



Local metaplectic correspondence and applications

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Abstract

We revisit the local metaplectic correspondence previously constructed and studied by Flicker and Kazhdan. After restoring and generalizing some of their results, we get several interesting applications to the representation theory of a Kazhdan–Patterson covering group over a p -adic field, including a full criterion of the irreducibility of the Bernstein–Zelevinsky product of two cuspidal representations, the classification of essentially square integrable and tempered representations, and more interestingly, the calculation of the Whittaker dimension of an irreducible representation based on a conjecture.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	Notation	4
2.2	Kazhdan–Patterson covering group	5
2.3	Metaplectic tensor product	6
3	Metaplectic correspondence	9
3.1	Definition	9
3.2	The case of a maximal split torus	12
3.3	Compatibility with parabolic induction and Jacquet module	13
3.4	Classification of essentially square integrable representations	16
3.5	Classification of essentially tempered representations	18
4	Calculation of Whittaker dimension	20
4.1	A general introduction of the Harish–Chandra germ expansion	20
4.2	Whittaker space and dimension	21
4.3	The main Whittaker dimensional formula for essentially square integrable representations	23
4.4	Evidence of Conjecture 4.6	27
5	Conjecture 14 of [30] for a Kazhdan–Patterson covering group	30
	References	31

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1 Introduction

Motivated by the famous program of Robert Langlands, a main topic in the representation theory is the study of the category of smooth representations of a p -adic reductive group, and functorial maps between such categories for different p -adic reductive groups over a fixed base field. We leave [8] for an expository introduction. Still motivated by the hope of generalizing Langlands' philosophy to larger classes of p -adic groups, the representation theory of finite central covering groups of p -adic reductive groups gradually became shining in the past decades. Correspondingly, we leave [19] for a historical introduction.

The main focus of this article is more concrete: we study a certain functorial map, called the local metaplectic correspondence, from the set of (certain) representations of a Kazhdan-Patterson covering group of a p -adic general linear group, to that of a p -adic general linear group. Our goal is, on the one hand, to restore and slightly generalize the existing theory, and on the other hand, to dig out interesting applications of this fascinating map.

We give a brief introduction of the above theory. Let F be a p -adic field and assume that the subgroup of F^\times of n -th roots of unity, denoted by μ_n , is of order n . We consider a certain n -fold covering group \widetilde{G}_r as a central extension of $G_r := \mathrm{GL}_r(F)$ by μ_n , constructed by Kazhdan-Patterson in their seminal work [32] (see §2.2 for more details). In particular when $r = 1$ we get a central extension of F^\times by μ_n twisted by a power of the n -th Hilbert symbol, and when $r = 2$ we get the cover considered by Kubota [34] (which is for $\mathrm{SL}_2(F)$ but can be extended to $\mathrm{GL}_2(F)$).

The (local) metaplectic correspondence (for essentially square integrable representations) is a map from the set of genuine essentially square integrable representations of \widetilde{G}_r to the set of essentially square integrable representations of G_r that are trivial on $\mu_n I_r \subset G_r$. Such a map is determined by a formula of Harish-Chandra characters of representations and constructed using a trace formula comparison. Historically, in the case $r = n = 2$ it first occurred as a map between two different classes of modular forms and is called the "Shimura correspondence" [45]. Using a trace formula comparison, Flicker [17] constructed the metaplectic correspondence for $r = 2$ and general n and Flicker-Kazhdan [18] generalized the above theory to general r and n .

Our starting point is the metaplectic correspondence for essentially square integrable representations of \widetilde{G}_r and G_r constructed in [18]. The first result of this article is to construct the metaplectic correspondence for essentially square integrable representations of Levi subgroups of \widetilde{G}_r and G_r as well (cf. §3.1). After that, we study its compatibility with parabolic induction and Jacquet module (cf. §3.3).

In the rest of Sect. 3, we focus on following applications of metaplectic correspondence, including:

- (cf. Proposition 3.14) For a cuspidal representation $\widetilde{\rho}$ of \widetilde{G}_r , we determine the exact value of the positive real number $s(\widetilde{\rho})$ (which is unique) such that $(\widetilde{\rho} \widetilde{\times} \widetilde{\rho} \nu^{s(\widetilde{\rho})})_{\widetilde{\omega}}$ is reducible. Here $\widetilde{\times}$ denotes the Bernstein-Zelevinsky product, $\nu(\cdot) := |\det(\cdot)|_F$ is an unramified character of G_r and $\widetilde{\omega}$ is a compatible genuine character of $Z(\widetilde{G}_{2r})$.
- (cf. Proposition 3.16) We classify all the essentially square integrable representations of \widetilde{G}_r as well as their image under the metaplectic correspondence, in the sense of the Zelevinsky classification (cf. [31]).
- (cf. Proposition 3.17, Proposition 3.20) We classify all the essentially tempered representations of \widetilde{G}_r via the Zelevinsky classification, and we study a possible metaplectic correspondence for essentially tempered representations.

