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CR eigenvalue estimate and Kohn-Rossi cohomology II

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Abstract

Let X be a weakly pseudoconvex, compact, and connected CR manifold with a transversal CR S^1-action of real dimension $2n-1$, where $n \geq 2$. The Fourier components of the Kohn-Rossi cohomology, with respect to the S^1-action, introduced by Hsiao-Li [6], are closely related to the embedding problem of CR manifolds. In this paper, we continue our previous study [14] and provide a sharp estimate for the asymptotic growth order, denoted as $O(m^q)$, of the dimension of the m-th Fourier components $H^{0,q}_{b,m}(X)$ of the Kohn-Rossi cohomology $H^{0,q}_b(X)$ as $m \to +\infty$. Together with our previous work [14], we present a comprehensive complete and sharp estimate for the growth order of the Fourier components $H^{0,q}_{b,m}(X)$ and $H^{n-1,q}_{b,m}(X)$ of the Kohn-Rossi cohomology $H^{0,q}_b(X)$ and $H^{n-1,q}_b(X)$ as $m \to \infty$. Additionally, we derive a Serre-type duality theorem for S^1-equivariant CR vector bundles.

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1 Introduction

The aim of this paper is to continue our previous study [14] of the estimate of the dimension of the Fourier components of the Kohn-Rossi cohomology with respect to the transversal CR S^1-action on a compact connected weakly pseudoconvex CR manifold. This is closely related to the embedding problem of a CR manifold, see [1, 5, 7] and references therein.

We work in the following setting, see the work of Hsiao-Li [6] for the fundamental construction. Let $(X, T^{1,0}X)$ be a compact connected weakly pseudoconvex CR manifold of real dimension $2n-1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on X. We assume that there is a transversal CR S^1-action on X. Let (L, h) be an S^1-equivariant Hermitian CR line bundle on X. Let $\Box_b$ be the associated $\bar\partial_b$-Laplace operator on $\Omega^{p,q}(X, L)$, with respect to a rigid Hermitian metric on X and the S^1-equivariant Hermitian metric h of L. Thanks to the S^1-action, we have the Fourier decomposition $\Omega^{p,q}(X, L) = \oplus_{m\in\mathbb{Z}}\Omega^{p,q}_m(X, L)$, where $\Omega^{p,q}_m(X, L)$ is the m-th Fourier component of $\Omega^{p,q}(X, L)$ with respect to the S^1-action, and the $\bar\partial_b$ operator acts on the graded algebra $\oplus_q\Omega^{p,q}_m(X, L)$. One can thus define the m-th Fourier component $H^{p,q}_{b,m}(X, L)$ of (p, q)-th Kohn-Rossi cohomology group $H^{p,q}_b(X, L)$. Let $\Box^{p,q}_{b,m}$ be the restriction of the $\bar\partial_b$-Laplace operator to the space $\Omega^{p,q}_m(X, L)$, which turns out to be a self-adjoint operator. Let $\mathcal{H}^{p,q}_{b,m,\leq\lambda}(X, L)$ be the linear span of the eigenforms of $\Box^{p,q}_{b,m}$ in $\Omega^{p,q}_m(X, L)$ with eigenvalues smaller than or equal to λ. By a Hodge type theory (see th work of Cheng-Hsiao-Tsai [3] for a nice presentation), $\mathcal{H}^{p,q}_{b,m,\leq 0}(X, L) := \mathcal{H}^{p,q}_{b,m}(X, L)$ is the space of $\Box^{p,q}_{b,m}$ harmonic forms, and isomorphic to $H^{p,q}_{b,m}(X, L)$. In particular, $H^{p,q}_{b,m}(X, L)$ is of finite dimension for every $m \in \mathbb{Z}$.

In [14], when L is a trivial line bundle, we extend the eigenvalue estimate technique of Berndtsson for $\bar\partial$-Laplace operator on compact Hermitian manifold to the setting of compact connected weakly pseudoconvex CR manifold with transversal CR S^1-action, getting the following

Theorem 1.1 ([14, Theorem 1.1]) *Let X be a compact connected weakly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$. Assume that X admits a transversal CR S^1-action. Then for m sufficiently large and $q = 0, 1, \ldots, n-1$, if $0 \leq \lambda \leq m$,*

$$\dim \mathcal{H}^{n-1,q}_{b,m,\leq\lambda} \leq C(\lambda+1)^q m^{n-1-q},$$

and if $1 \leq m \leq \lambda$,

$$\dim \mathcal{H}^{n-1,q}_{b,m,\leq\lambda} \leq C\lambda^{n-1}.$$

In particular, when $\lambda = 0$, we have that for m sufficiently large and $q = 0, 1, \ldots, n-1$,

$$\dim \mathcal{H}^{n-1,q}_{b,m}(X) \leq Cm^{n-1-q}.$$

Remark 1.1 Recently, there have been other extensions of Berndtsson's estimate. For instance, in [13], we extend Berndtsson's estimate to holomorphic line bundles with singular metrics. Additionally, H. Wang extends Berndtsson's estimate to non-compact, q-convex complex manifolds in [12] and to Nakano q-semipositive holomorphic Hermitian line bundles on compact complex manifolds in [11].

We also get a Serre type duality theorem as follows.

Theorem 1.2 ([14, Theorem 1.2])

Let X be a compact connected CR-manifold of real dimension $2n-1$, which admits a transversal CR S^1-action. Then we have the following conjugate linear isomorphism in the cohomologcial level

$$H^{p,q}_{b,m}(X) \simeq H^{n-1-p,n-1-q}_{b,-m}(X), 0 \leq p, q \leq n-1.$$

Combing Theorem 1.1 and Theorem 1.2, we get

Theorem 1.3 ([14, Theorem 1.3]) *Let X be a compact connected weakly pseudoconvex CR-manifold of real dimension $2n-1$, $n \geq 2$, which admits a transversal CR S^1-action. Then we have that for $q = 0, 1, \cdots, n-1$,*

$$\dim H^{0,q}_{b,-m}(X) \leq Cm^q, \qquad as \qquad m \to +\infty.$$

Remark 1.2 As pointed out in [14], combining Berndtsson's example [2, Proposition 4.2] and a Grauert tube type argument, we can see that the estimates of the growth order in Theorem 1.1 and Theorem 1.3 are sharp.

In this paper, we are concerning about the following question

Question 1.1 Whether we can get a sharp estimate of the growth order of $\dim H^{0,q}_{b,m}(X)$ $(q = 0, \ldots, n-1)$ as $m \to +\infty$?

To answer this question, we study the Kohn-Rossi cohomology valued in an S^1-equivariant CR Hermitian line bundle. By a careful study of the local behavior of an S^1-equivariant CR Hermitian line bundle, we see that the technique developed in our previous paper [13] can also be applied to get the following

Theorem 1.4 *Let X be a compact connected weakly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$. Assume that X admits a transversal CR S^1-action. Let (L, h_L) be a Hermitian rigid CR line bundle over X. Then for m sufficiently large and $q = 0, 1, \ldots, n-1$, if $0 \leq \lambda \leq m$,*

$$\dim \mathcal{H}^{n-1,q}_{b,m,\leq\lambda}(X, L) \leq C(\lambda+1)^q m^{n-1-q},$$

and if $1 \leq m \leq \lambda$,

$$\dim \mathcal{H}^{n-1,q}_{b,m,\leq\lambda}(X, L) \leq C\lambda^{n-1}.$$

Remark 1.3 Theorem 1.4 also holds for the case that L is an S^1-equivariant CR vector bundles, by applying the argument in [14].

Remark 1.4 The proof of Theorem 1.1 and Theorem 1.4 actually gives an upper bounde on the m-th Fourier component of the Szegö kernel $\Pi^{n-1,q}_{m,\leq\lambda}$ (for definition, see Definition 3.1) rather than just on the dimension of cohomology. The question of when there is a leading asymptotic term or expansion for $\Pi^{p,q}_{b,m}$ in the weakly pseudoconvex case is a natural and important question, and is closely related to the embedding problem of a weakly pseudoconvex CR manifold, see [9] and references therein.

Remark 1.5 It is worthy pointing out that if the compact connected weakly pseudoconvex CR manifold is the unit circle bundle of an semipositive orbifold bundle (see [3], see also [14]), our result corresponds to Berndtsson's estimate in the orbifold case.

In the above theorem, let $L = \det T^{1,0}X$, we can answer the Question 1.1 by the following

Corollary 1.5 *Let X be a compact connected weakly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$. Assume that X admits a transversal CR S^1-action. Then for m sufficiently large and $q = 0, 1, \ldots, n-1$, if $0 \leq \lambda \leq m$,*

$$\dim \mathcal{H}^{0,q}_{b,m,\leq\lambda}(X) \leq C(\lambda+1)^q m^{n-1-q},$$

and if $1 \leq m \leq \lambda$,

$$\dim \mathcal{H}^{0,q}_{b,m,\leq\lambda}(X) \leq C\lambda^{n-1}.$$

In particular, when $\lambda = 0$, we have

$$\dim H^{0,q}_{b,m}(X) \leq Cm^{n-1-q}$$

for $m \to +\infty$.

From Theorem 1.2, we can get the following

Theorem 1.6 *Let X be a compact connected weakly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$. Assume that X admits a transversal CR S^1-action. Then we have that for $q = 0, 1, \ldots, n-1$,*

$$\dim H^{n-1,q}_{b,-m}(X) \leq Cm^q, \qquad as \qquad m \to +\infty.$$

Remark 1.6 Examples in §7 show that Corollary 1.5 and Theorem 1.6 give sharp estimates of the growth order of the cohomology groups $H^{0,q}_{b,m}(X)$ and $H^{n-1,q}_{b,-m}(X)$ as $m \to +\infty$. In summary, Theorem 1.1, Theorem 1.3, Corollary 1.5 and Theorem 1.6 give a complete sharp estimates of the growth order of the cohomology groups $H^{0,q}_{b,m}(X)$ and $H^{n-1,q}_{b,-m}(X)$ as $m \to \infty$.

We also derive a Serre type duality theorem for S^1-equivariant vector bundle.

