to be the analogous subcategory of $\mathrm{Mor}_k(\mathcal{E})$ in which every morphism appearing in the diagram is an admissible epimorphism in $\mathcal{E}$. By again declaring all component-wise admissible exact sequences to be admissible, we obtain a candidate structure of exact category on $\mathrm{EMor}_k(\mathcal{E})$. There is a natural equivalence of categories between $\mathrm{EMor}_k(\mathcal{E})$ and $\mathrm{MMor}_k(\mathcal{E}^{op})$ which preserves their candidate exact structures. Thus dual versions of all results in this section apply to $\mathrm{EMor}_k(\mathcal{E})$; the reader can easily formulate the precise statements.

Our goal is to show that the above definitions give $\mathrm{MMor}_k(\mathcal{E})$ the structure of an exact category. The result is straightforward in the case of abelian categories.

Proposition 3.2 *Let $\mathcal{A}$ be an abelian category. Then* $\mathrm{MMor}_k(\mathcal{A})$ *is closed under extensions in* $\mathrm{Mor}_k(\mathcal{A})$. *In particular,* $\mathrm{MMor}_k(\mathcal{A})$ *is a fully exact subcategory of* $\mathrm{Mor}_k(\mathcal{A})$.

Proof Suppose we have a short exact sequence $X_\bullet \hookrightarrow Y_\bullet \twoheadrightarrow Z_\bullet$, where $(X_\bullet, \alpha_\bullet), (Z_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{A})$ and $(Y_\bullet, \beta_\bullet) \in \mathrm{Mor}_k(\mathcal{A})$. By the Snake Lemma, for each $1 \leq i \leq k$ we have a short exact sequence

$$0 \longrightarrow ker(\alpha_i) \longrightarrow ker(\beta_i) \longrightarrow ker(\gamma_i)$$

Since $ker(\alpha_i) = ker(\gamma_i) = 0$, it follows that $ker(\beta_i) = 0$ and β_i is a monomorphism for all i. Thus $(Y_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{A})$, and so $\mathrm{MMor}_k(\mathcal{A})$ is closed under extensions.

It is clear $\mathrm{MMor}_k(\mathcal{A})$ is a full additive subcategory of $\mathrm{Mor}_k(\mathcal{A})$, and that the candidate exact structure on $\mathrm{MMor}_k(\mathcal{A})$ agrees with that inherited from $\mathrm{Mor}_k(\mathcal{A})$. Thus $\mathrm{MMor}_k(\mathcal{A})$ is a fully exact subcategory of $\mathrm{Mor}_k(\mathcal{A})$. □

Proposition 3.3 *Let $\mathcal{E}$ be a small exact category. Then* $\mathrm{MMor}_k(\mathcal{E})$ *is exact.*

Proof Since $\mathcal{E}$ is small, by [5, Theorem A.1], there exists an abelian category $\mathcal{A}$ and a fully faithful exact functor $\iota : \mathcal{E} \to \mathcal{A}$ such that ι reflects exactness and $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. It is clear that ι induces an additive functor $\iota_* : \mathrm{Mor}_k(\mathcal{E}) \to \mathrm{Mor}_k(\mathcal{A})$, which remains fully faithful and sends objects of $\mathrm{MMor}_k(\mathcal{E})$ to $\mathrm{MMor}_k(\mathcal{A})$. Thus we may view $\mathrm{MMor}_k(\mathcal{E})$ as a full, additive subcategory of $\mathrm{MMor}_k(\mathcal{A})$; accordingly, we will suppress mention of the functor ι in our notation going forward.

We claim that $\mathrm{MMor}_k(\mathcal{E})$ is closed under extensions in $\mathrm{MMor}_k(\mathcal{A})$, hence is a fully exact subcategory. Let $X_\bullet \overset{f_\bullet}{\rightarrowtail} Y_\bullet \overset{g_\bullet}{\twoheadrightarrow} Z_\bullet$ be a short exact sequence in $\mathrm{MMor}_k(\mathcal{A})$, with $(X_\bullet, \alpha_\bullet), (Z_\bullet, \gamma_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$. We must show that $(Y_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$.

For each i, we have a short exact sequence $X_i \xhookrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ in $\mathcal{A}$. Thus $Y_i \in \mathcal{E}$, since $\mathcal{E}$ is closed under extensions. Since the inclusion functor $\iota : \mathcal{E} \to \mathcal{A}$ reflects exactness, the above short exact sequence is admissible in $\mathcal{E}$.

It remains to show that the monomorphisms β_i are admissible in $\mathcal{E}$. Consider the diagram

$$\begin{array}{ccccc}
X_i & \overset{\alpha_i}{\rightarrowtail} & X_{i+1} & \twoheadrightarrow & coker(\alpha_i) \\
\downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle \phi} \\
Y_i & \overset{\beta_i}{\hookrightarrow} & Y_{i+1} & \twoheadrightarrow & coker(\beta_i) \\
\downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_{i+1}} & & \downarrow{\scriptstyle \psi} \\
Z_i & \overset{\gamma_i}{\rightarrowtail} & Z_{i+1} & \twoheadrightarrow & coker(\gamma_i)
\end{array}$$

The first two columns are admissible and exact in $\mathcal{E}$ by the above remarks; we construct the third column by applying the Snake Lemma and deduce that it is a short exact sequence in $\mathcal{A}$. The monomorphisms α_i and γ_i are admissible in $\mathcal{E}$, hence $coker(\alpha_i), coker(\gamma_i) \in \mathcal{E}$. Since $\mathcal{E}$ is closed under extensions and ι reflects exactness, $coker(\beta_i) \in \mathcal{E}$ and the third column is an admissible short exact sequence in $\mathcal{E}$. Thus all the objects in the second row lie in $\mathcal{E}$, hence the second row is an admissible short exact sequence in $\mathcal{E}$. In particular, β_i is an admissible monomorphism in $\mathcal{E}$. Thus $(Y_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$.

It remains to show that the structure of exact category which $\mathrm{MMor}_k(\mathcal{E})$ inherits from $\mathrm{MMor}_k(\mathcal{A})$ agrees with the original exact structure, i.e. that which it inherited from $\mathrm{Mor}_k(\mathcal{E})$. This follows immediately from the fact that ι is exact and reflects exactness. □

Since verifying the axioms of an exact category only involves working with finitely many objects at a time, the smallness hypothesis in the previous proposition can be removed.

Lemma 3.4 *Let $(\mathcal{E}, \mathcal{S})$ be an exact category, and let $E \subseteq Ob(\mathcal{E})$ be a set of objects. Then there exists a small full subcategory $\mathcal{E}'$ of $\mathcal{E}$ containing E, such that $(\mathcal{E}', \mathcal{S}')$ is an exact category, where $\mathcal{S}'$ is the set of all kernel-cokernel pairs in $\mathcal{S}$ whose objects lie in $\mathcal{E}'$.*

Proof Given any full subcategory T of $\mathcal{E}$, let $C(T)$ (resp., $K(T)$) be the full subcategory of $\mathcal{E}$ consisting of the objects $coker(f)$ (resp., $ker(f)$), where f ranges over all morphisms in T which are admissible monomorphisms (resp., epimorphisms) in $\mathcal{E}$. In this definition we make a single choice of $coker(f)$ or $ker(f)$ for each morphism f, hence $C(T)$ and $K(T)$ are small if T is. For each $X \in Ob(T)$, we choose X to be the representative of both $ker(X \to 0)$ and $coker(0 \to X)$, so that T is a full subcategory of both $K(T)$ and $C(T)$. Finally, it is easily checked that if T is an additive subcategory of $\mathcal{E}$, then so are $C(T)$ and $K(T)$.

For any finite sequence $X_1, \cdots, X_n$ of objects in E, choose one object of $\mathcal{E}$ isomorphic to $\bigoplus_{i=1}^n X_i$, and let E_0 be full subcategory of $\mathcal{E}$ consisting of all chosen objects. Then E_0 is a small additive subcategory of $\mathcal{E}$ which can be chosen to contain E. For each $i > 0$, inductively define $E_i := K(C(E_{i-1}))$, and let $\mathcal{E}' := \bigcup_{i=0}^\infty E_i$. It is clear that $\mathcal{E}'$ is a small additive subcategory of $\mathcal{E}$ containing E.

It remains to show that $(\mathcal{E}', \mathcal{S}')$ is an exact category. It is immediate that all identity morphisms are admissible epimorphisms and monomorphisms. If f and g are two composable admissible monomorphisms in E_i, then $cok(f \circ g) \in E_{i+1}$ hence $f \circ g$ is an admissible monomorphism in $\mathcal{E}'$; by a dual argument, composition of admissible epimorphisms in $\mathcal{E}'$ also remain admissible. Similarly, if $f : X \rightarrowtail Y$ and $g : X \to Z$ are morphisms in E_i with f an admissible monomorphism, then by [5, Proposition 2.12] the pushout P of f along g in $\mathcal{E}$ fits into admissible exact sequences

$$X \overset{\begin{bmatrix} f \\ g \end{bmatrix}}{\rightarrowtail} Y \oplus Z \xrightarrow{\begin{bmatrix} g' & f' \end{bmatrix}} P$$

$$Z \overset{f'}{\rightarrowtail} P \twoheadrightarrow coker(f)$$

The first sequence shows that, up to isomorphism, $P \in E_{i+1}$. Since $coker(f) \in E_{i+1}$, we have that f' is an admissible monomorphism in $\mathcal{E}'$. By a dual argument, pull-backs preserve admissible epimorphisms in $\mathcal{E}'$. □

Proposition 3.5 *Let $\mathcal{E}$ be an exact category. Then* $\mathrm{MMor}_k(\mathcal{E})$ *is exact.*

Proof We let $\mathcal{S}$ denote the class of admissible exact sequences in $\mathcal{E}$. If $E \subseteq \mathcal{E}$ is any finite set of objects, let $(\mathcal{E}', \mathcal{S}')$ be the small exact category containing E constructed in Proposition 3.4. Then the inclusion functor $\mathcal{E}' \hookrightarrow \mathcal{E}$ is exact and induces a fully faithful functor $\mathrm{MMor}_k(\mathcal{E}') \hookrightarrow \mathrm{MMor}_k(\mathcal{E})$ which maps $\mathrm{MMor}_k(\mathcal{S}')$ into $\mathrm{MMor}_k(\mathcal{S})$. By Proposition 3.3, $(\mathrm{MMor}_k(\mathcal{E}'), \mathrm{MMor}_k(\mathcal{S}'))$ is an exact category.

To verify the exact category axioms, we need work only with finitely many objects of $\mathcal{E}$ at a time, hence exactness of $\mathrm{MMor}_k(\mathcal{E})$ can be verified inside $\mathrm{MMor}_k(\mathcal{E}')$. For instance, to verify that the push-out of the admissible monomorphism $f_\bullet : X_\bullet \rightarrowtail Y_\bullet$ along $g_\bullet : X_\bullet \to Z_\bullet$ is an admissible monomorphism, let $E = \{X_i, Y_i, Z_i \mid 1 \le i \le k+1\}$. Then the pushout of $f_\bullet$ along $g_\bullet$ exists and is an admissible monomorphism in $\mathrm{MMor}_k(\mathcal{E}')$, hence in $\mathrm{MMor}_k(\mathcal{E})$. Verification of the other axioms is analogous. □

We close this section by providing a convenient intrinsic description of the admissible monomorphisms and epimorphisms in the monomorphism category of an abelian category.

Proposition 3.6 *Let $\mathcal{A}$ be an abelian category and let $f_\bullet : (X_\bullet, \alpha_\bullet) \to (Y_\bullet, \beta_\bullet)$ be a morphism in* $\mathrm{MMor}_k(\mathcal{A})$*. $f_\bullet$ is an admissible epimorphism if and only if each f_i is an epimorphism. $f_\bullet$ is an admissible monomorphism if and only if each f_i is a monomorphism and each sub-diagram*

$$\begin{array}{ccc} X_i & \overset{\alpha_i}{\hookrightarrow} & X_{i+1} \\ \downarrow f_i & & \downarrow f_{i+1} \\ Y_i & \overset{\beta_i}{\hookrightarrow} & Y_{i+1} \end{array}$$

forms a pullback square in $\mathcal{A}$.

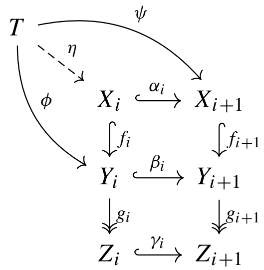
Proof If $f_\bullet$ is an admissible epimorphism, it follows immediately that each f_i is epic. Conversely, if each f_i is an epimorphism, then $f_\bullet$ is an epimorphism in $\mathrm{Mor}_k(\mathcal{A})$, hence it has a kernel $(K_\bullet, \iota_\bullet)$. To prove that $f_\bullet$ is an admissible epimorphism, we must show $K_\bullet \in \mathrm{MMor}_k(\mathcal{A})$. We have a commutative diagram

$$\begin{array}{ccc} K_i & \xrightarrow{\iota_i} & K_{i+1} \\ \downarrow & & \downarrow \\ X_i & \overset{\alpha_i}{\hookrightarrow} & X_{i+1} \end{array}$$

from which it is clear that ι_i is a monomorphism. Thus $K_\bullet \in \mathrm{MMor}_k(\mathcal{A})$.

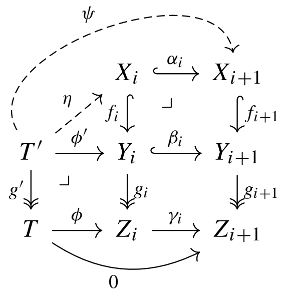
If $f_\bullet$ is an admissible monomorphism, then we have a short exact sequence $X_\bullet \overset{f_\bullet}{\hookrightarrow} Y_\bullet \overset{g_\bullet}{\twoheadrightarrow} Z_\bullet$ with $(Z_\bullet, \gamma_i) \in \mathrm{MMor}_k(\mathcal{A})$. It follows immediately that each f_i

is a monomorphism. To show X_i is a pullback, consider the commutative diagram with exact columns



where ψ and ϕ satisfy $f_{i+1}\psi = \beta_i\phi$. Postcomposing this equation with g_{i+1}, we see that $0 = g_{i+1} f_{i+1}\psi = g_{i+1}\beta_i\phi = \gamma_i g_i\phi$. Since γ_i is a monomorphism, $g_i\phi = 0$. By exactness of the first column there exists a unique $\eta : T \to X_i$ such that $\phi = f_i\eta$. An easy diagram chase yields $f_{i+1}\psi = f_{i+1}\alpha_i\eta$. Since f_{i+1} is a monomorphism, we have $\psi = \alpha_i\eta$, hence the top square is a pullback.

Conversely, assume each f_i is a monomorphism and each square in $f_\bullet$ is a pullback. Let $(Z_\bullet, \gamma_\bullet)$ be the cokernel of $f_\bullet$ in $\mathrm{Mor}_k(\mathcal{A})$. We must show that $Z_\bullet \in \mathrm{MMor}_k(\mathcal{A})$, i.e that each γ_i is monic. We shall construct the following commutative diagram:



We start with the rightmost two squares, which are commutative with exact columns. To show γ_i is a monomorphism, consider $\phi : T \to Z_i$ such that $\gamma_i\phi = 0$. Let T' be the pullback of ϕ along g_i; since g_i is an epimorphism, so is g'. We have that $g_{i+1}\beta_i\phi' = \gamma_i\phi g' = 0$, so by exactness of the right column $\beta_i\phi' = f_{i+1}\psi$ for some $\psi : T' \to X_{i+1}$. Since the top right square is a pullback, we obtain a morphism $\eta : T' \to X_i$ making the diagram commute. It follows that $\phi g' = g_i f_i\eta = 0$, hence $\phi = 0$. Thus γ_i is a monomorphism, $Z_\bullet \in \mathrm{MMor}_k(\mathcal{A})$, and $f_\bullet$ is an admissible monomorphism. □

Remark Both of the above criteria can fail when $\mathcal{A}$ is not abelian.

1) Let A be the path algebra of the A3 Dynkin quiver $1 \leftarrow 2 \rightarrow 3$, and let S_i be the simple module corresponding to vertex i. Let $\mathcal{E}$ be the full subcategory of mod-A obtained by removing all objects isomorphic to S_3. $\mathcal{E}$ is a full additive subcategory of mod-A which is closed under extensions and is therefore a fully exact subcategory of mod-A.

   Consider the objects $X_\bullet = S_1 \hookrightarrow \begin{smallmatrix} & S_2 & \\ S_1 & & S_3 \end{smallmatrix}$ and $Y_\bullet = 0 \hookrightarrow S_2$ in $\mathrm{MMor}_1(\mathcal{E})$. There is an obvious component-wise epimorphism $f_\bullet : X_\bullet \to Y_\bullet$ with kernel $K_\bullet = S_1 \hookrightarrow$

$S_1 \oplus S_3$. Since S_3 is not an object of $\mathcal{E}$, the monomorphism defining $K_\bullet$ has no cokernel in $\mathcal{E}$, hence is not admissible. Thus $K_\bullet \notin \mathrm{MMor}_1(\mathcal{E})$, and so $f_\bullet$ is not a distinguished epimorphism in this category.

An additive category is **weakly idempotent complete** if every split monomorphism has a cokernel (or, equivalently, every split epimorphism has a kernel). Using the dual of [5, Corollary 7.7], one can show that if $\mathcal{E}$ is weakly idempotent complete, then the epimorphism criterion in the above proposition holds.

2) Let B be the path algebra of the D4 Dynkin quiver $\begin{smallmatrix} 1 & 2 & 3 \\ \searrow & \downarrow & \swarrow \\ & 4 & \end{smallmatrix}$, and let S_i be the simple module corresponding to vertex i. Let $\mathcal{E}$ be the full subcategory of mod-B obtained by removing all objects isomorphic to S_3. As before, $\mathcal{E}$ is a fully exact subcategory of mod-B.

Let $X_\bullet = S_4 \hookrightarrow \begin{smallmatrix} S_1 \\ S_4 \end{smallmatrix}$ and $Y_\bullet = \begin{smallmatrix} S_2 \\ S_4 \end{smallmatrix} \hookrightarrow \begin{smallmatrix} S_1 & S_2 & S_3 \\ & S_4 & \end{smallmatrix}$ in $\mathrm{MMor}_1(\mathcal{E})$. The natural inclusions $f_i : X_i \hookrightarrow Y_i$ induce a monomorphism $f_\bullet : X_\bullet \hookrightarrow Y_\bullet$ in $\mathrm{MMor}_1(\mathcal{E})$, and it is clear that the commutative square defined by $f_\bullet$ is a pullback. The cokernel of $f_\bullet$ is $Z_\bullet = S_2 \hookrightarrow S_2 \oplus S_3$. Once again, $S_3 \notin \mathcal{E}$, hence the monomorphism defining $Z_\bullet$ is not admissible in $\mathcal{E}$ and so $Z_\bullet \notin \mathrm{MMor}_1(\mathcal{E})$. Therefore $f_\bullet$ is not an admissible monomorphism in $\mathrm{MMor}_1(\mathcal{E})$.

If every monomorphism in $\mathcal{E}$ is admissible, then the proof of monomorphism criterion in the above proposition holds with minimal changes. This is a very strong hypothesis; we do not know if there is a weaker one.

