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Global regularity results for a class of singular/degenerate fully nonlinear elliptic equations

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Abstract

We provide the Alexandroff–Bakelman–Pucci estimate and global $C^{1,\alpha}$-regularity for a class of singular/degenerate fully nonlinear elliptic equations. We also derive the existence of a viscosity solution to the Dirichlet problem with the associated operator.

Keywords Singular/degenerate fully nonlinear equations · Global regularity · Comparison principle

Mathematics Subject Classification Primary 35B65; Secondary 35J60 · 35J70 · 35D40

Contents

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1 Introduction

In this paper, we are concerned with the existence and global regularity results for viscosity solutions of a class of singular/degenerate fully nonlinear equations of the form

$$\begin{cases} \Phi(x,|Du|)F(D^2u) = f(x) & \text{in} \quad \Omega \\ u(x) = g(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $F : \mathcal{S}(n) \to \mathbb{R}$ is a uniformly (λ, Λ)-elliptic operator in the sense of (A1), $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ is a continuous map featuring degeneracy and singularity for the gradient described as in (A2), $f(\cdot)$ and $g(\cdot)$ are suitable regular functions in the sense of (A3), and Ω is a C^2-domain as in (A4). We recall that, as a consequence of Krylov-Safonov theory [32, 33], viscosity solutions to the homogeneous equation

$$F(D^2u) = 0 \quad \text{in} \quad B_1, \quad \text{where} \quad F \text{ is uniformly } \ (\lambda, \Lambda) \ \text{-elliptic},$$

belong to $C^{1,\overline{\alpha}}_{\text{loc}}(B_1)$ for a universal constant $\overline{\alpha} \equiv \overline{\alpha}(n, \lambda, \Lambda) \in (0, 1)$.

Some special cases of (1.1), which are singular or degenerate PDEs in non-divergence structure, have been widely studied in the past years. To be precise, the local $C^{1,\alpha}$-regularity results for degenerate fully nonlinear equations were developed in [2, 26] for $\Phi(x,t) = t^p$ with $p \geq 0$, in [15, 20] for $\Phi(x,t) = t^p + \mathfrak{a}(x)t^q$ with $0 \leq p \leq q$, in [11] for $\Phi(x,t) = t^{p(x)}$ with $\inf p(\cdot) > -1$, and in [6, 22] for $\Phi(x,t) = t^{p(x)} + \mathfrak{a}(x)t^{q(x)}$ with $0 \leq p(\cdot) \leq q(\cdot)$. On the other hand, comparison principle, Liouville type results, and the ABP estimate are found mostly for $\Phi(x,t) = t^p$ with $-1 < p < 0$; we refer to [8, 9, 18, 19, 25] for details. Recently, the ABP estimate for fully nonlinear models with unbalanced degeneracy was established in [6, 7]. Finally, for both singular and degenerate general operators which are considered in this paper, the local $C^{1,\alpha}$-regularity with the optimality was shown by the authors [5]. Global counterpart of such local regularity results can be found in [10] for $\Phi(x,t) = t^p$ with $p \geq 0$ and in [6] for $\Phi(x,t) = t^{p(x)} + \mathfrak{a}(x)t^{q(x)}$ with $0 \leq p(\cdot) \leq q(\cdot)$. It is noteworthy that the regularity theory for viscosity solutions to (1.1) plays a crucial role in the investigation of the free boundary problems of singular perturbation type [3, 7], of obstacle type [16, 17], and of one-phase Bernoulli type [14].

The goal of this paper is to investigate the global regularity, involving the ABP estimate and $C^{1,\alpha}$-estimate up to the boundary, for both singular and degenerate fully nonlinear elliptic equations in a unified way. To begin with, the ABP estimate in our setting reads as follows:

Theorem 1.1 (Alexandroff–Bakelman–Pucci estimate) *Suppose that $u \in C(\overline{\Omega})$ is a viscosity subsolution (resp. supersolution) of (1.1) in $\{x \in \Omega : u(x) > 0\}$ (resp. $\{x \in \Omega : u(x) < 0\}$) under the assumptions (A1)–(A2) (to be stated in Sect. 2). Suppose that $f \in L^n(\Omega) \cap C(\Omega)$. Then there exists a constant $c \equiv c(n, \lambda, i(\Phi), s(\Phi), L, \nu_0)$ such that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} g^+ + c \operatorname{diam}(\Omega) \left(\max\left\{ \|f^-\|_{L^n(\Gamma^+(u^+))}^{\frac{1}{i(\Phi)+1}}, \|f^-\|_{L^n(\Gamma^+(u^+))}^{\frac{1}{s(\Phi)+1}} \right\} + 1 \right), \tag{1.2}$$

(resp.

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} g^- + c \operatorname{diam}(\Omega) \left(\max\left\{ \|f^+\|_{L^n(\Gamma^+(u^-))}^{\frac{1}{i(\Phi)+1}}, \|f^+\|_{L^n(\Gamma^+(u^-))}^{\frac{1}{s(\Phi)+1}} \right\} + 1 \right).) \tag{1.3}$$

In particular, we have

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)} + c \operatorname{diam}(\Omega) \left(\max\left\{ \|f\|_{L^n(\Omega)}^{\frac{1}{i(\Phi)+1}}, \|f\|_{L^n(\Omega)}^{\frac{1}{s(\Phi)+1}} \right\} + 1 \right) \tag{1.4}$$

for some constant $c \equiv c(n, \lambda, i(\Phi), s(\Phi), L, \nu_0) > 0$.

We next establish the global $C^{1,\alpha}$-regularity result for viscosity solutions of Dirichlet problems.

Theorem 1.2 (Global $C^{1,\alpha}$-regularity) *Suppose the assumptions (A1)–(A4) (to be stated in Sect. 2) are in force. Let* α *be chosen to satisfy*

$$\alpha \in \begin{cases} (0, \overline{\alpha}) \cap \left(0, \frac{1}{1+s(\Phi)}\right] \cap (0, \beta_g) & \text{if } i(\Phi) \geq 0 \\ (0, \overline{\alpha}) \cap \left(0, \frac{1}{1+s(\Phi)-i(\Phi)}\right] \cap (0, \beta_g) & \text{if } -1 < i(\Phi) < 0. \end{cases} \tag{1.5}$$

For any viscosity solution u *of*

$$\begin{cases} \Phi(x, |Du|)F(D^2u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

there exists a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, \alpha)$ *such that* $u \in C^{1,\alpha}(\overline{\Omega})$ *and*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq c\left(1 + \|u\|_{L^\infty(\Omega)} + \|g\|_{C^{1,\beta_g}(\partial\Omega)} + \|f/\nu_0\|_{L^\infty(\Omega)}^{\frac{1}{1+i(\Phi)}}\right).$$

Corollary 1.3 *Suppose the assumptions of Theorem 1.2 are in force. Suppose further that* F *is convex (or concave) and* $g \in C^{1,1}(\partial\Omega)$*. Then* $u \in C^{1,\alpha}(\overline{\Omega})$*, where*

$$\alpha = \begin{cases} \frac{1}{1+s(\Phi)} & \text{if } i(\Phi) \geq 0 \\ \frac{1}{1+s(\Phi)-i(\Phi)} & \text{if } -1 < i(\Phi) < 0. \end{cases}$$

Corollary 1.3 immediately follows from Theorem 1.2, since viscosity solutions to convex/concave equations belong to $C^{1,1}_{\text{loc}}(\Omega)$ by Evans–Krylov theory [21, 30, 31]. We refer to [2, Corollary 3.2] and [6, Corollary 1.2] for similar results to Corollary 1.3.

The last main theorem concerning the solvability of the Dirichlet problem follows from Theorem 1.2 together with Perron's method.

Theorem 1.4 (Existence of a viscosity solution) *Suppose the assumptions (A1)–(A4) and (A5) (to be stated in Sect. 6) are in force. Then there exists a viscosity solution* $u \in C(\overline{\Omega})$ *of* (1.1).

Our strategy is to improve the global regularity of a viscosity solution u gradually. For this purpose, we begin with the ABP estimate to show the global boundedness of solutions. Then, by constructing an appropriate barrier function near the boundary, we capture the boundary behavior of solutions in terms of a distance function. The comparison with a distance function allows us to achieve a global Lipschitz estimate. In the end, we prove the approximation lemma by employing the compactness argument and then determine approximating linear functions in an iterative manner.

The main difficulty arises due to different behaviors of solutions relying on the sign of $i(\Phi)$ defined in (A2). To overcome such a challenge, we first discuss the degenerate case ($i(\Phi) \geq 0$) in Sect. 4 and then transport the regular properties to a viscosity solution of the singular case ($-1 < i(\Phi) < 0$) along with a suitable modification of equations in Sect. 5. In addition, the degenerate or singular character of PDEs leads to the lack of the comparison principle in general settings. Therefore, we formulate special types of the comparison principle: one is Lemma 4.2, where we exploit the smooth feature of barrier functions, and the other is Lemma

6.3, in which we approximate the equations to have a monotone property with respect to viscosity solution u of (1.1),

The paper is organized as follows. In Sect. 2, we present the assumptions (A1)–(A4) on the equation (1.1) and data to be used throughout the paper, and then collect preliminary results related to our main theorems. Section 3 is devoted to the proof of ABP estimate. The proofs for global $C^{0,1}$-estimate and $C^{1,\alpha}$-estimate of viscosity solutions u of (1.1) are provided in Sects. 4 and 5, respectively. Finally, in Sect. 6, we prove the comparison principle under an additional assumption (A5) to deduce the existence of a viscosity solution by Perron's method.

2 Preliminaries

Throughout the paper, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball of $\mathbb{R}^n$ with $n \geq 2$ centered at x_0 with positive radius r. If the center is clear in the context, we shall omit the center point by writing $B_r \equiv B_r(x_0)$. Moreover, $B_1 \equiv B_1(0) \subset \mathbb{R}^n$ denote the unit ball. We shall always denote by c a generic positive constant, possibly varying line to line, having dependencies on parameters using brackets, that is, for example $c \equiv c(n, i(\Phi), \nu_0)$ means that c depends only on $n, i(\Phi)$, and ν_0. For two positive functions f, g, we write $f \lesssim g$ when there exists a universal constant $c > 0$ such that $f \leq cg$.

For a measurable map $h : \Omega \to \mathbb{R}^n$ with $\gamma \in (0, 1]$ being a given number, we shall use the following notation for the Hölder semi-norm:

$$[h]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|h(x) - h(y)|}{|x - y|^{\gamma}}.$$

As in [24, Definition 2.1.1], we say that a function $h : (0, \infty) \to \mathbb{R}$ is *almost non-decreasing with constant* $L \geq 1$ if

$$h(s) \leq Lh(t) \quad \text{for all } 0 < s \leq t.$$

An *almost non-increasing function with constant* $L \geq 1$ can be defined in an analogous way.

We now state the main assumptions in the paper.

(A1) The operator $F : \mathcal{S}(n) \to \mathbb{R}$ is continuous and uniformly (λ, Λ)-elliptic in the sense that

$$\lambda \mathrm{tr}(N) \leq F(M + N) - F(M) \leq \Lambda \mathrm{tr}(N)$$

holds with some constants $0 < \lambda \leq \Lambda$ and $F(0) = 0$, whenever $M, N \in \mathcal{S}(n)$ with $N \geq 0$, where we denote by $\mathcal{S}(n)$ the set of $n \times n$ real symmetric matrices.