The above applications serve well as “appetizers”, and in Sect. 4 our “main dish” is to calculate the dimension of the Whittaker space of an irreducible representation of \widetilde{G}_r . Unlike the linear case, in general for a covering group such a Whittaker space does not satisfy the multiplicity one theorem. This causes additional difficulties in studying automorphic representations and local–global compatibilities, since usually the construction of a global L -function, in particular its factorization as an Euler product, is based on the uniqueness of Whittaker model (cf. [27] for the Rankin-Selberg integral for instance).

To achieve a reasonable theory of global automorphic representations, it is important to understand the dimension of a Whittaker space. This is illustrated in the work of Kaplan (cf. [29, Appendix A], [30]), using generalized doubling method to study the L -function of a pair of representations of a covering group of G_r ¹. A key idea in his argument is that, one need to pick a certain generalized Speh representation whose Whittaker dimension is one (cf. [30, Conjecture 14]). The consideration of such a generalized Speh representation dates back to the work of Suzuki [46] of distinguished representations. Then the Euler product factorization of the related generalized doubling integral follows from this multiplicity one fact.

We explain our idea of calculating the Whittaker dimension of an irreducible representation for a Kazhdan-Patterson covering group. In our case, the Whittaker functor is exact and compatible with the Bernstein-Zelevinsky product (cf. Proposition 4.2). Then using the Zelevinsky classification essentially we only need to calculate the Whittaker dimension of an essentially square integrable representation $\widetilde{\pi}$ of \widetilde{G}_r (cf. Remark 4.14). The Whittaker dimension of $\widetilde{\pi}$ can be re-explained by the Harish-Chandra germ function $c_{\widetilde{\pi}}$ of $\widetilde{\pi}$ evaluated at the identity, which is a well-known result in the linear case [41, 43] and also generalized to the covering group case [42]. Let π be the metaplectic lift of $\widetilde{\pi}$, then using the metaplectic correspondence the above question is reduced to calculating the Harish-Chandra germ function c_{π} of π evaluated at diagonal elements of order dividing n in G_r (cf. Corollary 4.3).

Let $\pi' = Z(\rho, [a, b])$ be the unique irreducible subrepresentation of the Bernstein-Zelevinsky product $\rho v^a \times \rho v^{a+1} \times \cdots \times \rho v^b$, where ρ is a cuspidal representation of G_{r_0} and $a \leq b$ are two integers. We propose a general conjecture (cf. Conjecture 4.6) about the value of $c_{\pi'}$ at a diagonal element of order dividing n . Our conjecture has its own interest, which dates back to the classical but difficult question of calculating special values of a Harish-Chandra germ function (cf. [26, Introduction]). Using this conjecture and Tadic’s determinantal formula, we are able to calculate the value of c_{π} at a diagonal element of order n , and thus the Whittaker dimension of $\widetilde{\pi}$ (cf. Theorem 4.7). Our dimensional formula seems to be neat enough.

We verify Conjecture 4.6 for the following two cases (cf. §4.4):

- $r_0 = 1$ and ρ is a character of F^\times ;
- $a = b$ and π' is cuspidal.

The first case is rather direct, since π' itself is a character. The proof of the second case is based on a rudiment usage of the simple type theory of Bushnell-Kutzko [12]. Hopefully our argument for the second case can be used to other groups and their supercuspidal representations.

Finally, in Sect. 5 we explain how our results could be used to study an analogue of [30, Conjecture 14] for a Kazhdan-Patterson covering group.

We end the introduction by comparing our result with other work of calculating the Whittaker dimension of an irreducible representation of a covering group. For a Kazhdan-

¹ We should warn the readers that in *loc. cit.* a different covering group of G_r is considered. But hopefully a similar theory could be established for a Kazhdan-Patterson covering group.

Patterson covering group, to the knowledge of the author, the only known cases² are principal series and their subquotients [32, 33, 46], as well as depth 0 cuspidal representations [7]. For general covering groups, still the known cases are confined to principal series and their subquotients, and depth 0 cuspidal representations. We refer to the work of F. Gao and his collaborators for more details, *cf.* [20–24].