3.2 Projective and injective objects

We shall classify the projective and injective objects of $\mathrm{MMor}_k(\mathcal{E})$. It will be convenient to introduce some notation.

Definition 3.7 For $X \in \mathcal{E}$ and $1 \leq i \leq k+1$, let $\chi_i(X)_\bullet \in \mathrm{Mor}_k(\mathcal{E})$ be given by $0 \to \cdots \to 0 \to X \xrightarrow{id_X} \cdots \xrightarrow{id_X} X$, where the first $i-1$ objects are 0, and the first X is in position i.

The following lemma, adapted from the proof of [5, Proposition 2.12], will be useful.

Lemma 3.8 *(Bühler [5]) Let* $\iota : X \rightarrowtail Y$ *be an admissible monomorphism in* $\mathcal{E}$*, and let* $f : X \to Z$ *be any morphism. Then* $\begin{bmatrix} \iota \\ f \end{bmatrix} : X \rightarrowtail Y \oplus Z$ *is an admissible monomorphism. Dually, if* $p : Y \twoheadrightarrow W$ *is an admissible epimorphism and* $g : Z \to W$ *is any morphism, then* $\begin{bmatrix} p & g \end{bmatrix} : Y \oplus Z \twoheadrightarrow W$ *is an admissible epimorphism.*

Proof We can factor $\begin{bmatrix} \iota \\ f \end{bmatrix}$ as the composition

$$X \xrightarrow{\begin{bmatrix} id_X \\ 0 \end{bmatrix}} X \oplus Z \xrightarrow[\sim]{\begin{bmatrix} id_X & 0 \\ f & id_Z \end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix} \iota & 0 \\ 0 & id_Z \end{bmatrix}} Y \oplus Z$$

Split monomorphisms and isomorphisms are admissible monomorphisms, as is the direct sum of two admissible monomorphisms [5, Proposition 2.9]. Thus $\begin{bmatrix} \iota \\ f \end{bmatrix}$ is the composition of three admissible monomorphisms.

The proof of the second statement is dual. □

Proposition 3.9 *Let $\mathcal{E}$ be an exact category. Then $(I_\bullet, \iota_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$ is injective (resp., projective) if and only if each I_i is injective (resp., projective) in $\mathcal{E}$ and each ι_i is split.*

Proof Take $(I_\bullet, \iota_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$ with each I_i injective and each ι_i split. Then we have $I_\bullet \cong \bigoplus_{i=1}^{k+1} \chi_i(I_i')_\bullet$, where $I_1' = I_1$ and $I_i' = coker(\iota_{i-1})$ for $i > 1$. Thus it suffices to show that $\chi_i(I)_\bullet$ is injective for every injective object I and each $1 \le i \le k+1$.

Fix I and i and suppose $f_\bullet : \chi_i(I)_\bullet \rightarrowtail (X_\bullet, \alpha_\bullet)$ is an admissible monomorphism; we shall define a retraction $r_\bullet$. We shall construct the following commutative diagram with admissible exact rows and columns:

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \xrightarrow{\quad id_I \quad} & I \\
\downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f} \\
X_{i-1} & \overset{\alpha_{i-1}^{k-i+2}}{\rightarrowtail} & X_{k+1} & \overset{p}{\twoheadrightarrow} & coker(\alpha_{i-1}^{k-i+2}) \\
\downarrow{\scriptstyle id_{X_{i-1}}} & & \downarrow & & \downarrow \\
X_{i-1} & \overset{\beta}{\rightarrowtail} & coker(f_{k+1}) & \twoheadrightarrow & coker(\beta)
\end{array}$$

In the case where $i = 1$, we define $X_0 = 0$. The first two rows and columns are clearly exact. Since $f_\bullet$ is an admissible monomorphism, $coker(f_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$, hence β is an admissible monomorphism and the third row is exact.

By [5, Exercise 3.7], the induced maps forming the third column are uniquely defined and form an admissible short exact sequence. By injectivity of I, f admits a retraction $r : coker(\alpha_{i-1}^{k-i+2}) \twoheadrightarrow I$. For $1 \le j \le k+1$, define $r_j : X_j \to I$ to be the composition $r_j = rp\alpha_j^{k+1-j}$. By the above diagram, $r_j = 0$ for $j \le i-1$; for such j we shall therefore view r_j as a morphism $X_j \to 0$. Furthermore, for each $1 \le j < k+1$, $r_j = r_{j+1}\alpha_j$, hence $r_\bullet : X_\bullet \to \chi_i(I)_\bullet$ is a morphism in $\mathrm{MMor}_k(\mathcal{E})$. The verification that $r_\bullet$ is a retraction of $f_\bullet$ is straightforward. Thus $\chi_i(I)_\bullet$ is injective.

Conversely, suppose $(I_\bullet, \iota_\bullet)$ is injective. To show each I_i is injective, consider the diagram in $\mathcal{E}$

$$\begin{array}{ccc}
I_i & & \\
{\scriptstyle f}\uparrow & & \\
X & \overset{g}{\rightarrowtail} & Y
\end{array}$$

We must find $h : Y \to I_i$ making the diagram commute. Note that g induces an admissible monomorphism $g_\bullet : \chi_i(X)_\bullet \rightarrowtail \chi_i(Y)_\bullet$. f also induces a morphism $f_\bullet : \chi_i(X)_\bullet \to I_\bullet$, where $f_j = 0$ for $j < i$, $f_i = f$, and $f_j = X \xrightarrow{f} I_i \rightarrowtail I_j$ for $j > i$. By injectivity of $I_\bullet$, we obtain an induced map $h_\bullet : \chi_i(Y)_\bullet \to I_\bullet$ such that $f_\bullet = h_\bullet g_\bullet$. Setting $h = h_i$, we have that $f = hg$, hence I_i is injective. It follows immediately that the ι_i are split.

We turn to the classification of the projective objects. To show that $(P_\bullet, \iota_\bullet)$, with P_i projective and ι_i split, is projective in $\mathrm{MMor}_k(\mathcal{E})$, it suffices to show that $\chi_i(P)_\bullet$ is projective for any i and any projective P. In fact, something stronger is true; we shall prove that $\chi_i(P)_\bullet$ is projective in $\mathrm{Mor}_k(\mathcal{E})$.

Let $p_\bullet : (X_\bullet, f_\bullet) \twoheadrightarrow \chi_i(P)_\bullet$ be an admissible epimorphism in $\mathrm{Mor}_k(\mathcal{E})$; we shall construct a section $s_\bullet$. Since P is projective, $p_i : X_i \twoheadrightarrow P$ admits a section s_i. For $j < i$ let $s_j = 0 \to X_j$, and for $j > i$ let $s_j = P \overset{s_i}{\rightarrowtail} X_i \xrightarrow{f_i^{j-i}} X_j$. It is easy to verify that $s_\bullet : \chi_i(P)_\bullet \to X_\bullet$ is a morphism in $\mathrm{MMor}_k(\mathcal{E})$ and a section of $p_\bullet$. Thus $\chi_i(P)_\bullet$ is projective in $\mathrm{Mor}_k(\mathcal{E})$, hence also in $\mathrm{MMor}_k(\mathcal{E})$.

Conversely, let $(P_\bullet, \iota_\bullet)$ be projective in $\mathrm{MMor}_k(\mathcal{E})$. To show that P_i is projective, consider the diagram in $\mathcal{E}$

$$\begin{array}{ccc} & & P_i \\ & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

We must find $h : P_i \to Y$ making the diagram commute.

We shall define objects $(X_\bullet, \alpha_\bullet), (Y_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$ and morphisms $f_\bullet : P_\bullet \to X_\bullet$, $g_\bullet : Y_\bullet \twoheadrightarrow X_\bullet$ such that $X_i = X$, $Y_i = Y$, $f_i = f$, and $g_i = g$. We start by defining $(X_\bullet, \alpha_\bullet)$ and $f_\bullet$. For all $1 \le j \le i$, let $X_j = X$ and $f_j = f\iota_j^{i-j}$. For all $1 \le j < i$ let α_j be the identity map on X. For $j \ge i$ we inductively define X_{j+1}, f_{j+1}, and α_j via the pushout

$$\begin{array}{ccc} P_j & \xrightarrow{\iota_j} & P_{j+1} \\ \downarrow f_j & & \downarrow f_{j+1} \\ X_j & \xrightarrow{\alpha_j} & X_{j+1} \end{array}$$

Admissible monomorphisms are stable under pushouts, hence α_i is an admissible monomorphism and $f_\bullet : P_\bullet \to X_\bullet$ is a morphism in $\mathrm{MMor}_k(\mathcal{E})$.

For $j \le i$, let $Y_j = Y$ and $g_j = g$. For $j > i$, let $Y_j = Y \oplus X_j$ and $g_j : Y_j \twoheadrightarrow X_j$ be given by $\begin{bmatrix} 0 & id_{X_j} \end{bmatrix}$. For $j < i$, let $\beta_j = id_Y$. Let $\beta_i = \begin{bmatrix} id_Y \\ \alpha_i g \end{bmatrix}$ and, for $j > i$, let $\beta_j = \begin{bmatrix} id_Y & 0 \\ 0 & \alpha_j \end{bmatrix}$. The direct sum of admissible monomorphisms is admissible, hence β_j is an admissible monomorphism for $j > i$. β_i is an admissible monomorphism by Lemma 3.8, therefore $Y_\bullet \in \mathrm{MMor}_k(\mathcal{E})$. It is clear that $g_\bullet : Y_\bullet \to X_\bullet$ is a morphism, that each g_i is an admissible epimorphism, and that $g_\bullet$ has kernel

$$ker(g) \overset{id}{\rightarrowtail} \cdots \overset{id}{\rightarrowtail} ker(g) \rightarrowtail Y \overset{id}{\rightarrowtail} \cdots \overset{id}{\rightarrowtail} Y \in \mathrm{MMor}_k(\mathcal{E})$$

Thus $g_\bullet$ is an admissible epimorphism.

By projectivity of $P_\bullet$, we obtain a morphism $h_\bullet : P_\bullet \to Y_\bullet$ such that $f_\bullet = g_\bullet h_\bullet$. Letting $h = h_i$, we have that $f = gh$, hence P_i is projective.

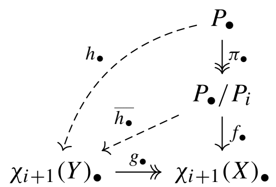
It remains to show that the ι_i are split. For any two indices $j > l$, denote $P_j/P_l := coker(\iota_l^{j-l})$. It suffices to show that each of the compositions $P_i \overset{\iota_i^{k+1-i}}{\rightarrowtail} P_{k+1}$ is split; this follows immediately if we show that P_{k+1}/P_i is projective for each $1 \le i \le k$.

Suppose we have an admissible epimorphism $g : Y \twoheadrightarrow X$ and any morphism $f : P_{k+1}/P_i \to X$; we shall construct a lift $h : P_{k+1}/P_i \to Y$. Define $P_\bullet/P_i$ to be the object in $\mathrm{MMor}_k(\mathcal{E})$ given by $0 \to \cdots \to 0 \to P_{i+1}/P_i \rightarrowtail \cdots \rightarrowtail P_{k+1}/P_i$, with the morphisms induced by the ι_j. There is a natural morphism $\pi_\bullet : P_\bullet \twoheadrightarrow P_\bullet/P_i$ with kernel

$$P_1 \rightarrowtail \cdots \rightarrowtail P_{i-1} \rightarrowtail P_i \overset{id}{\rightarrowtail} \cdots \overset{id}{\rightarrowtail} P_i \in \mathrm{MMor}_k(\mathcal{E})$$

Thus $\pi_\bullet$ is an admissible epimorphism. Moreover, f and g induce obvious morphisms $f_\bullet : P_\bullet/P_i \to \chi_{i+1}(X)_\bullet$, and $g_\bullet : \chi_{i+1}(Y)_\bullet \twoheadrightarrow \chi_{i+1}(X)_\bullet$.

Consider the following diagram:



By projectivity of $P_\bullet$, we can lift $f_\bullet \pi_\bullet$ to $h_\bullet : P_\bullet \to \chi_{i+1}(Y)_\bullet$. Furthermore, since $\chi_{i+1}(Y)_i = 0$, the composition $P_i \rightarrowtail P_j \xrightarrow{h_j} Y$ is zero for all $j > i$, hence h_j factors through $\overline{h_j} : P_j/P_i \to Y$. Defining $\overline{h_j} = 0$ for $j \leq i$, it follows that $h_\bullet = \overline{h_\bullet}\pi_\bullet$, hence $f_\bullet \pi_\bullet = g_\bullet \overline{h_\bullet} \pi_\bullet$. Since $\pi_\bullet$ is an epimorphism, we obtain $f_\bullet = g_\bullet \overline{h_\bullet}$, so the above diagram commutes. In particular, $\overline{h_{k+1}} : P_{k+1}/P_i \to Y$ is a lift of $f_{k+1} = f$, so P_{k+1}/P_i is projective, as claimed. □

It will also be helpful to have the following characterization of projectives and injectives in $\mathrm{Mor}_k(\mathcal{E})$.

Proposition 3.10 *Let $\mathcal{E}$ be an exact category. The object $(P_\bullet, \iota_\bullet) \in \mathrm{Mor}_k(\mathcal{E})$ is projective if and only if each P_i is projective in $\mathcal{E}$ and each ι_i is a split monomorphism. The object $(I_\bullet, \pi_\bullet) \in \mathrm{Mor}_k(\mathcal{E})$ is injective if and only if each I_i is injective in $\mathcal{E}$ and each π_i is a split epimorphism.*

Proof Let $(P_\bullet, \iota_\bullet)$ be projective in $\mathrm{Mor}_k(\mathcal{E})$. To show that P_i is projective, choose any admissible epimorphism $g : Y \twoheadrightarrow X$ in $\mathcal{E}$ and any morphism $f : P_i \to X$; we must construct $h : P_i \to Y$ such that $f = gh$. Define $\omega_i(X)_\bullet \in \mathrm{Mor}_k(\mathcal{E})$ to be

$$X \xrightarrow{id} \cdots \xrightarrow{id} X \to 0 \to \cdots 0$$

where X appears in the first i positions, and similarly for $\omega_i(Y)_\bullet$. We can extend f to a morphism $f_\bullet : P_\bullet \to \omega_i(X)_\bullet$ by setting $f_j := f\iota_j^{i-j}$ for $j \leq i$ and $f_j = 0$ for $j > i$; g extends to an admissible epimorphism $g_\bullet : \omega_i(X)_\bullet \twoheadrightarrow \omega_i(Y)_\bullet$ in the obvious way. By projectivity of $P_\bullet$, we obtain a lift $h_\bullet : P_\bullet \to Y_\bullet$ such that $f_\bullet = g_\bullet h_\bullet$. It follows that $f = h_i g$, hence P_i is projective.

To show that ι_i is a split monomorphism, define

$$P_\bullet^{\leq i} := P_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{i-1}} P_i \longrightarrow 0 \longrightarrow 0 \to \cdots \to 0$$

$$\widehat{P_\bullet^{\leq i}} := P_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{i-1}} P_i \xrightarrow{id} P_i \to 0 \to \cdots \to 0$$

There are natural morphisms $f_\bullet : P_\bullet \twoheadrightarrow P_\bullet^{\leq i}$ and $g_\bullet : \widehat{P_\bullet^{\leq i}} \twoheadrightarrow P_\bullet^{\leq i}$, both of which are admissible epimorphisms. By projectivity of $P_\bullet$, we obtain a map $r_\bullet : P_\bullet \to \widehat{P_\bullet^{\leq i}}$ such that $f_\bullet = g_\bullet r_\bullet$. For all $j \leq i$, we have that $f_j = id_{P_j} = g_j$, hence $r_j = id_{P_j}$. From the diagram

$$\begin{array}{ccc} P_i & \xrightarrow{\iota_i} & P_{i+1} \\ \downarrow{\scriptstyle r_i} & & \downarrow{\scriptstyle r_{i+1}} \\ P_i & \xrightarrow{id_{P_i}} & P_i \end{array}$$

we deduce that $r_{i+1}\iota_i = id_{P_i}$, hence ι_i is a split monomorphism.

For the reverse direction, it suffices to prove that $\chi_i(P)_\bullet$ is projective in $\mathrm{Mor}_k(\mathcal{E})$ for $1 \leq i \leq k+1$ and each $P \in \mathrm{Proj}(\mathcal{E})$. This claim was proved explicitly in our proof of Proposition 3.9.

Note that there is an equivalence of categories $\mathrm{Mor}_k(\mathcal{E})^{op} \xrightarrow{\sim} \mathrm{Mor}_k(\mathcal{E}^{op})$ given by $(X_\bullet, f_\bullet) \mapsto (X_{k+2-\bullet}, f^{op}_{k+1-\bullet})$. The characterization of injective objects thus follows from the characterization of projective objects. □

Remark Note that the projective objects of $\mathrm{MMor}_k(\mathcal{E})$ are precisely the projective objects of $\mathrm{Mor}_k(\mathcal{E})$. Dually, the injective objects of $\mathrm{EMor}_k(\mathcal{E})$ are precisely the injective objects of $\mathrm{Mor}_k(\mathcal{E})$.

If an exact category has enough injectives or projectives, so does its monomorphism category.

Proposition 3.11 *Let $\mathcal{E}$ be an exact category. If $\mathcal{E}$ has enough projectives (resp., injectives), then so does* $\mathrm{MMor}_k(\mathcal{E})$.

Proof Let $(X_\bullet, \alpha_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$, and suppose $\mathcal{E}$ has enough projectives. Then there exist projective objects P_i and admissible epimorphisms $p_i : P_i \twoheadrightarrow X_i$ for each $1 \leq i \leq k+1$. Let $P'_i = \bigoplus_{j=1}^{i} P_j = P'_{i-1} \oplus P_i$ and let $\iota_i : P'_i \rightarrowtail P'_{i+1}$ denote the canonical monomorphism. Then $(P'_\bullet, \iota_\bullet)$ is projective in $\mathrm{MMor}_k(\mathcal{E})$ by Proposition 3.9. Define $f_\bullet : P'_\bullet \to X_\bullet$ by $f_i := \begin{bmatrix}\alpha_1^{i-1} p_1 & \cdots & \alpha_{i-1}p_{i-1} & p_i\end{bmatrix} = \begin{bmatrix}\alpha_{i-1}f_{i-1} & p_i\end{bmatrix}$. Since p_i is an admissible epimorphism in $\mathcal{E}$, by Lemma 3.8 so is f_i, hence $f_\bullet$ is an admissible epimorphism in $\mathrm{Mor}_k(\mathcal{E})$. Let $g_\bullet : (K_\bullet, \beta_\bullet) \rightarrowtail (P'_\bullet, \iota_\bullet)$ be the kernel of $f_\bullet$. To show that $f_\bullet$ is admissible in $\mathrm{MMor}_k(\mathcal{E})$, we must show that $(K_\bullet, \beta_\bullet)$ lies in $\mathrm{MMor}_k(\mathcal{E})$.

