Mathematische Zeitschrift (2024) 306:51
https://doi.org/10.1007/s00209-024-03449-0

Mathematische Zeitschrift

Isometries in the symmetrized bidisc

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Received: 3 August 2023 / Accepted: 6 January 2024 / Published online: 16 February 2024

Abstract
We provide a characterization of isometries in the sense of the Carathéodory–Reiffen metric in the symmetrized bidisc.

1 Introduction

Let $D \subset \mathbb{C}^n$ be a domain. We say that a function $F_D : D \times \mathbb{C}^n \to [0, +\infty)$ is a *Finsler metric* if it is an upper semi-continuous function and

$$F_D(\lambda; tv) = |t| F_D(\lambda; v) \quad \text{for any } \lambda \in D, v \in \mathbb{C}^n, t \in \mathbb{C}.$$

Assume that $D_1 \subset \mathbb{C}^{n_1}$ and $D_2 \subset \mathbb{C}^{n_2}$ are domains with Finsler metrics F_{D_1} and F_{D_2}, respectively. We say that a C^1 mapping $\phi : D_1 \to D_2$ is an *isometry* if

$$F_{D_1}(\lambda; X) = F_{D_2}(\phi(\lambda); d_\lambda \phi(\lambda)(X)) \quad \text{for any } \lambda \in D_1, X \in \mathbb{C}^{n_1}, \tag{1}$$

where $d_\lambda \phi$ denotes the $\mathbb{R}$-differential of ϕ at λ. In case $n_1 = 1$ the equation (1) is equivalent with

$$F_{D_1}(\lambda; 1) = F_{D_2}\big(\phi(\lambda); \frac{\partial \phi}{\partial z}(\lambda) + \xi \frac{\partial \phi}{\partial \bar{z}}(\lambda)\big) \quad \text{for any } \lambda \in D_1, \xi \in \mathbb{T}, \tag{2}$$

where $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ denotes the unit circle.

The paper is motivated by the question whether an isometry is holomorphic or anti-holomorphic. This problem was considered in many papers and, it seems, that in case of invariant metrics (Kobayashi metric, Carathéodory metric, etc) is very difficult (see [2, 8, 10]).

We denote by $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$ the unit disc. For a domain $D \subset \mathbb{C}^n$ we define a biholomorphically invariant Finsler metric as follows

$$\gamma_D(\lambda, X) = \sup \left\{ \frac{|F'(\lambda) X|}{1 - |F(\lambda)|^2} : F : D \to \mathbb{D} \text{ holomorphic} \right\}$$

for any $\lambda \in D$ and any $X \in \mathbb{C}^n$. We call γ_D the Carathéodory-Reiffen (pseudo)metric for D (see e.g. [6], Chapter 2). According to the above definition a C^1-mapping $\phi : D_1 \to D_2$ is a

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γ*-isometry* if

$$\gamma_{D_2}(\phi(\lambda); d_\lambda\phi(X)) = \gamma_{D_1}(\lambda; X)$$

for any $\lambda \in D_1$ and any $X \in \mathbb{C}^{n_1}$.

Put

$$\pi : \mathbb{C}^2 \ni (z, w) \to (z + w, zw) \in \mathbb{C}^2$$

and $\mathbb{G}_2 = \pi(\mathbb{D}^2)$. We call $\mathbb{G}_2$ the symmetrized bidisc (see e.g. [1]). One of the main results of our paper is the following.

Theorem 1 *Let $\phi : \mathbb{D} \to \mathbb{G}_2$ be a mapping of class C^1. Assume that ϕ is a γ-isometry. Then ϕ is holomorphic or anti-holomorphic.*

As a corollary, we obtain the following result.