(A2) $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ is a continuous map satisfying the following properties:

1. There exist constants $s(\Phi) \geq i(\Phi) > -1$ such that the map $t \mapsto \Phi(x, t)/t^{i(\Phi)}$ is almost non-decreasing with constant $L \geq 1$ in $(0, \infty)$ and the map $t \mapsto \Phi(x, t)/t^{s(\Phi)}$ is almost non-increasing with constant $L \geq 1$ in $(0, \infty)$ for all $x \in \Omega$.
2. There exists constants $0 < \nu_0 \leq \nu_1$ such that $\nu_0 \leq \Phi(x, 1) \leq \nu_1$ for all $x \in \Omega$.

(A3) $f \in C(\Omega) \cap L^{\infty}(\Omega)$ and $g \in C^{1,\beta_g}(\partial\Omega)$ for some $\beta_g \in (0, 1)$.

(A4) $\Omega \subset \mathbb{R}^n$ is a bounded C^2-domain.

Before we proceed, we provide several remarks on the assumptions. To begin with, the Pucci extremal operators $P^{\pm}_{\lambda,\Lambda} : \mathcal{S}(n) \to \mathbb{R}$ are defined as

$$P^{+}_{\lambda,\Lambda}(M) := \Lambda \sum_{e_k>0} e_k + \lambda \sum_{e_k<0} e_k$$

and

$$P^{-}_{\lambda,\Lambda}(M) := \lambda \sum_{e_k>0} e_k + \Lambda \sum_{e_k<0} e_k,$$

where $\{e_k\}_{k=1}^n$ are the eigenvalues of the matrix M. The (λ, Λ)-ellipticity of the operator F via the Pucci extremal operators can be formulated as

$$P^{-}_{\lambda,\Lambda}(N) \leq F(M+N) - F(M) \leq P^{+}_{\lambda,\Lambda}(N)$$

for all $M, N \in \mathcal{S}(n)$.

Moreover, let us present some concrete examples of $\Phi(x,\xi)$ satisfying assumption (A2), together with their respective exponents $i(\Phi)$ and $s(\Phi)$:

(i) $\Phi(x,\xi) = |\xi|^p$ for $p > -1$: $i(\Phi) = s(\Phi) = p$.
(ii) $\Phi(x,\xi) = |\xi|^p + \mathfrak{a}(x)|\xi|^q$ for $-1 < p < q < \infty$ and $0 \leq \mathfrak{a} \in C(\Omega)$: $i(\Phi) = p$ and $s(\Phi) = q$.
(iii) $\Phi(x,\xi) = |\xi|^{p(x)}$ for $p \in C(\Omega)$ and $-1 < \inf_\Omega p(x) \leq \sup_\Omega p(x) < \infty$: $i(\Phi) = \inf_\Omega p(x)$ and $s(\Phi) = \sup_\Omega p(x)$.
(iv) $\Phi(x,\xi) = |\xi|^{p(x)} + \mathfrak{a}(x)|\xi|^{q(x)}$ for $p, q \in C(\Omega)$ with $-1 < \inf_\Omega\{p(x), q(x)\} \leq \sup_\Omega\{p(x), q(x)\} < \infty$ and $0 \leq \mathfrak{a} \in C(\Omega)$: $i(\Phi) = \inf_\Omega\{p(x), q(x)\}$ and $s(\Phi) = \sup_\Omega\{p(x), q(x)\}$.

Finally, the assumption (A4) was motivated by the approach developed in [10]. More precisely, we may assume that $0 \in \partial\Omega$, and there exist a ball $B = B_R(0)$ in $\mathbb{R}^n$ and $\phi \in C^2(\mathbb{R}^{n-1})$ such that $\phi(0) = 0$, $\nabla\phi(0) = 0$, and

$$\Omega \cap B \subset \{y \in B : y_n > \phi(y')\}, \quad \partial\Omega \cap B = \{y \in B : y_n = \phi(y')\}.$$

Definition 2.1 (*The ball condition*, [1, Definition 2.1]) Let Ω be a bounded domain in $\mathbb{R}^n$. We say that D satisfies the *exterior ball condition* (with radius r) if there exists $r > 0$ satisfying the following condition: for every $x \in \partial\Omega$, there exists a point $x^e \in \mathbb{R}^n \backslash \Omega$ such that $B_r(x^e) \subset \mathbb{R}^n \backslash \Omega$ and $x \in \partial B_r(x^e)$. Similarly, we can define the *interior ball condition*. Finally, we say that Ω satisfies the *ball condition* (with radius r) if Ω satisfies both the exterior and the interior ball condition (with radius r).

Lemma 2.2 [1, Lemma 2.2] *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then Ω is a $C^{1,1}$-domain if and only if Ω satisfies the ball condition.*

On the other hand, for any vector $\xi \in \mathbb{R}^n$, we consider a map $G_\xi : \Omega \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ defined by

$$G_\xi(x, p, M) := \Phi(x, |\xi + p|)F(M) - f(x)$$

under the assumptions prescribed in (A1)–(A3). In Sects. 4 and 5, we shall focus on viscosity solutions of the equation

$$G(x, Du, D^2u) := \Phi(x, |Du|)F(D^2u) - f(x) = 0 \text{ in } \Omega \tag{2.1}$$

or

$$G_\xi(x, Du, D^2u) = 0 \text{ in } \Omega. \tag{2.2}$$

We now provide the following definition of a viscosity solution u of the Eq. (2.1), which was introduced in [8, Definition 2.7] and [9, Definition 2.1].

Definition 2.3 A lower semicontinuous function v is called a *viscosity supersolution* of (2.1) if for any $x_0 \in \Omega$:

- either there exists $\delta > 0$ such that v is constant in $B_\delta(x_0)$ and $0 \le f(x)$ for all $x \in B_\delta(x_0)$,
- or for all $\varphi \in C^2(\Omega)$ such that $v - \varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$, one has

$$G(x_0, D\varphi(x_0), D^2\varphi(x_0)) \le 0.$$

In a similar way, an upper semicontinuous function w is called is a *viscosity subsolution* of (2.1) if for any $x_0 \in \Omega$:

- either there exists $\delta > 0$ such that w is constant in $B_\delta(x_0)$ and $0 \ge f(x)$ for all $x \in B_\delta(x_0)$,
- or for all $\varphi \in C^2(\Omega)$ such that $w - \varphi$ has a local maximum at x_0 and $D\varphi(x_0) \neq 0$, one has

$$G(x_0, D\varphi(x_0), D^2\varphi(x_0)) \ge 0.$$

We say that $u \in C(\Omega)$ is a *viscosity solution* of (2.1) if u is a viscosity supersolution and a subsolution simultaneously.

Remark 2.4 It is noteworthy that Definition 2.3 is necessary only for the case $-1 < i(\Phi) < 0$, due to the fact that $\Phi(x, |Du|)$ may not be defined when the gradient is zero. When $i(\Phi) \ge 0$, the classical definition of viscosity solutions coincides with Definition 2.3; see [12, 13] for example. Moreover, a viscosity solution of (2.2) can be understood as a viscosity solution of (2.1) by considering $\overline{u}(x) = u(x) + \xi \cdot x$.

We also recall a concept of superjet and subjet introduced in [13, Section 2].

Definition 2.5 Let $v : \Omega \to \mathbb{R}$ be an upper semicontinuos function and $w : \Omega \to \mathbb{R}$ be a lower semicontinuous function. For every $x_0 \in \Omega$, we define the *second order superjet* of v at x_0 by

$$\begin{aligned} J^{2,+}v(x_0) := \Big\{&(p, M) \in \mathbb{R}^n \times \mathcal{S}(n) : v(x) \le v(x_0) + \langle p, x - x_0\rangle \\ &+ \tfrac{1}{2}\langle M(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\Big\}\end{aligned}$$

and the *second order subjet* of w at x_0 by

$$\begin{aligned} J^{2,-}w(x_0) := \Big\{&(p, M) \in \mathbb{R}^n \times \mathcal{S}(n) : w(x) \ge w(x_0) + \langle p, x - x_0\rangle \\ &+ \tfrac{1}{2}\langle M(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\Big\}.\end{aligned}$$

(i) A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a *limiting superjet* of v at $x_0 \in \Omega$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in such a way that $(p_k, M_k) \in J^{2,+}v(x_k)$ and $\lim_{k\to\infty} v(x_k) = v(x_0)$.
(ii) A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a *limiting subjet* of w at $x \in B_1$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in such a way that $(p_k, M_k) \in J^{2,-}w(x_k)$ and $\lim_{k\to\infty} w(x_k) = w(x_0)$.

The following lemma is a consequence of stability results and 'cutting lemma'. To prove this, one can follow the lines of proof of [5, Theorem 4.1] or [22, Lemma 3.2].

Lemma 2.6 *Let $\{g_k\}_k$ be a sequence of Lipschitz continuous functions such that $g_k \to g_\infty$. Suppose that $\{u_k\}_k$ is a sequence of uniformly bounded continuous viscosity solutions of*

$$\begin{cases} \Phi_k(y, |\xi_k + Du_k|)F_k(D^2u_k) = f_k(y) \text{ in } B_1 \cap \{y_n > \phi(y')\} \\ u_k(y) = g_k(y) \quad\quad\quad \text{on } B_1 \cap \{y_n = \phi(y')\}, \end{cases}$$

where $\{\xi_k\}_k \subset \mathbb{R}^n$, $\{f_k\}_k \subset C(B_1 \cap \{y_n > \phi(y')\})$, and $\{F_k\}_k \subset C(\mathcal{S}(n), \mathbb{R})$ is uniformly (λ, Λ)-elliptic. Suppose further that $\xi_k \to \xi_\infty$, $f_k \to 0$ (uniformly), and $F_k \to F_\infty$. Then one can extract a subsequence from $\{u_k\}_k$ which converges uniformly to u_∞ on $\overline{B_1 \cap \{y_n > \phi(y')\}}$. Moreover, such a limit u_∞ satisfies

$$\begin{cases} F_\infty(D^2u_\infty) = 0 \text{ in } B_1 \cap \{y_n > \phi(y')\} \\ u_\infty(y) = g_\infty(y) \text{ on } B_1 \cap \{y_n = \phi(y')\}. \end{cases}$$

We finish this section by providing the interior regularity results shown in [5].

Theorem 2.7 [5, Theorem 1.1] *Let $u \in C(B_1)$ be a viscosity solution of*

$$\Phi(x, |Du|)F(D^2u) = f(x) \quad in \quad B_1,$$

under the assumptions (A1) and (A2) with $f \in L^\infty(B_1)$. Then $u \in C^{1,\beta}_{\text{loc}}(B_1)$ for all $\beta > 0$ satisfying

$$\beta \in \begin{cases} (0, \overline{\alpha}) \cap \left(\frac{1}{1+s(\Phi)}\right] & \text{if } i(\Phi) \geq 0 \\ (0, \overline{\alpha}) \cap \left(\frac{1}{1+s(\Phi)-i(\Phi)}\right] & \text{if } -1 < i(\Phi) < 0. \end{cases} \tag{2.3}$$

Moreover, for every β in (2.3), there exists a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, \beta)$ such that

$$\|u\|_{L^\infty(B_{1/2})} + \sup_{\substack{x,y \in \overline{B_{1/2}} \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x-y|^\beta} \leq c\left(1 + \|u\|_{L^\infty(B_1)} + \|f/\nu_0\|_{L^\infty(B_1)}^{\frac{1}{1+i(\Phi)}}\right).$$

3 Alexandroff–Bakelman–Pucci estimate

Before we develop $C^{0,1}$-regularity in Sect. 4 and $C^{1,\alpha}$-regularity in Sect. 5, our study on the global regularity starts with the Alexandroff–Bakelman–Pucci (ABP) estimate. In short, the ABP estimate controls the supremum of u over Ω in terms of the supremum of u on $\partial\Omega$ and the L^n-norm of f. In this section, we deduce an appropriate version of the ABP estimate for a viscosity subsolution of (1.1). We refer to [18, Theorem 1], [27, Theorem 1.1], [25, Theorem 1], [7, Theorem 8.6], and [6, Theorem 2.1] for ABP estimates in different settings.