Theorem 1.7 *Let X be a compact connected CR-manifold of real dimension $2n-1$, $n \geq 2$, which admits a transversal CR S^1-action. Let L be an S^1-equivariant CR vector bundle over X, and L^* be the dual bundle of L. Then we have the following conjugate linear isomorphism in the cohomologcial level*

$$H^{p,q}_{b,m}(X, L) \simeq H^{n-1-p,n-1-q}_{b,-m}(X, L^*), 0 \leq p, q \leq n-1.$$

Then as a direct consequence of Theorem 1.4 and Theorem 1.7, we have the following

Theorem 1.8 *Let X be a compact connected weakly pseudoconvex CR-manifold of real dimension $2n-1$, $n \geq 2$, which admits a transversal CR S^1-action. Let L be an S^1-equivariant CR line bundle over X. Then we have*

$$\dim H^{0,q}_{b,-m}(X, L^*) \leq Cm^q$$

for $0 \leq q \leq n-1$.

2 CR manifold with transversal CR S^1-action

Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n-1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on X. That is, $T^{1,0}X$ is a sub-bundle of the complexified tangent bundle $\mathbb{C}TX$ of rank $n-1$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = \mathcal{C}^\infty(X, T^{1,0}X)$.

We assume throughout this paper that, $(X, T^{1,0}X)$ is a compact connected CR manifold with a transversal CR S^1-action.

Denote by $e^{i\theta}$ $(0 \leq \theta < 2\pi)$ the S^1-action: $S^1 \times X \to X$, $(e^{i\theta}, x) \mapsto e^{i\theta} \circ x$. Set $X_{reg} = \{x \in X : \forall e^{i\theta} \in S^1, \text{ if } e^{i\theta} \circ x = x, \text{ then } e^{i\theta} = \mathrm{id}\}$. We call $x \in X_{reg}$ a regular point of the S^1-action. It is proved in [6] that X_{reg} is an open, dense subset of X, and thus the measure of $X \setminus X_{reg}$ is zero.

Let $T \in \mathcal{C}^\infty(X, TX)$ be the global real vector field induced by the S^1-action $e^{i\theta}$ $(\theta \in [0, 2\pi))$ given as follows

$$(Tu)(x) = \frac{\partial}{\partial \theta}(u(e^{i\theta} \circ x))\Big|_{\theta=0}, \; u \in \mathcal{C}^\infty(X).$$

We say that the S^1-action is CR if it preserves the CR structure of X, i.e.

$$[T, \mathcal{C}^\infty(X, T^{1,0}X)] \subset \mathcal{C}^\infty(X, T^{1,0}X)$$

where [,] is the Lie bracket between the smooth vector fields on X. Furthermore, we say that the S^1-action is transversal if for each $x \in X$,

$$\mathbb{C}T(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX.$$

Denote by ω_0 the global real 1-form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

Definition 2.1 For $x \in X$, the Levi form $\mathcal{L}_x$ associated with the CR structure is the Hermitian quadratic form on $T_x^{1,0}X$ defined as follows. For any $U, V \in T_x^{1,0}X$, pick $\mathcal{U}, \mathcal{V} \in \mathcal{C}^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(x) = U$, $\mathcal{V}(x) = V$. Set

$$\mathcal{L}_x(U, \overline{V}) = \frac{1}{2i} \langle [\mathcal{U}, \overline{\mathcal{V}}](x), \omega_0(x) \rangle$$

where [,] denotes the Lie bracket between smooth vector fields. Note that $\mathcal{L}_x$ does not depend on the choice of $\mathcal{U}$ and $\mathcal{V}$.

Definition 2.2 The CR structure on X is called (weakly) pseudoconvex at $x \in X$ if $\mathcal{L}_x$ is positive semi-definite. It is called strongly pseudoconvex at x if $\mathcal{L}_x$ is positive definite. If the CR structure is (strongly) pseudoconvex at every point of X, then X is called a (strongly) pseudoconvex CR manifold.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundle of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of (p, q)-forms by $T^{*p,q}X := \Lambda^p T^{*1,0}X \otimes \Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open subset. Let $\Omega^{p,q}(D)$ denote the space of smooth sections of $T^{*p,q}X$ over D and let $\Omega_0^{p,q}(D)$ be the subspace of $\Omega^{p,q}(D)$ whose elements have compact support in D.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$de^{i\theta_0} : \mathbb{C}T_xX \to \mathbb{C}T_{e^{i\theta_0}x}X$$

denote the differential map of $e^{i\theta_0}: X \to X$. By the property of transversal CR S^1-action, one can check that

$$\begin{aligned} de^{i\theta_0} &: T_x^{1,0}X \to T_{e^{i\theta_0}x}^{1,0}X, \\ de^{i\theta_0} &: T_x^{0,1}X \to T_{e^{i\theta_0}x}^{0,1}X, \\ de^{i\theta_0}(T(x)) &= T(e^{i\theta_0} \circ x). \end{aligned} \tag{1}$$

Let $(de^{i\theta_0})^* : \Lambda^{p+q}(\mathbb{C}T^*X) \to \Lambda^{p+q}(\mathbb{C}T^*X)$ be the pull-back of $de^{i\theta_0}$, $p, q = 0, 1, \ldots, n-1$. From (7), we can check that for every $p, q = 0, 1, \ldots, n-1$,

$$(de^{i\theta_0})^* : T^{*p,q}_{e^{i\theta_0}\circ x}X \to T_x^{*p,q}X. \tag{2}$$

Let $u \in \Omega^{p,q}(X)$, define Tu as follows. For any $X_1, \ldots, X_p \in T_x^{1,0}X$ and $Y_1, \ldots, Y_q \in T_x^{0,1}X$

$$Tu(X_1, \ldots, X_p; Y_1, \ldots, Y_q) := \frac{\partial}{\partial\theta}((de^{i\theta})^* u(X_1, \ldots, X_p; Y_1, \ldots, Y_q))|_{\theta=0}.$$

From (1) and (2), we have that $Tu \in \Omega^{p,q}(X)$ for all $u \in \Omega^{p,q}(X)$.

Let $D \subset X$ be an open set. We say that a function $u \in \mathcal{C}^\infty(D)$ is rigid if $Tu = 0$. We say a function $u \in \mathcal{C}^\infty(X)$ is Cauchy-Riemann (CR for short) if $\bar{\partial}_b u = 0$, and is rigid CR if $\bar{\partial}_b u = 0$ and $Tu = 0$.

3 S^1-equivariant CR Hermitian vector bundles

Definition 3.1 ([8])

Let X be a CR manifold. A smooth complex vector bundle (F, π, X) of rank r over X is called a CR vector bundle if F has the structure of a smooth CR manifold, the map $\pi : F \to X$ is a CR map, and for each point of X, there exists an open neighborhood U and a smooth trivialization of $F|_U$ that is a CR diffeomorphism (that is the map and its inverse are CR). We define a smooth CR section of F over an open subset D of X as a smooth section $s : D \to F$ that is a CR map. A CR frame of F over an open subset D of X is a smooth frame $\{f^1, \ldots, f^r\}$ of $F|_D$ where each f^k is a CR section.

Definition 3.2 ([8]) Let X be a CR manifold endowed with an S^1 action, and let (F, π, X) be a CR vector bundle of rank r over X. We say that the S^1 action on X can be lifted to F, that is there exists an S^1-action on F still denoted by $e^{i\theta}$ such that

$$\pi(e^{i\theta} \circ v(x)) = e^{i\theta} \circ x, \quad v(x) \in F_x, \quad x \in X.$$

A lifting is called a CR bundle lifting in F if for each $e^{i\theta}$, the map $e^{i\theta} : F \to F$ is a CR bundle map. Such a bundle is called an S^1-equivariant CR vector bundle.

Proposition 3.1 ([8, Proposition 2.7, Theorem 2.14]) *Let (F, π, X) be an S^1-equivariant CR vector bundle. Then in a neighborhood of each point, there exists a righd CR local frame of F. In particular, there exists an open cover $(U_j)_j$ of X and trivializing frames $\{f_j^1, \ldots, f_j^r\}$ on each U_j such that the corresponding transition matrices are rigid CR. Furthermore, on every S^1-equivariant CR bundle F over X, there is a S^1-equivariant hermitian metric on F.*

Let L be an S^1-equivariant CR line bundle over X. Let $(U_j)_j$ be an open covering and $(s_j)_j$ be a family of rigid CR frames s_j on U_j. Let s be a rigid CR frame of L on an open subset $D \subset X$. For any $u \in \Omega^{p,q}(X, L)$, write $u = \tilde{u} \otimes s$, with $\tilde{u} \in \Omega^{p,q}(D)$, we define $Tu := T\tilde{u} \otimes s$. Let $\overline{\partial}_b : \Omega^{p,q}(X, L) \to \Omega^{p,q+1}(X, L)$ be the tangential Cauchy-Riemann operator. Since the transition functions are rigid CR, Tu is well defined. Moreover, we have

$$T\overline{\partial}_b = \overline{\partial}_b T \text{ on } \Omega^{p,q}(X, L).$$

Let h^L be an S^1-equivairant Hermitian metric of L. If s is a local rigid CR frame of L on an open subset $D \subset X$, then the local weight of h^L with respect to s is the function $\Phi \in \mathcal{C}^\infty(D, \mathbb{R})$ for which

$$|s(x)|^2_{h_L} = e^{-\Phi(x)}, \quad x \in D.$$

Furthermore, from the S^1-equivariant property of h_L, we have that $T\Phi = 0$ on D.

Remark 3.1 As pointed out in [8, Example 1.16], $T^{1,0}X$ and $\det(T^{1,0}X)$ are both S^1-equivariant CR vector bundles on X, provided that X is a compact CR manifold with a locally free transversal CR S^1-action.