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2 Preliminaries

2.1 Notation

Throughout this article F will be a non-archimedean locally compact field of residue characteristic p , and beginning from Sect. 3 we will also assume F to be of characteristic 0. Let \mathfrak{o}_F be its ring of integers, let \mathfrak{p}_F be the maximal ideal of \mathfrak{o}_F and let \mathbb{F}_q be its residue field. Let $|\cdot|_F$ be the normalized absolute value of F .

We consider ℓ -groups in the sense of Bernstein-Zelevinsky [4]. The ℓ -groups we are considering in this paper will be the F -points of reductive groups as well as their finite central extensions. For an ℓ -group G and its closed subgroup H , we denote by $Z(G)$ the center of G and by $Z_G(H)$ the centralizer of H in G . In particular, we let $G_r = \mathrm{GL}_r(F)$ for a positive integer r .

By representations of an ℓ -group G , we always mean complex smooth representations. Similarly by characters we mean one-dimensional representations. We denote by $\mathrm{Rep}(G)$, $\mathrm{Irr}(G)$, $\mathrm{Temp}(G)$, $\mathrm{Sqrt}(G)$, $\mathrm{Cusp}(G)$ the sets of equivalence classes of finite length representations, irreducible representations, irreducible essentially tempered representations, irreducible essentially square integrable representations and irreducible cuspidal representations respectively. For a closed subgroup Z of $Z(G)$ and a character ω of Z , we denote by $\mathrm{Irr}_\omega(G)$, $\mathrm{Temp}_\omega(G)$, $\mathrm{Sqrt}_\omega(G)$, $\mathrm{Cusp}_\omega(G)$ the corresponding subsets of those equivalence classes of representations with central character restricted to Z being ω . For a representation π of H and a positive integer m , we let $m \cdot \pi$ be the direct sum of m -copies of π . If π is irreducible, then we let ω_π be its central character.

If G is a finite central extension of a reductive group over F (or even an open finite index subgroup of such a group), we let θ_π be the Harish-Chandra character of π as a locally integrable function on G , that is smooth on the semi-simple regular locus of G . This notation can also be generalized to any finite length representations of G by considering the sum of the Harish-Chandra characters of each irreducible subquotient. We also remark that if $\pi \in \mathrm{Irr}(G)$, then π is determined by θ_π . Moreover if $\pi \in \mathrm{Sqrt}(G)$, then π is determined by the restriction of θ_π to the elliptic locus (*cf.* for instance [16]).

For a non-negative integer m and an integer k , we denote by $\binom{m}{k}$ the binomial coefficient of x^k in $(1+x)^m$, and by $m!$ the factorial of m .

² Here, by “known” we mean that a concrete value could theoretically be obtained, although sometimes there exist combinatorial difficulties. Also see Remark 4.10.

2.2 Kazhdan-Patterson covering group

Fix a positive integer n . Let $\mu_n := \mu_n(F)$ be the group of n -th roots of unity in F . From now on we further assume that the cardinality of μ_n is n . We let $F^{\times n} := \{x^n \mid x \in F^\times\}$ be an open finite index subgroup of F^\times . We denote by

$$(\cdot, \cdot)_n : F^\times \times F^\times \rightarrow \mu_n$$

the n -th order Hilbert symbol. It is a bimultiplicative antisymmetric pairing that defines a non-degenerate bimultiplicative pairing on $F^\times/F^{\times n} \times F^\times/F^{\times n}$ (cf. [52, XIII.§5]).

For a (connected) reductive group G over F (we use G to denote its F -points by abuse of notation), we consider its central extension of ℓ -groups

$$0 \longrightarrow \mu_n \longrightarrow \tilde{G} \xrightarrow{\mathbf{p}} G \longrightarrow 0,$$

which is called an n -fold covering group. The set of equivalence classes of such central extensions is in bijection with the continuous cohomology group $H^2(G, \mu_n)$. More precisely, we fix a continuous section $s : G \rightarrow \tilde{G}$, that is, $\mathbf{p} \circ s = \text{id}$. Then the central extension \tilde{G} corresponds to the cohomology class of the 2-cocycle $\sigma : G \times G \rightarrow \mu_n$ satisfying

$$s(g_1)s(g_2) = s(g_1g_2)\sigma(g_1, g_2), \quad g_1, g_2 \in G.$$

In particular, we consider the case $G = G_r$. Let $\sigma_{\text{SL}_{r+1}}$ be the special 2-cocycle of $\text{SL}_{r+1}(F)$ considered by Matsumoto [37] with respect to the Steinberg symbol $(\cdot, \cdot)_n^{-1}$. Let $\sigma^{(0)}$ be the pull-back of $\sigma_{\text{SL}_{r+1}}$ via

$$G_r \rightarrow \text{SL}_{r+1}(F), \quad g \mapsto \text{diag}(\det(g)^{-1}, g).$$