Corollary 2 *Let $F : \mathbb{G}_2 \to \mathbb{G}_2$ be a mapping of class C^1 and a γ-isometry. Then F is holomorphic or anti-holomorphic.*

2 Proof of Theorem 1

For any $\lambda \in \mathbb{G}_2$, there exists an automorphism Φ of $\mathbb{G}_2$ such that $\Phi(\lambda) = (0, p)$, where $p \in [0, 1)$. Define a set $\Sigma = \{(2t, t^2) : t \in \mathbb{D}\}$ called the *royal set*. We have

$$\Sigma = \{\Phi(0) : \Phi \in \mathrm{Aut}(\mathbb{G}_2)\}.$$

Recall the following result (see [6], Chapter 7)

$$\gamma_{\mathbb{G}_2}((s, p); (X, Y)) = \max\left\{\frac{|(F_\eta)'_s(\lambda)X + (F_\eta)'_p(\lambda)Y|}{1 - |F(\lambda)|^2} : \eta \in \mathbb{T}\right\},$$

where $\lambda = (s, p) \in \mathbb{G}_2$ and $F_\eta(s, p) = \frac{2\eta p - s}{2 - \eta s}$, $\eta \in \mathbb{T}$. Note that

$$(F_\eta)'_s(s, p) = \frac{2\eta^2 p - 2}{(2 - \eta s)^2} \quad \text{and} \quad (F_\eta)'_p(s, p) = \frac{2\eta}{2 - \eta s}.$$

Hence,

$$\gamma_{\mathbb{G}_2}((s, p); (X, Y)) = \max\left\{\frac{|(\eta^2 p - 1)X + \eta(2 - \eta s)Y|}{2\big(1 - |p|^2 - \Re(\eta(s - \bar{s}p))\big)} : \eta \in \mathbb{T}\right\}.$$

In particular, we have

$$\gamma_{\mathbb{G}_2}((0, p); (X, Y)) = \frac{\max_{\omega\in\mathbb{T}} |X(1 - \omega^2 p) - 2Y\omega|}{2(1 - p^2)},$$

where $p \in [0, 1)$ and $(X, Y) \in \mathbb{C}^2$.

The main result of this section is the following.

Theorem 3 *Let $(X_1, Y_1), (X_2, Y_2) \in \mathbb{C}^2$ be fixed vectors and let $p \in (0, 1)$ be a fixed number. Assume that there exists a constant $C > 0$ such that*

$$\gamma_{\mathbb{G}_2}\big((0, p); (X_1, Y_1) + \xi(X_2, Y_2)\big) = C \quad \textit{for any } \xi \in \mathbb{T}. \tag{3}$$

Then $(X_1, Y_1) = 0$ or $(X_2, Y_2) = 0$.

We assume that $(X_1, Y_1) \neq 0$ and $(X_2, Y_2) \neq 0$. Consider polynomials

$$P_j(z) = pX_j z^2 + 2Y_j z - X_j, \quad z \in \mathbb{C},\ j = 1, 2,$$

and their duals,

$$Q_j(z) = z^2\overline{P_j(1/\bar{z})} = -\bar{X}_j z^2 + 2\bar{Y}_j z + p\bar{X}_j, \quad z \in \mathbb{C},\ j = 1, 2.$$

We have $P_j(z)Q_j(z) = z^2|P_j(z)|^2$ for any $z \in \mathbb{T}$. Note that the equality (3) means that

$$\frac{\max_{z\in\mathbb{T}} |P_1(z) + \xi P_2(z)|}{2(1-p^2)} = C \text{ for any } \xi \in \mathbb{T}.$$

It is easy to see that

$$C = \frac{\max_{z\in\mathbb{T}} \big(|P_1(z)| + |P_2(z)|\big)}{2(1-p^2)}.$$

We put

$$E = \{z \in \mathbb{T} : |P_1(z)| + |P_2(z)| = 2(1-p^2)C\}$$

and

$$E(\xi) = \{z \in \mathbb{T} : |P_1(z) + \xi P_2(z)| = 2(1-p^2)C\}.$$

We have divided the proof of Theorem 3 into a sequence of lemmas.