Write the admissible monomorphism $g_i : K_i \rightarrowtail P'_i = P'_{i-1} \oplus P_i$ as $g_i = \begin{bmatrix}\psi_i \\ -\varphi_i\end{bmatrix}$. We have an admissible short exact sequence

$$K_i \overset{\begin{bmatrix}\psi_i \\ -\varphi_i\end{bmatrix}}{\rightarrowtail} P'_{i-1} \oplus P_i \overset{\begin{bmatrix}\alpha_{i-1}f_{i-1} & p_i\end{bmatrix}}{\twoheadrightarrow} X_i$$

which gives rise to the bicartesian square:

$$\begin{array}{ccc} K_i & \overset{\psi_i}{\twoheadrightarrow} & P'_{i-1} \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \alpha_{i-1}f_{i-1}} \\ P_i & \overset{p_i}{\twoheadrightarrow} & X_i \end{array}$$

Since p_i is an admissible epimorphism, so is ψ_i. By projectivity of P'_{i-1}, the top row is split exact, hence $K_i \cong P'_{i-1} \oplus ker(\psi_i)$. Identifying the two, we can express ψ_i as $\begin{bmatrix}id & 0\end{bmatrix}$ and φ_i as $\begin{bmatrix}\tau_i & \theta_i\end{bmatrix}$ for some $\tau_i : P'_{i-1} \to P_i$ and $\theta_i : ker(\psi_i) \to P_i$. In particular, we can express $g_i : K_i \to P'_i$ as the matrix $\begin{bmatrix} id & 0 \\ -\tau_i & -\theta_i \end{bmatrix}$.

Let us express $\beta_{i-1} : K_{i-1} \to K_i = P'_{i-1} \oplus ker(\psi_i)$ as $\begin{bmatrix}\delta_{i-1} \\ \gamma_{i-1}\end{bmatrix}$. We can then rewrite the identity $g_i\beta_{i-1} = \iota_{i-1}g_{i-1}$ as the commutative diagram

$$\begin{array}{ccc} K_{i-1} & \xrightarrow{\begin{bmatrix}\delta_{i-1}\\ \gamma_{i-1}\end{bmatrix}} & P'_{i-1} \oplus ker(\psi_i) \\ \big\downarrow{\scriptstyle g_{i-1}} & \begin{bmatrix} id \\ 0 \end{bmatrix} & \big\downarrow{\scriptstyle \begin{bmatrix} id & 0 \\ -\tau_i & -\theta_i \end{bmatrix}} \\ P'_{i-1} & \rightarrowtail & P'_{i-1} \oplus P_i \end{array}$$

It follows that $\delta_{i-1} = g_{i-1}$. Since g_{i-1} is an admissible monomorphism, so is $\beta_{i-1} = \begin{bmatrix} g_{i-1} \\ \gamma_{i-1} \end{bmatrix}$. Thus $(K_\bullet, \beta_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$, and so $f_\bullet$ is an admissible epimorphism. Therefore $\mathrm{MMor}_k(\mathcal{E})$ has enough projectives.

Suppose now that $\mathcal{E}$ has enough injectives. Let $(X_\bullet, \alpha_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$; we shall construct an admissible monomorphism $g_\bullet : (X_\bullet, \alpha_\bullet) \rightarrowtail (I_\bullet, \iota_\bullet)$ for some injective object $(I_\bullet, \iota_\bullet)$.

Let $g_1 : X_1 \rightarrowtail I_1$ be an admissible morphism from X_1 to an injective object in $I_1 \in \mathcal{E}$; we shall define the remaining admissible monomorphsims g_i, injective objects I_i, and split monomorphisms ι_i inductively. Suppose we have constructed $g_i : X_i \rightarrowtail I_i$. Since $\alpha_i : X_i \rightarrowtail X_{i+1}$ is an admissible monomorphism, we can lift g_i to a morphism $\hat{g}_i : X_{i+1} \to I_i$. Since $\mathcal{E}$ has enough injectives, there exists an admissible monomorphism $h_{i+1} : coker(\alpha_i) \rightarrowtail I'_{i+1}$ for some injective object I'_{i+1}. We define $I_{i+1} := I_i \oplus I'_{i+1}$ and $g_{i+1} = \begin{bmatrix} \hat{g}_i & h_{i+1}\pi_{i+1} \end{bmatrix}$, where $\pi_{i+1} : X_{i+1} \twoheadrightarrow coker(\alpha_i)$ is the canonical map. Let $\iota_i : I_i \rightarrowtail I_{i+1}$ be the inclusion of I_i as a direct summand of I_{i+1}. Since I_i and I'_{i+1} are injective, so is I_{i+1}. It is clear that ι_i is split; it remains to check that g_{i+1} is an admissible monomorphism.

We have a commutative diagram with exact rows

$$\begin{array}{ccccc} X_i & \overset{\alpha_i}{\rightarrowtail} & X_{i+1} & \overset{\pi_{i+1}}{\twoheadrightarrow} & coker(\alpha_i) \\ \big\downarrow{\scriptstyle g_i} & & \big\downarrow{\scriptstyle g_{i+1}} & & \big\downarrow{\scriptstyle h_{i+1}} \\ I_i & \overset{\iota_i}{\rightarrowtail} & I_{i+1} & \twoheadrightarrow & I'_{i+1} \end{array}$$

It follows from the Five Lemma [5, Corollary 3.2] that g_{i+1} is an admissible monomorphism, hence $g_\bullet$, $I_\bullet$, and $\iota_\bullet$ are defined, and $g_\bullet$ is an admissible morphism in $\mathrm{Mor}_k(\mathcal{E})$.

To see that $g_\bullet$ is an admissible monomorphism in $\mathrm{MMor}_k(\mathcal{E})$, we must show that its cokernel $(Q_\bullet, \psi_\bullet)$ lies in $\mathrm{MMor}_k(\mathcal{E})$. We have a commutative diagram with exact columns:

$$\begin{array}{ccccc} X_i & \overset{\alpha_i}{\rightarrowtail} & X_{i+1} & \twoheadrightarrow & coker(\alpha_i) \\ \big\downarrow{\scriptstyle g_i} & & \big\downarrow{\scriptstyle g_{i+1}} & & \big\downarrow{\scriptstyle h_{i+1}} \\ I_i & \overset{\iota_i}{\rightarrowtail} & I_{i+1} & \twoheadrightarrow & I'_i \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ coker(g_i) & \overset{\psi_i}{\dashrightarrow} & coker(g_{i+1}) & \dashrightarrow & coker(h_{i+1}) \end{array}$$

Since the first two rows are exact, by the 3×3 Lemma [5, Corollary 3.6] the third row is also an admissible short exact sequence. In particular, ψ_i is an admissible monomorphism, hence $coker(g_\bullet) \in \mathrm{MMor}_k(\mathcal{E})$. Thus $g_\bullet$ is an admissible monomorphism. $I_\bullet$ is injective by Proposition 3.9, hence $\mathrm{MMor}_k(\mathcal{E})$ has enough injectives. □

We have arrived at the main result of this section:

Theorem 3.12 *Let $\mathcal{E}$ be a Frobenius exact category. Then* $\mathrm{MMor}_k(\mathcal{E})$ *is Frobenius exact.*

Proof Since $\mathrm{Proj}(\mathcal{E}) = \mathrm{Inj}(\mathcal{E})$, it follows immediately from Proposition 3.9 that $\mathrm{Proj}(\mathrm{MMor}_k(\mathcal{E})) = \mathrm{Inj}(\mathrm{MMor}_k(\mathcal{E}))$. Since $\mathcal{E}$ has enough projectives and injectives, by Proposition 3.11 so does $\mathrm{MMor}_k(\mathcal{E})$. □

Definition 3.13 Let $\mathcal{E}$ be a Frobenius exact category. For $N \geq 2$, define the N**-stable category of** $\mathcal{E}$, denoted $\mathrm{stab}_N(\mathcal{E})$, to be the stable category of $\mathrm{MMor}_{N-2}(\mathcal{E})$.

Note that when $N = 2$, we obtain the stable category of $\mathcal{E}$.

4 Acyclic projective-injective N-complexes

Throughout this section, let $\mathcal{A}$ denote a Gorenstein abelian category and let $\mathcal{E}$ denote the Frobenius exact subcategory $\mathrm{Gproj}(\mathcal{A})$. Consider the functor $F : C_N^{ac}(\mathrm{Proj}(\mathcal{A})) \to \mathrm{MMor}_{N-2}(\mathcal{E})$ given by

$$F(P^\bullet) = Z_1^0(P^\bullet) \hookrightarrow \cdots \hookrightarrow Z_{N-1}^0(P^\bullet)$$

In this section, we shall prove that F induces an equivalence $\overline{F}$ between $K_N^{ac}(\mathrm{Proj}(\mathcal{A}))$ and $\mathrm{stab}_N(\mathcal{E})$.

4.1 Properties of F

Since a priori F is only a functor into $\mathrm{Mor}_{N-2}(\mathcal{A})$, we must first prove that F actually takes values in $\mathrm{MMor}_{N-2}(\mathcal{E})$.

Proposition 4.1 *Let $(P^\bullet, d_P^\bullet) \in C_N^{ac}(\mathrm{Proj}(\mathcal{A}))$. Then for all $k \in \mathbb{Z}$ and $1 \leq i < N$, $Z_i^k(P^\bullet) \in \mathcal{E}$. The natural inclusion maps $Z_i^0(P^\bullet) \hookrightarrow Z_{i+1}^0(P^\bullet)$ are admissible monomorphisms in $\mathcal{E}$, hence $F(P^\bullet) \in \mathrm{MMor}_{N-2}(\mathcal{E})$.*

Proof Fix $1 \leq i < N$. To show that $Z_i^0(P^\bullet) \in \mathcal{E}$, let $Q \in \mathrm{Proj}(\mathcal{A})$ and $n > 0$. Note that Q has finite injective dimension $m \geq 0$, hence $\mathrm{Ext}_{\mathcal{A}}^{m+1}(M, Q) = 0$ for all $M \in \mathcal{A}$. We can convert $P^\bullet$ into a 2-complex $(\tilde{P}^\bullet, d_{\tilde{P}}^\bullet)$ by arranging the differentials into groups of i and $N - i$. More precisely, define

$$\tilde{P}^s = \begin{cases} P^{Nk} & s = 2k \\ P^{Nk+i} & s = 2k+1 \end{cases}, \quad d_{\tilde{P}}^s = \begin{cases} d_P^{Nk,i} & s = 2k \\ d_P^{Nk+i,N-i} & s = 2k+1 \end{cases}$$

Note that $\tilde{P}^\bullet$ is acyclic and $Z^0(\tilde{P}^\bullet) = Z_i^0(P^\bullet)$. Since, for all $k \in \mathbb{Z}$, $\tau_{\leq 0}(\tilde{P}^\bullet[k-1])$ is a projective resolution of $Z^k(\tilde{P}^\bullet)$, we have that

$$\begin{aligned} \mathrm{Hom}_{D^b(\mathcal{A})}(Z_i^0(P^\bullet), Q[n]) &= \mathrm{Hom}_{K^-(\mathcal{A})}(\tau_{\leq 0}(\tilde{P}^\bullet[-1]), Q[n]) \\ &= \mathrm{Hom}_{K(\mathcal{A})}(\tilde{P}^\bullet[-1], Q[n]) \\ &= \mathrm{Hom}_{K(\mathcal{A})}(\tilde{P}^\bullet[m-n], Q[m+1]) \\ &= \mathrm{Hom}_{K^-(\mathcal{A})}(\tau_{\leq 0}(\tilde{P}^\bullet[m-n]), Q[m+1]) \\ &= \mathrm{Hom}_{D^b(\mathcal{A})}(Z^{m-n+1}(\tilde{P}^\bullet), Q[m+1]) \\ &= \mathrm{Ext}_{\mathcal{A}}^{m+1}(Z^{m-n+1}(\tilde{P}^\bullet), Q) \\ &= 0 \end{aligned}$$

Thus $Z_i^0(P^\bullet) \in \mathcal{E}$ for all $1 \leq i < N$. Applying the same argument to $P^\bullet[k]$ shows that $Z_i^k(P^\bullet) \in \mathcal{E}$ for all $k \in \mathbb{Z}$.

A morphism in $\mathcal{E}$ is an admissible monomorphism if and only if it is a monomorphism in $\mathcal{A}$ with cokernel in $\mathcal{E}$. The map $\iota : Z_i^0(P^\bullet) \hookrightarrow Z_{i+1}^0(P^\bullet)$ is a monomorphism in $\mathcal{A}$ since it is the kernel of the restriction of $d_P^{0,i}$ to $Z_{i+1}^0(P^\bullet)$. Since $Z_{i+1}^0(P^\bullet) = B_{i+1}^0(P^\bullet)$, we obtain a short exact sequence $Z_i^0(P^\bullet) \overset{\iota}{\hookrightarrow} B_{i+1}^0(P^\bullet) \overset{d_P^{0,i}}{\twoheadrightarrow} B_1^i(P^\bullet)$. Since $B_1^i(P^\bullet) = Z_1^i(P^\bullet) \in \mathcal{E}$, ι is an admissible monomorphism in $\mathcal{E}$, and therefore $F(P^\bullet) \in \mathrm{MMor}_{N-2}(\mathcal{E})$. □

To prove that F is full, we introduce the following terminology.

Definition 4.2 Let $P^\bullet, Q^\bullet \in C_N(\mathcal{A})$. Let $n \in \mathbb{Z}$ and let $f^n : P^n \to Q^n$ be any morphism. We say f^n **preserves cycles** if the restriction of f^n to $Z_i^n(P^\bullet)$ has image in $Z_i^n(Q^\bullet)$ for each $1 \leq i \leq N-1$.

Similarly, we say f^n **preserves boundaries** if the restriction of f^n to $B_i^n(P^\bullet)$ has image in $B_i^n(Q^\bullet)$ for each $1 \leq i \leq N-1$.

Note that when $P^\bullet$ and $Q^\bullet$ are acyclic, the two notions are equivalent.

Proposition 4.3 *F is full.*

Proof Take $P^\bullet, Q^\bullet \in C_N^{ac}(\mathrm{Proj}(\mathcal{A}))$ and $f_\bullet : F(P^\bullet) \to F(Q^\bullet)$. Using the injectivity of Q^0, lift the map $Z_{N-1}^0(P^\bullet) \xrightarrow{f_{N-1}} Z_{N-1}^0(Q^\bullet) \hookrightarrow Q^0$ along the monomorphism $Z_{N-1}^0(P^\bullet) \hookrightarrow P^0$ to obtain a morphism $f^0 : P^0 \to Q^0$. Clearly, the restriction of f^0 to $Z_i^0(P^\bullet)$ is f_i, hence f^0 preserves cycles.

It thus suffices to extend f^0 to a morphism of complexes $f^\bullet : P^\bullet \to Q^\bullet$. We claim that, given a morphism $f^n : P^n \to Q^n$ which preserves cycles, we can construct maps $f^{n\pm1} : P^{n\pm1} \to Q^{n\pm1}$, both preserving cycles, such that $d_Q^i f^i = f^{i+1} d_P^i$ for $i = n-1, n$. Once this claim established, we can extend f^0 to $f^\bullet$ by induction, proving fullness.

Since f^n preserves cycles, we obtain an induced map on the images $\overline{f^n} : B_{N-1}^{n+1}(P^\bullet) \to B_{N-1}^{n+1}(Q^\bullet)$, which, by injectivity of Q^{n+1}, lifts to a map $f^{n+1} : P^{n+1} \to Q^{n+1}$. It follows immediately that $f^{n+1} d_P^n = d_Q^n f^n$. For $1 \leq i \leq N-2$, if we restrict both sides of this equation to $B_{i+1}^n(P^\bullet)$ and use the fact that f^n preserves boundaries, we see that f^{n+1} maps $B_i^{n+1}(P^\bullet)$ into $B_i^{n+1}(Q^\bullet)$. For $i = N-1$, note that by construction the restriction of f^{n+1} to $B_{N-1}^{n+1}(P^\bullet)$ is $\overline{f^n}$. Thus f^{n+1} preserves boundaries and therefore cycles.

Since f^n preserves boundaries, it restricts to a map from $B_{N-1}^n(P^\bullet)$ to $B_{N-1}^n(Q^\bullet)$. Using projectivity of P^{n-1}, we can lift this restriction to $f^{n-1} : P^{n-1} \to Q^{n-1}$. It follows that $f^n d_P^{n-1} = d_Q^{n-1} f^{n-1}$, hence f^{n-1} maps $Z_1^{n-1}(P^\bullet)$ into $Z_1^{n-1}(Q^\bullet)$. For $2 \leq i \leq N-1$, note that since f^n preserves cycles, the left side of this equation maps $Z_i^{n-1}(P^\bullet)$ into $Z_{i-1}^n(Q^\bullet)$. Postcomposing with $d_Q^{n,i-1}$, we get $d_Q^{n,i-1} f^n d_P^{n-1} = d_Q^{n-1,i} f^{n-1}$, hence the left side maps $Z_i^{n-1}(P^\bullet)$ to 0. The right side then shows that f^{n-1} maps $Z_i^{n-1}(P^\bullet)$ into $Z_i^{n-1}(Q^\bullet)$, hence f^{n-1} preserves cycles. □

To show that F is essentially surjective, it will be convenient to introduce the following terminology.

Definition 4.4 An N-**acyclic array** in $\mathcal{E}$ is the data of:

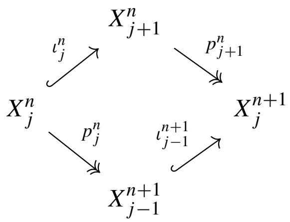
- objects X_j^n; $n \in \mathbb{Z}, 0 \leq j \leq N$

- monomorphisms $\iota_j^n : X_j^n \hookrightarrow X_{j+1}^n$; $n \in \mathbb{Z}, 0 \leq j < N$
- epimorphisms $p_j^n : X_j^n \twoheadrightarrow X_{j-1}^{n+1}$; $n \in \mathbb{Z}, 0 < j \leq N$

We shall write $\iota_j^{n,k} : X_j^n \hookrightarrow X_{j+k}^n$ for the composition $\iota_{j+k-1}^n \cdots \iota_j^n$ of k successive $\iota_\bullet^n$, beginning at ι_j^n, and similarly for $p_j^{n,k} : X_j^n \twoheadrightarrow X_{j-k}^{n+k}$.

The above data should satisfy the following three properties:

1) $X_0^n \cong 0$.
2) X_N^n is projective-injective.
3) For all $1 \leq j \leq N-1$, the diagram



commutes and forms a bicartesian square.

Given $X_\bullet \in \mathrm{MMor}_{N-2}(\mathcal{E})$, we say that the N-acyclic array $(X_j^n, \iota_j^n, p_j^n)$ **extends** $X_\bullet$ if $X_\bullet = (X_\bullet^0, \iota_\bullet^0)$.

Given $P^\bullet \in C_N^{ac}(\mathrm{Proj}(\mathcal{A}))$, it is easily verified that we obtain an N-cyclic array by defining $X_j^n = Z_j^n(P^\bullet)$ (here we take $Z_0^n(P^\bullet) = 0$ and $Z_N^n(P^\bullet) = P^n$), ι_j^n to be the inclusion of kernels, and p_j^n to be the morphism on kernels induced by d_P^n.

Proposition 4.5 *F is essentially surjective.*

Proof Let $(X_\bullet, \iota_\bullet) \in \mathrm{MMor}_{N-2}(\mathcal{E})$. The proof proceeds in two steps. First we prove that, given an N-acyclic array $(X_j^n, \iota_j^n, p_j^n)$ extending $X_\bullet$, there exists $P^\bullet \in C_N^{ac}(\mathrm{Proj}(\mathcal{A}))$ such that $F(P^\bullet) = X_\bullet$. In the second step, we shall construct such an N-acyclic array.