The explicit construction and calculation of $\sigma^{(0)}$ has been done in [3]. Finally for any $c \in \mathbb{Z}/n\mathbb{Z}$ let $\sigma^{(c)}$ be the 2-cocycle satisfying

$$\sigma^{(c)}(g_1, g_2) = \sigma^{(0)}(g_1, g_2) \cdot (\det(g_1), \det(g_2))_n^c, \quad g_1, g_2 \in G_r.$$

From now on we fix $c \in \mathbb{Z}$, and we call the n -fold cover \tilde{G}_r corresponding to $\sigma^{(c)}$ a *Kazhdan-Patterson covering group* of G_r . In particular \tilde{F}^\times will denote the n -fold cover of F^\times with respect to the 2-cocycle $(\cdot, \cdot)_n^c$.

For a closed subgroup H of G_r , let $\mathbf{p}^{-1}(H)$ or \tilde{H} be the preimage of H in \tilde{G} and let $H^{(n)} = \{h \in H \mid \det(h) \in F^{\times n}\}$ be an open normal subgroup of H of finite index.

Let β be a composition of r . It means that $\beta = (r_1, \dots, r_k)$ for certain k and positive integers r_i such that $r = r_1 + \dots + r_k$. Let G_β be the standard Levi subgroup of G_r with respect to β . So $G_\beta \simeq G_{r_1} \times \dots \times G_{r_k}$ with each G_{r_i} being regarded as a subgroup of G_r via the embedding $g_i \mapsto \text{diag}(I_{r_1}, \dots, I_{r_{i-1}}, g_i, I_{r_{i+1}}, \dots, I_{r_k})$.

We recall the following properties concerning a Kazhdan-Patterson covering group.

1. When $r = 1$, we have $\sigma^{(c)}(x, y) = (x, y)_n^c$ for $x, y \in F^\times$.
2. ([3, Theorem 11]) The restriction of $\sigma^{(c)}$ to $G_\beta \times G_\beta$ is given by

$$\begin{aligned} &\sigma^{(c)}(\text{diag}(x_1, \dots, x_k), \text{diag}(y_1, \dots, y_k)) \\ &= \prod_{i=1}^k \sigma^{(c)}(x_i, y_i) \cdot \prod_{1 \leq i < j \leq k} (\det(x_i), \det(y_j))_n^{c+1} (\det(x_j), \det(y_i))_n^c, \quad x_i, y_i \in G_{r_i}. \end{aligned} \tag{2.1}$$

3. ([15, §2.1, Lemma 1]) The center of \widetilde{G}_r is given by

$$Z(\widetilde{G}_r) = \mathfrak{p}^{-1}(\{\lambda I_r \mid \lambda^{2rc+r-1} \in F^{\times n}\}). \tag{2.2}$$

4. By a direct calculation,

$$G_\beta^{(n)} \cap \mathfrak{p}(Z(\widetilde{G}_r)) = G_r^{(n)} \cap \mathfrak{p}(Z(\widetilde{G}_r)) = F^{\times n} I_r. \tag{2.3}$$

We fix a faithful character $\epsilon : \mu_n \rightarrow \mathbb{C}^\times$. For a closed subgroup H of G_r , a representation $\widetilde{\pi}$ of \widetilde{H} is called ϵ -genuine (or simply genuine if ϵ is clear) if the restriction of $\widetilde{\pi}$ to μ_n acts by ϵ . Notice that if a representation of \widetilde{H} is not genuine, then it factors through an n' -fold covering group with n' dividing n . So essentially we only need to stick to ϵ -genuine representations. We also remark that the H -conjugation on \widetilde{H} and $\widetilde{\pi}$, as well as the twist $\widetilde{\pi} \cdot \chi$ for χ being a character of H are well-defined in the usual sense.

2.3 Metaplectic tensor product

In this subsection, we rephrase the construction of metaplectic tensor product for irreducible representations in [31, §4]. Our presentation here is rather down-to-earth, which in particular generalizes the previous works of Kable [28], Mezo [39] and Takeda [48, 49].