Lemma 4 *Assume that $z_0 \in E$. Then $P_1(z_0)P_2(z_0) \neq 0$.*

Proof Assume that $P_1(z_0) = 0$ and that $X_1X_2 \neq 0$. Then

$$|P_1(z)| + |P_2(z)| \leq |P_2(z_0)| \quad \text{for any } z \in \mathbb{T}.$$

Put $Q(z) = z^2|P_2(z_0)|^2 - P_2(z)Q_2(z)$. Then Q is a holomorphic polynomial of degree ≤ 4 such that

$$\frac{Q(z)}{z^2} \geq |P_1(z)|\big(|P_2(z_0)| + |P_2(z)|\big) \geq 0 \quad \text{for any } z \in \mathbb{T}.$$

Then by the Fejér–Riesz theorem (see [4, 7], see also [1, 3]) there exists a polynomial R of degree ≤ 2 such that

$$Q(z) = z^2|R(z)|^2 \quad \text{for any } z \in \mathbb{T}.$$

Hence, P_1 has a double zero at z_0. We get

$$P_1(z) = pX_1(z - z_0)^2$$

and, therefore, $pX_1z_0^2 = -X_1$. So, $X_1 = 0$. A contradiction. □

Lemma 5 *We have $E(\xi) \subset E$ for any $\xi \in \mathbb{T}$. Moreover, $E(\xi_1) \cap E(\xi_2) = \varnothing$ when $\xi_1 \neq \xi_2$. In particular, the set E is infinite.*

Proof It suffices to note that the equality $|a + \xi b| = |a| + |b|$ where $a, b \in \mathbb{C}\setminus\{0\}$ and $\xi \in \mathbb{T}$ is equivalent with $\xi = |ab|/(b\bar{a})$. Now we use the previous Lemma and get that $z_0 \in E$ is such that $z_0 \in E(\xi)$ if and only if

$$\xi = \frac{|P_1(z_0)P_2(z_0)|}{\overline{P_1(z_0)}P_2(z_0)}.$$

□

Lemma 6 *Assume that the equation*

$$|X_1 pz^2 + 2Y_1 z - X_1| + |X_2 pz^2 + 2Y_2 z - X_2| = 1 \tag{4}$$

has at least 9 solutions in $\mathbb{T}$. *Then at least one of the following conditions hold:*

(1) $X_1 = X_2 = 0$;
(2) $P_j(z) = pX_j(z-\zeta_j)^2$, $j = 1, 2$, *where* $|X_1| = |X_2| = \frac{1}{2p+2}$, $\zeta_1 = -\zeta_2 = \frac{\epsilon i}{\sqrt{p}}$, *and* $\epsilon \in \{-1, 1\}$.

Proof From the equality $|P_1(z)| + |P_2(z)| = 1$ we get $|P_2(z)|^2 = (1 - |P_1(z)|)^2$ and, therefore,

$$4|P_1(z)|^2 = (1 + |P_1(z)|^2 - |P_2(z)|^2)^2.$$

We have

$$4z^2 P_1(z) Q_1(z) = (z^2 + P_1(z)Q_1(z) - P_2(z)Q_2(z))^2$$

has at least 9 different solutions. Hence, it holds for any $z \in \mathbb{C}$. Using that

$$P_1(z)Q_1(z) - P_2(z)Q_2(z) = p(|X_2|^2 - |X_1|^2) + \dots$$

we have $|X_1| = |X_2|$. Moreover, there exists a polynomial R_1 such that $P_1 Q_1 = R_1^2$.

Let $P_1(z) = pX_1(z-\zeta_1)(z-\xi_1)$. Note that $pX_1\zeta_1\xi_1 = -X_1$. Hence, $X_1 = 0$ or $\zeta_1\xi_1 = -\frac{1}{p}$. Assume that $X_1 \neq 0$. Note that

$$Q_1(z) = p\overline{X_1}(1 - \bar{\zeta}_1 z)(1 - \bar{\xi}_1 z).$$

So, $P_1 Q_1 = R_1^2$ if and only if $\zeta_1 = \xi_1$ or $\bar{\zeta}_1 = 1/\xi_1$. Since $\zeta_1\xi_1 = -\frac{1}{p}$, we get $\zeta_1 = \xi_1$ and, therefore, $\zeta_1^2 = -\frac{1}{p}$ and $P_1(z) = pX_1(z-\zeta_1)^2$. By similar arguments, we get $P_2(z) = pX_2(z-\zeta_2)^2$. Putting these equalities to (4) we get $\zeta_2 = -\zeta_1$ and $|X_1| = |X_2| = \frac{1}{2p+2}$. □

Proof of Theorem 3 For the proof, we apply Lemma 6 to the polynomials $\tilde{P}_j = P_j/(2C(1-p^2))$ with $\tilde{X}_j = X_j/(2C(1-p^2))$, $\tilde{Y}_j = Y_j/(2C(1-p^2))$, where $j = 1, 2$. Then we have $\tilde{P}_1(z) = p\tilde{X}_1(z-\zeta_0)^2$ and $\tilde{P}_2(z) = p\tilde{X}_2(z+\zeta_0)^2$, where $|\zeta_0| = \frac{1}{\sqrt{p}}$ and, $|\tilde{X}_1| = |\tilde{X}_2| = \frac{1}{2+2p}$.