Given an N-acyclic array $(X_j^n, \iota_j^n, p_j^n)$ extending $X_\bullet$, define maps

$$d^n := \iota_{N-1}^{n+1} p_N^n : X_N^n \to X_N^{n+1}$$

We claim that $(X_N^\bullet, d^\bullet) \in C_N^{ac}(\mathrm{Proj}(\mathcal{A}))$. By assumption, all p and ι commute, so we have that $d^{n,j} = \iota_{N-j}^{n+j,j} p_N^{n,j}$ for all $1 \leq j \leq N$. In particular, $d^{n,N}$ factors through $X_0^{n+N} = 0$, hence $X_N^\bullet \in C_N(\mathcal{A})$. Each X_N^n is projective-injective by assumption.

To show that $X_N^\bullet$ is acyclic, note that

$$\begin{aligned} Z_j^n(X_N^\bullet) &= ker(d^{n,j}) = ker(\iota_{N-j}^{n+j,j} p_N^{n,j}) \\ &= ker(p_N^{n,j}) \\ B_j^n(X_N^\bullet) &= im(d^{n-N+j,N-j}) = im(\iota_j^{n,N-j} p_N^{n-N+j,N-j}) \\ &= X_j^n \end{aligned}$$

Thus we must show that $X_j^n = ker(p_N^{n,j})$. Since the composition of bicartesian squares is bicartesian, the commutative square

$$\begin{array}{ccccc} & & X_{j+k}^n & & \\ & \overset{\iota_j^{n,k}}{\nearrow} & & \overset{p_{j+k}^{n,j}}{\searrow} & \\ X_j^n & & & & X_k^{n+j} \\ & \underset{p_j^{n,j}}{\searrow} & & \underset{\iota_0^{n+j,k}}{\nearrow} & \\ & & X_0^{n+j} = 0 & & \end{array}$$

is bicartesian for all $1 \leq j \leq N-1$, $1 \leq k \leq N-j$. This yields an exact sequence

$$0 \longrightarrow X_j^n \overset{\iota_j^{n,k}}{\hookrightarrow} X_{j+k}^n \overset{p_{j+k}^{n,j}}{\twoheadrightarrow} X_k^{n+j} \longrightarrow 0$$

Taking $k = N - j$, we obtain that $X_j^n = ker(p_N^{n,j})$, as desired. Therefore $X_N^\bullet$ is acyclic.

Taking $n = 0$ and $k = 1$ in the above exact sequence, we see that the morphism $Z_j^0(X_N^\bullet) \hookrightarrow Z_{j+1}^0(X_N^\bullet)$ is precisely $X_j^0 \overset{\iota_j^0}{\hookrightarrow} X_{j+1}^0$. Thus $F(X_N^\bullet) = X_\bullet$. Thus $P^\bullet := X_N^\bullet$ satisfies the desired properties.

We must now construct an N-acyclic array extending $(X_\bullet, \iota_\bullet)$. For $1 \leq j \leq N-1$, let $X_j^0 = X_j$ and let $X_0^0 = 0$. For $1 \leq j \leq N-2$, let $\iota_j^0 = \iota_j$ and let $\iota_0^0 : 0 \hookrightarrow X_1$ be the zero map. Define $\iota_{N-1}^0 : X_{N-1}^0 \hookrightarrow X_N^0$ to be the inclusion of X_{N-1}^0 into a projective-injective object X_N^0.

Suppose for some $n \geq 0$ we have constructed, for all j, X_j^n and ι_j^n. Define $X_0^{n+1} = 0$ and $p_1^n : X_1^n \twoheadrightarrow 0$. Next, inductively define X_j^{n+1}, i_{j-1}^{n+1}, and p_{j+1}^n for $1 \leq j \leq N-1$ via iterated pushouts

$$\begin{array}{ccccc} & & X_{j+1}^n & & \\ & \overset{\iota_j^n}{\nearrow} & & \overset{p_{j+1}^n}{\searrow} & \\ X_j^n & & & & X_j^{n+1} \\ & \underset{p_j^n}{\searrow} & & \underset{\iota_j^{n+1}}{\nearrow} & \\ & & X_{j-1}^{n+1} & & \end{array} \qquad (1)$$

Since $\mathcal{E}$ is an exact category, it follows immediately that the newly defined maps ι are admissible monomorphisms, and the maps p are admissible epimorphisms by the dual of [5, Proposition 2.15]. Finally, define $\iota_{N-1}^{n+1} : X_{N-1}^{n+1} \hookrightarrow X_N^{n+1}$ to be an inclusion of X_{N-1}^{n+1} into a projective-injective object X_N^{n+1}. Note that we have now constructed X_j^{n+1}, ι_j^{n+1}, and p_j^n for all j. Proceeding inductively, we can define X_j^n, ι_j^n, and p_j^n for all $n \geq 0$ and for all j.

For $n \leq 0$, the construction is dual. Having defined X_j^n and ι_j^n for all j, define $p_N^{n-1} : X_N^{n-1} \twoheadrightarrow X_{N-1}^n$ to be a surjection from a projective-injective object X_N^{n-1}. Then X_j^{n-1}, i_j^{n-1}, and p_j^{n-1} are defined via iterated pullbacks for $N-1 \geq j \geq 1$. Finally, define $X_0^{n-1} = 0$ and ι_0^{n-1} to be the zero map.

It is immediate that $(X_j^n, \iota_j^n, p_j^n)$ satisfies properties 1 and 2 of Definition 4.4. To see that property 3 holds, note that each commutative square in (1) is, by construction, either a pullback ($n < 0$) or pushout ($n \geq 0$). But since the ι are admissible monomorphisms and the p are admissible epimorphisms, any such pullback or pushout square is automatically bicartesian, for instance by [5, Proposition 2.12]. Thus the data we have constructed form an N-acyclic array which extends $(X_\bullet, \iota_\bullet)$. □

The category $C_N^{ac}(\text{Proj}(\mathcal{A}))$ inherits the structure of an exact category from $C_N(\mathcal{A})$.

Proposition 4.6 *$C_N^{ac}(\text{Proj}(\mathcal{A}))$ is a fully exact subcategory of $C_N(\mathcal{A})$. An object $P^\bullet \in C_N^{ac}(\text{Proj}(\mathcal{A}))$ is projective (resp., injective) if and only if it is projective (resp., injective) in $C_N(\mathcal{A})$. Thus $C_N^{ac}(\text{Proj}(\mathcal{A}))$ is Frobenius exact.*

Proof $C_N^{ac}(\text{Proj}(\mathcal{A}))$ is clearly a full, additive subcategory of $C_N(\mathcal{A})$. Given a chainwise-split short exact sequence $X^\bullet \rightarrowtail Y^\bullet \twoheadrightarrow Z^\bullet$ with $X^\bullet, Z^\bullet \in C_N^{ac}(\text{Proj}(\mathcal{A}))$ and $Y^\bullet \in C_N(\mathcal{A})$, it is clear that $Y^n \in \text{Proj}(\mathcal{A})$ for all $n \in \mathbb{Z}$. Since $X^\bullet$ and $Z^\bullet$ are acyclic, it follows immediately from the long exact sequence in homology that $Y^\bullet$ is acyclic. Thus $C_N^{ac}(\text{Proj}(A))$, together with the class of all chainwise split exact sequences, is a fully exact subcategory of $C_N(\mathcal{A})$. The proof of [15, Theorem 2.1] applies without change to $C_N^{ac}(\text{Proj}(\mathcal{A}))$, hence the projective and injective objects are direct sums of complexes of the form $\mu_N^n(P)$, where $P \in \text{Proj}(\mathcal{A})$. The second and third statements follow immediately. □

Proposition 4.7 $F : C_N^{ac}(\text{Proj}(\mathcal{A})) \to \text{MMor}_{N-2}(\mathcal{E})$ *preserves short exact sequences.*

Proof Consider a chainwise split exact sequence $P^\bullet \overset{f^\bullet}{\rightarrowtail} Q^\bullet \overset{g^\bullet}{\twoheadrightarrow} R^\bullet$ in $C_N^{ac}(\text{Proj}(\mathcal{A}))$. Applying the Snake Lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^0 & \overset{f^0}{\hookrightarrow} & Q^0 & \overset{g^0}{\twoheadrightarrow} & R^0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_P^{0,j}} & & \downarrow{\scriptstyle d_Q^{0,j}} & & \downarrow{\scriptstyle d_R^{0,j}} & & \\ 0 & \longrightarrow & P^j & \overset{f^j}{\hookrightarrow} & Q^j & \overset{g^j}{\twoheadrightarrow} & R^j & \longrightarrow & 0 \end{array}$$

we obtain an exact sequence

$$0 \to Z_j^0(P^\bullet) \hookrightarrow Z_j^0(Q^\bullet) \to Z_j^0(R^\bullet) \overset{\phi}{\to} coker(d_P^{0,j})$$

It remains to show that the connecting morphism ϕ is zero.

We briefly recall the construction of ϕ. Let X be the pullback

$$\begin{array}{ccc} X & \overset{p}{\twoheadrightarrow} & Z_j^0(R^\bullet) \\ \downarrow{\scriptstyle \iota} & & \downarrow \\ Q^0 & \overset{g^0}{\twoheadrightarrow} & R^0 \end{array}$$

From this diagram we see that $g^j \circ d_Q^{0,j}\iota = 0$, hence $d_Q^{0,j}\iota$ factors through $ker(g^j) = f^j$. Write $d_Q^{0,j}\iota$ as $X \overset{\alpha}{\to} P^j \overset{f^j}{\hookrightarrow} Q^j$ for a unique map α. Then ϕ is given by the induced map on cokernels

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(p) & \hookrightarrow & X & \overset{p}{\twoheadrightarrow} & Z_j^0(R^\bullet) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi} & & \\ & & P^0 & \overset{d_P^{0,j}}{\longrightarrow} & P^j & \twoheadrightarrow & coker(d_P^{0,j}) & \longrightarrow & 0 \end{array}$$

Thus for ϕ to be zero, we must show that α factors through $im(d_P^{0,j})$.

Since $P^\bullet \rightarrowtail Q^\bullet \twoheadrightarrow R^\bullet$ is chainwise split exact, for each n we can write $Q^n \cong P^n \oplus R^n$, with f^n and g^n becoming the canonical inclusion and projection maps, respectively. Using this decomposition, we can express

$$\begin{aligned} \iota &= \begin{bmatrix} \iota_1 \\ \iota_2 \end{bmatrix} \\ d_Q^{0,j} &= \begin{bmatrix} d_P^{0,j} & \beta \\ 0 & d_R^{0,j} \end{bmatrix} \\ d_Q^{j,N-j} &= \begin{bmatrix} d_P^{j,N-j} & \gamma \\ 0 & d_R^{j,N-j} \end{bmatrix} \end{aligned}$$

Note that $d_R^{0,j}\iota_2 = d_R^{0,j} g^0 \iota = d_R^{0,j} p = 0$. It follows that

$$d_Q^{0,j}\iota = \begin{bmatrix} d_P^{0,j} & \beta \\ 0 & d_R^{0,j} \end{bmatrix} \begin{bmatrix} \iota_1 \\ \iota_2 \end{bmatrix} = \begin{bmatrix} d_P^{0,j}\iota_1 + \beta\iota_2 \\ 0 \end{bmatrix}$$

hence $\alpha = d_P^{0,j}\iota_1 + \beta\iota_2$. Furthermore,

$$0 = d_Q^{j,N-j} \circ d_Q^{0,j}\iota = \begin{bmatrix} d_P^{j,N-j} & \gamma \\ 0 & d_R^{j,N-j} \end{bmatrix} \begin{bmatrix} d_P^{0,j}\iota_1 + \beta\iota_2 \\ 0 \end{bmatrix} = \begin{bmatrix} d_P^{j,N-j}\beta\iota_2 \\ 0 \end{bmatrix}$$

We have that $\beta\iota_2$ factors through $Z^j_{N-j}(P^\bullet) = im(d_P^{0,j})$, hence so does $\alpha = d_P^{0,j}\iota_1 + \beta\iota_2$. Thus $\phi = 0$ and so $0 \to Z^0_j(P^\bullet) \to Z^0_j(Q^\bullet) \to Z^0_j(R^\bullet) \to 0$ is exact for each j. □

Corollary 4.8 *F descends to a functor $\overline{F} : K_N^{ac}(\mathrm{Proj}(\mathcal{A})) \to \mathrm{stab}_N(\mathcal{E})$ of triangulated categories.*

Proof By Proposition 3.9, for any $i \in \mathbb{Z}$, $F(\mu^i_N(P))$ is projective-injective in $\mathrm{MMor}_{N-2}(\mathcal{E})$. Thus F preserves projective-injective objects and so descends to a functor $\overline{F}$ between the stable categories. Since F preserves exact sequences and projective-injective objects, it follows immediately that $\overline{F}$ preserves distinguished triangles and the suspension functor, hence is a functor of triangulated categories. □

4.2 Properties of $\overline{F}$

In this section, we shall prove that $\overline{F}$ is an equivalence of categories. Most of our work will be to show that $\overline{F}$ is faithful. The following terminology will be convenient for the proof.

Definition 4.9 Let $f^\bullet : P^\bullet \to Q^\bullet$ be a morphism in $K_N^{ac}(\mathrm{Proj}(\mathcal{A}))$. Given a family of morphisms $h^i : P^i \to Q^{i-N+1}$, we define the sum

$$S_h(n, j, k) := \sum_{i=n+j}^{n+k-1} d_Q^{\circ, n-i+N-1} h^i d_P^{n,i-n} : P^n \to Q^n$$

whenever the h^i appearing in the formula are defined. To understand this expression, note that $f^\bullet$ is null-homotopic if and only if h^i is defined for all $i \in \mathbb{Z}$ and $f^n = S_h(n, 0, N)$ for each $n \in \mathbb{Z}$. Increasing the second parameter removes terms from the start of the sum, and decreasing the third parameter removes terms from the end of the sum.

We define a **homotopy (of $f^\bullet$) at** n to be a sequence of N maps $(h^n, h^{n+1}, \dots, h^{n+N-1})$ such that $f^n = S_h(n, 0, N)$. We define a **seed (of $f^\bullet$) at** n to be a sequence of $N-1$ maps $(h^n, h^{n+1}, \dots, h^{n+N-2})$ such that $f^n|_{Z^n_{N-1}(P^\bullet)} = S_h(n, 0, N-1)|_{Z^n_{N-1}(P^\bullet)}$.

The following lemma is trivial when $N = 2$.

Lemma 4.10 *Let $f^\bullet : P^\bullet \to Q^\bullet$ be a morphism in $K^{ac}_N(\mathrm{Proj}(\mathcal{A}))$. If $\overline{F}(f) = 0$, then there exists a seed of $f^\bullet$ at 0.*

Proof Since $\overline{F}(f) = 0$, we have a diagram in $\mathcal{E}$

$$\begin{array}{ccccccc}
Z^0_1(P^\bullet) & \hookrightarrow & Z^0_2(P^\bullet) & \hookrightarrow & \cdots & \hookrightarrow & Z^0_{N-1}(P^\bullet) \\
\downarrow & & \downarrow & & & & \downarrow \\
I_1 & \hookrightarrow & I_1 \oplus I_2 & \hookrightarrow & \cdots & \hookrightarrow & \bigoplus_{j=1}^{N-1} I_j \\
\downarrow & & \downarrow & & & & \downarrow \\
Z^0_1(Q^\bullet) & \hookrightarrow & Z^0_2(Q^\bullet) & \hookrightarrow & \cdots & \hookrightarrow & Z^0_{N-1}(Q^\bullet)
\end{array}$$

where the horizontal maps are canonical inclusions, the I_j are projective-injective, and the jth pair of vertical maps composes to $f^0|_{Z^0_j(P^\bullet)}$. For $1 \le j \le N-1$, let $a_j : Z^0_{N-1}(P^\bullet) \to I_j$ and $b_j : I_j \to Z^0_{N-1}(Q^\bullet)$ denote the components of the rightmost vertical maps, so that we have $f^0|_{Z^0_{N-1}(P^\bullet)} = \sum_{j=1}^{N-1} b_j a_j$.

For each $1 \le i \le N-1$, by commutativity of the top rows we have that a_i factors through $Z^0_{N-1}(P^\bullet)/Z^0_{i-1}(P^\bullet)$. (For the degenerate case $i = 1$ we let $Z^0_0(P^\bullet) = 0$.) By injectivity of I_i, we obtain a commutative diagram

$$\begin{array}{ccccc}
Z^0_{N-1}(P^\bullet) & \twoheadrightarrow & Z^0_{N-1}(P^\bullet)/Z^0_{i-1}(P^\bullet) & \xhookrightarrow{\overline{d_P^{0,i-1}}} & P^{i-1} \\
& \searrow^{a_i} & \downarrow{\scriptstyle \overline{a_i}} & \swarrow^{\alpha^{i-1}} & \\
& & I_i & &
\end{array}$$

Thus $a_i = \alpha^{i-1} d_P^{0,i-1}|_{Z^0_{N-1}(P^\bullet)}$ for $1 \le i \le N-1$.

Dually, by commutativity of the bottom rows, b_i factors through $Z^0_i(Q^\bullet)$, which by acyclicity of $Q^\bullet$ is equal to $B^0_i(Q^\bullet)$. By projectivity of I_i, we obtain a map $\beta^{i-1} : I_i \to Q^{i-N}$ such that $b_i = d_Q^{i-N,-i+N} \beta^{i-1}$.

Define $h^i = \beta^i \alpha^i : P^i \to Q^{i-N+1}$ for $0 \le i \le N-2$. Then we have

$$\begin{aligned}
f^0|_{Z^0_{N-1}(P^\bullet)} &= \sum_{i=0}^{N-2} b_{i+1} a_{i+1} = \sum_{i=0}^{N-2} d_Q^{\circ,-i+N-1} h^i d_P^{0,i}|_{Z^0_{N-1}(P^\bullet)} \\
&= S_h(0, 0, N-1)|_{Z^0_{N-1}(P^\bullet)}
\end{aligned}$$

Thus $(h^0, \dots, h^{N-2})$ is a seed of $f^\bullet$ at 0. □

If $(h^n, \dots, h^{n+N-1})$ is a homotopy of $f^\bullet : P^\bullet \to Q^\bullet$ at n, it is clear that the shortened tuple $(h^n, \dots, h^{n+N-2})$ is a seed at n, since the last term of $f^n = S_h(n, 0, N)$ vanishes on $Z^n_{N-1}(P^\bullet)$. The next lemma establishes a converse.