We keep the notations of the last subsection. Let H_i be a closed subgroup of G_{r_i} for each $i = 1, \dots, k$ and let $H = H_1 \times \dots \times H_k$ be a closed subgroup of G_β . Such an H is called *block compatible* if the groups \widetilde{H}_i are pairwise commutative as subgroups in \widetilde{H} for $i = 1, \dots, k$. In this case, the group \widetilde{H} is isomorphic to $\widetilde{H}_1 \times \dots \times \widetilde{H}_k / \Xi$ with

$$\Xi = \{(\zeta_1, \dots, \zeta_k) \in \mu_n \times \dots \times \mu_n \mid \zeta_1 \dots \zeta_k = 1\}.$$

If H is block compatible, then for each i and a genuine representation $\widetilde{\pi}_i$ of \widetilde{H}_i , the tensor product

$$\widetilde{\pi}_1 \otimes \dots \otimes \widetilde{\pi}_k$$

defines a representation of $\widetilde{H}_1 \times \dots \times \widetilde{H}_k$ that descends to a representation of \widetilde{H} , which we denote by

$$\widetilde{\pi}_1 \widetilde{\otimes} \dots \widetilde{\otimes} \widetilde{\pi}_k.$$

We remark that in general H is usually not block compatible, but the group $H_1^{(n)} \times \dots \times H_k^{(n)}$ is block compatible by (2.1).

Now for each $i = 1, \dots, k$ let γ_i be a composition of r_i and let $\gamma = \gamma_1 + \dots + \gamma_k$ be a composition of r as a refinement of β . Given $\widetilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G}_{\gamma_i})$ for each i , our goal is to construct a representation in $\text{Irr}_\epsilon(\widetilde{G}_\gamma)$ as some kind of “tensor product” of $\widetilde{\pi}_i$. This cannot be achieved directly, since $G_\gamma = G_{\gamma_1} \times \dots \times G_{\gamma_k}$ is not block compatible. But instead, its open and finite index subgroup

$$G_{\gamma, \beta}^{(n)} := G_{\gamma_1}^{(n)} \times \dots \times G_{\gamma_k}^{(n)}$$

is block compatible. In particular this notation is valid for $\gamma = \beta$. So we may first restrict each $\widetilde{\pi}_i$ to $\widetilde{G}_{\gamma_i}^{(n)}$, which we denote by $\widetilde{\pi}_i^{(n)}$. Then we take the tensor product $\widetilde{\pi}_1^{(n)} \widetilde{\otimes} \dots \widetilde{\otimes} \widetilde{\pi}_k^{(n)}$ as a representation of $\widetilde{G}_{\gamma, \beta}^{(n)}$. Finally we consider the induction $\text{Ind}_{\widetilde{G}_{\gamma, \beta}^{(n)}}^{\widetilde{G}_\gamma}(\widetilde{\pi}_1^{(n)} \widetilde{\otimes} \dots \widetilde{\otimes} \widetilde{\pi}_k^{(n)})$ and take one of its irreducible component as the “tensor product” we want. Bootstrapping the above procedure gives us as the following theorem.

Theorem 2.1 [31, Proposition 3.10 and §4.2]

Let $\tilde{\pi} \in \text{Irr}_\epsilon(\widetilde{G_\gamma})$ and let $\tilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G_{\gamma_i}})$ for each $i = 1, \dots, k$.

1. The center of $Z(\widetilde{G_\beta})$ is $Z(\widetilde{G_r})Z(\widetilde{G_{\beta,\beta}^{(n)}})$, and $[Z(\widetilde{G_\beta}) : Z(\widetilde{G_r})Z(\widetilde{G_{\beta,\beta}^{(n)}})]$ is the square of a positive integer n_β .
2. Let $\tilde{\tau}$ be any irreducible component of $\tilde{\pi}^{(n)} := \tilde{\pi}|_{\widetilde{G_{\gamma,\beta}^{(n)}}}$, then we have

$$\tilde{\pi}^{(n)} \simeq n_\beta \cdot \bigoplus_{g \in \widetilde{G_\gamma} / Z(\widetilde{G_\beta})\widetilde{G_{\gamma,\beta}^{(n)}}} \tilde{\tau}^g, \tag{2.4}$$

where those $\tilde{\tau}^g$ on the right-hand side are pairwise inequivalent. We let $m_\beta := n_\beta \cdot [\widetilde{G_\gamma} : Z(\widetilde{G_\beta})\widetilde{G_{\gamma,\beta}^{(n)}}]$ which is the total multiplicity of $\tilde{\pi}^{(n)}$.