Let L be an S^1-equivariant CR bundle with an S^1-equivariant Hermitian metric h_L. For $m \in \mathbb{Z}$, define

$$\Omega^{p,q}_m(X, L) := \{u \in \Omega^{p,q}(X, L) : Tu = imu\}.$$

Let $(\cdot|\cdot)_{h_L}$ be the L^2 inner product on $\Omega^{p,q}(X, L)$ induced by h_L, $\langle\cdot|\cdot\rangle$ and let $\|\cdot\|_{h_L}$ denote the corresponding norm. Let s be a local rigid CR frame of L on an open subset $D \subset X$. For $u = \tilde{u} \otimes s$, $v = \tilde{v} \otimes s \in \Omega^{p,q}_0(D, L)$, we have

$$(u|v)_{h_L} = \int_X \langle\tilde{u}|\tilde{v}\rangle e^{-\Phi(x)} dv_X$$

where dv_X is the volume form on X induced by the S^1-equivariant Hermitian metric $\langle\cdot|\cdot\rangle$ on X. Let $L^2_{(p,q),m}(X, L)$ be the completion of $\Omega^{p,q}_m(X, L)$ with respect to $(\cdot|\cdot)_{h_L}$. For $m \in \mathbb{Z}$, let

$$Q^{p,q}_m : L^2_{(p,q)}(X, L) \to L^2_{(p,q),m}(X, L)$$

be the orthogonal projection with respect to $(\cdot|\cdot)_{h_L}$. Let $\bar{\partial}^*_b : \Omega^{p,q+1}(X, L) \to \Omega^{p,q}(X, L)$ be the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot|\cdot)_{h_L}$. Since $\langle\cdot|\cdot\rangle$ and h_L are S^1-equivariant, we can check that

$$T\bar{\partial}^*_b = \bar{\partial}^*_b T \text{ on } \Omega^{p,q}_m(X, L), q = 0, 1, \cdots, n-1,$$

and then

$$\bar{\partial}^*_b : \Omega^{p,q+1}_m(X, L) \to \Omega^{p,q}(X, L), \forall m \in \mathbb{Z}.$$

Put

$$\Box^{p,q}_b := \bar{\partial}_b\bar{\partial}^*_n + \bar{\partial}^*_b\bar{\partial}_b : \Omega^{p,q}(X, L) \to \Omega^{p,q}(X, L).$$

Then we have

$$T\Box^{p,q}_b = \Box^{p,q}_b T \text{ on } \Omega^{p,q}(X, L),\ p, q = 0, 1, \cdots, n-1,$$

and

$$\Box^{p,q}_b : \Omega^{p,q}_m(X, L) \to \Omega^{p,q}_m(X, L), \forall m \in \mathbb{Z}.$$

We will write $\Box^{p,q}_{b,m}$ to denote the restriction of $\Box^{p,q}_{b}$ on the space $\Omega^{p,q}_m(X, L)$. For every $m \in \mathbb{Z}$, we extend $\Box^{p,q}_{b,m}$ to $L^2_{(p,q),m}(X, L)$ in the sense of distribution by

$$\Box^{p,q}_{b,m} : \mathrm{Dom}(\Box^{p,q}_{b,m}) \to L^2_{(p,q),m}(X, L),$$

where $\mathrm{Dom}(\Box^{p,q}_{b,m}) = \{u \in L^2_{(p,q),m}(X, L) : \Box^{p,q}_{b,m} u \in L^2_{(p,q),m}(X, L)\}$. The following follows from Kohn's L^2-estimate (e.g. see [4, Theorem 8.4.2]).

Theorem 3.2 [3, Theorem 3.1] *For every $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, there exists a constant C_s such that*

$$\|u\|_{s+1} \leq C_s\big(\|\Box^{(p,q)}_{b} u\|_s + \|Tu\|_s + \|u\|_s\big), \forall u \in \Omega^{p,q}(X, L),$$

where $\|\cdot\|_s$ denotes the standard sobolev norm of order s on X.

Theorem 3.3 [3, Corollary 3.2] *Fix $m \in \mathbb{Z}$, for every $s \in \mathbb{N}_0$, there is a constant $C_{s,m}$ such that*

$$\|u\|_{s+1} \leq C_{s,m}\big(\|\Box^{(p,q)}_{b,m} u\|_s + \|u\|_s\big), \forall u \in \Omega^{p,q}_m(X, L).$$

Theorem 3.4 ([3, Lemma 3.4 and Proposition 3.5]) *Fix $m \in \mathbb{Z}$, $\Box^{(p,q)}_{b,m} : Dom(\Box^{(p,q)}_{b,m}) \subset L^2_{(p,q),m}(X, L) \to L^2_{(p,q),m}(X, L)$, is a self-adjoint operator. The spectrum of $\Box^{(p,q)}_{b,m}$ denoted by $Spec(\Box^{(p,q)}_{b,m})$ is a discrete subset of $[0, +\infty)$. For every $\lambda \in Spec(\Box^{(p,q)}_{b,m})$ the eigenspace with respect to λ*

$$\mathcal{H}^{p,q}_{b,m,\lambda}(X, L) = \Big\{u \in Dom(\Box^{(p,q)}_{b,m}) : \Box^{(p,q)}_{b,m} u = \lambda u\Big\}$$

is finite dimensional with $\mathcal{H}^{p,q}_{b,m,\lambda}(X, L) \subset \Omega^{p,q}_m(X, L)$ and for $\lambda = 0$ we denote by $\mathcal{H}^{p,q}_{b,m}(X, L)$ the harmonic space $\mathcal{H}^{p,q}_{b,m,0}(X, L)$ for brevity and then we have the Dolbeault isomorphism

$$\mathcal{H}^{p,q}_{b,m}(X, L) \simeq H^{p,q}_{b,m}(X, L).$$

In particular, we have

$$\dim H^{p,q}_{b,m}(X, L) < \infty, \forall m \in \mathbb{Z}, \forall\, 0 \leq p, q \leq n-1.$$

For $\lambda \geq 0$, we collect the eigenspace of $\Box^{(p,q)}_{b,m}$ whose eigenvalue is less than or equal to λ and define

$$\mathcal{H}^{p,q}_{b,m,\leq\lambda} := \oplus_{\sigma\leq\lambda}\mathcal{H}^{p,q}_{b,m,\sigma}(X, L),$$
$$\mathscr{L}^{p,q}_{b,m,\leq\lambda} := \mathrm{Ker}\overline{\partial}_b \cap \mathcal{H}^{p,q}_{b,m,\leq\lambda}.$$

Definition 3.1 The Szegö kernel function of the space $\mathscr{L}^{p,q}_{b,m,\leq\lambda}$ is defined as

$$\Pi^{p,q}_{m,\leq\lambda}(x) := \sum_{j=1}^{d_m} |g_j(x)|^2,$$

where $\{g_j\}_{j=1}^{d_m}$ is any orthonormal basis for the space $\mathscr{L}^{p,q}_{b,m,\leq\lambda}$.

It is easy to see that

$$\dim \mathscr{L}^{p,q}_{b,m,\leq\lambda} = \int_X \Pi^{p,q}_{m,\leq\lambda} dv_X. \tag{3}$$

The extremal function $S^{p,q}_{m,\leq\lambda}$ for $y \in X$ is defined by

$$S^{p,q}_{m,\leq\lambda}(y) := \sup_{u \in \mathscr{L}^{p,q}_{b,m,\leq\lambda}, \|u\|=1} |u(y)|^2.$$

The next lemma is classical in Bergman's theory of reproducing kernels.

Lemma 3.2 [c.f. [2]]
For any $y \in X$,

$$S^{p,q}_{m,\leq\sigma}(y) \leq \Pi^{p,q}_{m,\leq\sigma}(y) \leq \binom{n-1}{p}\binom{n-1}{q} S^{p,q}_{m,\leq\sigma}(y).$$

In particular,

$$\int_X S^{p,q}_{m,\leq\sigma}(y) dv_X \leq \dim \mathscr{L}^{p,q}_{b,m,\leq\lambda} \leq \binom{n-1}{p}\binom{n-1}{q} \int_X S^{p,q}_{m,\leq\sigma}(y) dv_X.$$

For the proof of the above Lemma, we refer to [2, Page 308, Lemma 4.1].

4 Canonical local coordinates

In this section, we draw the local picture for compact connected CR manifolds with transversal CR S^1-action. The following result is due to Baouendi-Rothschild-Treves [1].

Theorem 4.1 (c.f. [1]) Let X be a compact CR manifold of $\dim_X = 2n-1$, $n \geq 2$ with a transversal CR S^1-action. Let $\langle\cdot|\cdot\rangle$ be the given T-rigid Hermitian metric on X. For any point $x_0 \in X$, there exists local coordinates $(x_1, \cdots, x_{2n-1}) = (z, \theta) = (z_1, \cdots, z_{n-1}, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n-1$, $x_{2n-1} = \theta$, defined in some small neighborhood $D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\}$ of x_0 such that

$$T = \frac{\partial}{\partial\theta}$$
$$Z_j = \frac{\partial}{\partial z_j} + i\frac{\partial\varphi(z)}{\partial z_j}\frac{\partial}{\partial\theta}, \ j = 1, \cdots, n-1,$$

where $\{Z_j(x)\}_{j=1}^{n-1}$ form a basis of $T^{1,0}_x X$ for each $x \in D$ and $\varphi(z) \in \mathcal{C}^\infty(D, \mathbb{R})$ is independent of θ. Moreover, on D we can take (z, θ) and φ so that $(z(x_0), \theta(x_0)) = (0, 0)$ and $\varphi(z) = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z|^3)$, $\forall (z, \theta) \in D$, where $\{\lambda_j\}_{j=1}^{n-1}$ are the eigenvalues of Levi-form of X at x_0 with respect to the given T-rigid Hermitian metric on X.

Remark 4.1 It was proved in [6] that if $x_0 \in X_{reg}$, δ can be taken to be π, and if x_0 is not a regular point, say $x_0 \in X_k$, δ can be taken to be any positive number smaller than $\frac{\pi}{k}$.

It was pointed out in [14, Proposition 4.2] that in Theorem 4.1, if we assume that X is weakly pseudoconvex, then $i\partial\overline{\partial}\varphi \geq 0$ as a $(1, 1)$-form on $\widetilde{D}$.