There exists $\xi_0 \in \mathbb{T}$ such that $\xi_0\tilde{X}_2 = -\tilde{X}_1$. Then

$$1 = \max_{z\in\mathbb{T}} |\tilde{P}_1(z) + \xi_0\tilde{P}_2(z)| = \max_{z\in\mathbb{T}} p|\tilde{X}_1| \cdot |(z-\zeta_0)^2 - (z+\zeta_0)^2|. \tag{5}$$

And, therefore, we have

$$1 = 4p|\tilde{X}_1| \cdot |\zeta_0| = \frac{2\sqrt{p}}{1+p}.$$

Hence, we get $p = 1$. A contradiction with the condition $p \in [0; 1)$. □

Lemma 7 *Let* $f : \mathbb{D} \to \mathbb{C}$ *be a* C^1 *function such that for any* $z \in \mathbb{D}$ *we have* $\frac{\partial f}{\partial z}(z) = 0$ *or* $\frac{\partial f}{\partial \bar{z}}(z) = 0$. *Then* f *is holomorphic or anti-holomorphic in* $\mathbb{D}$.

Proof Put $h = \frac{\partial f}{\partial z}$. Then h is a continuous function on $\mathbb{D}$ and on a set $\{z \in \mathbb{D} : h(z) \neq 0\}$ we have $\frac{\partial f}{\partial \bar{z}} \equiv 0$. Hence, h is holomorphic on $\mathbb{D}\setminus h^{-1}(0)$. By Radó's theorem, h is holomorphic in $\mathbb{D}$. If $\{z \in \mathbb{D} : h(z) = 0\}$ is a discrete, locally finite set, then f is holomorphic in $\mathbb{D}$. For otherwise it is equal to $\mathbb{D}$ and, therefore, f is an anti-holomorphic function. □

Proof of Theorem 1 It suffices to show that for any $(X_1, Y_1), (X_2, Y_2) \in \mathbb{C}^2$, any $(s, p) \in \mathbb{G}_2$, and any constant $C > 0$ such that

$$\gamma_{\mathbb{G}_2}\big((s, p); (X_1, Y_1) + \xi(X_2, Y_2)\big) = C \quad \text{for any } \xi \in \mathbb{T}, \tag{6}$$

we have $(X_1, Y_1) = 0$ or $(X_2, Y_2) = 0$. There exists an automorphism Φ of $\mathbb{G}_2$ such that $\Phi(s, p) = (0, \tilde{p})$, where $\tilde{p} \geq 0$. Then

$$\begin{aligned} &\gamma_{\mathbb{G}_2}\big((s, p); (X_1, Y_1) + \xi(X_2, Y_2)\big) = \\ &\quad \gamma_{\mathbb{G}_2}\big((0, \tilde{p}); \Phi'(s, p)(X_1, Y_1) + \xi\Phi'(s, p)(X_2, Y_2)\big). \end{aligned} \tag{7}$$

Since $\det \Phi' \neq 0$, we get $(X_1, Y_1) = 0$ or $(X_2, Y_2) = 0$. □

3 Carathéodory isometries

For a domain $D \subset \mathbb{C}^n$, we define another biholomorphically invariant function. For any $\lambda_1, \lambda_2 \in D$ we put

$$c_D(\lambda_1, \lambda_2) = \sup\{\rho(F(\lambda_1), F(\lambda_2)) : F : D \to \mathbb{D} \text{ holomorphic}\}$$

We say that a mapping $\Phi : D_1 \to D_2$ between domains D_1, D_2 is a Carathéodory isometry if

$$c_{D_1}(\lambda_1, \lambda_2) = c_{D_2}(\Phi(\lambda_1), \Phi(\lambda_2)) \quad \text{for any } \lambda_1, \lambda_2 \in D_1.$$