Lemma 4.11 *Let $f^\bullet : P^\bullet \to Q^\bullet$ be a morphism in $K^{ac}_N(\mathrm{Proj}(\mathcal{A}))$. Suppose there exists a seed $(h^n, \dots, h^{n+N-2})$ of $f^\bullet$ at n. Then there exists h^{n+N-1} such that:*

- *$(h^n, \ldots, h^{n+N-1})$ is a homotopy at n.*
- *$(h^{n+1}, \ldots, h^{n+N-1})$ is a seed at $n+1$.*

There also exists h^{n-1} such that:

- *$(h^{n-1}, h^n, \ldots, h^{n+N-2})$ is a homotopy at $n-1$.*
- *$(h^{n-1}, h^n, \ldots, h^{n+N-3})$ is a seed at $n-1$.*

Proof Let $\psi = f^n - S_h(n, 0, N-1)$. Since $(h^n, \ldots, h^{n+N-2})$ is a seed at n, we have $\psi \mid_{Z^n_{N-1}(P^\bullet)} = 0$, hence ψ factors through $P^n / Z^n_{N-1}(P^\bullet)$. Note that $P^n / Z^n_{N-1}(P^\bullet) \cong B_1^{n+N-1}(P^\bullet) = Z_1^{n+N-1}(P^\bullet) \in \mathcal{E}$. By injectivity of Q^n, we obtain

$$\begin{array}{ccccc} P^n & \twoheadrightarrow & P^n/Z^n_{N-1}(P^\bullet) & \xhookrightarrow{\overline{d_P^{n,N-1}}} & P^{n+N-1} \\ & \underset{\psi}{\searrow} & \downarrow{\scriptstyle \overline{\psi}} & \underset{h^{n+N-1}}{\swarrow} & \\ & & Q^n & & \end{array}$$

Thus

$$\begin{aligned} f^n &= S_h(n, 0, N-1) + \psi = S_h(n, 0, N-1) + h^{n+N-1} d_P^{n,N-1} \\ &= S_h(n, 0, N) \end{aligned}$$

so $(h^n, \ldots, h^{n+N-1})$ is a homotopy at n.

To see that $(h^{n+1}, \ldots, h^{n+N-1})$ is a seed at $n+1$, note that

$$f^{n+1} d_P^n = d_Q^n f^n = d_Q^n S_h(n, 0, N) = S_h(n+1, 0, N-1) d_P^n$$

Since $d_P^n : P^n \twoheadrightarrow Z^{n+1}_{N-1}(P^\bullet)$ is an epimorphism, we can cancel it on the right to obtain $f^{n+1}|_{Z^{n+1}_{N-1}(P^\bullet)} = S_h(n+1, 0, N-1)|_{Z^{n+1}_{N-1}(P^\bullet)}$, as desired.

To construct h^{n-1}, let $\varphi = f^{n-1} - S_h(n-1, 1, N)$. Note that

$$\begin{aligned} d_Q^{n-1} \varphi &= d_Q^{n-1} f^{n-1} - d_Q^{n-1} S_h(n-1, 1, N) \\ &= (f^n - S_h(n, 0, N-1)) d_P^{n-1} = 0 \end{aligned}$$

where the last equality holds because $(h^n, \ldots, h^{n+N-1})$ is a seed at n. Thus φ factors through $Z_1^{n-1}(Q^\bullet)$, and by projectivity of P^{n-1} we obtain

$$\begin{array}{ccccc} & & P^{n-1} & & \\ & \overset{h^{n-1}}{\swarrow} & \downarrow{\scriptstyle \varphi} & & \\ Q^{n-N} & \underset{d_Q^{\circ,N-1}}{\twoheadrightarrow} & Z_1^{n-1}(Q^\bullet) & \hookrightarrow & Q^{n-1} \end{array}$$

Thus

$$\begin{aligned} f^{n-1} &= \varphi + S_h(n-1, 1, N) = d_Q^{\circ,N-1} h^{n-1} + S_h(n-1, 1, N) \\ &= S_h(n-1, 0, N) \end{aligned}$$

hence $(h^{n-1}, \ldots, h^{n+N-2})$ is a homotopy at $n-1$. It follows immediately that $(h^{n-1}, \ldots, h^{n+N-3})$ is a seed at $n-1$. □

We are now ready to prove the main theorem of this section.

Theorem 4.12 *$\overline{F} : K_N^{ac}(\mathrm{Proj}(\mathcal{A})) \to \mathrm{stab}_N(\mathcal{E})$ is an equivalence.*

Proof Let $f^\bullet : P^\bullet \to Q^\bullet$ be a morphism in $K_N^{ac}(\mathrm{Proj}(\mathcal{A}))$ such that $\overline{F}(f) = 0$. By Lemmas 4.10 and 4.11, we can inductively define maps $h^i : P^i \to Q^{i-N+1}$ for all $i \in \mathbb{Z}$ such that $(h^n, \ldots, h^{n+N-1})$ is a homotopy at n for every $n \in \mathbb{Z}$. Thus f is null-homotopic, and so $\overline{F}$ is faithful.

$\overline{F}$ is defined via a commutative diagram of functors

$$\begin{array}{ccc} C_N^{ac}(\mathrm{Proj}(\mathcal{A})) & \xrightarrow{F} & \mathrm{MMor}_{N-2}(\mathcal{E}) \\ \downarrow & & \downarrow \\ K_N^{ac}(\mathrm{Proj}(\mathcal{A})) & \xrightarrow{\overline{F}} & \mathrm{stab}_N(\mathcal{E}) \end{array}$$

By Propositions 4.3 and 4.5, F is full and essentially surjective, and the same is clearly true for the projection $\mathrm{MMor}_{N-2}(\mathcal{E}) \to \mathrm{stab}_N(\mathcal{E})$. It follows immediately that $\overline{F}$ is full and essentially surjective, hence an equivalence. □

5 The *N*-singularity category

Throughout this section, let $\mathcal{A}$ be a Gorenstein abelian category and let $\mathcal{E} = \mathrm{Gproj}(\mathcal{A})$.

There is a fully faithful additive functor $G : \mathrm{Mor}_{N-2}(\mathcal{A}) \hookrightarrow C_N^b(\mathcal{A})$ given by interpreting the object $(X_\bullet, \alpha_\bullet) \in \mathrm{Mor}_{N-2}(\mathcal{A})$ as an N-complex concentrated in degrees 1 through $N-1$. In this section, we shall show that G induces an equivalence $\overline{G}$ between $\mathrm{stab}_N(\mathcal{E})$ and $D_N^s(\mathcal{A})$.

Proposition 5.1 *G induces a functor $\overline{G} : \mathrm{stab}_N(\mathcal{E}) \to D_N^s(\mathcal{A})$ of triangulated categories.*

Proof Let G' denote the composition

$$\mathrm{MMor}_{N-2}(\mathcal{E}) \hookrightarrow \mathrm{MMor}_{N-2}(\mathcal{A}) \overset{G}{\hookrightarrow} C_N^b(\mathcal{A}) \to D_N^b(\mathcal{A}) \to D_N^s(\mathcal{A})$$

Recall that the projective-injective objects of $\mathcal{E}$ are precisely the projective objects of $\mathcal{A}$. By Proposition 3.9, G maps projective objects in $\mathrm{MMor}_{N-2}(\mathcal{E})$ to perfect complexes, hence G' sends projective objects to zero. Thus G' induces an additive functor $\overline{G} : \mathrm{stab}_N(\mathcal{E}) \to D_N^s(\mathcal{A})$.

If $X_\bullet \rightarrowtail Y_\bullet \twoheadrightarrow Z_\bullet$ is admissible in $\mathrm{MMor}_{N-2}(\mathcal{E})$, apply G to obtain a short exact sequence in $C_N^b(\mathcal{A})$. By [15, Proposition 3.7], there is a corresponding distinguished triangle $G(X_\bullet) \to G(Y_\bullet) \to G(Z_\bullet) \to \Sigma G(X_\bullet)$ in $D_N^b(\mathcal{A})$, hence in $D_N^s(\mathcal{A})$.

Consider an admissible exact sequence $X_\bullet \rightarrowtail I_{X_\bullet} \twoheadrightarrow \Omega^{-1}X_\bullet$, with $I_{X_\bullet}$ injective. This induces a triangle $G(X_\bullet) \to 0 \to G(\Omega^{-1}X_\bullet) \xrightarrow{\phi_X} \Sigma G(X_\bullet)$ in $D_N^s(\mathcal{A})$, which defines a natural isomorphism $\phi : \overline{G}\Omega^{-1} \xrightarrow{\sim} \Sigma\overline{G}$. Since every distinguished triangle in $\mathrm{stab}_N(\mathcal{E})$ is isomorphic to one arising from an admissible short exact sequence in $\mathrm{MMor}_{N-2}(\mathcal{E})$, it follows easily that $(\overline{G}, \phi)$ is a triangulated functor. □

The functor G also gives a canonical embedding of $\mathrm{Mor}_{N-2}(\mathcal{A})$ into $D_N^b(\mathcal{A})$. With some extra hypotheses on $\mathcal{A}$, this is a corollary of [15, Theorem 4.2]; however, the proof below is valid for an arbitrary abelian category (which need not be Gorenstein).

Proposition 5.2 *The composition $\mathrm{Mor}_{N-2}(\mathcal{A}) \overset{G}{\hookrightarrow} C_N^b(\mathcal{A}) \to D_N^b(\mathcal{A})$ is fully faithful. In particular, the restriction of this functor to $\mathrm{Mor}_{N-2}(\mathcal{E})$ is fully faithful.*

Proof Let $(X_\bullet, \alpha_\bullet), (Y_\bullet, \beta_\bullet) \in \mathrm{Mor}_{N-2}(\mathcal{A})$.

To prove fullness, take a morphism $h : G(X_\bullet) \to G(Y_\bullet)$ in $D^b_N(\mathcal{A})$. Write h as the span $G(X_\bullet) \xleftarrow{s^\bullet} M^\bullet \xrightarrow{g^\bullet} G(Y_\bullet)$, where $s^\bullet$ is a quasi-isomorphism. Since $G(X_\bullet)$ is concentrated in degrees 1 through $N-1$, the natural map $\iota^\bullet : \sigma_{\leq N-1}M^\bullet \hookrightarrow M^\bullet$ is also a quasi-isomorphism; thus h can be written as $G(X_\bullet) \xleftarrow{s^\bullet\iota^\bullet} \sigma_{\leq N-1}M^\bullet \xrightarrow{g^\bullet\iota^\bullet} G(Y_\bullet)$. Let $f_\bullet : X_\bullet \to Y_\bullet$ be given by $f_i = H^i_{N-i}(g^\bullet) \circ H^i_{N-i}(s^\bullet)^{-1}$.

To see that $f_\bullet$ defines a morphism in $\mathrm{Mor}_{N-2}(\mathcal{A})$, consider for each $1 \leq i \leq N-1$ the commutative diagrams

$$\begin{array}{ccc} Z^i_{N-i}(M^\bullet) & \xrightarrow{\pi^i} & H^i_{N-i}(M^\bullet) \\ \downarrow{\scriptstyle s^i\iota^i} & & {\scriptstyle H^i_{N-i}(s^\bullet)}\downarrow\wr \\ Z^i_{N-i}(G(X_\bullet)) & \twoheadrightarrow & H^i_{N-i}(G(X_\bullet)) \end{array}, \quad \begin{array}{ccc} Z^i_{N-i}(M^\bullet) & \xrightarrow{\pi^i} & H^i_{N-i}(M^\bullet) \\ \downarrow{\scriptstyle g^i\iota^i} & & {\scriptstyle H^i_{N-i}(g^\bullet)}\downarrow \\ Z^i_{N-i}(G(Y_\bullet)) & \twoheadrightarrow & H^i_{N-i}(G(Y_\bullet)) \end{array} \tag{2}$$

Note that $Z^i_{N-i}(G(X_\bullet)) = H^i_{N-i}(G(X_\bullet)) = X_i$, and similarly for Y_i. Thus the lower morphisms in both diagrams are just the identity maps on X_i and Y_i. In particular, $s^i\iota^i$ is an epimorphism. We also have that

$$f_i \circ s^i\iota^i = H^i_{N-i}(g^\bullet)H^i_{N-i}(s^\bullet)^{-1} \circ s^i\iota^i = H^i_{N-i}(g^\bullet)\pi^i = g^i\iota^i \tag{3}$$

It follows that, for $1 \leq i < N-1$,

$$f_{i+1}\alpha_i \circ s^i\iota^i = f_{i+1}s^{i+1}\iota^{i+1}d^i_M = g^{i+1}\iota^{i+1}d^i_M = \beta_i g^i\iota^i = \beta_i f_i \circ s^i\iota^i$$

Since $s^i\iota^i$ is an epimorphism, we conclude that $f_{i+1}\alpha_i = \beta_i f^i$, hence $f_\bullet$ is a morphism. From Equation (3) it follows immediately that $h = G(f_\bullet)$ in $D^b_N(\mathcal{A})$. Thus the functor is full.

To prove faithfulness, let $f_\bullet : X_\bullet \to Y_\bullet$ be such that $G(f_\bullet) = 0$ in $D^b_N(\mathcal{A})$. Then there is a quasi-isomorphism $s^\bullet : M^\bullet \to G(X_\bullet)$ such that $G(f_\bullet)s^\bullet = 0$ in $K^b_N(\mathcal{A})$. Define as above the quasi-isomorphism $\iota^\bullet : \sigma_{\leq N-1}M^\bullet \hookrightarrow M^\bullet$; it follows that $G(f_\bullet)s^\bullet\iota^\bullet = 0$ in $K^b_N(\mathcal{A})$. Since $G(Y_\bullet)$ is concentrated in degrees 1 through $N-1$, it is easily checked that the only null-homotopic morphism of complexes from $\sigma_{\leq N-1}M^\bullet$ to $G(Y_\bullet)$ is the zero map. Thus $G(f_\bullet)s^\bullet\iota^\bullet = 0$ in $C^b_N(\mathcal{A})$; that is, $f_i s^i\iota^i = 0$ for all $1 \leq i \leq N-1$.

Note that the left square in (2) remains valid for all $1 \leq i \leq N-1$. In particular, $s^i\iota^i : Z^i_{N-i}(M^\bullet) \twoheadrightarrow X_i$ is an epimorphism. Thus $f_i = 0$ for all i. Since $f_\bullet = 0$, the functor is faithful. □

We shall prove the following theorem via a sequence of lemmas.

Theorem 5.3 $\overline{G} : \mathrm{stab}_N(\mathcal{E}) \to D^s_N(\mathcal{A})$ *is an equivalence.*

First, it will be helpful to more easily express morphisms in $D_N(\mathcal{A})$. The following proposition is completely analogous to the known result for $N = 2$. It holds for any abelian category and does not require the Gorenstein hypothesis.

Lemma 5.4 *Let* $X^\bullet \in K_N(\mathcal{A})$, $P^\bullet \in K^-_N(\mathrm{Proj}(\mathcal{A}))$, $I^\bullet \in K^+_N(\mathrm{Inj}(\mathcal{A}))$. *Let* $f : P^\bullet \to X^\bullet$ *and* $g : X^\bullet \to I^\bullet$ *be morphisms in* $D_N(\mathcal{A})$. *Then* f *and* g *can be represented by morphisms in* $K_N(\mathcal{A})$.

Proof Express f as the span $P^\bullet \xleftarrow{p^\bullet} Q^\bullet \xrightarrow{h^\bullet} X^\bullet$, where $p^\bullet$ is a quasi-isomorphism. Then $p^\bullet$ fits into a triangle $\Sigma^{-1}C^\bullet \to Q^\bullet \xrightarrow{p^\bullet} P^\bullet \to C^\bullet$ in $K_N(\mathcal{A})$, where $C^\bullet$ is an acyclic

N-complex. By [15, Lemma 3.3], $\mathrm{Hom}_{K_N(\mathcal{A})}(P^\bullet, C^\bullet) = 0$. Since the last map in the above triangle is zero, the map $p^\bullet$ admits a section $s^\bullet : P^\bullet \to Q^\bullet$ in $K_N(\mathcal{A})$. It follows that the span representing f is equivalent to $P^\bullet \xleftarrow{id} P^\bullet \xrightarrow{h^\bullet s^\bullet} X^\bullet$, hence f is equal to the morphism of complexes $h^\bullet s^\bullet$.

Similarly, express g as a cospan $X^\bullet \xrightarrow{e^\bullet} J^\bullet \xleftarrow{i^\bullet} I^\bullet$, where $i^\bullet$ is a quasi-isomorphism. Extend $i^\bullet$ to the triangle $D^\bullet \to I^\bullet \xrightarrow{i^\bullet} J^\bullet \to \Sigma D^\bullet$ in $K_N(\mathcal{A})$, for some acyclic $D^\bullet$. Again by [15, Lemma 3.3], there are no nonzero morphisms from $D^\bullet$ to $I^\bullet$, hence $i^\bullet$ admits a retraction $r^\bullet$ in $K_N(\mathcal{A})$. Thus g is equal to the span $X^\bullet \xrightarrow{r^\bullet e^\bullet} I^\bullet \xleftarrow{id} I^\bullet$, hence $g = r^\bullet e^\bullet$. □

Lemma 5.5 *Let* $X^\bullet \in K_N^b(\mathrm{Gproj}(\mathcal{A}))$, $P^\bullet \in K_N^b(\mathrm{Proj}(\mathcal{A}))$. *Let* $n \in \mathbb{Z}$, *and suppose that* $X^i = 0$ *for all* $i \leq n$ *and* $P^j = 0$ *for all* $j > n$. *(That is,* $P^\bullet$ *is entirely to the left of* $X^\bullet$*.) Then* $\mathrm{Hom}_{D_N(\mathcal{A})}(X^\bullet, P^\bullet) = 0$.

Proof Let us first consider the case where both complexes are concentrated in a single degree: we must show that $\mathrm{Hom}_{D_N^b(\mathcal{A})}(X, P[m]) = 0$ for any $X \in \mathrm{Gproj}(\mathcal{A})$, $P \in \mathrm{Proj}(\mathcal{A})$, $m > 0$. Let $Q^\bullet$ be a projective resolution of X (as a 2-complex). Define an N-complex $(\widetilde{Q}^\bullet, d^\bullet_{\widetilde{Q}})$ by

$$\widetilde{Q}^{kN+j} = \begin{cases} Q^{2k} & j = 0 \\ Q^{2k+1} & 0 < j < N \end{cases}, \quad \text{for any } k \in \mathbb{Z}$$

with differential

$$d_{\widetilde{Q}}^{kN+j} = \begin{cases} d_Q^{2k} & j = 0 \\ id_{Q^{2k+1}} & 1 \leq j < N-1 \\ d_Q^{2k+1} & j = N-1 \end{cases}, \quad \text{for any } k \in \mathbb{Z}$$

It is straightforward to check that $\widetilde{Q}^\bullet$ is quasi-isomorphic to X (viewed as an N-complex concentrated in degree 0), and

$$\mathrm{Hom}_{K_N(\mathcal{A})}(\widetilde{Q}^\bullet, P[m]) = \begin{cases} \mathrm{Ext}^{2k}_{\mathcal{A}}(X, P) & m = Nk \text{ for some } k > 0 \\ \mathrm{Ext}^{2k-1}_{\mathcal{A}}(X, P) & m = Nk-1 \text{ for some } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $X \in \mathrm{Gproj}(\mathcal{A})$, $\mathrm{Ext}^i_{\mathcal{A}}(X, P) = 0$ for all $i > 0$, hence we have that $\mathrm{Hom}_{K_N(\mathcal{A})}(\widetilde{Q}^\bullet, P[m]) = 0$ for all $m > 0$. It follows from Lemma 5.4 that $\mathrm{Hom}_{D_N(\mathcal{A})}(X, P[m]) = 0$ for all $m > 0$.