Fix $x_0 \in X$ and let $D = \widetilde{D} \times (-\delta, \delta) \subset \mathbb{C}^{n-1} \times \mathbb{R}$ be a canonical local patch with canonical coordinates (z, θ, φ) such that (z, θ, φ) is trivial at x_0. The T-rigid Hermitian metric on D

induces an Hermitian metric on $T^{*1,0}$ in a standard way. Up to a coordinate transformation if necessary, we can find orthonormal frame $\{e^j\}_{j=1}^{n-1}$ of $T^{*1,0}$ with respect to the fixed T-rigid Hermitian metric such that on D, we have $e^j(x) = e^j(z) = dz_j + O(|z|)$, $\forall x = (z,\theta) \in D$, $j = 1, \cdots, n-1$. Moreover, if we denote by dv_X the volume form with respect to the T-rigid Hermitian metric on $\mathbb{C}TX$, then on D we have $dv_X = \lambda(z)dv(z)d\theta$ with $\lambda(z) \in \mathcal{C}^\infty(\widetilde{D}, \mathbb{R})$ which does not depend on θ and $dv(z) = 2^{n-1}dx_1 \cdots dx_{2n-2}$. We identify $\widetilde{D}$ with an open subset of $\mathbb{C}^{n-1}$ with complex coordinates $z = (z_1, \cdots, z_{n-1})$. Since $\{dz_j\}_{j=1}^{n-1}$ is a frame of $T^{*1,0}D$ over D, we will treat them as the frame of $T^{*1,0}\widetilde{D}$ which is the bundle of $(1,0)$-forms over the domain $\widetilde{D}$. Let $(g^{\bar{k}j}(z))$ be the induced Hermitian metric on $T^{*1,0}\widetilde{D}$. It induces Hermitian metrics on $T^{1,0}\widetilde{D}$ and $T^{*,p,q}\widetilde{D}$ in a canonical way. We denote by the induced Hermitian metric on $T^{1,0}\widetilde{D}$ by ω. Moreover, on $\widetilde{D}$, $\omega = \sum_{j=1}^{n-1} e^j \wedge \overline{e}^j$, and $\omega(x_0) = \sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j$. Then the volume form on $\widetilde{D}$ is given by $\omega^{n-1} := \frac{\omega^{n-1}}{(n-1)!} = \lambda(z)dv(z)$. For the detailed discussions, we refer to [6].

5 Local representations of $\overline{\partial}_b$, $\overline{\partial}_b^*$ and $\Box_{b,m}^{(p,q)}$

Fix $x_0 \in X$ and choose canonical local patch D near x_0 with canonical coordinate (z, θ, φ) such that (z, θ, φ) is trivial at x_0. Write $D = \tilde{D} \times (-\delta, \delta)$, $\tilde{D} = \{z \in \mathbb{C}^{n-1} : |z| < \varepsilon\}$. In this section, we always see $\tilde{D}$ as an open subset of $\mathbb{C}^{n-1}$ with the complex coordinates $z = (z_1, \cdots, z_{n-1})$. We choose the fixed Hermitian metric on $T^{*0,1}\tilde{D}$ induced by a T-rigid Hermitian metric $\langle\cdot|\cdot\rangle$ on D, and extend it to $T^{*p,q}\tilde{D}$. We still use the notation $\langle\cdot|\cdot\rangle$ to denote the Hermitian metric on $T^{*p,q}\tilde{D}$. Let $u \in \Omega_m^{p,q}(X, L)$. Let e be a local rigid CR frame of L on D, and Φ be the local weight of the Hermitian metric h_L of L. From the definition of $\Omega_m^{p,q}(X, L)$ we have that $Tu = imu$. Then on D, $u = \tilde{u}e^{im\theta} \otimes e$ with $\tilde{u}(z) \in \Omega^{p,q}(\tilde{D})$ and $\tilde{u}(z) = \sum'_{|I|=p,|J|=q} \tilde{u}_{IJ} dz_I \wedge d\bar{z}_J$. Since h_L is S^1-equivariant, we have $T\Phi = 0$, i.e. $\Phi(z,\theta) = \Phi(z)$ is independent of θ on D.

Similar with [6, Lemma 2.11], we have the following local representation of the operators mentioned above.

Lemma 5.1 *For all $u \in \Omega_m^{p,q}(X, L)$, on D we have*

$$\overline{\partial}_b u = e^{im\theta}e^{-m\varphi}\overline{\partial}(e^{m\varphi}e^{-im\theta}u),\ \overline{\partial}_b^* u = e^{im\theta}e^{-m\varphi}\overline{\partial}^{*,2m\varphi+\Phi}(e^{m\varphi}e^{-im\theta}u)$$
$$\Box_{b,m}^{(p,q)}u = e^{im\theta}e^{-m\varphi}\Box_{2m\varphi+\Phi}^{(p,q)}(e^{m\varphi}e^{-im\theta}u).$$

Proof The proof is a minor modification of that in the proof of [6, Lemma 2.11]. For the sake of completeness, we include the proof here.

Write $u = \sum'_{|I|=p,|J|=q} u_{IJ} dz_I \wedge d\bar{z}_J \otimes e$. Then $\bar{\partial}_b u = \sum_j \sum'_{IJ} \left(\frac{\partial u_{IJ}}{\partial \bar{z}_j} - i\frac{\partial\varphi(z)}{\partial\bar{z}_j}\frac{\partial u_{IJ}}{\partial\theta}\right) d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J \otimes e$. Since $Tu = imu$, we have $\frac{\partial u_{IJ}}{\partial\theta} = imu_{IJ}$ on D for every I, J. Thus

$$\begin{aligned}\bar{\partial}_b u &= \sum_{|I|=p,|J|=q}' \sum_{j=1}^{n-1} \left(\frac{\partial u_{IJ}}{\partial\bar{z}_j} + m\frac{\partial\varphi(z)}{\partial\bar{z}_j}u_{IJ}\right) d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J \otimes e \\ &= e^{im\theta} \sum_{|I|=p,|J|=q}' \sum_{j=1}^{n-1} \left(\frac{\partial\tilde{u}_{IJ}}{\partial\bar{z}_j} + m\frac{\partial\varphi(z)}{\partial\bar{z}_j}\tilde{u}_{IJ}\right) d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J \otimes e. \qquad (4)\end{aligned}$$

Set $v(z) := e^{m\varphi}\tilde{u}(z) = \sum'_{|I|=p,|J|=q} v_{I,J}(z)dz_I \wedge d\bar{z}_J \otimes e$. Then

$$\frac{\partial v_{I,J}}{\partial \bar{z}_j} = \frac{\partial}{\partial \bar{z}_j}\left(\tilde{u}_{IJ}e^{m\varphi}\right) = e^{m\varphi}\left(\frac{\partial \tilde{u}_{IJ}}{\partial \bar{z}_j} + m\frac{\partial \varphi(z)}{\partial \bar{z}_j}\tilde{u}_{IJ}\right). \tag{5}$$

Substituting (5) to (4), we can get the first identity of Lemma 5.1.

Since $\bar{\partial}_b^* u \in \Omega_m^{p,q-1}(X, L)$ on D, we write $\bar{\partial}_b^* u = e^{im\theta}\tilde{v}(z) \otimes e$, $\tilde{v}(z) \in \Omega^{p,q-1}(\tilde{D})$. Take $\chi(\theta) \in \mathcal{C}_0^\infty((-\delta,\delta))$ with $\int_{-\delta}^{\delta}\chi(\theta)d\theta = 1$. Let $g \in \Omega_0^{p,q-1}(\tilde{D})$. We have

$$\begin{aligned}&(\bar{\partial}_b^* u|e^{-2m\varphi(z)}g(z)\chi(\theta)e^{im\theta} \otimes e) = (e^{im\theta}\tilde{v}(z) \otimes e|e^{-2m\varphi(z)}g(z)\chi(\theta)e^{im\theta} \otimes e)\\ &\quad= (\tilde{v}(z)|g(z))_{2m\varphi+\Phi}.\end{aligned} \tag{6}$$

On the other hand, from the first identity of Lemma 5.1, we have

$$\begin{aligned}&(\bar{\partial}_b^* u|e^{-2m\varphi(z)}g(z)\chi(\theta)e^{im\theta} \otimes e) = (u|\bar{\partial}_b(e^{-2m\varphi(z)}g(z)\chi(\theta)e^{im\theta}) \otimes e)\\ &\quad= (u|\chi(\theta)e^{im\theta}e^{-m\varphi(z)}\bar{\partial}(e^{-m\varphi}g(z)) \otimes e) + (u|(-i)\chi'(\theta)e^{im\theta}e^{-2m\varphi(z)}\bar{\partial}\varphi \wedge g(z) \otimes e)\\ &\quad= (e^{m\varphi}\tilde{u}(z)|\bar{\partial}(e^{-m\varphi(z)}g(z)))_{2m\varphi+\Phi} = (\bar{\partial}^{*,2m\varphi+\Phi}(e^{m\varphi(z)}\tilde{u})|e^{-m\varphi(z)}g(z))_{2m\varphi+\Phi}\\ &\quad= (e^{-m\varphi(z)}\bar{\partial}^{*,2m\varphi+\Phi}(e^{m\varphi(z)}\tilde{u})|g(z))_{2m\varphi+\Phi}.\end{aligned} \tag{7}$$

Combining (6) and (7), we obtain that

$$(\tilde{v}(z)|g(z))_{2m\varphi+\Phi} = (e^{-m\varphi(z)}\bar{\partial}^{*,2m\varphi+\Phi}(e^{m\varphi(z)}\tilde{u})|g(z))_{2m\varphi+\Phi}.$$

This gives the second identity in Lemma 5.1, and the third identity follows directly from the the above two identities.

□

Based on Lemma 5.1, we go a little bit further by direct computations to get the following

Lemma 5.2 *Suppose that $u \in \Omega_m^{p,q}(X, L)$ satisfies $\Box_{b,m}^{(p,q)} u = \lambda u$. Let e be a local CR rigid frame of L on D. We define $\tilde{u} \otimes e := e^{m\varphi}e^{-im\theta}u$, then $\tilde{u} \in \Omega_m^{p,q}(\tilde{D})$ and the following equality holds on $\tilde{D}$:*

$$\Box_{2m\varphi+\Phi}^{(p,q)}\tilde{u} = \lambda\tilde{u}.$$

Furthermore, for any $u \in \mathcal{H}_{b,m,\le\sigma}^{p,q}(X, L)$, we get a form $\tilde{u} \in \mathcal{H}_{2m\varphi+\Phi,\le\sigma}^{p,q}(\tilde{D}, L)$, where $\mathcal{H}_{b,m,\le\sigma}^{p,q}(X, L)$ (resp. $\mathcal{H}_{2m\varphi+\Phi,\le\sigma}^{p,q}(\tilde{D}, L)$) is the linear span of the eigenforms of $\Box_{b,m}^{(p,q)}$ (resp. $\Box_{2m\varphi+\Phi}^{(p,q)}$) with eigenvalue less than or equal to σ on X (resp. on $\tilde{D}$).