To prove ABP estimates, we need to define the notion of upper contact set of a function u:

Definition 3.1 For $v : \Omega \to \mathbb{R}$ and $R > 0$, the *upper contact set* is defined by

$$\Gamma^+(v, \Omega) = \left\{x \in \Omega : \exists p \in \mathbb{R}^n \text{ such that } u(y) \leq u(x) + \langle p, y-x\rangle \text{ for all } y \in \Omega\right\},$$

$$\Gamma^+_R(v, \Omega) = \left\{x \in \Omega : \exists p \in \overline{B_R(0)} \text{ such that } u(y) \leq u(x) + \langle p, y-x\rangle \text{ for all } y \in \Omega\right\}.$$

We are now ready to prove the ABP estimate.

Proof of Theorem 1.1 The proof consists of two parts. In the first part, we prove the above theorem for the viscosity subsolution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. In the second part, we consider $u \in C(\overline{\Omega})$ via approximation based on the sup convolutions.

Part 1. Suppose that the subsolution u belongs to $C^2(\Omega) \cap C(\overline{\Omega})$. Let us define

$$R_0 \equiv R_0(u) := \frac{1}{\operatorname{diam}(\Omega)} \left(\sup_{x\in\Omega} u(x) - \sup_{x\in\partial\Omega} u^+ \right).$$

The purpose is to obtain a certain estimate on R_0 in terms of $\| f^- \|_{L^n(\Gamma^+(u^+))}$ and **data**, from which the estimate (1.2) follows. Applying [27, Lemma 3.1], for all $R < R_0$, we find

$$\int_{B_R(0)} g(z)\,dz \le \int_{\Gamma_R^+(u^+)} g(Du)\left|\det\left(D^2u\right)\right| dx \quad (\forall g \in C(\mathbb{R}^n),\ g \ge 0) \tag{3.1}$$

and

$$D^2u \le 0 \text{ on } \Gamma_R^+(u^+) \subset \{x \in \Omega : u(x) > 0\}. \tag{3.2}$$

Let us now discuss the behavior of Du in the set $\Gamma_R^+(u^+)$. Let $x_0 \in \Gamma_R^+(u^+)$ be any point. If $Du(x_0) \neq 0$, then we are able to take u as a test function in the definition of viscosity subsolution. In turn, we have

$$\Phi(x_0, |Du(x_0)|)F(D^2u(x_0)) \ge f(x_0).$$

Then we observe that

$$-f^-(x_0) \le f(x_0) \le \Phi(x_0, |Du(x_0)|)F(D^2u(x_0)) \le \Phi(x_0, |Du(x_0)|)P^+_{\lambda,\Lambda}(D^2u(x_0)).$$

Recalling $D^2u(x_0) \le 0$ by (3.2), we find $P^+_{\lambda,\Lambda}(D^2u(x_0)) = \lambda \operatorname{tr}(D^2u(x_0))$ and

$$\left(\frac{-\operatorname{tr}(D^2u(x_0))}{n}\right)^n \le \left(\frac{f^-(x_0)}{n\lambda\Phi(x_0, |Du(x_0)|)}\right)^n. \tag{3.3}$$

If $Du(x_0) = 0$ and $D^2u(x_0) \neq 0$, then x_0 is a critical point of u. On the other hand, recalling again (3.2), we have $D^2u(x_0) < 0$, which means that x_0 is a non-degenerate critical point of u. However, the set of non-degenerate critical points of u is countable since $u \in C^2(\Omega)$.

Let us recall also the following classical inequality,

$$\det(A)\det(B) \le \left(\frac{\operatorname{tr}(AB)}{n}\right)^n \text{ for all } A, B \in \mathcal{S}(n) \text{ with } A, B \ge 0.$$

In turn, the last display together with (3.2) and (3.3) yields

$$\left|\det D^2u(x)\right| \le \left(\frac{f^-(x)}{n\lambda\Phi(x, |Du(x)|)}\right)^n \tag{3.4}$$

for all $x \in \Gamma_R^+(u^+)\backslash\mathcal{U}$, where $\mathcal{U} = \{x \in \Gamma_R^+(u^+) : Du(x) = 0\}$. We now consider two steps depending on the sign of $i(\Phi)$.

Step 1: $i(\Phi) \ge 0$. Let us select $g(z) = \min\left\{|z|^{i(\Phi)n}, |z|^{s(\Phi)n}\right\}$ in (3.1). In turn, recalling (3.4), we find

$$I_1 := \int\limits_{B_R(0)} \min\left\{|z|^{i(\Phi)n}, |z|^{s(\Phi)n}\right\} dz$$
$$\leq \int\limits_{\Gamma_R^+(u^+)\setminus\mathcal{U}} \min\left\{|Du|^{i(\Phi)n}, |Du|^{s(\Phi)n}\right\}\left(\frac{f^-}{n\lambda\Phi(x,|Du|)}\right)^n dx$$
$$\leq \frac{L^n}{n^n\lambda^n\nu_0^n}\int\limits_{\Gamma_R^+(u^+)} (f^-)^n\, dx,$$

where we have used (A2). On the other hand, by co-area formula, we have

$$I_1 = \int_0^R \min\left\{t^{i(\Phi)n}, t^{s(\Phi)n}\right\}\int\limits_{\partial B_t(0)} dS dt = n\omega_n\int_0^R \min\left\{t^{i(\Phi)n}, t^{s(\Phi)n}\right\} t^{n-1}\, dt$$
$$= \begin{cases} \frac{\omega_n R^{(i(\Phi)+1)n}}{i(\Phi)+1} - \frac{\omega_n(s(\Phi)-i(\Phi))}{(i(\Phi)+1)(s(\Phi)+1)} & \text{if } R\geq 1\\ \frac{\omega_n R^{(s(\Phi)+1)n}}{s(\Phi)+1} & \text{if } R<1. \end{cases}$$

Combining the last two displays, we arrive at (1.2).

Step 2: $-1 < i(\Phi) < 0$. In this case, we select

$$g(z) = \left(\frac{|z|}{|z|+\delta}\right)^{-i(\Phi)n}\min\left\{|z|^{i(\Phi)n}, |z|^{s(\Phi)n}\right\}$$

for an arbitrary number $\delta > 0$. Clearly, $g \in C(\mathbb{R}^n)$ and so we have

$$I_2(\delta) := \int\limits_{B_R(0)} \left(\frac{|z|}{|z|+\delta}\right)^{-i(\Phi)n}\min\left\{|z|^{i(\Phi)n}, |z|^{s(\Phi)n}\right\} dz$$
$$\leq \int\limits_{\Gamma_R^+(u^+)\setminus\mathcal{U}} \left(\frac{|Du|}{|Du|+\delta}\right)^{-i(\Phi)n}\min\left\{|Du|^{i(\Phi)n}, |Du|^{s(\Phi)n}\right\}\left(\frac{f^-}{n\lambda\Phi(x,|Du|)}\right)^n dx$$
$$\leq \frac{L^n}{n^n\lambda^n\nu_0^n}\int\limits_{\Gamma_R^+(u^+)} (f^-)^n\, dx,$$

where we have used again (A2) and the fact that $i(\Phi) < 0$. By using co-area formula and recalling that $-1 < i(\Phi) < 0$, we get

$$I_2(\delta) = n\omega_n\int_0^R \left(\frac{t}{t+\delta}\right)^{-i(\Phi)n}\min\left\{t^{i(\Phi)n}, t^{s(\Phi)n}\right\} t^{n-1}\, dt.$$

By applying Lebesgue's dominated convergence theorem, we conclude

$$\lim_{\delta\to 0^+} I_2(\delta) == n\omega_n\int_0^R \min\left\{t^{i(\Phi)n}, t^{s(\Phi)n}\right\} t^{n-1}\, dt$$
$$= \begin{cases} \frac{\omega_n R^{(i(\Phi)+1)n}}{i(\Phi)+1} - \frac{\omega_n(s(\Phi)-i(\Phi))}{(i(\Phi)+1)(s(\Phi)+1)} & \text{if } R\geq 1,\\ \frac{\omega_n R^{(s(\Phi)+1)n}}{s(\Phi)+1} & \text{if } R<1. \end{cases}$$

Combining the last two displays, we get (1.2).

Part 2. Let $u \in C(\overline{\Omega})$. Since we have ABP estimates for $u \in C^2(\Omega) \cap C(\overline{\Omega})$, the remainder of the proof can be argued similarly as in the proof of [27, Theorem 1.1] or [18, Theorem 1]. □

4 Local Lipschitz estimates up to the boundary

By Theorem 1.1, any viscosity solution of (1.1) is bounded in $L^\infty(\Omega)$ under the assumptions (A1)–(A3). In this section, to derive further Lipschitz estimates up to the boundary as in [6, 10], we consider a bounded viscosity solution of

$$\begin{cases} \Phi(y, |Du|)F(D^2u) = f(y) \text{ in } B \cap \{y_n > \phi(y')\} \\ u(y) = g(y) \qquad \text{on } \ B \cap \{y_n = \phi(y')\}, \end{cases} \tag{4.1}$$

where the function ϕ is introduced in Sect. 2.

Remark 4.1 [Smallness regime] Here we verify that, for a bounded viscosity solution u of

$$\begin{cases} \Phi(y, |\xi + Du|)F(D^2u) = f(y) \text{ in } B \cap \{y_n > \phi(y')\} \\ u(y) = g(y) \qquad \text{on } B \cap \{y_n = \phi(y')\}, \end{cases} \tag{4.2}$$

we are able to assume

$$\|u\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le 1, \ \|g\|_{C^{1,\beta_g}(B_1\cap\{y_n=\phi(y')\})} \le 1, \ \text{and } \|f\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le \varepsilon_0 \tag{4.3}$$

for some constant $\varepsilon_0 \in (0, 1)$ small enough, and also $\nu_0 = \nu_1 = 1$ in (A2). In order to consider the problem in a smallness regime as in (4.3), for a fixed ball $B_r(x) \subset B$, we define $\bar{u} : B_1 \cap \{y_n > \bar{\phi}(y')\} \to \mathbb{R}$ by

$$\bar{u}(y) := \frac{u(ry + x)}{K}$$

for a function $\bar{\phi}$ and positive constants $K \ge 1 \ge r$ to be determined later. It can be seen that $\bar{u}$ is a viscosity solution of

$$\begin{cases} \bar{\Phi}(y, |\bar{\xi} + D\bar{u}|)\bar{F}(D^2\bar{u}) = \bar{f}(y) \text{ in } B_1 \cap \{y_n > \bar{\phi}(y')\} \\ \bar{u}(y) = \bar{g}(y) \qquad \text{on } B_1 \cap \{y_n = \bar{\phi}(y')\}, \end{cases} \tag{4.4}$$

where

$$\bar{F}(M) := \frac{r^2}{K} F\left(\frac{K}{r^2}M\right), \quad \bar{\Phi}(y, t) := \frac{\Phi\left(ry + x, \frac{K}{r}t\right)}{\Phi\left(ry + x, \frac{K}{r}\right)}, \quad \bar{f}(y) := \frac{r^2}{\Phi\left(ry + x, \frac{K}{r}\right)K} f(ry + x),$$
$$\bar{\xi} := \frac{r}{K}\xi, \quad \bar{\phi}(y') := \frac{\phi(ry' + x') - x_n}{r}, \text{ and } \bar{g}(y) := \frac{g(ry + x)}{K}.$$