The full result follows immediately, since every bounded N-complex is a finite iterated extension of single-term complexes. □

Lemma 5.6 $\overline{G}$ *is faithful.*

Proof Let $(X_\bullet, \alpha_\bullet), (Y_\bullet, \beta_\bullet) \in \mathrm{stab}_N(\mathcal{E})$, and let $f_\bullet : X_\bullet \to Y_\bullet$ be a fixed representative of a morphism in $\mathrm{stab}_N(\mathcal{E})$. Suppose $\overline{G}(f_\bullet) = 0$.

We first show that $G(f_\bullet)$ factors in $C_N^b(\mathcal{A})$ as $G(X_\bullet) \xrightarrow{g^\bullet} I^\bullet \xrightarrow{h^\bullet} G(Y_\bullet)$ for some bounded complex of projectives $I^\bullet$. Since $\overline{G}(f_\bullet) = 0$ in $D_N^s(\mathcal{A})$, there exists a morphism with perfect cone $s : \overline{G}(Y_\bullet) \to M^\bullet$ in $D_N^b(\mathcal{A})$ such that $s \circ \overline{G}(f_\bullet) = 0$. Let $P^\bullet$ denote the cocone of $s^\bullet$;

we obtain a morphism of triangles in $D^b_N(\mathcal{A})$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{G}(X_\bullet) & \xrightarrow{id} & \overline{G}(X_\bullet) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \overline{G}(f_\bullet)} & & \downarrow \\
\Sigma^{-1}M^\bullet & \longrightarrow & P^\bullet & \xrightarrow{b} & \overline{G}(Y_\bullet) & \xrightarrow{s} & M^\bullet
\end{array}$$

Changing the bottom row up to isomorphism, we may assume that $P^\bullet$ is a bounded complex of projectives. Note that for each $i \in \mathbb{Z}$, we have a chainwise split exact sequence $\tau_{\geq i}P^\bullet \hookrightarrow P^\bullet \twoheadrightarrow \tau_{\leq i-1}P^\bullet$, where τ denotes the sharp truncation. We obtain the following morphisms of triangles in $D^b_N(\mathcal{A})$:

$$\begin{array}{ccccc}
0 & \longrightarrow & \overline{G}(X_\bullet) & \xrightarrow{id} & \overline{G}(X_\bullet) \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} \\
\tau_{\geq N}P^\bullet & \hookrightarrow & P^\bullet & \twoheadrightarrow & \tau_{\leq N-1}P^\bullet \\
\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\
0 & \longrightarrow & \overline{G}(Y_\bullet) & \xrightarrow{id} & \overline{G}(Y_\bullet)
\end{array}
\qquad
\begin{array}{ccccc}
\overline{G}(X_\bullet) & \xrightarrow{id} & \overline{G}(X_\bullet) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle c} & & \downarrow \\
\tau_{\geq 1}\tau_{\leq N-1}P^\bullet & \hookrightarrow & \tau_{\leq N-1}P^\bullet & \twoheadrightarrow & \tau_{\leq 0}P^\bullet \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle d} & & \downarrow \\
\overline{G}(Y_\bullet) & \xrightarrow{id} & \overline{G}(Y_\bullet) & \longrightarrow & 0
\end{array}$$

The lower left square of the left diagram clearly commutes in $K^b_N(\mathcal{A})$, hence also in $D^b_N(\mathcal{A})$ by Lemma 5.4. This induces the morphism d. The upper right square of the right diagram commutes by Lemma 5.5 and thus induces the map g. The maps c and h are defined in the obvious ways, and the commutativity of the remaining squares in both diagrams is immediate. Consequently, $\overline{G}(f_\bullet) = ba = dc = hg$, so $\overline{G}(f_\bullet)$ factors through the complex $I^\bullet := \tau_{\geq 1}\tau_{\leq N-1}P^\bullet$. $I^\bullet$ has projective terms and is concentrated in degrees 1 through $N-1$, hence $\overline{G}(X_\bullet)$, $\overline{G}(Y_\bullet)$, and $I^\bullet$ all lie in the image of $\mathrm{Mor}_{N-2}(\mathcal{A})$, which by Proposition 5.2 is a full subcategory of $D^b_N(\mathcal{A})$. Thus the morphisms $g = g^\bullet$, $h = h^\bullet$ can be expressed as morphisms of complexes and $G(f_\bullet) = h^\bullet g^\bullet$ in $C^b_N(\mathcal{A})$.

It remains to construct $(I'_\bullet, \iota_\bullet) \in \mathrm{Proj}(\mathrm{MMor}_{N-2}(\mathcal{E}))$ and a factorization $X_\bullet \xrightarrow{\hat{g}_\bullet} I'_\bullet \xrightarrow{\hat{h}_\bullet} Y_\bullet$ of $f_\bullet$. Define $I'_i := \bigoplus_{j=1}^{i} I^j = I'_{i-1} \oplus I^i$, and let $\iota_i : I'_i \hookrightarrow I'_i \oplus I^{i+1}$ be given by $\begin{bmatrix} id \\ d^i_I\pi_i \end{bmatrix}$, where $\pi_i : I'_i \twoheadrightarrow I^i$ is the canonical projection. It is clear that $(I'_\bullet, \iota_\bullet) \in \mathrm{Proj}(\mathrm{MMor}_{N-2}(\mathcal{E}))$, since each I'_i is projective-injective in $\mathcal{E}$ and each ι_i is a (necessarily split) monomorphism. Define $\hat{h}_\bullet : I'_\bullet \to Y_\bullet$ by $\hat{h}_i := h^i\pi_i$; it is straightforward to check that $\hat{h}_\bullet$ is a morphism in $\mathrm{MMor}_{N-2}(\mathcal{E})$.

We shall inductively construct a family $\hat{g}_i : X_i \to I'_i$ such that $\pi_i\hat{g}_i = g^i$ for all $1 \leq i \leq N-1$ and $\iota_{i-1}\hat{g}_{i-1} = \hat{g}_i\alpha_{i-1}$ for all $2 \leq i \leq N-1$. Let $\hat{g}_1 = g^1$; note that $\pi_1 : I'_1 \twoheadrightarrow I^1$ is the identity map, so the desired equation holds. Next, suppose that $\hat{g}_{i-1}$ has been constructed; by injectivity of I'_{i-1} we may lift $\hat{g}_{i-1}$ to $\phi_i : X_i \to I'_{i-1}$ such that $\hat{g}_{i-1} = \phi_i\alpha_{i-1}$. Define $\hat{g}_i : X_i \to I'_{i-1} \oplus I^i$ to be $\begin{bmatrix} \phi_i \\ g^i \end{bmatrix}$; it easy to verify that $\hat{g}_i$ satisfies both of the desired equations. Thus the morphism $\hat{g}_\bullet : X_\bullet \to I'_\bullet$ is defined. Furthermore, we have that $\hat{h}_i\hat{g}_i = h^i\pi_i\hat{g}_i = h^ig^i = f_i$, hence $f_\bullet = \hat{h}_\bullet\hat{g}_\bullet$. Thus $f_\bullet = 0$ in $\mathrm{stab}_N(\mathcal{E})$ and $\overline{G}$ is faithful. □

To prove fullness, we need a better understanding of how to express morphisms in $D^s_N(\mathcal{A})$.

Lemma 5.7 *Let $(X_\bullet, \alpha_\bullet), (Y_\bullet, \beta_\bullet) \in \mathrm{MMor}_{N-2}(\mathcal{E})$. Then the natural map $\mathrm{Hom}_{D_N^b(\mathcal{A})}(\overline{G}(X_\bullet), \overline{G}(Y_\bullet)) \to \mathrm{Hom}_{D_N^s(\mathcal{A})}(\overline{G}(X_\bullet), \overline{G}(Y_\bullet))$ is surjective. That is, any morphism $\overline{G}(X_\bullet) \to \overline{G}(Y_\bullet)$ in $D_N^s(\mathcal{A})$ can be represented by a span of the form*

$$\overline{G}(X_\bullet) \xleftarrow{id} \overline{G}(X_\bullet) \xrightarrow{g} \overline{G}(Y_\bullet)$$

where g is a morphism in $D_N^b(\mathcal{A})$.

Proof Any morphism in $\mathrm{Hom}_{D_N^s(\mathcal{A})}(\overline{G}(X_\bullet), \overline{G}(Y_\bullet))$ can be represented by a span $\overline{G}(X_\bullet) \xleftarrow{s} M^\bullet \xrightarrow{f} \overline{G}(Y_\bullet)$, where s and f are morphisms in $D_N^b(\mathcal{A})$ and s fits into a triangle $M^\bullet \xrightarrow{s} \overline{G}(X_\bullet) \xrightarrow{t} I^\bullet \to \Sigma M^\bullet$ with $I^\bullet \in D_N^{perf}(\mathcal{A})$. Since each projective object in $\mathcal{A}$ has finite injective dimension, by changing $I^\bullet$ up to isomorphism in $D_N^b(\mathcal{A})$, we may assume without loss of generality that it is a bounded N-complex of injectives. By Lemma 5.4 we can represent t by a morphism of complexes $t^\bullet$. Changing $M^\bullet$ up to isomorphism in $D_N^b(\mathcal{A})$, we can also assume that $M^\bullet$ is the cocone of $t^\bullet$ in $K_N^b(\mathcal{A})$, hence $M^\bullet \xrightarrow{s^\bullet} \overline{G}(X_\bullet) \xrightarrow{t^\bullet} I^\bullet \to \Sigma M^\bullet$ is a triangle in $K_N^b(\mathcal{A})$. Note that if $I^\bullet = 0$, then $s^\bullet : M^\bullet \xrightarrow{\sim} \overline{G}(X_\bullet)$ is an isomorphism in $K_N^b(\mathcal{A})$ and we are done; we thus assume that $I^\bullet$ is nonzero.

By Theorem 4.12, there exists an acyclic N-complex $P^\bullet$ of projectives such that $X_\bullet = Z_\bullet^0(P^\bullet)$. Let $\hat{X}^\bullet$ be the N-complex

$$\hat{X}^\bullet = 0 \to X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{N-1} \hookrightarrow P^0 \to P^1 \to \cdots$$

where X_1 is in degree 1. It is straightforward to check that $\hat{X}^\bullet$ is acyclic. For any integer $m \geq N$, there is a natural morphism of N-complexes $p^\bullet : \tau_{\leq m}\hat{X}^\bullet \twoheadrightarrow \overline{G}(X_\bullet)$. We claim that for sufficiently large $m \geq N$, there is a morphism of N-complexes $r^\bullet : \tau_{\leq m}\hat{X}^\bullet \to M^\bullet$ satisfying $p^\bullet = s^\bullet r^\bullet$, and an equivalence of morphisms in $D_N^s(\mathcal{A})$:

$$\overline{G}(X_\bullet) \xleftarrow{s^\bullet} M^\bullet \xrightarrow{f} \overline{G}(Y_\bullet) = \overline{G}(X_\bullet) \xleftarrow{p^\bullet} \tau_{\leq m}\hat{X}^\bullet \xrightarrow{fr^\bullet} \overline{G}(Y_\bullet)$$

Let k be the maximum integer such that I^k is nonzero, and choose $m \geq max(N, k+N)$. We have a triangle in $K_N^+(\mathcal{A})$

$$\tau_{>m}\hat{X}^\bullet \to \hat{X}^\bullet \to \tau_{\leq m}\hat{X}^\bullet \to \Sigma\tau_{>m}\hat{X}^\bullet$$

arising from the chain-wise split exact sequence of complexes. All nonzero terms of $\tau_{>m}\hat{X}^\bullet$ and $\Sigma\tau_{>m}\hat{X}^\bullet$ occur in degrees greater than k, hence $\mathrm{Hom}_{K_N^+(\mathcal{A})}(\tau_{>m}\hat{X}^\bullet, I^\bullet) = 0 = \mathrm{Hom}_{K_N^+(\mathcal{A})}(\Sigma\tau_{>m}\hat{X}^\bullet, I^\bullet)$. Since $\hat{X}^\bullet$ is acyclic, $\mathrm{Hom}_{K_N^+(\mathcal{A})}(\hat{X}^\bullet, I^\bullet) = 0$ by [15, Lemma 3.3]. Applying the functor $\mathrm{Hom}_{K_N^+(\mathcal{A})}(-, I^\bullet)$ to the triangle, we see that $\mathrm{Hom}_{K_N^b(\mathcal{A})}(\tau_{\leq m}\hat{X}^\bullet, I^\bullet) = 0$.

The kernel of $p^\bullet$ is $J^\bullet := \tau_{\leq m}((\tau_{\geq 0}P^\bullet)[-N]) \in K_N^b(\mathrm{Proj}(\mathcal{A}))$; the chainwise split exact sequence $J^\bullet \hookrightarrow \tau_{\leq m}\hat{X}^\bullet \xrightarrow{p^\bullet} \overline{G}(X_\bullet)$ induces a triangle in $K_N^b(\mathcal{A})$. Since $\mathrm{Hom}_{K_N^b(\mathcal{A})}(\tau_{\leq m}\hat{X}^\bullet, I^\bullet) = 0$, we obtain a morphism of triangles in $K_N^b(\mathcal{A})$:

$$\begin{array}{ccccccc} J^\bullet & \longrightarrow & \tau_{\leq m}\hat{X}^\bullet & \xrightarrow{p^\bullet} & \overline{G}(X_\bullet) & \longrightarrow & \Sigma J^\bullet \\ \downarrow{\scriptstyle \Sigma^{-1}q^\bullet} & & \downarrow & & \downarrow{\scriptstyle t^\bullet} & & \downarrow{\scriptstyle q^\bullet} \\ \Sigma^{-1}I^\bullet & \longrightarrow & 0 & \longrightarrow & I^\bullet & \xrightarrow{id} & I^\bullet \end{array}$$

which in turn yields

$$\begin{array}{ccccccc}
J^\bullet & \longrightarrow & \tau_{\leq m}\hat{X}^\bullet & \xrightarrow{p^\bullet} & \overline{G}(X_\bullet) & \longrightarrow & \Sigma J^\bullet \\
\downarrow{\scriptstyle \Sigma^{-1}q^\bullet} & & \downarrow{\scriptstyle r^\bullet} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle q^\bullet} \\
\Sigma^{-1}I^\bullet & \longrightarrow & M^\bullet & \xrightarrow{s^\bullet} & \overline{G}(X_\bullet) & \xrightarrow{t^\bullet} & I^\bullet
\end{array}$$

Since $s^\bullet$ and $p^\bullet = s^\bullet r^\bullet$ both have perfect cones, it follows from the octahedron axiom that $r^\bullet$ does as well. The desired equivalence of roofs $f(s^\bullet)^{-1} = (fr^\bullet)(s^\bullet r^\bullet)^{-1} = (fr^\bullet)(p^\bullet)^{-1}$ follows immediately.

Furthermore, since $J^\bullet \in K_N^b(\mathrm{Proj}(\mathcal{A}))$ is concentrated in degrees N through m and $\overline{G}(Y_\bullet)$ is concentrated in degrees 1 through $N-1$, $\mathrm{Hom}_{K_N^b(\mathcal{A})}(J^\bullet, \overline{G}(Y_\bullet)) = 0 = \mathrm{Hom}_{D_N^b(\mathcal{A})}(J^\bullet, \overline{G}(Y_\bullet))$. We obtain a morphism of triangles in $D_N^b(\mathcal{A})$:

$$\begin{array}{ccccccc}
J^\bullet & \longrightarrow & \tau_{\leq m}\hat{X}^\bullet & \xrightarrow{p^\bullet} & \overline{G}(X_\bullet) & \longrightarrow & \Sigma J^\bullet \\
\downarrow & & \downarrow{\scriptstyle fr^\bullet} & & \downarrow{\scriptstyle g} & & \downarrow \\
0 & \longrightarrow & \overline{G}(Y_\bullet) & \xrightarrow{id} & \overline{G}(Y_\bullet) & \longrightarrow & 0
\end{array}$$

Therefore we have an equivalence of morphisms

$$\overline{G}(X_\bullet) \xleftarrow{p^\bullet} \tau_{\leq m}\hat{X}^\bullet \xrightarrow{fr^\bullet} \overline{G}(Y_\bullet) = \overline{G}(X_\bullet) \xleftarrow{id} \overline{G}(X_\bullet) \xrightarrow{g} \overline{G}(Y_\bullet)$$

□

Corollary 5.8 *$\overline{G}$ is full.*

Proof Let $X_\bullet, Y_\bullet \in \mathrm{stab}_N(\mathcal{E})$, and let $g : \overline{G}(X_\bullet) \to \overline{G}(Y_\bullet)$ be a morphism in $D_N^s(\mathcal{A})$. By Lemma 5.7, g can be taken to be a morphism in $D_N^b(\mathcal{A})$, and by Proposition 5.2, $g = G(f_\bullet)$ for some $f_\bullet : X_\bullet \to Y_\bullet$ in $\mathrm{MMor}_{N-2}(\mathcal{E})$. Let $\overline{f}_\bullet$ denote the image of $f_\bullet$ in $\mathrm{stab}_N(\mathcal{E})$. By the construction of $\overline{G}$, $\overline{G}(\overline{f}_\bullet) = G(f_\bullet) = g$. Thus $\overline{G}$ is full. □

It remains to show that $\overline{G}$ is essentially surjective. Recall the objects $\chi_i(X)_\bullet \in \mathrm{MMor}_{N-2}(\mathcal{E})$ of Definition 3.7. We shall also use the formula in [15, Lemma 2.6] describing the action of Σ on the complexes $\mu_r^s(X)$ in the homotopy category.

Lemma 5.9 *$\overline{G}$ is essentially surjective, hence an equivalence of triangulated categories.*

Proof By Proposition 5.1, Lemma 5.6 and Corollary 5.8, $\overline{G}$ is a fully faithful functor of triangulated categories, hence its essential image $Im(\overline{G})$ is a triangulated subcategory of $D_N^s(\mathcal{A})$.

Let $\mathcal{S} = \{\mu_i^k(X) \mid k \in \mathbb{Z}, 1 \leq i \leq N-1, X \in \mathcal{E}\}$, and let $\mathcal{T}$ denote the smallest isomorphism-closed triangulated subcategory of $D_N^s(\mathcal{A})$ containing $\mathcal{S}$. We claim that $\mathcal{T} = D_N^s(\mathcal{A})$.

By Theorem 2.1, for any $Y \in \mathcal{A}$, there is a short exact sequence $P \hookrightarrow X \twoheadrightarrow Y$ where $P \in \mathcal{A}$ has finite projective dimension and $X \in \mathcal{E}$. Interpreting these objects as N-complexes in degree 0 induces a distinguished triangle in $D_N^b(\mathcal{A})$ and thus in $D_N^s(\mathcal{A})$, where P becomes 0. Therefore in $D_N^s(\mathcal{A})$, $Y \cong X \in \mathcal{S}$. It follows that any N-complex of length 1 lies in $\mathcal{T}$.

Now, suppose for a contradiction that $X^\bullet \in D_N^s(\mathcal{A})$ is a bounded N-complex of minimum possible length such that $X^\bullet \notin \mathcal{T}$. Clearly $X^\bullet \neq 0$; suppose m is the largest integer such that $X^m \neq 0$. Then we have a triangle $\mu_1^m(X^m) \to X^\bullet \to \tau_{<m}X^\bullet \to \Sigma\mu_1^m(X^m)$ in $D_N^s(\mathcal{A})$ arising from the natural short exact sequence of complexes. But $\mu_1^m(X^m) \in \mathcal{T}$ since it has length 1 and $\tau_{<m}X^\bullet \in \mathcal{T}$ since it has length less than $X^\bullet$. It follows that $X^\bullet \in \mathcal{T}$, a contradiction. Thus $\mathcal{T} = D_N^s(\mathcal{A})$.