3. We call a character $\tilde{\omega}$ of $Z(\widetilde{G_r})$ compatible (with respect to $(\tilde{\pi}_1, \dots, \tilde{\pi}_k)$) if the following compatibility condition is satisfied

$$\prod_{i=1}^k \omega_{\tilde{\pi}_i}(s(\lambda I_r)) = \tilde{\omega}(s(\lambda I_r)) \text{ for any } \lambda \in F^{\times n}. \tag{2.5}$$

Then we have

$$\text{Ind}_{\widetilde{G_{\gamma,\beta}^{(n)}}}^{\widetilde{G_\gamma}} (\tilde{\pi}_1^{(n)} \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k^{(n)}) \simeq M_\beta \cdot \bigoplus_{\tilde{\omega}} (\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k) \tilde{\omega}, \tag{2.6}$$

where $\tilde{\omega}$ ranges over all the characters of $Z(\widetilde{G_r})$ satisfying (2.5), and $(\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k) \tilde{\omega}$ is an irreducible representation of $\widetilde{G_\gamma}$ whose central character restricted to $Z(\widetilde{G_r})$ is $\tilde{\omega}$, and $M_\beta := n_\beta \cdot \prod_{i=1}^k m_{(r_i)}$. Here each $m_{(r_i)}$ is defined as in (2) by taking $\beta = (r_i)$ correspondingly.

4. • For two genuine characters $\tilde{\omega}, \tilde{\omega}'$ of $Z(\widetilde{G_r})$ satisfying (2.5), there exists a character χ of $G_\gamma / G_\gamma^{(n)}$ such that

$$(\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}'} \simeq (\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}} \cdot \chi.$$

- Let $\tilde{\pi}'_i \in \text{Irr}_\epsilon(\widetilde{G_{\gamma_i}})$ for $i = 1, \dots, k$ such that $(\tilde{\pi}_1, \dots, \tilde{\pi}_k; \tilde{\omega})$ and $(\tilde{\pi}'_1, \dots, \tilde{\pi}'_k; \tilde{\omega})$ satisfy (2.5), then

$$(\tilde{\pi}'_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}'_k)_{\tilde{\omega}} \simeq (\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$$

if and only if $\tilde{\pi}'_i \simeq \tilde{\pi}_i \chi_i$ for a certain character χ_i of $G_{\gamma_i} / G_{\gamma_i}^{(n)}$ for each i .

- Every $\tilde{\pi} \in \text{Irr}_\epsilon(\widetilde{G_\gamma})$ is of the form $(\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$ for certain $\tilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G_{\gamma_i}})$ for each i and a compatible genuine character $\tilde{\omega}$ of $Z(\widetilde{G_r})$.

Thus for $\tilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G_{\gamma_i}})$ and a character $\tilde{\omega}$ of $Z(\widetilde{G_r})$ satisfying (2.5), we define $(\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$ the metaplectic tensor product with respect to $(\tilde{\pi}_1, \dots, \tilde{\pi}_k; \tilde{\omega})$ as in the theorem. It is clear that $(\tilde{\pi}_1 \tilde{\omega} \cdots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}} \in \mathcal{C}_\epsilon(\widetilde{G_\gamma})$ if and only if $\tilde{\pi}_i \in \mathcal{C}_\epsilon(\widetilde{G_{\gamma_i}})$ for each $i = 1, \dots, k$, where \mathcal{C} could be Temp, Sqrt and Cusp.

For any positive integer m we define $d_F(m) := m \cdot |m|_F^{-1/2}$. So $d_F : \mathbb{Z}_{>0} \rightarrow \mathbb{Q}_{>0}$ is a multiplicative function. In particular $d_F(m) = m$ if $\text{gcd}(m, p) = 1$.

For a positive integer r_0 , we define $d'_{r_0} = \text{gcd}(n, r_0)$ and $d_{r_0} = \text{gcd}(n, 2r_0c + r_0 - 1)$. We emphasize that d_{r_0} depends on n, r_0 as well as c .

We have the following basic lemma.

Lemma 2.2 [52, Corollary of Proposition II.9, Proposition XIII.6]

Assume m divides n . Then we have $[F^\times : F^{\times m}] = d_F(m)^2$, and moreover $d_F(m)$ is an integer.

In practice, it is useful to know the exact value of the multiplicities $n_\beta, m_\beta, M_\beta$ considered in the theorem.