Note that $\bar{F}$ is still a uniformly (λ, Λ)-elliptic operator, the map $t \mapsto \bar{\Phi}(y, t)/t^{i(\Phi)}$ is almost non-decreasing and the map $t \mapsto \bar{\Phi}(y, t)/t^{s(\Phi)}$ is almost non-increasing with the same constants $L \ge 1$ and $s(\Phi) \ge i(\Phi) > -1$ as in (A2), and $\bar{\Phi}(y, 1) = 1$ for all $y \in B_1$. It is immediate from the choice of r that $\|D^2\bar{\phi}\|_\infty \le \|D^2\phi\|_\infty$ and

$$\|\bar{g}\|_{C^{1,\beta_g}(B_1\cap\{y_n=\bar{\phi}(y')\})} \le \frac{1}{K}\|g\|_{C^{1,\beta_g}(\partial\Omega)}.$$

Moreover, the assumption (A2) implies

$$\|\bar{f}\|_{L^\infty(B_1\cap\{y_n>\bar{\phi}(y')\})} \le \frac{Lr^{2+i(\Phi)}}{\nu_0 K^{1+i(\Phi)}} \|f\|_{L^\infty(\Omega)} .$$

By recalling $i(\Phi) > -1$ and setting

$$K := 2\left(1 + \|u\|_{L^\infty(\Omega)} + \|g\|_{C^{1,\beta_g}(\partial\Omega)} + \left[\frac{L}{\nu_0}\|f\|_{L^\infty(\Omega)}\right]^{\frac{1}{1+i(\Phi)}}\right)$$

and

$$r := \varepsilon_0^{\frac{1}{2+i(\Phi)}},$$

we see that $\bar{u}$ solves the Eq. (4.4) under the smallness regime in (4.3).

If we have special conditions on a viscosity supersolution (or subsolution), then we can apply the comparison principle without an additional structure condition such as (A5) in Sect. 6. See Sect. 6 for more comments on the comparison principle.

Lemma 4.2 (Comparison principle I) *Let $f_1, f_2 \in C(\overline{\Omega})$ with $f_1 > f_2$ and $v \in C(\overline{\Omega})$ be a viscosity subsolution of $\Phi(y, |Du|)F(D^2u) = f_1(y)$ in Ω. Moreover, let $w \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a viscosity supersolution of $\Phi(y, |Du|)F(D^2u) = f_2(y)$. If $v \le w$ on $\partial\Omega$, then $v \le w$ in Ω.*

Proof By contradiction, we suppose that

$$\max_{x\in\overline{\Omega}}(v(x) - w(x)) > 0$$

and the maximum is achieved at a point $\hat{x} \in \Omega$. Since v is a viscosity subsolution, $w \in C^2(\Omega)$ and $v - w$ has a local maximum at $\hat{x}$, the definition of viscosity subsolutions yields

$$\Phi(\hat{x}, |Dw(\hat{x})|)F(D^2w(\hat{x})) \ge f_1(\hat{x}).$$

On the other hand, since w is a viscosity supersolution, we have

$$\Phi(\hat{x}, |Dw(\hat{x})|)F(D^2w(\hat{x})) \le f_2(\hat{x}),$$

which leads to the contradiction. □

The following lemma describes the boundary behavior of a viscosity solution u in terms of a distance function d. Indeed, our approach to obtain the boundary regularity (without utilizing a change of variables) was strongly inspired by [10].

Lemma 4.3 *Let $g \in C^{1,\beta_g}(\partial\Omega)$. Let d be the distance to the hypersurface $\{y_n = \phi(y')\}$. Then for every $r \in (0, 1)$ and $\gamma \in (0, 1)$, there exists $\delta_0 > 0$ depending on $\|f\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})}$, λ, Λ, Ω, r, L, ν_0, and $\mathrm{Lip}_g(\partial\Omega)$ such that for every $0 < \delta < \delta_0$, if u is a viscosity solution of (4.1) with $\|u\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le 1$, then*

$$|u(y', y_n) - g(y')| \le \frac{6}{\delta}\frac{d(y)}{1 + d(y)^\gamma} \quad \textit{in } B_r(0) \cap \{y_n > \phi(y')\}.$$

Proof We separate two cases: (i) $g \equiv 0$, (ii) g is not identically zero.

(i) ($g \equiv 0$) In this case, we have

$$|u(y', y_n) - g(y')| \leq \|u\|_\infty \leq 1$$

and so we will only consider the smaller set $\Omega_\delta := \{y \in \Omega : d(y) < \delta\}$. Moreover, we choose $\delta_1 > 0$ such that if $d(y) < \delta_1$, then d belongs to C^2 and $|D^2 d| \leq K$ for some universal constant $K > 0$.
The proof relies on the construction of upper and lower barriers. For this purpose, we define a function $w \in C^2(\Omega_\delta)$ by

$$w(y) = \begin{cases} \frac{2}{\delta} \frac{d(y)}{1+d^\gamma(y)} & \text{for } |y| < r \\ \frac{2}{\delta} \frac{d(y)}{1+d^\gamma(y)} + \frac{1}{(1-r)^3}(|y| - r)^3 & \text{for } |y| \geq r. \end{cases}$$

By following the argument in [10, Lemma 2.2], we have $w \geq u$ on $\partial(B_1 \cap \{y_n > \phi(y')\} \cap \Omega_\delta)$. Moreover, it is easily checked that $|Dw| \geq 1/(4\delta)$ when $\delta \leq (1-r)/12$ and so if we choose $\delta < 1/4$, then $|Dw| \geq 1$. Moreover, we can calculate

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) \leq -2\gamma\delta^{\gamma-2}\lambda \frac{1+\gamma}{(1+\delta^\gamma)^3} + \frac{2}{\delta} nK\Lambda + \frac{6n\Lambda}{(1-r)^2} \lesssim -\delta^{\gamma-2} + \delta^{-1}.$$

Since $\gamma - 2 < -1 < 0$, we can further choose $\delta \in (0, 1)$ small enough so that $\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) < 0$. Then, by recalling (A2),

$$\Phi(x, |Dw|)\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) \lesssim -L\nu_0 |Dw|^{i(\Phi)}(\delta^{\gamma-2} - \delta^{-1}) \leq -L\nu_0(\delta^{\gamma-i(\Phi)-2} - \delta^{-1-i(\Phi)}).$$

Since $\gamma - i(\Phi) - 2 < -1 - i(\Phi) < 0$, we finally choose $\delta \in (0, 1)$ small enough so that

$$\Phi(x, |Dw|)\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) < -\|f\|_\infty - 1.$$

By applying Lemma 4.2, we conclude that

$$u \leq w = \frac{2}{\delta} \frac{d(y)}{1 + d^\gamma(y)} \quad \text{in } B_r(0) \cap \{y_n > \phi(y')\}.$$

The lower bound for u can be obtained in a similar argument.

(ii) (g is not identically zero) This case follows from the same argument as in [10, Lemma 2.2].

□

The main result in this section is the following boundary Lipschitz estimate, whose proof relies on Lemma 4.3 and the Ishii–Jensen Lemma [13, Theorem 3.2].

Theorem 4.4 (Lipschitz estimates for $\xi = 0$) *Let g be a Lipschitz continuous function. Suppose that u satisfies (4.1) with $\|u\|_{L^\infty(B_1 \cap \{y_n > \phi(y')\})} \leq 1$. Then for every $r \in (0, 1)$, we have $u \in C^{0,1}(B_r \cap \{y_n > \phi(y')\})$ and*

$$\|u\|_{C^{0,1}(B_r \cap \{y_n > \phi(y')\})} \leq C(n, \lambda, \Lambda, i(\Phi), s(\Phi), r, L, \mathrm{Lip}_g(\partial\Omega), \|f\|_{L^\infty(\Omega)}). \tag{4.5}$$

Proof Let $r_1 \in (r, 1)$ be fixed. For $x_0 \in B_r \cap \{y_n > \phi(y')\}$, we define

$$\Psi(x, y) := u(x) - u(y) - M\omega(|x - y|) - L\left(|x - x_0|^2 + |y - x_0|^2\right),$$

where

$$\omega(s) := \begin{cases} s - \omega_0 s^{3/2} & \text{if } s \le s_0 := (2/(3\omega_0))^2 \\ \omega(s_0) & \text{if } \ s \ge s_0. \end{cases}$$

We claim that for $L, M \gg 1$ large enough,

$$\Psi(x, y) \le 0 \quad \text{for all } (x, y) \in (B_{r_1} \cap \overline{\Omega}) \times (B_{r_1} \cap \overline{\Omega}). \tag{4.6}$$

Note that this inequality implies the desired Lipschitz estimate.

First of all, suppose that $y \in B_{r_1} \cap \{y_n = \phi(y')\}$. Then by Lemma 4.3, there exists a constant $K_0 > 0$ such that

$$|u(z) - g(z')| \le K_0 d(z, \partial\Omega) \quad \text{for } z \in B_{r_1} \cap \{y_n > \phi(y')\},$$

which implies that

$$\begin{aligned} |u(x) - u(y)| &\le |u(x', x_n) - u(x', \phi(x'))| + |u(x', \phi(x')) - u(y', \phi(y'))| \\ &\le K_0 d(x, \partial\Omega) + \mathrm{Lip}_g(\partial\Omega)|x' - y'| \le (K_0 + \mathrm{Lip}_g(\partial\Omega))|x - y|. \end{aligned}$$

Therefore, if we choose $M/3 \ge K_0 + \mathrm{Lip}_g(\partial\Omega)$, then

$$\Psi(x, y) \le M\left(\frac{|x - y|}{3} - \omega(|x - y|)\right) - L\left(|x - x_0|^2 + |y - x_0|^2\right) \le 0.$$

We now prove (4.6) by contradiction; suppose that there exists some point $(\hat{x}, \hat{y}) \in (B_{r_1} \cap \overline{\Omega}) \times (B_{r_1} \cap \overline{\Omega})$ such that

$$\Psi(\hat{x}, \hat{y}) = \max_{(B_{r_1} \cap \overline{\Omega}) \times (B_{r_1} \cap \overline{\Omega})} \Psi(x, y) > 0.$$

Here, we also choose $L > \max\left\{8/(r_1 - r)^2, 1/(2(r + r_1))\right\}$. Then we can easily check that

(i) $\hat{x} \ne \hat{y}$;
(ii) $\hat{x}, \hat{y} \in B_{r_1} \cap \{y_n > \phi(y')\}$;
(iii) $\hat{x}, \hat{y} \in B_{(r_1+r)/2}$.