We now claim $\mathcal{S}$ is contained in $Im(\overline{G})$; once this is proved, it follows immediately that $Im(\overline{G}) = \mathcal{T} = D_N^s(\mathcal{A})$, hence $\overline{G}$ is an equivalence.

We first show that $\mathcal{S}' = \{\mu_i^k(X) \mid 1 \leq i \leq k \leq N-1, X \in \mathcal{E}\}$, consisting of all elements of $\mathcal{S}$ which are concentrated in degrees 1 through $N-1$, is contained in $Im(\overline{G})$. Fix $X \in \mathcal{E}$. It is immediate that $\mu_i^{N-1}(X) = \overline{G}(\chi_i(X)_\bullet)$ for each $1 \leq i \leq N-1$. For $1 \leq i \leq k \leq N-1$, we have a short exact sequence of N-complexes $\mu_{N-1-k}^{N-1}(X) \hookrightarrow \mu_{N-1-k+i}^{N-1}(X) \twoheadrightarrow \mu_i^k(X)$ which induces a triangle in $D_N^s(\mathcal{A})$. Since the first two members of this triangle lie in $Im(\overline{G})$, so does $\mu_i^k(X)$. Thus $\mathcal{S}' \subseteq Im(\overline{G})$.

For any $\mu_i^k(X) \in \mathcal{S}$, there is a unique $x \in \mathbb{Z}$ such $k = xN + r$, where $0 \leq r < N$. Then $\Sigma^{2x}\mu_i^k(X) \cong \mu_i^k(X)[xN] = \mu_i^r(X)$. If $i \leq r$, then $\mu_i^r(X) \in \mathcal{S}'$. Otherwise, $0 \leq r < i$, hence $\Sigma^{-1}(\mu_i^r(X)) = \mu_{N-i}^{N-(i-r)}(X) \in \mathcal{S}'$. In either case, $\Sigma^y\mu_i^k(X) \in Im(\overline{G})$ for some value of y, hence $\mu_i^k(X) \in Im(\overline{G})$. Thus $\mathcal{S} \subseteq Im(\overline{G})$, hence $\overline{G}$ is essentially surjective. □

6 Calabi–Yau properties of stab$_N$(mod-A)

In this section we let A be an associative algebra over a field F. We shall assume that A is finite-dimensional and self-injective. Fix an integer $N \geq 2$. Under these hypotheses, the category mod-A is Frobenius exact, hence stab$_N$(mod-A) (hereafter abbreviated as stab$_N(A)$) is a triangulated category by Theorem 3.12.

It is known that stab$_N(A)$ possesses a Serre functor. (See [26] for case $N = 3$ and [28] for general N.) The goal of this section is to obtain a sufficient condition for stab$_N(A)$ to be fractionally Calabi–Yau. In order to obtain a useful description of the Serre functor on stab$_N(A)$, we must first introduce several other functors.

6.1 The minimal monomorphism functor

The **minimal monomorphism** construction was introduced in [26] for $N = 3$ and [29] for general N. To simplify notation in this section, we shall let $k = N - 2$.

Definition 6.1 Let $(X_\bullet, \alpha_\bullet) \in \mathrm{Mor}_k(A)$. Define $(\mathrm{Mimo}_\bullet(X), m_\bullet(X)) \in \mathrm{MMor}_k(A)$ as follows. For $1 \leq i \leq k$, let $ker(\alpha_i) \hookrightarrow J_{i+1}(X)$ denote the injective hull of $ker(\alpha_i)$, and choose a lift $\omega_i : X_i \to J_{i+1}(X)$ of this map. Let $J_1(X) = 0$. For $1 \leq i \leq k+1$, let $I_i(X) := \bigoplus_{j=1}^{i} J_j(X)$, so that $I_1(X) = 0$ and $I_i(X) = J_i(X) \oplus I_{i-1}(X)$. Define $\mathrm{Mimo}_i(X) := X_i \oplus I_i(X)$ and let $m_i(X) : \mathrm{Mimo}_i(X) \to \mathrm{Mimo}_{i+1}(X)$ be given by

$$m_i(X) := \begin{bmatrix} \alpha_i & 0 \\ \omega_i & 0 \\ 0 & 1 \end{bmatrix} : X_i \oplus I_i(X) \hookrightarrow X_{i+1} \oplus J_{i+1}(X) \oplus I_i(X)$$

Given $f_\bullet : X_\bullet \to Y_\bullet$, define $\mathrm{Mimo}_\bullet(f) : \mathrm{Mimo}_\bullet(X) \to \mathrm{Mimo}_\bullet(Y)$ inductively as follows. Define $\mathrm{Mimo}_1(f) := f_1 : X_1 \to Y_1$. Suppose that we have defined $\mathrm{Mimo}_{i-1}(f) : X_{i-1} \oplus I_{i-1}(X) \to Y_{i-1} \oplus I_{i-1}(Y)$ to be of the form $\begin{bmatrix} f_{i-1} & 0 \\ \phi_{i-1} & \psi_{i-1} \end{bmatrix}$. Define $\begin{bmatrix} \phi_i & \psi_i \end{bmatrix} : X_i \oplus I_i(X) \to I_i(Y)$ to be a lift of the map

$$X_{i-1} \oplus I_{i-1}(X) \xrightarrow{\mathrm{Mimo}_{i-1}(f)} Y_{i-1} \oplus I_{i-1}(Y) \xhookrightarrow{m_{i-1}(Y)} Y_i \oplus I_i(Y) \twoheadrightarrow I_i(Y)$$

along the injection $m_{i-1}(X) : X_{i-1} \oplus I_{i-1}(X) \hookrightarrow X_i \oplus I_i(X)$. Then define $\mathrm{Mimo}_i(f) : X_i \oplus I_i(X) \to Y_i \oplus I_i(Y)$ by the matrix $\begin{bmatrix} f_i & 0 \\ \phi_i & \psi_i \end{bmatrix}$.

In the above definition, it is clear that each $m_i(X)$ is a monomorphism, and that the map $\mathrm{Mimo}_\bullet(f)$ is a morphism in $\mathrm{MMor}_k(A)$. Note also that we have a morphism $\mathrm{Mimo}_\bullet(X) \twoheadrightarrow X_\bullet$ given by component-wise projection onto $X_\bullet$. We now state some basic properties of this construction.

Proposition 6.2 *1) For any object* $X_\bullet \in \mathrm{Mor}_k(A)$, $\mathrm{Mimo}_\bullet(X)$ *is independent, up to isomorphism in* $\mathrm{MMor}_k(A)$, *of the choice of the maps* ω_i.
2) For any morphism $f_\bullet : X_\bullet \to Y_\bullet$ *in* $\mathrm{Mor}_k(A)$, *the image of* $\mathrm{Mimo}_\bullet(f)$ *in* $\mathrm{stab}_N(A)$ *is independent of the choice of maps* ϕ_i *and* ψ_i.
3) Mimo *acts as the identity on both objects and morphisms in* $\mathrm{MMor}_k(A)$.
4) Mimo *defines a functor* $\mathrm{Mor}_k(A) \to \mathrm{stab}_N(A)$ *which descends to functors* $\underline{\mathrm{Mor}_k}(A) \to \mathrm{stab}_N(A)$ *and* $\overline{\mathrm{Mor}_k}(A) \to \mathrm{stab}_N(A)$.
5) $\mathrm{Mimo} : \underline{\mathrm{Mor}_k}(A) \to \mathrm{stab}_N(A)$ *is right adjoint to the inclusion functor.*

Proof 1) It is proved in [29, Lemma 2.3] that the projection $\mathrm{Mimo}_\bullet(X) \twoheadrightarrow X_\bullet$ is a right minimal approximation of $X_\bullet$ in $\mathrm{MMor}_k(A)$, hence is unique up to isomorphism in $\mathrm{MMor}_k(A)$. In particular, any two choices of the maps ω_i in the construction of $\mathrm{Mimo}_\bullet(X_\bullet)$ yield isomorphic objects.
2) Given $f_\bullet : X_\bullet \to Y_\bullet$ and two different choices in the construction of $\mathrm{Mimo}_\bullet(f)$, it is easy to check that their difference factors through the projective-injective object $I_1(Y) \hookrightarrow I_2(Y) \hookrightarrow \cdots \hookrightarrow I_{k+1}(Y)$.
3) If $X_\bullet \in \mathrm{MMor}_k(A)$, then $ker(\alpha_i) = 0$ for all i. Thus $I_i(X) = 0$ and $\mathrm{Mimo}_\bullet(X) = X_\bullet$. The statement about morphisms is immediate.
4) The first statement is easily verified. For the second statement, note that by Propositions 3.9 and 3.10 the projective objects of $\mathrm{Mor}_k(A)$ are precisely the projective-injective objects of $\mathrm{MMor}_k(A)$, hence are preserved by Mimo. Thus the functor $\mathrm{Mimo} : \mathrm{Mor}_k(A) \to \mathrm{stab}_N(A)$ kills projectives and so descends to $\underline{\mathrm{Mor}_k}(A)$. Similarly, the injective objects in $\mathrm{Mor}_k(A)$ are component-wise projective-injective with all maps split epimorphisms; such objects are mapped to projective-injective objects by Mimo, hence Mimo also descends to $\overline{\mathrm{Mor}_k}(A)$.
5) Let $\iota : \mathrm{stab}_N(A) \hookrightarrow \underline{\mathrm{Mor}_k}(A)$ denote the inclusion functor. Let $X_\bullet \in \underline{\mathrm{Mor}_k}(A)$, $Y_\bullet \in \mathrm{stab}_N(A)$. Define natural transformations

$$\epsilon : \iota \circ \mathrm{Mimo} \to 1_{\underline{\mathrm{Mor}_k}(A)} \qquad \eta : 1_{\mathrm{stab}_N(A)} \to \mathrm{Mimo} \circ \iota$$

as follows. Let $\epsilon_{X_\bullet} : \mathrm{Mimo}_\bullet(X) \to X_\bullet$ be the component-wise projection onto $X_\bullet$, and let $\eta_{Y_\bullet} : Y_\bullet \to \mathrm{Mimo}_\bullet(Y) = Y_\bullet$ be the identity map. It follows immediately from definitions that ϵ and η are indeed natural transformations; it remains to verify that they satisfy the triangle identities.

That $(\epsilon\iota) \circ (\iota\eta) = id_\iota$ is immediate. To see that $(\mathrm{Mimo}\,\epsilon) \circ (\eta\,\mathrm{Mimo}) = id_{\mathrm{Mimo}}$, evaluate at $X_\bullet$ and note that the left-hand side simplifies to $\mathrm{Mimo}_\bullet(\epsilon_X) : \mathrm{Mimo}_\bullet(X) \to \mathrm{Mimo}_\bullet(X)$. We can choose this map to be the identity map. Thus the pair (ι, Mimo) is an adjunction. □

6.2 Cokernel and rotation functors

Throughout this section, we shall let $k = N - 2$ to simplify notation.

Definition 6.3 For $(X_\bullet, \alpha_\bullet) \in \mathrm{MMor}_k(A)$, define

$$\mathrm{Cok}_\bullet(X) := X_{k+1} \twoheadrightarrow coker(\alpha_1^k) \twoheadrightarrow coker(\alpha_2^{k-1}) \twoheadrightarrow \cdots \twoheadrightarrow coker(\alpha_k)$$

For $f_\bullet : X_\bullet \to Y_\bullet$, let $\mathrm{Cok}_\bullet(f) : \mathrm{Cok}_\bullet(X) \to \mathrm{Cok}_\bullet(Y)$ be given by the component-wise induced maps on the cokernels.

It is clear that Cok defines a functor $\mathrm{MMor}_k(A) \to \mathrm{Mor}_k(A)$ which sends projective-injective objects to injective objects. Thus Cok descends to a functor $\mathrm{stab}_N(A) \to \overline{\mathrm{Mor}_k}(A)$. Though we shall not need this fact, we note that Cok also defines an exact equivalence between $\mathrm{MMor}_k(A)$ and $\mathrm{EMor}_k(A)$ which descends to a triangulated equivalence between the respective stable categories.

Definition 6.4 Define the **rotation functor** to be the composition

$$R = \mathrm{Mimo} \circ \mathrm{Cok} : \mathrm{stab}_N(A) \to \overline{\mathrm{Mor}_k}(A) \to \mathrm{stab}_N(A)$$

The rotation construction was first defined in [26] for $N = 3$ and later generalized to arbitrary N in [28]. Our formulation differs slightly in that it is defined on $\mathrm{stab}_N(A)$ rather than $\mathrm{Mor}_{N-2}(\mathrm{stab}(A))$. On $\mathrm{stab}_N(A)$, the rotation functor can be somewhat difficult to work with, but it simplifies considerably when expressed in terms of complexes.

Recall the triangulated equivalence $\overline{G} : \mathrm{stab}_N(A) \to D_N^s(A)$ defined in Proposition 5.1. Note that $\overline{G}$ extends to a functor $\overline{\mathrm{Mor}_k}(A) \to D_N^s(A)$.

Proposition 6.5 *There is an isomorphism* $\Sigma[-1] \circ \overline{G} \cong \overline{G} \circ R$ *of functors* $\mathrm{stab}_N(A) \to D_N^s(A)$.

Proof Let $(X_\bullet, \alpha_\bullet) \in \mathrm{stab}_N(A)$. The short exact sequence in $C_N^b(A)$

$$\overline{G}(X_\bullet) \hookrightarrow \mu_N^{N-1}(X_{N-1}) \twoheadrightarrow \overline{G}(\mathrm{Cok}_\bullet(X))[1]$$

induces a triangle in $D_N^s(A)$. The middle term is null-homotopic, so we have an isomorphism $\overline{G}(\mathrm{Cok}_\bullet(X))[1] \xrightarrow{\sim} \Sigma(\overline{G}(X_\bullet))$ in $D_N^s(A)$; since the above exact sequence is natural in $X_\bullet$, so is this isomorphism. Applying $[-1]$ yields a natural isomorphism $\overline{G} \circ \mathrm{Cok} \cong \Sigma[-1] \circ \overline{G}$.

Applying $\overline{G}$ to the short exact sequence in $\mathrm{MMor}_k(A)$

$$I_\bullet(\mathrm{Cok}(X)) \rightarrowtail \mathrm{Mimo}_\bullet(\mathrm{Cok}(X)) \twoheadrightarrow \mathrm{Cok}_\bullet(X)$$

we obtain a triangle in $D_N^s(A)$. The left term is mapped to $D_N^{perf}(A)$, hence vanishes; we obtain an isomorphism $\overline{G}R(X_\bullet) \cong \overline{G}(\mathrm{Cok}_\bullet(X))$ which is clearly natural in $X_\bullet$. Thus $\overline{G} \circ R \cong \overline{G} \circ \mathrm{Cok} \cong \Sigma[-1] \circ \overline{G}$. □

6.3 Upper triangular matrices

Throughout this section, we shall let $n = N - 1$ to simplify notation.

Let $B = T_n(A)$ denote the F-algebra of $n \times n$ upper-triangular matrices with entries in A. We write $E_{i,j}$ for the matrix with 1_A in the (i, j)-th position (that is, row i and column j) and 0's everywhere else.

Given $X \in \text{mod-}B$, we can create the following object in $\text{Mor}_{n-1}(A)$:

$$XE_{1,1} \xrightarrow{r_{E_{1,2}}} XE_{2,2} \xrightarrow{r_{E_{2,3}}} \cdots \xrightarrow{r_{E_{n-1,n}}} XE_{n,n}$$

More explicitly, there is an equivalence $M_r : \text{mod-}B \xrightarrow{\sim} \text{Mor}_{n-1}(A)$ given by $M_r(X) = (XE_{\bullet,\bullet}, r_{E_{\bullet,\bullet+1}})$ [29, Lemma 1.3]. The inverse of M_r is given by $M_r^{-1}(X_\bullet, f_\bullet) = \bigoplus_{i=1}^n X_i$, where $E_{i,i}$ acts as projection onto the i-th coordinate and $E_{i,i+j}$ acts as f_i^j.

Similarly, there is an equivalence $M_l : B\text{-mod} \xrightarrow{\sim} \text{Mor}_{n-1}(A^{op})$ which is given by $M_l(X) = (E_{n+1-\bullet,n+1-\bullet}X, l_{E_{n-\bullet,n+1-\bullet}})$. Its inverse is given by $M_l^{-1}(X_\bullet, f_\bullet) = \bigoplus_{i=1}^n X_i$, where $E_{i,i}$ acts as projection onto X_{n+1-i} and $E_{i-j,i}$ acts as f_{n+1-i}^j.

It is easy to check that $M_r(B) \cong \bigoplus_{i=1}^n \chi_i(A)_\bullet \cong M_l(B)$ has injective dimension 1 in $\text{Mor}_{n-1}(A)$, hence B is Gorenstein. (Recall the definition of $\chi_i(A)_\bullet$ from Sect. 3.2.) The following proposition allows us to identify the monomorphism and epimorphism categories of A with the Gorenstein projective and Gorenstein injective B-modules, respectively. (See Sect. 2.6 for the definition of a Gorenstein injective module.)

Proposition 6.6 ([29, Corollary 4.1, 4.2]) *The functors M_r and M_l restrict to the following exact equivalences:*

1) $M_r : \text{Gproj}(B) \xrightarrow{\sim} \text{MMor}_{n-1}(A)$

2) $M_l : \text{Gproj}(B^{op}) \xrightarrow{\sim} \text{MMor}_{n-1}(A^{op})$

3) $M_r : \text{Ginj}(B) \xrightarrow{\sim} \text{EMor}_{n-1}(A)$

4) $M_l : \text{Ginj}(B^{op}) \xrightarrow{\sim} \text{EMor}_{n-1}(A^{op})$

Each of the above equivalences descends to a triangulated equivalence between the respective stable categories.

Proof It is clear that M_r and M_l are exact equivalences. Once 1)-4) have been established, it is also clear that the functors descend to triangulated equivalences between the stable categories. All that is needed is to show that each functor has the appropriate image.

1) Let $(X_\bullet, \alpha_\bullet) \in \text{Mor}_{n-1}(A)$. Since $M_r(B) \cong \bigoplus_{i=1}^n \chi_i(A)_\bullet$, it suffices to prove that $X_\bullet \in \text{MMor}_{n-1}(A)$ if and only if $\text{Ext}^1(X_\bullet, \chi_i(A)_\bullet) = 0$ for all $1 < i \le n$. (Since $\chi_1(A)_\bullet$ is injective, $\text{Ext}^1(X_\bullet, \chi_1(A)_\bullet) = 0$ for any $X_\bullet$.) Let $\overline{\chi}_i(A)_\bullet$ denote the cokernel of the natural inclusion $\chi_i(A)_\bullet \hookrightarrow \chi_1(A)_\bullet$. Define a complex in $C^b(\text{Mor}_{n-1}(A))$

$$I^\bullet(i) = \; \cdots \to 0 \to \chi_1(X)_\bullet \twoheadrightarrow \overline{\chi}_i(X)_\bullet \to 0 \to \cdots$$

with $\chi_1(X)_\bullet$ in degree 0. $I^\bullet(i)$ is an injective resolution of $\chi_i(A)_\bullet$, hence $\text{Ext}^1(X_\bullet, \chi_i(A)_\bullet) = \text{Hom}_{K^b(\text{MMor}_{n-1}(A))}(X_\bullet, I^\bullet(i)[1])$. Note that a morphism of complexes $X_\bullet \to I^\bullet(i)[1]$ is the same data as a morphism $f_{i-1} : X_{i-1} \to A$; such a morphism is null-homotopic if and only if f_{i-1} factors through α_{i-1}^j for all $1 \le j \le n-i+1$.