Proposition 2.3 1.

$$n_\beta = \frac{\prod_{i=1}^k [F^\times : F^{\times n/d'_i}]^{1/2}}{[F^{\times n/d_r} : F^{\times n}]^{1/2}} = \frac{d_F(n^k)}{d_F(d_r \cdot \prod_{i=1}^k d'_{r_i})}.$$

2.

$$m_\beta = \frac{[F^\times : F^{\times n}]^{k/2} \cdot \prod_{i=1}^k [F^\times : F^{\times d'_{r_i}}]^{1/2}}{[F^{\times n/d_r} : F^{\times n}]^{1/2}} = \frac{d_F(n^k \cdot \prod_{i=1}^k d'_{r_i})}{d_F(d_r)}.$$

3.

$$M_\beta = \frac{[F^\times : F^{\times n}]^k}{[F^{\times n/d_r} : F^{\times n}]^{1/2} \cdot \prod_{i=1}^k [F^{\times n/d_{r_i}} : F^{\times n}]^{1/2}} = \frac{d_F(n^{2k})}{d_F(d_r \cdot \prod_{i=1}^k d_{r_i})}.$$

Proof Using Lemma 2.2 we only need to prove the left part of each equation. We have

$$n_\beta^2 = [Z(\widetilde{G_\beta}) : Z(\widetilde{G_r})Z(\widetilde{G_{\beta,\beta}})] = [Z(\widetilde{G_\beta}) : Z(\widetilde{G_{\beta,\beta}})]/[Z(\widetilde{G_r}) : Z(\widetilde{G_r}) \cap Z(\widetilde{G_{\beta,\beta}})].$$

By definition,

$$[Z(\widetilde{G_\beta}) : Z(\widetilde{G_{\beta,\beta}})] = [Z(G_\beta) : Z(G_{\beta,\beta})] = \prod_{i=1}^k [F^\times : F^{\times n/d'_{r_i}}],$$

and by (2.2) and (2.3) we have

$$[Z(\widetilde{G_r}) : Z(\widetilde{G_r}) \cap Z(\widetilde{G_{\beta,\beta}})] = [F^{\times n/d_r} : F^{\times n}].$$

So indeed we proved (1). By definition

$$[G_\gamma : Z(\widetilde{G_\beta})G_{\gamma,\beta}^{(n)}] = [G_\gamma : Z(G_\beta)G_{\gamma,\beta}^{(n)}] = \prod_{i=1}^k [G_{\gamma_i} : Z(G_{r_i})G_{\gamma_i}^{(n)}] = \prod_{i=1}^k [F^\times : F^{\times d'_{r_i}}],$$

so using (1) we also proved (2). Finally (3) follows from (1) and (2). □

Finally the following corollary will be used later on.

Corollary 2.4 Let $\tilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G_{\gamma_i}})$ for $i = 1, \dots, k$ and let $\tilde{\omega}$ be a character of $Z(\widetilde{G_r})$ satisfying (2.5), then

$$\prod_{i=1}^k d_F(d_{r_i}) \cdot (\tilde{\pi}_1^{(n)} \tilde{\otimes} \dots \tilde{\otimes} \tilde{\pi}_k^{(n)}) \simeq d_F(d_r) \cdot (\tilde{\pi}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\pi}_k)_{\tilde{\omega}}^{(n)}.$$

Proof First we notice that the restriction $(\tilde{\pi}_1 \otimes \cdots \otimes \tilde{\pi}_k)_{\tilde{\omega}}^{(n)}$ does not depend on the choice of $\tilde{\omega}$, which follows from Theorem 2.1.(4). Then restricting (2.6) to $\widetilde{G}_{\gamma, \beta}^{(n)}$ and using the Mackey formula, we have

$$[\widetilde{G}_{\gamma} : \widetilde{G}_{\gamma, \beta}^{(n)}] \cdot (\tilde{\pi}_1^{(n)} \otimes \cdots \otimes \tilde{\pi}_k^{(n)}) \simeq M_{\beta} \cdot [F^{\times n/d_r} : F^{\times n}] \cdot (\tilde{\pi}_1 \otimes \cdots \otimes \tilde{\pi}_k)_{\tilde{\omega}}^{(n)}.$$

Notice that

$$[\widetilde{G}_{\gamma} : \widetilde{G}_{\gamma, \beta}^{(n)}] = [G_{\gamma} : G_{\gamma, \beta}^{(n)}] = [F : F^{\times n}]^k,$$

then our result follows from the Krull-Schmidt theorem and Proposition 2.3.(3). □

3 Metaplectic correspondence

In this section we recall and update the metaplectic correspondence studied by Flicker and Kazhdan [18].