Thus, by applying Ishii–Jensen Lemma [13, Theorem 3.2], we see that, for every $\varepsilon > 0$ sufficiently small, there exist $X, Y \in \mathcal{S}(n)$ such that

$$\begin{aligned} &(M\omega'(|\hat{x} - \hat{y}|)\hat{a} + 2L(\hat{x} - x_0), X) \in \overline{J}^{2,+} u(\hat{x}), \\ &(M\omega'(|\hat{x} - \hat{y}|)\hat{a} - 2L(\hat{y} - x_0), -Y) \in \overline{J}^{2,-} u(\hat{y}), \\ &\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le M\begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L + \varepsilon)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \end{aligned} \tag{4.7}$$

where

$$Z = \omega''(|\hat{x} - \hat{y}|)\hat{a} \otimes \hat{a} + \frac{\omega'(|\hat{x} - \hat{y}|)}{|\hat{x} - \hat{y}|}(I - \hat{a} \otimes \hat{a}) \quad \text{for } \hat{a} := \frac{\hat{x} - \hat{y}}{|\hat{x} - \hat{y}|}.$$

For simplicity, we write $q_x := M\omega'(|\hat{x} - \hat{y}|)\hat{a} + 2L(\hat{x} - x_0)$ and $q_y := M\omega'(|\hat{x} - \hat{y}|)\hat{a} - 2L(\hat{y} - x_0)$. We first choose $\omega_0 > 0$ small enough so that $s_0 \ge 2 > r + r_1$. Note that $t \mapsto \omega'(t)$ is decreasing on $t \in [0, s_0]$. If we choose $M > 0$ large enough so that

$2L(r+r_1) \le M\omega'(r+r_1)/2$, then we have $2L|\hat{x}-x_0|, 2L|\hat{y}-x_0| \le M\omega'(|\hat{x}-\hat{y}|)/2$. In particular, we obtain

$$\frac{M}{2}\omega'(|\hat{x}-\hat{y}|) \le |q_x|, |q_y| \le 2M\omega'(|\hat{x}-\hat{y}|), \tag{4.8}$$

and by the choice of L, we also know that $|q_x|, |q_y| \ge 1$.

On the other hand, for X and Y, we will use the matrix inequality (4.7). First, by evaluating a vector of the form (ξ, ξ) for any $\xi \in \mathbb{R}^n$, we have

$$(X+Y)\xi \cdot \xi \le 6L|\xi|^2,$$

which implies that any eigenvalues of $X+Y$ are less than $6L$. Moreover, by applying the matrix inequality (4.7) for $(\hat{a}, -\hat{a})$, we observe

$$(X+Y)\hat{a}\cdot\hat{a} \le 4M\omega''(|\hat{x}-\hat{y}|) + 6L = -3M\omega_0|\hat{x}-\hat{y}|^{-1/2} + 6L.$$

In other words, at least one eigenvalue of $X+Y$ is less than $-3M\omega_0|\hat{x}-\hat{y}|^{-1/2}+6L$. Therefore, by the definition of the Pucci operator, we have

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}(X+Y) &\le \lambda(-3M\omega_0|\hat{x}-\hat{y}|^{-1/2}+6L) + 6\Lambda(n-1)L \\ &= -3\lambda M\omega_0|\hat{x}-\hat{y}|^{-1/2} + 6[\Lambda(n-1)+\lambda]L.\end{aligned}$$

We now employ the definition of limiting superjet and limiting subjet:

$$\begin{aligned}\Phi(\hat{x}, |q_x|)F(X) &\ge f(\hat{x}) \ge -\|f\|_\infty, \\ \Phi(\hat{y}, |q_y|)F(-Y) &\le f(\hat{y}) \le \|f\|_\infty.\end{aligned}$$

Since $|q_x|, |q_y| \ge 1$, an application of (A2) and (4.8) yields that

$$\Phi(\hat{x}, |q_x|) \ge L\nu_0|q_x|^{i(\Phi)}, \quad \Phi(\hat{y}, |q_y|) \ge L\nu_0|q_y|^{i(\Phi)}.$$

Moreover, (A1) shows that

$$F(X) - F(-Y) \le \mathcal{P}^+_{\lambda,\Lambda}(X+Y) \le -3\lambda M\omega_0|\hat{x}-\hat{y}|^{-1/2} + 6[\Lambda(n-1)+\lambda]L.$$

Combining these results, we have

$$-\|f\|_\infty\left(|q_x|^{-i(\Phi)} + |q_y|^{-i(\Phi)}\right) \le -3\lambda M\omega_0|\hat{x}-\hat{y}|^{-1/2} + 6[\Lambda(n-1)+\lambda]L.$$

We now split into two cases depending on the sign of $i(\Phi)$:

(i) ($i(\Phi) \ge 0$) Since $|q_x|, |q_y| \ge 1$ and $|\hat{x}-\hat{y}| \le 1$, we conclude that

$$3\lambda M\omega_0 \le 2\|f\|_\infty + 6[\Lambda(n-1)+\lambda]L,$$

which does not hold for sufficiently large $M > 0$.

(ii) ($-1 < i(\Phi) < 0$) Recalling that $|q_x|, |q_y| \le 2M\omega'(|\hat{x}-\hat{y}|) \le 2M$, we derive

$$3\lambda M\omega_0 \le 2\|f\|_\infty(2M)^{-i(\Phi)} + 6[\Lambda(n-1)+\lambda]L.$$

Since $-i(\Phi) < 1$, this inequality does not hold for sufficiently large $M > 0$.

This finishes the proof of (4.6). □

On the other hand, for a modified Eq. (2.2), we can prove the boundary Lipschitz estimate, provided that $|\xi|$ is large. In short, the boundary Lipschitz estimates hold when either

(i) $\xi = 0$ with $i(\Phi) > -1$ (Lemma 4.3 and Theorem 4.4) or
(ii) $|\xi|$ is large with $i(\Phi) \geq 0$ (Lemma 4.5 and Theorem 4.6).

Lemma 4.5 *Let g be Lipschitz continuous on $\partial\Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Then for every $r \in (0, 1)$ and $\gamma \in (0, 1)$, there exists $\delta > 0$ depending on λ, Λ, $s(\Phi)$, r, and $\mathrm{Lip}_g(\partial\Omega)$ such that for $0 \leq b < \delta/6$, any viscosity solution u of*

$$\begin{cases} \Phi(y, |\xi + bDu|)F(D^2u) = f(y) \ \text{in } B_1 \cap \{y_n > \phi(y')\} \\ u(y) = g(y) \qquad\qquad\qquad\qquad\quad \text{on } B_1 \cap \{y_n = \phi(y')\} \end{cases} \tag{4.9}$$

with

$$\|u\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \leq 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \leq \varepsilon_0$$

satisfies

$$|u(y', y_n) - g(y')| \leq \frac{6}{\delta}\frac{d(y)}{1 + d(y)^\gamma} \quad \text{in } B_r \cap \{y_n > \phi(y')\}.$$

Proof As in the proof of Lemma 4.3, we may suppose that $g \equiv 0$ and construct a barrier function in a local domain $\Omega_\delta := \{y \in \Omega : d(y) < \delta\}$. If $b = 0$, then there is no degeneracy with respect to the gradient Du and so the result holds. Thus, we may assume that $b > 0$.

We now define a function $w \in C^2(\Omega_\delta)$ by

$$w(y) = \begin{cases} \frac{2}{\delta}\frac{d(y)}{1+d^\gamma(y)} & \text{for } |y| < r \\ \frac{2}{\delta}\frac{d(y)}{1+d^\gamma(y)} + \frac{1}{(1-r)^3}(|y| - r)^3 & \text{for } |y| \geq r. \end{cases}$$

We recall that $w \geq u$ on $\partial(B \cap \{y_n > \phi(y')\} \cap \Omega_\delta)$ and

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) \leq -2\gamma\delta^{\gamma-2}\lambda\frac{1+\gamma}{(1+\delta^\gamma)^3} + \frac{2}{\delta}nK\Lambda + \frac{6n\Lambda}{(1-r)^2} \lesssim -\delta^{\gamma-2} + \delta^{-1}.$$

On the other hand, since

$$Dw(y) = \begin{cases} \frac{2}{\delta}\frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2}Dd & \text{for } |y| < r \\ \frac{2}{\delta}\frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2}Dd + \frac{y}{|y|}\frac{3}{(1-r)^3}(|y| - r)^2 & \text{for } |y| \geq r, \end{cases}$$

we have $|Dw| \leq 3/\delta$ provided that $\delta \leq (1 - r)/3$. As a consequence, we derive

$$\frac{1}{2} \leq |\xi + bDw| \leq \frac{3}{2} \quad \text{for } 0 < b < \frac{\delta}{6},$$

and so we conclude that

$$\Phi(y, |\xi + bDw|)F(D^2w) < -\|f\|_\infty - 1 \quad \text{for sufficiently small } \delta > 0.$$

Lemma 4.2 yields the upper bound for u, and the remaining part can be done as in Lemma 4.3. □

Note that Lemma 4.5 holds for any $i(\Phi) > -1$, while Theorem 4.6 holds only for the degenerate case, $i(\Phi) \geq 0$.

Theorem 4.6 (Lipschitz estimates for large $|\xi|$; degenerate case) *Let g be Lipschitz continuous on $\partial\Omega$ and $\xi \in \mathbb{R}^n$. Assume that u is a viscosity solution of*

$$\begin{cases} \Phi(y, |\xi + Du|)F(D^2u) = f(y) \ \text{in } B_1 \cap \{y_n > \phi(y')\} \\ u(y) = g(y) \qquad\qquad\qquad\qquad \text{on } B_1 \cap \{y_n = \phi(y')\} \end{cases} \tag{4.10}$$

with

$$\|u\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le \varepsilon_0.$$

Then for all $r \in (0,1)$*, there exists* $p_0 = p_0(\lambda, \Lambda, n, i(\Phi), s(\Phi), r, \varepsilon_0, \mathrm{Lip}_g(\partial\Omega)) > 0$*, such that if* $|\xi| > p_0$*, then* $u \in C^{0,1}(B_r \cap \{y_n > \phi(y')\})$ *and we have the estimate*

$$\|u\|_{C^{0,1}(B_r\cap\{y_n>\phi(y')\})} \le C(\lambda, \Lambda, n, i(\Phi), s(\Phi), r, \varepsilon_0, \mathrm{Lip}_g(\partial\Omega)).$$

Proof Since the proof is similar to the one of Theorem 4.4, here we concentrate on the differences.

(i) We first need to show that if x or y belongs to $B_{r_1} \cap \{y_n = \phi(y')\}$, then $\Psi(x, y) \le 0$. In the case of Theorem 4.4, this result immediately followed from Lemma 4.3. In a similar manner, it is enough to apply Lemma 4.5 for a solution u. More precisely, if u is a solution of (4.10), then u solves

$$\begin{cases} \tilde{\Phi}(y, |\xi/|\xi| + bDu|)\, F(D^2u) = \tilde{f}(y) & \text{in } B_1 \cap \{y_n > \phi(y')\} \\ u(y) = g(y) & \text{on } B_1 \cap \{y_n = \phi(y')\}, \end{cases}$$

where $\tilde{\Phi}(y,t) := \Phi(y, |\xi|t)/\Phi(y, |\xi|)$, $\tilde{f}(y) := f(y)/\Phi(y, |\xi|)$ and $b := 1/|\xi|$. Thus, if we choose $p_0 > \max\{1, 6/\delta\}$, then we can apply Lemma 4.5 for u.