Suppose $X_\bullet \in \text{MMor}_{n-1}(A)$. Since α_{i-1}^j is a monomorphism and A is injective, any morphism $f_{i-1} : X_{i-1} \to A$ admits a factorization $f_{i-1} = g_{i-1+j}\alpha_{i-1}^j$, hence $\text{Ext}^1(X_\bullet, \chi_i(A)_\bullet) = 0$. Conversely, if α_{i-1} is not injective for some $1 < i \le n$, then there is a nonzero morphism $ker(\alpha_{i-1}) \to A$ which can be lifted to a morphism $f_{i-1} : X_{i-1} \to A$. Since f_{i-1} is nonzero on $ker(\alpha_{i-1})$, it cannot factor through α_{i-1},

hence f_{i-1} defines a nonzero element of $\mathrm{Ext}^1(X_\bullet, \chi_i(A)_\bullet)$. Thus M_r identifies $\mathrm{Gproj}(B)$ with $\mathrm{MMor}_{n-1}(A)$.
2) Since $M_l(B) \cong \bigoplus_{i=1}^n \chi_i(A)_\bullet$, the proof is identical to 1).
3) By Proposition 6.7 below, $M_r \cong D_* M_l D$. The result then follows from 2).
4) The result follows from Proposition 6.7 and 1). □

6.4 Duality and the Nakayama functor

In this section, we continue to write $n = N - 1$.

It will be convenient to introduce some notation. If $F : \text{mod-}A \to \mathcal{C}$ is a covariant functor (into any category $\mathcal{C}$), there is an induced functor $F_* : \mathrm{Mor}_{n-1}(A) \to \mathrm{Mor}_{n-1}(\mathcal{C})$ given by $F(X_\bullet, \alpha_\bullet) = (F(X_\bullet), F(\alpha_\bullet))$. Given a contravariant functor $G : (\text{mod-}A)^{op} \to \mathcal{C}$, we likewise obtain a functor $G_* : \mathrm{Mor}_{n-1}(A)^{op} \to \mathrm{Mor}_{n-1}(\mathcal{C})$, this time given by $G_*(X_\bullet, \alpha_\bullet) = (G(X_{n+1-\bullet}), G(\alpha_{n-\bullet}))$.

Recall the Nakayama functor ν_A, defined in Section 2.6 to be the composition of the dualities $D = \mathrm{Hom}_F(-, F)$ and $\mathrm{Hom}_A(-, A)$. Note that both of the induced functors D_* and $\mathrm{Hom}_A(-, A)_*$ define dualities $\mathrm{Mor}_{n-1}(A)^{op} \xrightarrow{\sim} \mathrm{Mor}_{n-1}(A^{op})$ which identify the monomorphism subcategory with the epimorphism subcategory, and vice versa. It follows that the equivalence $\nu_{A*} = D_* \mathrm{Hom}_A(-, A)_* : \mathrm{Mor}_{n-1}(A) \xrightarrow{\sim} \mathrm{Mor}_{n-1}(A)$, preserves both $\mathrm{MMor}_{n-1}(A)$ and $\mathrm{EMor}_{n-1}(A)$ and descends to the corresponding stable categories.

In contrast with the behavior of ν_{A*}, recall that ν_B restricts to an equivalence $\mathrm{Gproj}(B) \xrightarrow{\sim} \mathrm{Ginj}(B)$; it is therefore worth investigating the relationship between these two functors. Before we express ν_B in the language of the monomorphism category, it will be helpful to first translate the F-linear duality on B.

Proposition 6.7 *There is an isomorphism $D_* \circ M_l \cong M_r \circ D$ of functors $(B\text{-mod})^{op} \to \mathrm{Mor}_{n-1}(A)$. Similarly, $M_l \circ D \cong D_* \circ M_r$.*

Proof Let $X \in B$-mod. The left A-module map $l_{E_{i,i}} : X \twoheadrightarrow E_{i,i}X$ yields a monomorphism $l^*_{E_{i,i}} : D(E_{i,i}X) \hookrightarrow DX$ whose image is $(DX)E_{i,i}$. We have a commutative diagram in mod-A.

$$\begin{array}{ccc} D(E_{i-1,i-1}X) & \xrightarrow{l^*_{E_{i-1,i}}} & D(E_{i,i}X) \\ \sim\downarrow l^*_{E_{i-1,i-1}} & & \sim\downarrow l^*_{E_{i,i}} \\ (DX)E_{i-1,i-1} & \xrightarrow{r_{E_{i-1,i}}} & (DX)E_{i,i} \end{array}$$

hence $l^*_{E_{\bullet,\bullet}} : D_* M_l(X) \xrightarrow{\sim} M_r D(X)$ is an isomorphism which is easily verified to be natural in X.

The second isomorphism follows immediately by precomposing with D and postcomposing with D_*. □

Proposition 6.8 *There is an isomorphism $M_r \circ \nu_B \cong \mathrm{Cok}\, \nu_{A*} \circ M_r$ of functors $\mathrm{Gproj}(B) \to \mathrm{EMor}_{n-1}(A)$.*

Proof It is enough to show that $D_* M_r \nu_B \cong D_* \mathrm{Cok}\, \nu_{A*} M_r$. By Proposition 6.7, we have that

$$D_* M_r \nu_B \cong M_l D \nu_B \cong M_l \mathrm{Hom}_B(-, B)$$

Since ν_A is exact, it is easily verified that $\mathrm{Cok}\, \nu_{A*} \cong \nu_{A*}\, \mathrm{Cok}$, hence

$$D_* \,\mathrm{Cok}\, \nu_{A*} M_r \cong D_* \nu_{A*}\, \mathrm{Cok}\, M_r \cong \mathrm{Hom}_A(-, A)_*\, \mathrm{Cok}\, M_r$$

It thus suffices to construct $\zeta : M_l \,\mathrm{Hom}_B(-, B) \xrightarrow{\sim} \mathrm{Hom}_A(-, A)_*\, \mathrm{Cok}\, M_r$, an isomorphism of functors $\mathrm{Gproj}(B)^{op} \to \mathrm{MMor}_{n-1}(A^{op})$.

Let $X \in \mathrm{Gproj}(B)$. Note that $E_{i,i} \mathrm{Hom}_B(X, B)$ consists of precisely those homomorphisms with image in $E_{i,i} B$. Thus

$$M_l \,\mathrm{Hom}_B(X, B) = (\mathrm{Hom}_B(X, E_{n+1-\bullet,n+1-\bullet} B), l_{E_{n-\bullet,n+1-\bullet}})$$

A direct computation shows that

$$\mathrm{Hom}_A(-, A)_*\, \mathrm{Cok}\, M_r(X) = (\mathrm{Hom}_A(XE_{n,n}/XE_{n-\bullet,n}, A), \pi^*_{n-\bullet})$$

where $\pi_i : XE_{n,n}/XE_{i-1,n} \twoheadrightarrow XE_{n,n}/XE_{i,n}$ is the canonical projection. (Here we define $XE_{0,n}$ to be 0.)

Given $f \in \mathrm{Hom}_B(X, E_{i,i} B)$, note that the restriction of f to $XE_{n,n}$ has image in $E_{i,i} BE_{n,n} = E_{i,n} B = E_{i,n} A$, which is canonically isomorphic to A as an (A, A)-bimodule. Furthermore, $f(XE_{i-1,n}) \subseteq E_{i,i} BE_{i-1,n} = 0$, hence the restriction descends to a map

$$\overline{f\,|_{XE_{n,n}}} : XE_{n,n}/XE_{i-1,n} \to E_{i,n} A \cong A$$

Let $\zeta_{X,i} : \mathrm{Hom}_B(X, E_{i,i} B) \to \mathrm{Hom}_A(XE_{n,n}/XE_{i-1,n}, A)$ be the map sending f to $\overline{f\,|_{XE_{n,n}}}$.

To show that $\zeta_{X,i}$ is injective, let $f \in ker(\zeta_{X,i})$ and let $x \in X$. Since $\zeta_{X,i}(f) = 0$, then $f(XE_{n,n}) = 0$ and so $f(x)E_{j,n} = f(xE_{j,n}E_{n,n}) = 0$ for all $j \leq n$. The map $r_{E_{j,n}} : BE_{j,j} \hookrightarrow BE_{n,n}$ is injective for all $j \leq n$; it follows from the above equation that $f(x)E_{j,j} = 0$ for all $j \leq n$, hence $f(x) = 0$. Thus $f = 0$ and $\zeta_{X,i}$ is injective.

To see that $\zeta_{X,i}$ is surjective, take any $g \in \mathrm{Hom}_A(XE_{n,n}/XE_{i-1,n}, A)$. Define $f : X \to E_{i,i} B$ by $f(x) = \sum_{j=i}^{n} g(xE_{j,n})E_{i,j}$. A direct computation shows that for any $1 \leq r \leq s \leq n$,

$$f(xE_{r,s}) = g(xE_{r,n})E_{i,s} = f(x)E_{r,s}$$

It follows that f is a right B-module morphism and $\zeta_{X,i}(f) = g$. Thus $\zeta_{X,i}$ is an isomorphism for each i.

It is easily checked that $\zeta_{X,n+1-\bullet}$ is a morphism in $\mathrm{MMor}_{n-1}(A^{op})$ and is natural in X, hence the two functors are isomorphic. □

6.5 Serre duality

The inclusion functor $\underline{\mathrm{Gproj}(B)} \hookrightarrow \underline{\text{mod-}B}$ possesses a right adjoint $P : \underline{\text{mod-}B} \to \underline{\mathrm{Gproj}(B)}$ [21, Lemma 6.3.6]. We have already seen that Mimo plays an analogous role in the monomorphism category, so it is no surprise that the two functors are related.

Proposition 6.9 *There is an isomorphism* $M_r \circ P \cong \mathrm{Mimo} \circ M_r$ *of functors* $\underline{\text{mod-}B} \to \mathrm{stab}_N(A)$.

Proof Let $\iota_1 : \mathrm{stab}_N(A) \hookrightarrow \underline{\mathrm{Mor}_{N-2}(A)}$ and $\iota_2 : \underline{\mathrm{Gproj}(B)} \hookrightarrow \underline{\text{mod-}B}$ be the inclusion functors. It is clear that $\iota_1 M_r = M_r \iota_2$. By Proposition 6.2, Mimo is right adjoint to ι_1; it follows that both P and $M_r^{-1}\, \mathrm{Mimo}\, M_r$ are right adjoint to ι_2, hence $P \cong M_r^{-1}\, \mathrm{Mimo}\, M_r$. The result follows. □

We are ready to describe the Serre functors on $\mathrm{stab}_N(A)$ and $D^s_N(A)$. We shall write Ω_A, Ω_B, and Ω_N to denote the syzygy functors on $\mathrm{stab}(A)$, $\mathrm{stab}(B)$ and $\mathrm{stab}_N(A)$, respectively. Recall that since A is self-injective, ν_A is exact and so lifts to $D^s_N(A)$.

Theorem 6.10 *$\Omega_N R\nu_{A_*}$ is a Serre functor on $\mathrm{stab}_N(A)$. $[-1]\nu_A$ is a Serre functor on $D^s_N(A)$.*

Proof By [21, Corollary 6.4.10], $\underline{\mathrm{Gproj}(B)}$ has Serre functor $S := \Omega_B P\nu_B$. Thus $M_r S M_r^{-1}$ is a Serre functor for $\mathrm{stab}_N(A)$ and $\overline{G} M_r S M_r^{-1} \overline{G}^{-1}$ is a Serre functor for $D^s_N(A)$. Then

$$\begin{aligned} M_r S M_r^{-1} &= M_r \Omega_B P \nu_B M_r^{-1} \\ &\cong \Omega_N M_r P \nu_B M_r^{-1} \\ &\cong \Omega_N \operatorname{Mimo} M_r \nu_B M_r^{-1} && \text{Proposition 6.9} \\ &\cong \Omega_N \operatorname{Mimo} \operatorname{Cok} \nu_{A*} && \text{Proposition 6.8} \\ &= \Omega_N R \nu_{A*} \end{aligned}$$

and

$$\begin{aligned} \overline{G} M_r S M_r^{-1} \overline{G}^{-1} &\cong \overline{G} \Omega_N R \nu_{A*} \overline{G}^{-1} \\ &\cong \Sigma^{-1} \overline{G} R \nu_{A*} \overline{G}^{-1} \\ &\cong \Sigma^{-1} \Sigma[-1] \overline{G} \nu_{A*} \overline{G}^{-1} && \text{Proposition 6.5} \\ &\cong [-1]\nu_A \end{aligned}$$

where the isomorphism $\overline{G}\nu_{A*} \cong \nu_A \overline{G}$ follows immediately from exactness of ν_A. □

When the order of the Nakayama automorphism is known, one obtains a description of the fractional Calabi–Yau dimension of the N-stable category. (See Sect. 2.2 for definitions.)

Corollary 6.11 *Suppose the Nakayama automorphism of A has order r. Let $s = lcm(N, r)$ and $t = \frac{s}{N}$. If $N > 2$, then $\mathrm{stab}_N(A)$ is $(-2t, s)$-Calabi–Yau. $\mathrm{stab}(A)$ is $(-r, r)$-Calabi–Yau.*

Proof It suffices to check that $D^s_N(A)$ has the appropriate Calabi–Yau property. We have that $\nu_A^r \cong id$, hence $\nu_A^s \cong id$. Then

$$([-1]\nu_A)^s \cong [-s] = [-tN] \cong \Sigma^{-2t}$$

For $N = 2$, we have $\Sigma = [1]$, hence $([-1]\nu_A)^r \cong \Sigma^{-r}$. □

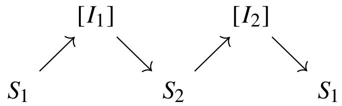
Corollary 6.12 *Suppose A is symmetric. Then $\mathrm{stab}(A)$ is (-1)-Calabi–Yau and $\mathrm{stab}_N(A)$ is $(-2, N)$-Calabi–Yau for all $N > 2$.*

Proof Since A is symmetric, $\nu_A = id$ hence $r = 1$. The statement follows. □

The above integer pairs need not be minimal. The presence of additional relations between the functors Ω, ν_{A*} and R may allow $\mathrm{stab}_N(A)$ to be (x, y)-Calabi–Yau for smaller values of x and y; see below for a concrete example.

6.6 An example

Let F be any field, let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$, and let $A = FQ/rad^2(FQ)$. Then A is self-injective with four indecomposable modules: the simple modules S_1 and S_2 and their two-dimensional injective hulls I_1 and I_2.



The Auslander-Reiten quiver of A.
Vertices in brackets are projective-injective.

Fix some $N \geq 2$. For any integers $i, j \geq 0$ satisfying $1 \leq i + j \leq N - 1$, define objects $X(i, j)$ and $Y(i, j)$ in $\mathrm{MMor}_{N-2}(A)$ by

$$X(i, j) := 0 \to \cdots \to 0 \to S_1 \to \cdots \to S_1 \to I_1 \to \cdots \to I_1$$
$$Y(i, j) := 0 \to \cdots \to 0 \to S_2 \to \cdots \to S_2 \to I_2 \to \cdots \to I_2$$

Here each sequence has exactly i simples and j projective-injectives, and each morphism is the canonical inclusion.

In mod-A, every monomorphism from an indecomposable module M into a direct sum $Y \oplus Z$ factors through either Y or Z, so $(M_\bullet, \alpha_\bullet) \in \mathrm{MMor}_{N-2}(A)$ is indecomposable if and only if each M_i is indecomposable. Thus the indecomposable objects of $\mathrm{MMor}_{N-2}(A)$ are precisely the $X(i, j)$ and $Y(i, j)$. The indecomposable projective-injectives are precisely the objects $X(0, j)$ and $Y(0, j)$.

The Nakayama automorphism of A has order 2, so by Corollary 6.11, $\mathrm{stab}_N(A)$ is $(-4, 2N)$-Calabi–Yau if N is odd and $(-2, N)$ if $N > 2$ is even. However, it is easy to check that $\nu_{A*} \cong \Omega \cong \Omega^{-1}$ on $\mathrm{stab}_N(A)$ for any N. It follows from Proposition 6.5 that R and Ω^{-1} commute, since the corresponding functors Σ and $\Sigma[-1]$ commute in $D^s_N(A)$. Thus $\mathrm{stab}_N(A)$ has Serre functor $S = \Omega R \nu_{A*} \cong R$, and $D^s_N(A)$ has Serre functor $\Sigma[-1]$. In particular,

$$S^N \cong \Omega^{-N+2} \cong \begin{cases} \Omega^{-1} & N \text{ odd} \\ id & N \text{ even} \end{cases}$$

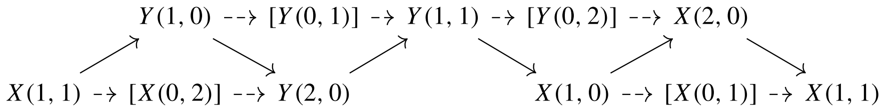
Thus for $N > 2$, $\mathrm{stab}_N(A)$ is $(1, N)$-Calabi–Yau for odd N and $(0, N)$-Calabi–Yau for even N.

A straightforward computation shows that for any $i > 0$,

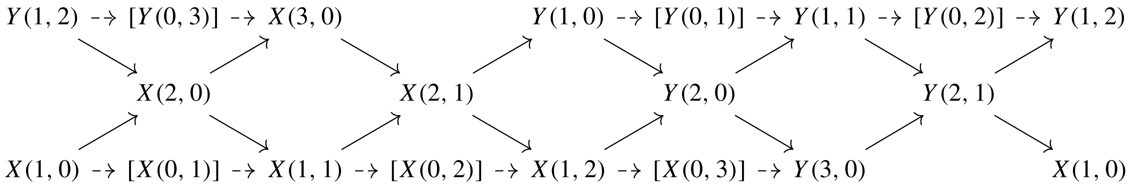
$$S(X(i, j)) = \begin{cases} Y(i, j-1) & j > 0 \\ X(N-i, i-1) & j = 0 \end{cases}$$
$$S(Y(i, j)) = \begin{cases} X(i, j-1) & j > 0 \\ Y(N-i, i-1) & j = 0 \end{cases}$$
$$\Omega(X(i, j)) = Y(i, j)$$
$$\Omega(Y(i, j)) = X(i, j)$$

It follows immediately that S^n is not isomorphic to any power of Ω for any $0 < n < N$.

We conclude by providing the Auslander-Reiten quiver of $\mathrm{MMor}_{N-2}(A)$ for representative values of N.



The Auslander-Reiten quiver of $\mathrm{MMor}_3(A)$.
Vertices in brackets are projective-injective.



The Auslander-Reiten quiver of $\mathrm{MMor}_4(A)$.
Vertices in brackets are projective-injective.

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