We say that Φ is local c-isometry if for any $\lambda_0 \in D_1$ there exists a neighborhood $U_0 \subset D_1$ of λ_0 such that

$$c_{D_1}(\lambda_1, \lambda_2) = c_{D_2}(\Phi(\lambda_1), \Phi(\lambda_2)) \quad \text{for any } \lambda_1, \lambda_2 \in U_0.$$

Note that any c-isometry is a local c-isometry and any local c-isometry is a γ-isometry (see e.g. [10]). In particular, we have

Theorem 8 *Let $\phi : \mathbb{D} \to \mathbb{G}_2$ be a local c-isometry of class C^1. Then ϕ is holomorphic or anti-holomorphic.*

4 Applications

In [5] the authors gave a proof of the description of automorphisms of the symmetrized bidisc. One of the main step in the proof is to show that for any automorphism $\Phi : \mathbb{G}_2 \to \mathbb{G}_2$ we have $\Phi(\Sigma) \subset \Sigma$. We show this property by using our approach.

Corollary 9 *Let $F : \mathbb{G}_2 \to \mathbb{G}_2$ be a holomorphic mapping such that F is a γ-isometry at* 0. *Then $F(0) \in \Sigma$. In particular, if F is a γ-isometry on Σ, then $F(\Sigma) \subset \Sigma$.*

Proof Assume that $F(0) = (0, p)$, where $p \in (0, 1)$. Then

$$\gamma_{\mathbb{G}_2}(0; (X, Y)) = \gamma_{\mathbb{G}_2}((0, p); a_1 X + b_1 Y, a_2 X + b_2 Y) \quad \text{for any } X, Y \in \mathbb{C},$$

where $a_1 = \frac{\partial F_1}{\partial s}(0)$, $b_1 = \frac{\partial F_1}{\partial p}(0)$, $a_2 = \frac{\partial F_2}{\partial s}(0)$, $b_2 = \frac{\partial F_2}{\partial p}(0)$. Take pairs $(X, Y) = (1, \xi)$, where $\xi \in \mathbb{T}$. Recall that

$$\gamma_{\mathbb{G}_2}(0; (X, Y)) = \frac{|X|}{2} + |Y|.$$

In this way we get a contradiction. □

Now we can prove a Vigué type result (see e.g. [9]).

Corollary 10 *Let $F : \mathbb{G}_2 \to \mathbb{G}_2$ be a holomorphic mapping such that F is a γ-isometry at 0. Then F is an automorphism of $\mathbb{G}_2$.*

Proof By the above Corollary, we may assume that $F(0) = 0$. Then

$$\frac{|X|}{2} + |Y| = \frac{|a_1 X + b_1 Y|}{2} + |a_2 X + b_2 Y| \quad \text{for any } X, Y \in \mathbb{C},$$

where $a_1 = \frac{\partial F_1}{\partial s}(0)$, $b_1 = \frac{\partial F_1}{\partial p}(0)$, $a_2 = \frac{\partial F_2}{\partial s}(0)$, $b_2 = \frac{\partial F_2}{\partial p}(0)$.

By taking $X = 0$ and later $Y = 0$ we get $\frac{|b_1|}{2} + |b_2| = 1$ and $\frac{|a_1|}{2} + |a_2| = \frac{1}{2}$. So,

$$\begin{aligned} \frac{|X|}{2} + |Y| = \frac{|a_1 X + b_1 Y|}{2} + |a_2 X + b_2 Y| \leq \\ \frac{|a_1 X| + |b_1 Y|}{2} + |a_2 X| + |b_2 Y| \leq \frac{|X|}{2} + |Y|. \end{aligned} \tag{8}$$

We get $a_1 b_1 = 0$ and $a_2 b_2 = 0$. Therefore, $a_1 = 0$, $|a_2| = \frac{1}{2}$, $b_2 = 0$, $|b_1| = 2$ or $b_1 = 0$, $|b_2| = 1$, $a_2 = 0$, $|a_1| = 1$. We have $|\det F'(0)| = 1$. From the Cartan theorem we get that F is an automorphism. □

Remark 11 The author thanks the anonymous referee for her/his helpful comments that improved the presentation of the results.

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