(ii) We next follow the contradiction argument of Theorem 4.4 and the difference occurs when we employ the definition of limiting superjet and subjet:

$$\Phi(\hat{x}, |\xi + q_x|)F(X) \ge -\|f\|_\infty,$$
$$\Phi(\hat{y}, |\xi + q_y|)F(-Y) \le \|f\|_\infty.$$

This is due to the difference between Eqs. (4.1) and (4.10), but we are still able to derive a contradiction. Recalling that $|q_x|, |q_y| \le 2M\omega'(|\hat{x} - \hat{y}|) \le 2M$, if we choose $p_0 > 3M$, then we have

$$|\xi + q_x|, |\xi + q_y| \ge M.$$

Combining this estimate with

$$-\|f\|_\infty \left(|\xi + q_x|^{-i(\Phi)} + |\xi + q_y|^{-i(\Phi)}\right) \le -3\lambda M\omega_0|\hat{x} - \hat{y}|^{-1/2} + 6[\Lambda(n-1) + \lambda]L,$$

we conclude that

$$3\lambda M\omega_0 \le 2\|f\|_\infty M^{-i(\Phi)} + 6[\Lambda(n-1) + \lambda]L,$$

which is a contradiction. In this step, we have exploited the condition $i(\Phi) \ge 0$.

□

5 Global $C^{1,\alpha}$-regularity

We start this section with several reductions of the proof of Theorem 1.2. First of all, by recalling Theorem 2.7 which provides the interior $C^{1,\alpha}$-estimate of viscosity solutions, it is enough to develop the pointwise boundary $C^{1,\alpha'}$-estimate. Next, by following the proof of [5, Theorem 1.1], we shall consider the degenerate case ($i(\Phi) \ge 0$) first, and then utilize this result for the singular case ($-1 < i(\Phi) < 0$).

We would like to emphasize that the optimal exponent α' (given by (5.1)), which appears in the pointwise boundary estimate, does not depend on $\overline{\alpha}$ from the Krylov–Safonov theory. Moreover, we can observe that the optimal exponent β (given by (2.3)) for interior estimates is independent of the choice of β_g. In the end, the optimal exponent $\alpha = \min\{\alpha', \beta\}$ for the global estimate should satisfy a stronger condition (1.5), which is a combination of interior and boundary estimates. A similar consequence can be found in [4, Theorem 1.1].

Lemma 5.1 (Pointwise boundary $C^{1,\alpha'}$-estimate; degenerate case) *Suppose the assumptions (A1)–(A4) are in force with $i(\Phi) \geq 0$. Let α' be chosen to satisfy*

$$\alpha' \in \left(0, \frac{1}{1+s(\Phi)}\right] \cap (0, \beta_g). \tag{5.1}$$

Then there exist constants $\varepsilon_0 \in (0,1)$, $\rho \in (0, 1/2)$, and $C_0 > 0$ depending on α', n, λ, Λ, $\|D^2\phi\|_{L^\infty(\Omega)}$, $\|g\|_{C^{1,\beta_g}(\partial\Omega)}$, $i(\Phi)$, and $s(\Phi)$ such that for any $\xi \in \mathbb{R}^n$ and a viscosity solution u of

$$\begin{cases} \Phi(y, |Du|)F(D^2u) = f(y) & \text{in } B_1(x) \cap \{y_n > \phi(y')\} \\ u(y) = g(y) & \text{on } B_1(x) \cap \{y_n = \phi(y')\}, \end{cases}$$

the following holds: if

$$\|u\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq \varepsilon_0,$$

then there exists an affine function $l(y) = a + b\cdot(y-x)$ with $|a| + |b| \leq C_0$ such that for each $0 < r \leq \rho$,

$$\|u - l\|_{L^\infty(B_r(x)\cap\{y_n>\phi(y')\})} \leq Cr^{1+\alpha'}$$

for some universal constant $C > 0$.

Before we prove Lemma 5.1 by using the induction, we first show the approximation lemma.

Lemma 5.2 (Approximation lemma; degenerate case) *Suppose (A1)–(A4) hold true with $i(\Phi) \geq 0$ and $\nu_0 = \nu_1 = 1$. Let $\xi \in \mathbb{R}^n$ be an arbitrary vector and $u \in C(B_1(x) \cap \{y_n > \phi(y')\})$ be a viscosity solution of*

$$\begin{cases} \Phi(y, |\xi + Du|)F(D^2u) = f(y) & \text{in } B_1(x) \cap \{y_n > \phi(y')\} \\ u(y) = g(y) & \text{on } B_1(x) \cap \{y_n = \phi(y')\}, \end{cases} \tag{5.2}$$

satisfying $\|u\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq 1$ and $\|g\|_{C^{1,\beta_g}(B_1(x)\cap\{y_n=\phi(y')\})} \leq 1$. Then for any $\mu > 0$, there exists a constant $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, i(\Phi), L, \mu) > 0$ such that if

$$\|f\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq \varepsilon_0,$$

then one can find a viscosity solution h of an uniformly (λ, Λ)-elliptic equation

$$\begin{cases} \mathcal{F}(D^2h) = 0 & \text{in } B_{3/4}(x) \cap \{y_n > \phi(y')\} \\ h = g & \text{on } B_{3/4}(x) \cap \{y_n = \phi(y')\} \end{cases} \tag{5.3}$$

such that

$$\|u - h\|_{L^\infty(B_{1/2}(x)\cap\{y_n>\phi(y')\})} \leq \mu.$$

Proof By contradiction, we suppose the conclusion of the lemma fails. Therefore, there exist $\mu_0 > 0$ and sequences of functions $\{F_k\}_{k=1}^{\infty}$, $\{\Phi_k\}_{k=1}^{\infty}$, $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$, $\{u_k\}_{k=1}^{\infty}$, and a sequence of vectors $\{\xi_k\}_{k=1}^{\infty}$ such that

(C1) $F_k \in C(\mathcal{S}(n), \mathbb{R})$ is uniformly (λ, Λ)-elliptic;
(C2) for $\Phi_k \in C(B_1 \times [0, \infty), [0, \infty))$, the map $t \mapsto \Phi_k(x, t)/t^{i(\Phi)}$ is almost non-decreasing and the map $t \mapsto \Phi_k(x, t)/t^{s(\Phi)}$ is almost non-increasing with constant $L \geq 1$, and $\Phi_k(y, 1) = 1$ for all $y \in B_1(x) \cap \{y_n > \phi(y')\}$;
(C3) $f_k \in C(B_1(x) \cap \{y_n > \phi(y')\})$ with $\|f_k\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq 1/k$;
(C4) $u_k \in C(B_1(x) \cap \{y_n > \phi(y')\})$ with $\|u\|_{L^\infty(B_1(x)\cap\{y_n>\phi(y')\})} \leq 1$ solves the equation

$$\begin{cases} \Phi_k(y, |\xi_k + Du_k|)F_k(D^2u_k) = f_k(y) & \text{in } B_1(x) \cap \{y_n > \phi(y')\} \\ u_k(y) = g_k(y) & \text{on } B_1(x) \cap \{y_n = \phi(y')\} \end{cases}$$

with $\|g_k\|_{C^{1,\beta_g}(B_1(x)\cap\{y_n=\phi(y')\})} \leq 1$, but

$$\|u_k - h\|_{L^\infty(B_{1/2}(x)\cap\{y_n>\phi(y')\})} > \mu_0 \quad \text{for any} \quad k \in \mathbb{N}, \tag{5.4}$$

for any h satisfying (5.3).

The condition (C1) implies that F_k converges to some uniformly (λ, Λ)-elliptic operator $F_\infty \in C(\mathcal{S}(n), \mathbb{R})$. Similarly, the condition (C4) implies that g_k converges to g_∞ uniformly. For a further discussion, we consider two cases:

(i) ($\{\xi_k\}_{k=1}^{\infty}$ is bounded) Upto a subsequence, ξ_k converges to some vector ξ_∞. Then we consider a sequence $\{\tilde{u}_k\}_{k=1}^{\infty} := \{u_k + x \cdot \xi_k\}_{k=1}^{\infty}$ satisfying

$$\begin{cases} \Phi_k(y, |D\tilde{u}_k|)F(D^2\tilde{u}_k) = f_k(y) & \text{in } B_1(x) \cap \{y_n > \phi(y')\} \\ \tilde{u}_k(y) = \tilde{g}_k(y) & \text{on } B_1(x) \cap \{y_n = \phi(y')\} \end{cases}$$

for $\tilde{g}_k(x) := g_k(x) + x \cdot \xi_k$. Therefore, we can apply Theorem 4.4 for $\tilde{u}_k$ and so by Arzela–Ascoli theorem, we conclude that $u_k \to u_\infty$ uniformly in $B_r(x) \cap \{y_n > \phi(y')\}$ for any $0 < r < 1$. Then by Lemma 2.6, u_∞ satisfies

$$\begin{cases} F_\infty(D^2u_\infty) = 0 \text{ in } B_{3/4}(x) \cap \{y_n > \phi(y')\} \\ u_\infty(y) = g_\infty(y) \text{ on} B_{3/4}(x) \cap \{y_n = \phi(y')\}, \end{cases}$$

which leads to the contradiction with (5.4) (choose $h = u_\infty, g = g_\infty$, and $\mathcal{F} = F_\infty$).

(ii) ($\{\xi_k\}_{k=1}^{\infty}$ is unbounded) In this case, for the constant $p_0 > 0$ chosen in Theorem 4.6, we may assume $|\xi_k| > p_0$ and $|\xi_k| \to \infty$ (up to a subsequence). Thus, we can apply Theorem 4.6 for u_k and so by Arzela-Ascoli theorem, we conclude that $u_k \to u_\infty$ uniformly in $B_r(x) \cap \{y_n > \phi(y')\}$ for any $0 < r < 1$. Again by Lemma 2.6, we conclude that

$$\begin{cases} F_\infty(D^2u_\infty) = 0 \text{ in } B_{3/4}(x) \cap \{y_n > \phi(y')\} \\ u_\infty(y) = g_\infty(y) \text{ on } B_{3/4}(x) \cap \{y_n = \phi(y')\}, \end{cases}$$

which leads to the contradiction with (5.4) (choose $h = u_\infty, g = g_\infty$, and $\mathcal{F} = F_\infty$).

□

Remark 5.3 Let us summarize the boundary regularity results for uniformly elliptic fully nonlinear equations with Dirichlet boundary conditions. To be precise, suppose that h is a viscosity solution of an uniformly (λ, Λ)-elliptic equation

$$\begin{cases} F(D^2h) = 0 & \text{in } B_1^+ \\ h = g & \text{on } B_1', \end{cases}$$

where $g \in C^{1,\beta_g}(B_1')$. Then h enjoys the boundary local $C^{1,\beta'}$-estimate for some $\beta' \in (0,1)$, due to Milakis and Silvestre in [34, Proposition 2.2], and Winter in [35, Theorem 3.1]. However, we cannot guarantee that the Hölder exponent β' coincides with β_g. Recently, [4, Theorem 2.1] derived the boundary local $C^{1,\beta''}$-estimate for $\beta'' = \min\{\overline{\alpha}, \beta_g\}$, and the pointwise boundary C^{1,β_g}-estimate in the sense that there exists a linear function l which approximates h in C^{1,β_g}-manner.