Now we recall the so-called Siu's $\partial\bar{\partial}$-formula. Let (L, h) be a holomorphic Hermitian line bundle over a compact complex n-fold (X, ω) and α be a L-valued (n, q)-form. The Hodge-$*$ operator is defined by the formula

$$\alpha \wedge \overline{*\alpha} = |\alpha|^2\omega_n, \tag{8}$$

where $\omega_n = \omega^n/n!$. We define an $(n-q, n-q)$-form T_α associated to α in a local trivialization as

$$T_\alpha = c_{n-q}\gamma \wedge \overline{\gamma}e^{-\psi}, \tag{9}$$

where $\gamma = *\alpha$, $c_{n-q} = i^{(n-q)^2}$ and ψ defines the metric of L. Note that the form T_α is well defined globally.

Lemma 5.3 (c.f. [2]) *Let α be an L-valued (n,q)-form. If α is $\overline{\partial}$-closed*

$$i\partial\overline{\partial}(T_\alpha\wedge\omega_{q-1})\geq\big(-2Re\langle\Box\alpha,\alpha\rangle+\langle\Theta_L\wedge\Lambda\alpha,\alpha\rangle-c|\alpha|^2\big)\omega_n,$$

where Θ_L is the curvature of (L,h) and locally Θ_L can be written as $\Theta_L=i\partial\overline{\partial}\psi$ if ψ is the local potential of h, i.e. $h=e^{-\psi}$. The constant c is equal to zero if $\overline{\partial}\omega_{q-1}=\overline{\partial}\omega_q=0$.

6 A preparation for localization procedure: the scaling technique

In this section, we prepare necessary tools for the localization precedure needed in the proof of Theorem 1.4. This is a modification of results in [6].

Fix $x_0\in X$, we take canonical local patch $D=\widetilde{D}\times(-\delta,\delta)=\{(z,\theta):|z|<\varepsilon,|\theta|<\delta\}$ with canonical coordinates (z,θ,φ) such that (z,θ,φ) is trivial at x_0. In this section, we identify $\widetilde{D}$ with an open subset of $\mathbb{C}^{n-1}=\mathbb{R}^{2n-2}$ with complex coordinates $z=(z_1,\cdots,z_{n-1})$. Let $L_1\in T^{1,0}\widetilde{D},\cdots,L_{n-1}\in T^{1,0}\widetilde{D}$ be the dual frame of $e^1,\cdots,e^{n-1}$ with respect to the T-rigid Hermitian metric $\langle\cdot|\cdot\rangle$. Let ω be the induced Hermitian metric on $T^{1,0}\widetilde{D}$.

Let $\Omega^{p,q}(\widetilde{D})$ be the space of smooth (p,q)-forms on $\widetilde{D}$ and let $\Omega_0^{p,q}(\widetilde{D})$ be the subspace of $\Omega^{p,q}(\widetilde{D})$ whose elements have compact support in $\widetilde{D}$. Let $\big(\cdot\,|\,\cdot\big)_{2\varphi}$ be the weighted inner product on the space $\Omega_0^{p,q}(\widetilde{D})$ defined as follows:

$$\big(f|g\big)=\int_{\widetilde{D}}\langle f|g\rangle e^{-2\varphi(z)}\lambda(z)dv(z)$$

where $f,g\in\Omega_0^{p,q}(\widetilde{D})$. We denote by $L^2_{(p,q)}(\widetilde{D},2\varphi)$ the completion of $\Omega_0^{p,q}(\widetilde{D})$ with respect to $\big(\cdot\,|\,\cdot\big)_{2\varphi}$. For $r>0$, let $\widetilde{D}_r=\{z\in\mathbb{C}^{n-1}:|z|<r\}$. Here $\{z\in\mathbb{C}^{n-1}:|z|<r\}$ means that $\{z\in\mathbb{C}^{n-1}:|z_j|<r,j=1,\cdots,n-1\}$. For $m\in\mathbb{N}$, let F_m be the scaling map $F_m(z)=\big(\frac{z_1}{\sqrt{m}},\cdots,\frac{z_{n-1}}{\sqrt{m}}\big)$, $z\in\widetilde{D}_{\log m}$. From now on, we assume m is sufficiently large such that $F_m(\widetilde{D}_{\log m})\subset\subset\widetilde{D}$. We define the scaled bundle $F_m^*T^{*p,q}\widetilde{D}$ on $\widetilde{D}_{\log m}$ to be the bundle whose fiber at $z\in\widetilde{D}_{\log m}$ is

$$F_m^*T^{*p,q}\widetilde{D}|_z=\Big\{\sum_{|I|=p,|J|=q}{}^{'}a_{IJ}e^I\big(\frac{z}{\sqrt{m}}\big)\wedge\overline{e^J}\big(\frac{z}{\sqrt{m}}\big):\\ a_{IJ}\in\mathbb{C},I,J\ \text{ strictly increasing}\Big\}.$$

We take the Hermitian metric $\langle\cdot|\cdot\rangle_{F_m^*}$ on $F_m^*T^{*p,q}\widetilde{D}$ so that at each point $z\in\widetilde{D}_{\log m}$,

$$\Big\{e^I\big(\frac{z}{\sqrt{m}}\big)\wedge\overline{e^J}\big(\frac{z}{\sqrt{m}}\big):|I|=p,|J|=q,I,J\ \text{ strictly increasing}\Big\}$$

is an orthonormal frame for $F_m^*T^{*p,q}\widetilde{D}$ on $\widetilde{D}_{\log m}$.

Let $F_m^*\Omega_0^{p,q}(\widetilde{D}_r)$ denote the space of smooth sections of $F_m^*\Omega^{p,q}(\widetilde{D}_r)$ whose elements have compact support in $\widetilde{D}_r$. Given $f\in\Omega^{p,q}(\widetilde{D}_r)$. We write $f=\sum'_{|J|=q}f_{IJ}e^I\wedge\overline{e^J}$. We define the scaled form $F_m^*f\in F^*\Omega^{p,q}(\widetilde{D}_{\log m})$ by

$$F_m^*f(z)=\sum_{IJ}{}^{'}f_{IJ}\big(\frac{z}{\sqrt{m}}\big)e^I\big(\frac{z}{\sqrt{m}}\big)\wedge\overline{e^J}\big(\frac{z}{\sqrt{m}}\big),\ z\in\widetilde{D}_{\log m}.$$

For brevity, we denote $F_m^*f(z)$ by $f(\frac{z}{\sqrt{m}})$. Let P be a partial differential operator of order one on $F_m\widetilde{D}_{\log m}$ with $\mathcal{C}^\infty$ coefficients. We write $P=\sum_{j=1}^{2n-2}a_j(z)\frac{\partial}{\partial x_j}$. The scaled partial differential operator $P_{(m)}$ on $\widetilde{D}_{\log m}$ is given by $P_{(m)} = \sum_{j=1}^{2n-2} F_m^* a_j \frac{\partial}{\partial x_j}$. Let $f \in \mathcal{C}^\infty(F_m(\widetilde{D}_{\log m}))$. We can check that

$$P_{(m)}(F_m^* f) = \frac{1}{\sqrt{m}} F_m^*(Pf). \tag{10}$$

Let $\overline{\partial} : \Omega^{p,q}(\widetilde{D}) \to \Omega^{p,q+1}(\widetilde{D})$ be the Cauchy-Riemann operator and we have

$$\overline{\partial} = \sum_{j=1}^{n-1} \overline{e^j}(z) \wedge \overline{L}_j + \sum_{j=1}^{n-1} (\overline{\partial}\overline{e^j})(z) \wedge (\overline{e^j}(z)\wedge)^*$$

where $(\overline{e^j}(z)\wedge)^* : T^{*p,q}\widetilde{D} \to T^{*p,q-1}\widetilde{D}$ is the adjoint of $\overline{e^j}(z)\wedge$ with respect to the Hermitian metric $\langle\cdot|\cdot\rangle$ on $T^{*p,q}\widetilde{D}$, $j = 1, \cdots, n-1$. That is

$$\langle e^j(z) \wedge u | v\rangle = \langle u | (e^j(z)\wedge)^* v\rangle$$

for all $u \in T^{*p,q-1}\widetilde{D}, v \in T^{*p,q}\widetilde{D}$. The scaled differential operator $\overline{\partial}_{(m)} : F_m^*\Omega^{p,q}(\widetilde{D}_{\log m}) \to F_m^*\Omega^{p,q+1}(\widetilde{D}_{\log m})$ is given by

$$\overline{\partial}_{(m)} = \sum_{j=1}^{n-1} \overline{e^j}(\frac{z}{\sqrt{m}}) \wedge \overline{L}_{j,(m)} + \sum_{j=1}^{n-1} \frac{1}{\sqrt{m}} (\overline{\partial}\overline{e^j})(\frac{z}{\sqrt{m}}) \wedge (\overline{e^j}(\frac{z}{\sqrt{m}}))^*. \tag{11}$$

Similarly, $(\overline{e^j}(\frac{z}{\sqrt{m}})\wedge)^* : F_m^* T^{*p,q}\widetilde{D} \to F_m^* T^{*p,q-1}\widetilde{D}$ is the adjoint of $\overline{e^j}(\frac{z}{\sqrt{m}})\wedge$ with respect to $\langle\cdot|\cdot\rangle_{F_m^*}$, $j = 1, \cdots, n-1$. From (10) and (11), $\overline{\partial}_{(m)}$ satisfies that

$$\overline{\partial}_{(m)} F_m^* f = \frac{1}{\sqrt{m}} F_m^*(\overline{\partial} f),\ \forall f \in \Omega^{p,q}(F_m(\widetilde{D}_{\log m})).$$