Proof of Lemma 5.1 By the smallness regime in Remark 4.1, we may assume that $u \in C(\Omega)$ is a viscosity solution with

$$\|u\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le 1,\ \|g\|_{C^{1,\beta_g}(B_1\cap\{y_n=\phi(y')\})} \le 1,\ \|f\|_{L^\infty(B_1\cap\{y_n>\phi(y')\})} \le \varepsilon_0,$$

and $\nu_0 = \nu_1 = 1$. As in [5, 6, 11], the proof is based on the induction argument: we claim that there exist universal constants $0 < \rho \ll 1$, $C_0 > 1$ and a sequence of affine functions

$$l_k(y) := a_k + b_k \cdot (y - x),$$

where $\{a_k\}_{k=1}^\infty \subset \mathbb{R}$ and $\{b_k\}_{k=1}^\infty \subset \mathbb{R}^n$ satisfy, for every $k \in \mathbb{N}$,

(E1) $\sup_{y\in B_{\rho^k}(x)\cap\{y_n>\phi(y')\}} |u(y) - l_k(y)| \le \rho^{k(1+\alpha')}$;

(E2) $|a_k - a_{k-1}| \le C_0\rho^{(k-1)(1+\alpha')}$ and $|b_k - b_{k-1}| \le C_0\rho^{(k-1)\alpha'}$.

(i) (Initial step) Without loss of generality, we may assume $x = 0$. Let h be the approximation function coming from Lemma 5.2 for a constant $\mu > 0$ to be determined later. Then, by the pointwise boundary estimate for uniformly elliptic fully nonlinear operators obtained in [4, Theorem 2.1], there exist an affine function l_1 and a universal constant $C_0 > 0$ such that

$$\sup_{y\in B_\rho\cap\{y_n>\phi(y')\}} |h(y) - l_1(y)| \le C_0\rho^{1+\beta_g} \quad \text{for every } 0 < \rho \le 1/2,$$

and

$$|l_1(0)| + |Dl_1(0)| \le C_0.$$

Then the triangle inequality yields that

$$\sup_{y\in B_\rho\cap\{y_n>\phi(y')\}} |u(y) - l_1(y)| \le C_0\rho^{1+\beta_g} + \mu.$$

We now select a universal constant $0 < \rho \ll 1$ small enough so that

$$C_0\rho^{\beta_g} \le \frac{1}{2}\rho^{\alpha'} \quad \text{and} \quad \rho^{1-\alpha'(1+s(\Phi))} \le 1, \tag{5.5}$$

which is possible due to the choice of α'. In a sequel, we choose a constant $\mu := \rho^{1+\alpha'}/2$ and set $a_0 = 0$, $b_0 = 0$, $a_1 = l_1(0)$, and $b_1 = Dl_1(0)$, which completes the proof of the initial step.

(ii) (Iterative procedure) We now suppose that (E1) and (E2) hold true for $k \ge 1$. We then verify (E1) and (E2) for $k + 1$. For this purpose, we define a rescaled function

$$u_k(y) := \frac{u(\rho^k y) - l_k(\rho^k y)}{\rho^{k(1+\alpha')}}.$$

Then u_k satisfies

$$\begin{cases} \Phi_k(y, |\xi_k + Du_k|)F(D^2u_k) = f_k(y) & \text{in } B_1 \cap \{y_n > \phi_k(y')\} \\ u_k(y) = g_k(y) & \text{on} B_1 \cap \{y_n = \phi_k(y')\}, \end{cases}$$

where

$$F_k(M) := \rho^{k(1-\alpha')} F(\rho^{k(\alpha'-1)} M), \quad \Phi_k(y,t) := \frac{\Phi(\rho^k y, \rho^{k\alpha'} t)}{\Phi(\rho^k y, \rho^{k\alpha'})},$$
$$f_k(y) := \frac{\rho^{k(1-\alpha')}}{\Phi(\rho^k y, \rho^{k\alpha'})} f(\rho^k y), \quad g_k(y) := \frac{g(\rho^k y) - l_k(\rho^k y)}{\rho^{k(1+\alpha')}},$$
$$\phi_k(y') := \rho^{-k}\phi(\rho^k y'), \text{ and } \xi_k := \rho^{-k\alpha'} b_k.$$

It can be easily checked that

(a) F_k satisfies (A1) with the same constants (λ, Λ);
(b) Φ_k satisfies (A2) with the same constants $(i(\Phi), s(\Phi))$ and $\Phi_k(y, 1) \equiv 1$;
(c) $\|u_k\|_{L^\infty(B_1 \cap \{y_n > \phi_k(y')\})} \leq 1$ by the induction hypothesis;
(d) $\|f_k\|_{L^\infty(B_1 \cap \{y_n > \phi_k(y')\})} \leq L\varepsilon_0 \rho^{k(1-\alpha'(1+s(\Phi)))} \leq L\varepsilon_0$ by (5.5);
(e) $\|D^2\phi_k\|_{L^\infty(B_1 \cap \{y_n > \phi_k(y')\})} \leq \rho^k \|D^2\phi\|_{L^\infty(B_1 \cap \{y_n > \phi(y')\})} \leq \|D^2\phi\|_{L^\infty(B_1 \cap \{y_n > \phi(y')\})}$.

Moreover, for g_k, we can compute

$$\begin{aligned}|Dg_k(y) - Dg_k(z)| = \rho^{-k\alpha'} |Dg(\rho^k y) - Dg(\rho^k z)| &\leq \|g\|_{C^{1,\beta_g}(\partial\Omega)} \cdot \rho^{-k\alpha'} |\rho^k (y-z)|^{\beta_g} \\ &\leq \|g\|_{C^{1,\beta_g}(\partial\Omega)} \cdot |y-z|^{\beta_g}.\end{aligned}$$

By recalling the fact that $u = g$ on $\partial\Omega$ together with the induction hypothesis, we observe that

$$\|g_k\|_{C^{1,\beta_g}(B_1 \cap \{y_n > \phi_k(y')\})} \leq \|g\|_{C^{1,\beta_g}(\partial\Omega)} \leq 1.$$

Hence, we now apply Lemma 5.2 for u_k and then follow the argument in the initial step to ensure the existence of an affince function $\overline{l}(y) := \overline{a} + \overline{b} \cdot y$ such that

$$\sup_{y \in B_\rho \cap \{y_n > \phi_k(y')\}} |u_k(y) - \overline{l}(y)| \leq \rho^{1+\alpha'} \quad \text{and} \quad |\overline{a}|, |\overline{b}| \leq C_0.$$

By scaling back, we conclude that

$$\sup_{y \in B_{\rho^{k+1}} \cap \{y_n > \phi(y')\}} |u(y) - l_{k+1}(y)| \leq \rho^{(k+1)(1+\alpha')},$$

where

$$l_{k+1}(y) := l_k(y) + \rho^{k(1+\alpha')} \cdot \overline{l}(\rho^{-k} y).$$

Here note that

$$|a_{k+1} - a_k| = \rho^{k(1+\alpha')} |\overline{a}| \leq C_0 \rho^{k(1+\alpha')},$$
$$|b_{k+1} - b_k| = \rho^{k\alpha'} |\overline{b}| \leq C_0 \rho^{k\alpha'}.$$

Therefore, (E1) and (E2) hold for $k+1$.

□

Proof of Theorem 1.2 We first consider the degenerate case, i.e., $i(\Phi) \geq 0$. We note that, by applying Lemma 5.1, a viscosity solution can be approximated by an affine function with an error of order $r^{1+\alpha'}$ at boundary points. By following the argument in the proof of [4, Theorem 1.1], we can derive the desired global $C^{1,\alpha}$-estimate with α satisfying (1.5).

On the other hand, for the singular case ($i(\Phi) < 0$), we employ the idea of [5, Theorem 1.1]. Indeed, we claim that Lemma 5.1 still holds for the singular case. Theorem 4.4 guarantees that

$$\|u\|_{C^{0,1}(B_{3/4}\cap\{y_n>\phi(y')\})} \le c,$$

for a universal constant $c > 0$. Then u is a viscosity solution of

$$\begin{cases} \tilde{\Phi}(y, |Du|)F(D^2u) = \tilde{f}(y) & \text{in } B_{3/4}\cap\{y_n > \phi(y')\} \\ u(y) = g(y) & \text{on } B_{3/4}\cap\{y_n = \phi(y')\}, \end{cases}$$

where

$$\tilde{\Phi}(y,t) := t^{-i(\Phi)}\Phi(y,t) \quad \text{and} \quad \tilde{f}(y) := |Du|^{-i(\Phi)} f(y).$$

Here $\tilde{\Phi}$ satisfies the condition (A2) with $i(\tilde{\Phi}) = 0$, $s(\tilde{\Phi}) = s(\Phi) - i(\Phi)$, and

$$\|\tilde{f}\|_{L^\infty(\Omega)} \le c^{-i(\Phi)}\varepsilon_0.$$

Thus, one can repeat the argument in the proof of Lemma 5.1 to obtain the global $C^{1,\alpha}$-estimate. □

6 Comparison principle and Perron's method

The purpose of this section is to study the classical result of comparison principle and as a consequence, to deduce the existence of a viscosity solution to (1.1) by Perron's method. Nevertheless, the assumptions (A1)–(A4) are not sufficient to obtain the aforementioned results. Therefore, we require an additional assumption (A5) which guarantees the comparison principle for approximated Dirichlet problems. Before we precisely state this new assumption, we summarize known results regarding the comparison principle.

Remark 6.1 (Comparison principle) Let $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ be a *proper* map. In other words, H satisfies

$$H(x,r,p,X) \le H(x,r,p,Y) \quad \text{whenever} \quad X \le Y, \tag{6.1}$$

$$H(x,r,p,X) \le H(x,s,p,X) \quad \text{whenever} \quad s \le r. \tag{6.2}$$

Then we say that H satisfies the *comparison principle* if the following holds:

Let $v \in \mathrm{USC}(\overline{\Omega})$ [resp. $w \in \mathrm{LSC}(\overline{\Omega})$] be a subsolution [resp. supersolution] of $H = 0$ in Ω and $v \le w$ on $\partial\Omega$. Then $v \le w$ in $\overline{\Omega}$.

We refer to [8, 13, 23, 28, 29] for several sufficient conditions of the comparison principle. In short, H satisfies the comparison principle if $H(x,r,p,X)$ is independent of x, and one of the following conditions holds:

(i) $H(x,r,p,X)$ is strictly decreasing in r and H is degenerate elliptic (i.e., H satisfies (6.1)), or
(ii) $H(x,r,p,X)$ is non-increasing in r and H is uniformly elliptic.

It is noteworthy that the condition that H is independent of x can be relaxed to some extra structural conditions on H, which display a kind of smoothness on H with respect to x-variable; see [13, 23, 28] for details.

We now consider a proper map

$$H(x, u, Du, D^2u) := \Phi(x, |Du|)F(D^2u) - f(x).$$

It is easily checked that H is degenerate elliptic, nonincreasing in r, and H depends on x. In view of the previous remark, we cannot expect the comparison principle for H and Perron's method for the associated Dirichlet problem. To overcome this challenge, we will impose an additional structure condition on H and approximate the map H to ensure the strict monotonicity with respect to r-variable:

(i)(A5) There exists a continuous function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$ and

$$\Phi(x, \theta|x-y|)F(X) - \Phi(y, \theta|x-y|)F(-Y) \le \omega(\theta|x-y|^2 + |x-y|),$$

whenever $\theta > 0$, $x, y \in \Omega$, $X, Y \in \mathcal{S}(n)$ and

$$-3\theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\theta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{6.3}$$

(ii) Let us consider the approximated problem given by

$$H_\varepsilon(x, u, Du, D^2u) := \Phi(x, |Du|)F(D^2u) - f(x) - \varepsilon u = 0 \quad \text{in} \quad \Omega \tag{6.4}$$

for $\varepsilon > 0$. Clearly, H_ε is strictly decreasing in r.

Remark 6.2 We would like to provide a concrete example of (Φ, F) that satisfies the condition (A5). Indeed, suppose that F is degenerate elliptic and Φ is independent of x-variable, that is, $\Phi(x, \xi) \equiv \Phi(\xi)$. Then for X, Y satisfying the relation (6.3), we have $F(X) \le F(-Y)$. Therefore, we observe that

$$\Phi(x, \theta|x-y|)F(X) - \Phi(y, \theta|x-y|)F(-Y) = \Phi(|x-y|)(F(X) - F(-Y)) \le 0.$$

We also refer to [8, Condition 2], [13, Condition (3.14)], and [6, Remark 2.2] for the corresponding assumptions in different settings.

We are now ready to present the second version of a comparison principle, which can be seen as a variant of [6, Theorem 2.3].

Lemma 6.3 (Comparison principle II) *Suppose that the assumptions (A1), (A2), (A5) are in force and $f \in C(\overline{\Omega})$. Then H_ε satisfies the comparison principle:*

Let v and w be a viscosity subsolution and a supersolution of (6.4), respectively. If $v \le w$ on $\partial\Omega$, then $v \le w$ in Ω.

Proof By contradiction, we suppose that

$$L_0 := \sup_{x \in \overline{\Omega}} (v(x) - w(x)) > 0.$$

For any $\theta > 0$, we define

$$L_\theta := \sup_{x, y \in \overline{\Omega}} \left[v(x) - w(y) - (\theta/2)|x-y|^2 \right]$$

and clearly $L_\theta \ge L_0$. Suppose that the maximum L_θ is attained at a point $(x_\theta, y_\theta) \in \overline{\Omega} \times \overline{\Omega}$. It implies from [13, Lemma 3.1] that

$$\lim_{\theta \to \infty} \theta |x_\theta - y_\theta|^2 = 0.$$

This one and the fact that $v \le w$ on $\partial\Omega$ yield that $x_\theta, y_\theta \in \Omega$ for $\theta > 0$ large enough. At this moment, we are able to apply Ishii–Jensen lemma, [13, Teorem 3.2], to ensure that there exist a limiting super-jet $(\theta(x_\theta - y_\theta), X_\theta)$ of v at x_θ and a limiting sub-jet $(\theta(x_\theta - y_\theta), -Y_\theta)$ of w at y_θ so that

$$-3\theta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\theta & 0 \\ 0 & Y_\theta \end{pmatrix} \le 3\theta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and

$$\begin{cases} \Phi\left(x_\theta, \theta|x_\theta - y_\theta|\right) F(X_\theta) - \varepsilon v(x_\theta) \ge f(x_\theta) \\ \Phi\left(y_\theta, \theta|x_\theta - y_\theta|\right) F(-Y_\theta) - \varepsilon w(y_\theta) \le f(y_\theta). \end{cases}$$

By using the relation $L_0 = \lim_{\theta\to\infty}[v(x_\theta) - w(y_\theta)]$ and the assumption (A5), we have, for sufficiently large $\theta > 0$,

$$\begin{aligned} \frac{\varepsilon L_0}{2} &\le \varepsilon[v(x_\theta) - w(y_\theta)] \\ &\le f(y_\theta) - f(x_\theta) + \Phi\left(x_\theta, \theta|x_\theta - y_\theta|\right) F(X_\theta) - \Phi\left(y_\theta, \theta|x_\theta - y_\theta|\right) F(-Y_\theta) \\ &\le f(y_\theta) - f(x_\theta) + \omega(\theta|x_\theta - y_\theta|^2 + |x_\theta - y_\theta|). \end{aligned}$$

Since $f \in C(\overline{\Omega})$ and $\omega(0+) = 0$, we arrive at a contradiction when $\theta \to \infty$. □

We now turn our attention to showing the existence of viscosity sub/supersolutions to (6.4); we refer to [6, Lemma 2.2] for a similar result.

Lemma 6.4 (Existence of sub/supersolutions) *Suppose the assumptions (A1)–(A4) are in force. Then for every $\varepsilon \in (0, 1)$, there exist a viscosity subsolution $v_\varepsilon \in C(\overline{\Omega})$ and a viscosity supersolution $w_\varepsilon \in C(\overline{\Omega})$ of (6.4) with $v_\varepsilon = w_\varepsilon = g$ on $\partial\Omega$. Moreover, there exists a positive constant $c \equiv c(n, \lambda, \Lambda, \nu_0, L, r, \mathrm{diam}(\Omega), \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)})$ such that*

$$-c \le v_\varepsilon \le w_\varepsilon \le c \quad \textit{for any } 0 < \varepsilon < 1.$$

Proof Let $z \in \partial\Omega$ be a fixed point. There exists a point $x_z \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $\overline{B_r(x_z)} \cap \overline{\Omega} = \{z\}$ with $r = |z - x_z|$ since Ω satisfies the exterior ball condition; see Lemma 2.2. We now consider a function $v_z : \overline{\Omega} \to [0, \infty)$ defined by

$$v_z(x) := K\left(r^{-\kappa_0} - |x - x_z|^{-\kappa_0}\right)$$

for positive constants $\kappa_0 := (n\Lambda + 1)/\lambda$ and $K \ge \min\left\{1, R^{\kappa_0+1}/\kappa_0\right\}$ to be determined later, with $R := r + \mathrm{diam}(\Omega)$. Note that $v_z(z) = 0$, $v_z > 0$ in Ω, and direct calculations yield that

$$Dv_z(x) = K\kappa_0 \frac{x - x_z}{|x - x_z|^{\kappa_0+2}}$$

and

$$D^2 v_z(x) = K\kappa_0 \frac{I}{|x - x_z|^{\kappa_0+2}} - K\kappa_0(\kappa_0 + 2) \frac{(x - x_z) \otimes (x - x_z)}{|x - x_z|^{\kappa_0+4}}.$$

Due to the choice of κ_0, we have

$$F(D^2 v_z(x)) \le \frac{K\kappa_0}{|x - x_z|^{\kappa_0+2}} \left((n-1)\Lambda - (\kappa_0 + 1)\lambda\right) \le -\frac{K\kappa_0}{|x - x_z|^{\kappa_0+2}}.$$

On the other hand, for a fixed $\delta \in (0, 1)$, we further define

$$v_{z,\delta}(x) := g(z) + \delta + M_\delta v_z(x),$$

where the constant $M_\delta \geq 1$ can be chosen so that $v_{z,\delta} \geq g$ on $\partial\Omega$. This is possible because $K \geq 1$ and g is continuous on $\partial\Omega$. Indeed, M_δ depends only on the modulus of continuity of g, and is independent of z. Then we see that

$$\begin{aligned}
&\Phi(x, |Dv_{z,\delta}(x)|)F(D^2 v_{z,\delta}(x)) - \varepsilon v_{z,\delta}(x) \\
&\leq \frac{\nu_0}{L} \min\left\{\left(\frac{K\kappa_0}{|x - x_z|^{\kappa_0+1}}\right)^{i(\Phi)}, \left(\frac{K\kappa_0}{|x - x_z|^{\kappa_0+1}}\right)^{s(\Phi)}\right\} \cdot \left(-M_\delta \frac{K\kappa_0}{|x - x_z|^{\kappa_0+2}}\right) + \varepsilon \|g\|_{L^\infty(\partial\Omega)} .
\end{aligned}$$

Here if we let $A := K\kappa_0/|x - x_z|^{\kappa_0+1}$, then

$$\begin{aligned}
\min\left\{A^{i(\Phi)}, A^{s(\Phi)}\right\} &= A^{-1} \min\left\{A^{i(\Phi)+1}, A^{s(\Phi)+1}\right\} \\
&\geq \left(\frac{K\kappa_0}{r^{\kappa_0+1}}\right)^{-1} \min\left\{\left(\frac{K\kappa_0}{R^{\kappa_0+1}}\right)^{i(\Phi)+1}, \left(\frac{K\kappa_0}{R^{\kappa_0+1}}\right)^{s(\Phi)+1}\right\}.
\end{aligned}$$

Note that we need to select K so that $K\kappa_0 \geq R^{\kappa_0+1}$. Since $0 < i(\Phi) + 1 \leq s(\Phi) + 1$, we conclude that

$$\min\left\{A^{i(\Phi)}, A^{s(\Phi)}\right\} \geq \left(\frac{\kappa_0}{r^{\kappa_0+1}}\right)^{-1} \left(\frac{\kappa_0}{R^{\kappa_0+1}}\right)^{i(\Phi)+1} K^{i(\Phi)}.$$

Hence, we deduce that

$$\begin{aligned}
&\Phi(x, |Dv_{z,\delta}(x)|)F(D^2 v_{z,\delta}(x)) - \varepsilon v_{z,\delta}(x) \\
&\qquad \leq -\frac{\nu_0}{L} \left(\frac{\kappa_0}{r^{\kappa_0+1}}\right)^{-1} \left(\frac{\kappa_0}{R^{\kappa_0+1}}\right)^{i(\Phi)+1} \frac{\kappa_0}{R^{\kappa_0+2}} K^{i(\Phi)+1} + \|g\|_{L^\infty(\partial\Omega)} .
\end{aligned}$$

Therefore, we can choose $K = K(n, \lambda, \Lambda, \nu_0, L, r, \mathrm{diam}(\Omega), \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)})$ large enough so that

$$\Phi(x, |Dv_{z,\delta}(x)|)F(D^2 v_{z,\delta}(x)) - \varepsilon v_{z,\delta}(x) \leq -\|f\|_{L^\infty(\Omega)} \quad \text{in } \Omega,$$

i.e., $v_{z,\delta}$ is a viscosity supersolution to (6.4).

Finally, we define

$$w_\varepsilon(x) := \inf\left\{v_{z,\delta}(x) : z \in \partial\Omega \text{ and } \delta \in (0, 1)\right\}.$$

It is easy to check that w_ε is a viscosity supersolution to (6.4) in Ω and enjoys the boundary condition $w_\varepsilon = g$ on $\partial\Omega$. Moreover, it immediately follows from the construction of w_ε that

$$w_\varepsilon \leq C(n, \lambda, \Lambda, \nu_0, L, \mathrm{diam}(\Omega), \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}) \quad \text{in } \Omega$$

for any $\varepsilon \in (0, 1)$. The existence of a viscosity subsolution v_ε and its lower bound can be shown in a similar manner. Finally, since $v_\varepsilon = w_\varepsilon = g$ on $\partial\Omega$, Lemma 6.3 implies that $v_\varepsilon \leq w_\varepsilon$ in Ω. □

Proof of Theorem 1.4 An application of Perron's method [13, Theorem 4.1] together with Lemma 6.3 and Lemma 6.4 yields the existence of a viscosity solution u_ε to the approximated equation (6.4) with the boundary condition $u_\varepsilon = g$.

We now understand u_ε as a viscosity solution of

$$\begin{cases} \Phi(x, |Du_\varepsilon|)F(D^2u_\varepsilon) = f_\varepsilon(x) & \text{in} \quad \Omega \\ u_\varepsilon(x) = g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

where $f_\varepsilon(x) := f(x) + \varepsilon u_\varepsilon(x)$. Here note that $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^\infty(\Omega)$ by Lemma 6.4. Then, by applying [5, Lemma 3.1] and Theorem 4.4 (or just by applying the stronger result Theorem 1.2), we have that $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Therefore, we can extract a uniformly converging subsequence such that $u_{\varepsilon_j} \to u_\infty$ when $\varepsilon_j \to 0$, and by Lemma 2.6, we conclude that u_∞ solves (1.1). □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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