Let $\big(\cdot|\cdot\big)_{2mF_m^*\varphi + F_m^*\Phi}$ be the weighted inner product on the space $F_m^*\Omega_0^{p,q}(\widetilde{D}_{\log m})$ defined as follows:

$$\big(f|g\big)_{2mF_m^*\varphi + F_m^*\Phi} = \int_{\widetilde{D}_{\log m}} \langle f|g\rangle_{F_m^*} e^{-2mF_m^*\varphi - F_m^*\Phi} \lambda(\frac{z}{\sqrt{m}}) dv(z).$$

Let $\overline{\partial}^*_{(m)} : F_m^*\Omega^{p,q+1}(\widetilde{D}_{\log m}) \to F_m^*\Omega^{p,q}(\widetilde{D}_{\log m})$ be the formal adjoint of $\overline{\partial}_{(m)}$ with respect to the weighted inner product $\big(\cdot|\cdot\big)_{2mF_m^*\varphi + F_m^*\Phi}$. Let $\overline{\partial}^{*,2m\varphi+\Phi} : \Omega^{p,q+1}\widetilde{D} \to \Omega^{p,q}(\widetilde{D})$ be the formal adjoint of $\overline{\partial}$ with respect to the weighted inner product $\big(\cdot|\cdot\big)_{2m\varphi+\Phi}$. Then we also have

$$\overline{\partial}^*_{(m)} F_m^* f = \frac{1}{\sqrt{m}} F_m^*(\overline{\partial}^{*,2m\varphi+\Phi} f),\ \forall f \in \Omega^{p,q}(F_m(\widetilde{D}_{\log m})).$$

We now define the scaled complex Laplacian $\Box^{(p,q)}_{(m)} : F_m^*\Omega^{p,q}(\widetilde{D}_{\log m}) \to F_m^*\Omega^{p,q}(\widetilde{D}_{\log m})$ which is given by $\Box^{(p,q)}_{(m)} = \overline{\partial}^*_{(m)}\overline{\partial}_{(m)} + \overline{\partial}_{(m)}\overline{\partial}^*_{(m)}$. Then we can see that

$$\Box^{(p,q)}_{(m)} F_m^* f = \frac{1}{m} F_m^*(\Box^{(p,q)}_{2m\varphi+\Phi} f),\ \forall f \in \Omega^{p,q}(F_m(\widetilde{D}_{\log m})). \tag{12}$$

Here

$$\Box^{(p,q)}_{2m\varphi+\Phi} = \overline{\partial}\overline{\partial}^{*,2m\varphi+\Phi} + \overline{\partial}^{*,2m\varphi+\Phi}\overline{\partial} : \Omega^{p,q}(\widetilde{D}) \to \Omega^{p,q}(\widetilde{D})$$

is the complex Laplacian with respect to the given Hermitian metric on $T^{*p,q}(\widetilde{D})$ and weight function $2m\varphi(z)+\Phi$ on $\widetilde{D}$.

Since $2mF_m^*\varphi = 2\Phi_0(z)+\frac{1}{\sqrt{m}}O(|z|^3)$ and $F_m^*\Phi = \Phi(0)+\frac{1}{\sqrt{m}}O(|z|)$, $\forall z \in \widetilde{D}_{\log m}$, where $\Phi_0(z)=\sum_{j=1}^{n-1}\lambda_j|z_j|^2$, we have

$$\lim_{m\to\infty}\sup_{\widetilde{D}_{\log m}} |\partial_z^\alpha(2mF_m^*\varphi+F_m^*\Phi-2\Phi_0)|=0,\ \forall\alpha\in\mathbb{N}_0^{2n-2}.$$

Consider $\mathbb{C}^{n-1}$. Let $\langle\cdot|\cdot\rangle_{\mathbb{C}^{n-1}}$ be the Hermitian metric with constant coefficients on $T^{*p,q}\mathbb{C}^{n-1}$, such that at the origin, it is equal to $\omega(0)$. Let $(\,\cdot\mid\cdot\,)_{2\Phi_0}$ be the L^2 inner product on $\Omega_0^{p,q}(\mathbb{C}^{n-1})$ given by

$$(f|g)_{2\Phi_0}=\int_{\mathbb{C}^{n-1}}\langle f|g\rangle e^{-2\Phi_0(z)}\lambda(0)dv(z),\ f,g\in\Omega_0^{p,q}(\mathbb{C}^{n-1}),$$

where $\lambda(0)$ is the value of the function $\lambda(z)$ at x_0.

Put

$$\Box_{2\Phi_0}^{(p,q)}=\overline{\partial}\,\overline{\partial}^{*,2\Phi_0}+\overline{\partial}^{*,2\Phi_0}\overline{\partial}:\Omega^{p,q}(\mathbb{C}^{n-1})\to\Omega^{p,q}(\mathbb{C}^{n-1}),\tag{13}$$

where $\overline{\partial}^{*,2\Phi_0}$ is the formal adjoint of $\overline{\partial}$ with respect to $(\,\cdot\mid\cdot\,)_{2\Phi_0}$.

It is not difficult to check that

$$\Box_{(m)}^{(p,q)}=\Box_{2\Phi_0}^{p,q}+\varepsilon_m\mathcal{P}_m\tag{14}$$

on $\widetilde{D}_{\log m}$, where $\mathcal{P}$ is a second order partial differential operator and all the coefficients of $\mathcal{P}_m$ are uniformly bounded with respect to m in $\mathcal{C}^\mu(\widetilde{D}_{\log m})$-norm for every $\mu\in\mathbb{N}_0$ and ε_m is a sequence tending to zero as $m\to\infty$.

From Gårding's inequality together with Sobolev estimates for elliptic operator $\Box_{(m)}^{(p,q)}$, one can get the following

Proposition 6.1 (c.f. [2]) *Let $u\in F_m^*\Omega^{p,q}(\widetilde{D}_{\log m})$. For every $r>0$ with $\widetilde{D}_r\subset\subset\widetilde{D}_{\log m}$, and every $k\in\mathbb{N}^+$ and $k>\frac{n-1}{2}$, there is a constant $C_{r,k}$ independent of m such that*

$$|u(0)|^2\le C_{r,k}\Big(\|u\|^2_{2mF_m^*\varphi+F_m^*\Phi,\widetilde{D}_r}+\|(\Box_{(m)}^{(p,q)})^k u\|_{2mF_m^*\varphi+F_m^*\Phi,\widetilde{D}_r}\Big).$$

7 Proof of the Theorem 1.4

The proof is a modification of that in [14], which is an adaption of Berndtsson's technique [2] to CR setting.

Proof of Theorem 1.4 **Step 1.** Fix a point $x_0\in X$. From Sect. 4, up to a coordinate transformation, we can choose a canonical local patch $D=\widetilde{D}\times(-\delta,\delta)=\{(z,\theta):|z|<\varepsilon,|\theta|<\delta\}$ with canonical coordinates (z,θ,φ) such that (z,θ,φ) is trivial at x_0 and the metric ω induced by the T-rigid Hermitian metric on X be the Hermitian metric satisfies $\omega=\frac{i}{2}\partial\overline{\partial}|z|^2=:\beta$ at x_0. Let e be a local CR rigid frame of L on D, and Φ be the local weight of the Hermitian metric h_L of L. Since h_L is rigid, we have $T\Phi=0$, i.e. $\Phi(z,\theta)=\Phi(z)$ is independent of θ on D. Let $u\in\mathcal{H}^{n-1,q}_{b,m,\le\lambda}(X,L)$ such that $\|u\|=1$ and $\overline{\partial}_b u=0$. Set $\widetilde{u}\otimes e=e^{m\varphi}e^{-im\theta}u$ on

$\widetilde{D}$, then from Lemma 5.1 and Lemma 5.2, we know that $\widetilde{u} \in \mathcal{H}^{n-1,q}_{2m\varphi,\leq\lambda}(\widetilde{D})$ and $\overline{\partial}\widetilde{u} = 0$. By the definition and Lemma 5.1, it is easy to show that

$$|u|^2 = |\widetilde{u}|^2 e^{-2m\varphi-\Phi}, \tag{15}$$

$$|\Box^{(n-1,q)}_{b,m} u|^2 = |\Box^{(n-1,q)}_{2m\varphi+\Phi}\widetilde{u}|^2 e^{-2m\varphi-\Phi}. \tag{16}$$

□

The aim of this step is to generalize Berndtsson's submean value inequality [2, Theorem 2.1] to the CR setting.

Theorem 7.1 *Under above notations, and under the assumption of Theroem 1.1, we have that for* $r < \lambda^{-1/2}$ *and* $r < c_0$*,*

$$\int_{|z|<r} \left[\widetilde{u}\right]^2 \omega_{n-1} \leq C r^{2q} (\lambda + 1)^q \int_X |u|^2.$$

The constant c_0 *and* C *are independent of* m, λ *amd the point* x_0.

To proceed, we construct a trivial holomorphic Hermitian line bundle ($L := \widetilde{D} \times \mathbb{C}, h := e^{-2m\varphi-\Phi}$) over $\widetilde{D}$. From (15) and (16), one can identify $\widetilde{u}$ with an L-valued $(n-1, q)$ form on $\widetilde{D}$, i.e. a section of the bundle $\Omega^{n-1,q} \otimes L$ over $\widetilde{D}$, and $\Box^{(n-1,q)}_{2m\varphi+\Phi}$ with the formal $\overline{\partial}$-Laplacian operator on $\widetilde{D}$ with respect to the induced Hermitian metric ω and the Hermitian metric h_L of L on $\widetilde{D}$. For this consideration, we make the following notations throughout this section

$$\left[\widetilde{u}\right]^2 := |\widetilde{u}|^2 e^{-2m\varphi-\Phi} \tag{17}$$

$$\left[\Box^{(n-1,q)}_{2m\varphi+\Phi}\widetilde{u}\right]^2 := |\Box^{(n-1,q)}_{2m\varphi+\Phi}\widetilde{u}|^2 e^{-2m\varphi-\Phi}. \tag{18}$$

Since X is pseudoconvex, then from [14, Proposition 4.2] we have that $i\Theta_L = i\partial\overline{\partial}\varphi \geq 0$. From Lemma 5.3, we get that

$$i\partial\overline{\partial}(T_{\widetilde{u}} \wedge \omega_{q-1}) \geq (-2Re\langle\Box^{n-1,q}_{2m\varphi+\Phi}\widetilde{u}, \widetilde{u}\rangle - c\left[\widetilde{u}\right]^2\rangle)\omega_{n-1}. \tag{19}$$

For $r > 0$ small, we define

$$\sigma(r) := \int_{|z|<r} \left[\widetilde{u}\right]^2 \omega_{n-1} = \int_{|z|<r} T_{\widetilde{u}} \wedge \omega_q =: s^2(r),$$

$$\lambda(r) := \Big(\int_{|z|<r} \left[\Box^{n-1,q}_{2m\varphi+\Phi}\widetilde{u}\right]^2\Big)^{1/2}.$$

From Cauchy's inequality, we get that

$$\int_{|z|<r} \left[\Box^{n-1,q}_{2m\varphi+\Phi}\widetilde{u}\right]\left[\widetilde{u}\right] \leq \lambda(r)\sigma(r)^{1/2}.$$

Without loss of generality, we may assume that $\lambda \geq 1$.
From (19) we see that

$$\int_{|z|<r} (r^2 - |z|^2) i\partial\overline{\partial}(T_{\widetilde{u}} \wedge \omega_{q-1}) \geq -cr^2\sigma(r) - 2r^2 \int_{|z|<r} \left[\Box^{n-1,q}_{2m\varphi+\Phi}\widetilde{u}\right]\left[\widetilde{u}\right]\omega_{n-1}. \tag{20}$$

Applying Stokes' formula to the left hand side of (20), we get that

$$2\int_{|z|<r} iT_{\widetilde{u}} \wedge \omega_{q-1} \wedge \beta$$

$$\leq \int_{|z|=r} iT_{\widetilde{u}} \wedge \omega_{q-1} \wedge \partial|z|^2 + cr^2\sigma(r) + 2r^2\sigma(r)^{1/2}\lambda(r). \tag{21}$$

Since ω is smooth and $\omega(0) = \beta$, up to shrinking the local patch if necessary, we have that

$$(1 - O(r))\omega \leq \beta \leq (1 - O(r))\omega. \tag{22}$$

Note that if $\omega = \beta$, the boundary term in (21) can be estimated by an integral with respect to surface measure

$$r \int_{|z|=r} [\widetilde{u}]^2 dS,$$

and this implies that in our case

$$\int_{|z|=r} iT_{\widetilde{u}} \wedge \omega_{q-1} \wedge \partial|z|^2 \leq r(1 - O(r)) \int_{|z|=r} \left[\widetilde{u}\right]^2 (\omega_{n-1}/\beta_{n-1}) dS. \tag{23}$$

However,

$$\int_{|z|=r} \left[\widetilde{u}\right]^2 (\omega_{n-1}/\beta_{n-1}) dS = \sigma'(r). \tag{24}$$

From (21), (22) and (24), by incorporating the term $cr^2\sigma(r)$ in $O(r)\sigma(r)$, we get that

$$2q(1 - O(r))\sigma(r) \leq r\sigma'(r) + 2r^2\sigma(r)^{1/2}\lambda(r). \tag{25}$$

Dividing by $2rs(r)$ to both sides of (25), we obtain

$$q(1/r - O(1))s(r) \leq s'(r) + r\lambda(r). \tag{26}$$

We are going to prove

$$s(r) \leq Cr^k\lambda^{k/2}$$

for $k \leq q$ by induction over k.

The statement is trivial for $k = 0$. In fact, from (15) and (17), we have that

$$\sigma(r) = \int_{|z|<r} \left[\widetilde{u}\right]^2 \omega_{n-1} = \frac{1}{2\delta} \int_{|z|<r, -\delta \leq \theta \leq \delta} |u|^2 dv_X \leq \frac{1}{2\delta},$$

since we have assumed that $\|u\| = 1$.

Now we assume that it has been proved for a certain value of $k < q$. Then (26) implies

$$(k + 1)(1/r - O(1))s(r) \leq s'(r) + r\lambda(r). \tag{27}$$

Since $\widetilde{u} \in \mathcal{H}^{n-1,q}_{2m\varphi+\Phi,\leq\lambda}(\widetilde{D})$, the form $\Box^{n-1,q}_{2m\varphi+\Phi}\widetilde{u}$ also lies in $\mathcal{H}^{n-1,q}_{2m\varphi+\Phi,\leq\lambda}(\widetilde{D})$, then by the induction hypothesis we get that

$$\lambda(r) \leq Cr^k\lambda^{k/2+1}. \tag{28}$$

From (27) and (28), we obtain that

$$(k + 1)(1/r - O(1))s(r) \leq s'(r) + Cr^{k+1}\lambda^{k/2+1}. \tag{29}$$

Set

$$\Psi(r) = (k + 1) \int (1/r - O(1)) dr \sim (k + 1) \log r$$

and multiply (29) by the integrating factor $e^{-\Psi(r)}$. The result is that

$$(se^{-\Psi})' \geq -C\lambda^{k/2+1}.$$

Integrate this inequality from r to $\lambda^{-1/2}$. Since $e^{-\Psi} \sim 1/r^{k+1}$, we get that

$$r^{-(k+1)}s(r) \leq C\lambda^{k/2+1/2} + s(\lambda^{-1/2})\lambda^{k/2+1/2} \leq C\lambda^{k/2+1/2}.$$

By induction, we obtain that

$$s(r) \leq Cr^q\lambda^{q/2}.$$

After squaring both sides, we obtain that

$$\int_{|z|<r} \left[\widetilde{u}\right]^2 \omega_{n-1} \leq Cr^{2q}\lambda^q. \tag{30}$$

Go through the proof given above line by line, one can see that the constant C only depends on the local coordinates, c in Siu's formula (which depends only on the metric ω), $O(1)$ and δ, but from the compactness of X, one can get a uniform constant C independent of r, m, λ and the point x_0. The proof of Theorem 7.1 is complete.

Step 2. In the sequel, we shall use the scaling technique in Sect. 6.

For any form $u \in \Omega^{n-1,q}_{b,m}(X, L)$, we express u in terms of the trivialization and local canonical coordinates on D and write $\widetilde{u} \otimes e = e^{m\varphi}e^{-im\theta}u$ on $\widetilde{D}$ as in Step 1. Firstly we assume that $\lambda \leq m$. Put

$$\widetilde{u}^{(m)}(z) = F_m^*\widetilde{u}(z) = \widetilde{u}\left(\frac{z}{\sqrt{m}}\right),$$

so that $\widetilde{u}^{(m)}$ is defined for $|z| < 1$ if m is large enough.

We also have the scaled Laplacian $\Box^{(n-1,q)}_{(m)}$, and from (12), it satisfies

$$m\Box^{(n-1,q)}_{(m)}\widetilde{u}^{(m)} = F_m^*(\Box^{(n-1,q)}_{2m\varphi+\Phi}\widetilde{u}) =: (\Box^{(n-1,q)}_{2m\varphi+\Phi}\widetilde{u})^{(m)}.$$

From (14), $\Box^{(n-1,q)}_{(m)}$ converges to a m-independent elliptic operator as $m \to \infty$ on a neighborhood of $|z| \leq 1$.

Therefore, from Proposition 6.1, we obtain that

$$|u(0)|^2 = [\widetilde{u}]^2(0) \leq C_k\Big(\int_{|z|<1} \left[\widetilde{u}^{(m)}\right]^2 \omega^{(m)}_{n-1} + \int_{|z|<1} \left[\Box^{(n-1,q)}_{(m)})^k\widetilde{u}^{(m)}\right]^2 \omega^{(m)}_{n-1}\Big), \tag{31}$$

for m sufficiently large and $k > \frac{n-1}{2}$, where $C_{r,k}$ in Proposition 6.1 depends on r and k, but here $C_{r,k} = C_{1,k} =: C_k$ only depends on k since $r = 1$ in (31).

By coordinate transformation formula, we have that

$$\int_{|z|<1} \left[\widetilde{u}^{(m)}\right]^2 \omega^{(m)}_{n-1} = m^{n-1}\int_{|z|<\frac{1}{\sqrt{m}}} \left[\widetilde{u}\right]^2 \omega_{n-1},$$

and

$$\int_{|z|<1} \left[\Box^{(n-1,q)}_{(m)})^k\widetilde{u}^{(m)}\right]^2 \omega^{(m)}_{n-1} = m^{n-1-2k}\int_{|z|<\frac{1}{\sqrt{m}}} \left[(\Box^{n-1,q}_{2m\varphi+\Phi})^k\widetilde{u}\right]^2 \omega_{n-1}.$$

From (30) in Step 1, we get that

$$m^{n-1}\int_{|z|<\frac{1}{\sqrt{m}}} \left[\widetilde{u}\right]^2 \omega_{n-1} \leq Cm^{n-1-q}(\lambda+1)^q, \tag{32}$$

and

$$m^{n-1-2k}\int_{|z|<\frac{1}{\sqrt{m}}}\left[(\Box_{2m\varphi+\Phi}^{n-1,q})^k\widetilde{u}\right]^2\omega_{n-1}\le Cm^{n-1-q}(\lambda+1)^q(\lambda/m)^{2k}. \tag{33}$$

Combining (31), (32) and (33), we obtain that

$$|u(0)|^2\le Cm^{n-1-q}(\lambda+1)^q.$$

Secondly, if $\lambda\ge m$, we apply the above procedure to the scaling $\widetilde{u}^{(\lambda)}$ instead, and trivially get

$$|u(0)|^2\le C\lambda^{n-1}.$$

Step 3. Since $\overline{\partial}_b$ commutes with $\Box_{b,m}^{(p,q)}$, we have the following exact sequence

$$0\to\mathscr{Z}_{b,m,\le\lambda}^{n-1,q}\xrightarrow{inclusion}\mathcal{H}_{b,m,\le\lambda}^{n-1,q}\xrightarrow{\overline{\partial}_b}\mathscr{Z}_{b,m,\le\lambda}^{n-1,q+1}.$$

Thus we obtain that

$$\dim\mathcal{H}_{b,m,\le\lambda}^{n-1,q}\le\dim\mathscr{Z}_{b,m,\le\lambda}^{n-1,q}+\dim\mathscr{Z}_{b,m,\le\lambda}^{n-1,q+1}. \tag{34}$$

From Lemma 3.2, we see that, for any $y\in X$

$$\dim\mathscr{Z}_{b,m,\le\lambda}^{n-1,q}\le\binom{n-1}{p}\binom{n-1}{q}\int_X S_{m,\le\lambda}^{n-1,q}(y)dv_X\le Cm^{n-1-q}(\lambda+1)^q \tag{35}$$

with $\lambda\le m$, and

$$\dim\mathscr{Z}_{b,m,\le\lambda}^{n-1,q}\le\binom{n-1}{p}\binom{n-1}{q}\int_X S_{m,\le\lambda}^{n-1,q}(y)dv_X\le C\lambda^{n-1} \tag{36}$$

with $\lambda\ge m$.

From (34), (35) and (36), we obtain that for $\lambda\le m$

$$\begin{aligned}\dim\mathcal{H}_{b,m,\le\lambda}^{n-1,q}&\le C\left(m^{n-1-q}(\lambda+1)^q+m^{n-2-q}(\lambda+1)^{q+1}\right)\\&\le Cm^{n-1-q}(\lambda+1)^q,\end{aligned}$$

and for $\lambda\ge m$,

$$\dim\mathcal{H}_{b,m,\le\lambda}^{n-1,q}\le C\lambda^n.$$

In conclusion, we complete the proof of the Theorem 1.4. □

Remark 7.1 Let $L=\det T^{1,0}X$, then $\Omega^{n-1,q}(X,\det T^{1,0}X)=\Omega^{0,q}(X)$, Corollary 1.5 is a direct consequence of Theorem 1.4, and Theorem 1.6 is a direct consequenc of Theorem 1.2 and Corollary 1.5.

In the following, we will show that the estimate of the growth order of $\dim H_{b,m}^{0,q}(X)$ in Corollary 1.5 and thus the estimate of the growth order of $\dim H_{b,-m}^{n-1,q}(X)$ in Theorem 1.6 are sharp. For $0\le q\le n-1$, let T_1 be an abelian variety of dimension $n-1-q$ and T_2 be a complex tori of dimension q. Let E be a strictly positive line bundle over T_1, and let L be the pull-back of E' to $M:=T_1\times T_2$. It is easy to see that

$$\dim H^{0,q}(M,L^m)\ge\dim H^{0,q}(T_2)\cdot\dim H^{0,0}(T_1,E^m)\ge cm^{n-1-q},$$

by noting that $H^{0,q}(T_2) \simeq H^{q,0}(T_2) = H^{0,0}(T_2, K_{T_2}) = \mathbb{C}$ (since K_{T_2} is trivial), and $\dim H^{0,0}(T_1, E^m) \geq cm^{n-1-q}$.

Let X be the circle bundle $\{v \in L^* : |v|^2_{h^{-1}} = 1\}$ over M. X is a real hypersurface in the complex manifold L^* which is the boundary of the disc bundle $D = \{v \in L^* : |v|^2_{h^{-1}} < 1\}$, with the defining function $\rho = |v|^2_{h^{-1}} - 1$. The Levi form of ρ restricted to the complex tangent plane of X coincides with the pull-back of Θ (i.e. the curvature of L which is semipositive) through the canonical projection $\pi : X \to M$. It is a well-known fact (e.g. see [14, Remark 3.1]) that

- the space $\Omega^{p,q}_m(X)$ can be identified with the space $\Omega^{p,q}(M, L^m)$,
- for each integer m, we get a subcomplex $(\Omega^{p,\bullet}_m(X), \overline{\partial}_b)$ which is isomorphic to the Dolbeault complex $(\Omega^{p,\bullet}(M, L^m), \overline{\partial})$, thus we get that the Kohn-Rossi cohomology group $H^{p,q}_{b,m}(X)$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(M, L^m)$.

Then $\dim H^{0,q}_{b,m}(X) \geq cm^{n-1-q}$, this shows that the estimate of the growth order of $\dim H^{0,q}_{b,m}(X)$ in Corollary 1.5, and thus the estimates of the growth order of $\dim H^{n-1,q}_{b,-m}(X)$ in Theorem 1.6 are sharp.

8 Serre type duality theorem for S^1-equivariant CR Hermitian vector bundles

Let X be a compact connected CR-manifold of real dimension $2n - 1$, $n \geq 2$, which admits a transversal CR S^1-action. Let (L, h_L) be an S^1-equivariant CR Hermitian vector bundle over X. Let ω_0 be the global 1-form associated to the S^1-action. In the following, we prove the Theorem 1.7 and Theorem 1.8. We follow the counterpart for complex manifold case in [10].

As in [14], we define the Hodge-$*$ operator in the CR level by the following

$$\langle u|v\rangle dv_X = u \wedge *\overline{v} \wedge \omega_0, \tag{37}$$

where $u, v \in \Omega^{p,q}_m(X)$, dv_X is the volume form on X defined in §4. Let $D = \widetilde{D} \times (-\delta, \delta)$ be a canonical local patch with canonical coordinates (z, θ, φ) such that z, θ, φ, and let $(f^1, \cdots, f^r)$ be a local rigid CR frame of L, and $(g_{j\bar{k}})$ be the Hermitian metric h_L of L on D. Let L^* be the dual bundle of L and $((f^1)^*, \cdots, (f^r)^*)$ be the dual rigid CR frame of $(f^1, \cdots, f^r)$ for L^* on D. Then $(g^{\bar{j}k})$ is the induced Hermitian metric of L^* on D. For any $u \in \Omega^{p,q}_m(X, L)$, we write $u = \sum u_j f^j$, where $u_j \in \Omega^{p,q}_m(D)$. Put

$$u^* = \sum g_{j\bar{k}} * \bar{u}_k (f^j)^*. \tag{38}$$

It ican be checked that $u^* \in \Omega^{n-p,n-q}_{-m}(X, L^*)$ and the definition of u^* is independend of the local frame.

From [14, Proposition 8.3], we know that $*$ is a complex linear map and $**u = (-1)^{p+q}u$, then one can derive that

$$u = (-1)^{p+q} \sum g^{\bar{j}k} * \overline{u^*_j} f^k$$

where $u^*_j = \sum g_{j\bar{k}} * \bar{u}_k$. It can be checked that this is also independent of the local frame. Thus the map $u \to u^*$ maps $\Omega^{p,q}_m(X, L)$ onto $\Omega^{n-p,n-q}_{-m}(X, L^*)$ bijectively, and this is a

$\mathbb{R}$-linear map. Furthermore, for $u \in \Omega_m^{p,q}(X, L)$ and $v^* \in \Omega_{-m}^{n-p,n-q}(X, L^*)$,

$$\sum_j v_j^* \wedge u_j \wedge \omega_0$$

is a globally defined volume form on X. Now we define an inner product of $u \in H_{b,m}^{p,q}(X, L)$ and $v^* \in H_{b,-m}^{n-p,n-q}(X, L^*)$ by

$$\langle v^*, u \rangle = \int_X \sum_j u_j \wedge v_j^* \wedge \omega_0. \tag{39}$$

From (37) and (38), one can see that

$$\langle v^*, u \rangle = \int_X \langle u|v \rangle dv_X = (u, v). \tag{40}$$

One can also check that

$$(u^*|v^*) = (u|v), \tag{41}$$

for any $u^*, v^* \in \Omega_{-m}^{n-p,n-q}(X, L^*)$.

For $u \in \Omega_m^{p,q-1}(X, L)$ and $v^* \in \Omega_{-m}^{n-1-p,n-1-q}(X, L^*)$, $\sum_j u_j \wedge v_j^*$ is a smooth $(n-1, n-2)$-form on X. Hence

$$\int_X \bar{\partial}_b(\sum u_j \wedge v_j^* \wedge \omega_0) = \int_M d(\sum_j u_j \wedge v_j^* \wedge \omega_0) = 0.$$

This implies that

$$\bar{\partial}_b(\sum_j u_j \wedge v_j^* \wedge \omega_0) = \sum_j \bar{\partial}_b u_j \wedge v_j^* \wedge \omega_0 - (-1)^{p+q} \sum_j u_j \wedge \bar{\partial}_b v_j^* \wedge \omega_0$$

on X, thus

$$\langle v^*, \bar{\partial}_b u \rangle = (-1)^{p+q} \langle \bar{\partial}_b v^*, u \rangle.$$

On the other hand, we have

$$\langle v^*, \bar{\partial}_b u \rangle = (\bar{\partial}_b u, v) = (u, \bar{\partial}_b^* v) = ((\bar{\partial}_b^* v)^*, u^*) = (\bar{\partial}_b^* v, u).$$

hence, we get that

$$\bar{\partial}_b v^* = (-1)^{p+q} (\bar{\partial}_b^* v)^* \tag{42}$$

for $v^* \in \Omega_{-m}^{n-1-p,n-1-q}(X, L^*)$.

Similarly, from

$$\begin{aligned}(v^*, \bar{\partial}_b^*(u^*)) &= (\bar{\partial}_b(v^*), u^*) = (-1)^{p+q}((\bar{\partial}_b^* v)^*, u^*) \\ &= (-1)^{p+q}(\bar{\partial}_b^* v, u) = (-1)^{p+q}(v^*, (\bar{\partial}_b u)^*),\end{aligned}$$

we have that

$$\bar{\partial}_b^*(u^*) = (-1)^{p+q}(\bar{\partial} u)^*$$

for $u \in \Omega_m^{p,q-1}(X, L)$, thus

$$\bar{\partial}_b^*(v^*) = (-1)^{p+q+1}(\bar{\partial} v)^* \tag{43}$$

for $v^* \in \Omega^{n-p,n-q}_{-m}(X, L^*)$. From (42) and (43), we get that v^* is harmonic if and only if v is harmonic. Thus $v \to v^*$ maps $\mathcal{H}^{p,q}_m(X, L)$ bijectively onto $\mathcal{H}^{n-p,n-q}_{-m}(X, L^*)$.

It is easy to see that the inner product (39) is not degenerate. Thus we complete the proof of Theorem 1.7. Theorem 1.8 is a direct consequence of Theorem 1.7 and Theorem 1.4.

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