3.1 Definition

Fix a character $\omega : Z(G_r) \rightarrow \mathbb{C}^{\times}$ and a genuine character $\tilde{\omega} : Z(\widetilde{G}_r) \rightarrow \mathbb{C}^{\times}$ such that

$$\tilde{\omega}(s(z^n)) = \omega(z) \quad \text{for every } z \in Z(G_r). \tag{3.1}$$

Notice that the existence of such $\tilde{\omega}$ is equivalent to $\omega|_{\mu_n} = 1$, and conversely ω is totally determined by $\tilde{\omega}$. Consider the map

$$G_r \rightarrow \widetilde{G}_r, \quad x \mapsto x^* = u(x)s(x)^n,$$

where $u(x) = \pm 1$ is defined exactly as in [18, §4]. The key point is that this “star” map is equivariant under G_r -conjugacy (cf. [33, Theorem 4.1]), thus it induces a map from the set of conjugacy classes of \widetilde{G}_r to that of G_r .

Let $\beta = (r_1, \dots, r_k)$ be a composition of r . An element $g \in G_{\beta}$ is called

- *semi-simple* if it is diagonalizable as a matrix in $GL_r(\overline{F})$ with \overline{F} being the algebraic closure of F ;
- *semi-simple regular* if the roots of its characteristic polynomial are pairwise different;
- *elliptic* if $F[X]/(P_g(X))$ is a field of degree r over F , where $P_g(X)$ denotes the characteristic polynomial of g .

For $g = \text{diag}(g_1, \dots, g_k) \in G_{\beta}$ semi-simple and $\alpha_{i1}, \dots, \alpha_{ir_i}$ the roots of the characteristic polynomial of g_i for each i , we define

$$D^{G_{\beta}}(g) = \prod_{i=1}^k \prod_{\substack{1 \leq j < l \leq r_i, \\ \alpha_{ij} \neq \alpha_{il}}} \frac{|\alpha_{ij} - \alpha_{il}|_F^2}{|\alpha_{ij}\alpha_{il}|_F}$$

the Weyl discriminant of g , and $\Delta^{G_{\beta}}(g) = D^{G_{\beta}}(g)^{1/2}$.

For $\pi \in \text{Irr}(G_{\beta})$ and $\tilde{\pi} \in \text{Irr}_{\epsilon}(\widetilde{G}_{\beta})$, we say that $\tilde{\pi}$ *lifts to* π (or π is a *lift* of $\tilde{\pi}$) if the following relation is satisfied:

$$\Delta^{G_{\beta}}(x^n) \cdot \theta_{\tilde{\pi}}(x^*) = \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t \in G_{\beta} | t^n = x^n\}} \Delta^{G_{\beta}}(t) \cdot \theta_{\pi}(t) \cdot \epsilon(x^*/t^*), \tag{3.2}$$

where $\theta_{\tilde{\pi}}$ and θ_{π} denote the corresponding Harish-Chandra characters, and $x \in G_{\beta}$ such that x^n is semi-simple regular.

Remark 3.1 We mention the difference between (3.2) and that in [18]. In *ibid.* they assume the restriction of $\omega_{\tilde{\pi}}$ to $Z(\widetilde{G}_r)$ to be $\tilde{\omega}$ and the restriction of ω_{π} to $Z(G_r)$ to be ω , whereas we release this restriction. Moreover the relation of Harish-Chandra characters in *ibid.* is of the following form

$$\Delta^{G_{\beta}}(x^n) \cdot \theta_{\tilde{\pi}}(x^*) = \frac{n}{d_F(d_r) |n|_F^{r/2}} \sum_{t, \tilde{z}} \Delta^{G_{\beta}}(t) \cdot \theta_{\pi}(t) \cdot \tilde{\omega}(\tilde{z}),$$

where the sum ranges over $\{t \in G_{\beta}/Z(G_r), \tilde{z} \in Z(\widetilde{G}_r) \mid t^* \tilde{z} = x^*\}$. To verify that the two formulae coincide, we notice that for such a pair (t, \tilde{z}) we have $\tilde{z} \in Z(\widetilde{G}_r) \cap \widetilde{G}_{\beta}^{(n)}$. Thus by (2.3), we may choose a representative t such that $t^n = x^n$ and the corresponding $\tilde{z} \in \mu_n$. Or in other words, the above set is in bijection with $\{t \in G_{\beta}/\mu_n I_r \mid t^n = x^n\}$. Since $\tilde{\omega}(\tilde{z}) = \tilde{\omega}(x^*/t^*) = \epsilon(x^*/t^*)$ and ω_{π} is trivial on $\mu_n I_r$, the two expressions coincide.

Remark 3.2 Using [18, Proposition 3], the Harish-Chandra character $\theta_{\tilde{\pi}}$ has support contained in $Z(\widetilde{G}_r) \cup_{T \subset G_{\beta}} \widetilde{T}^{(n)}$, where T in the union ranges over all the maximal torus of G_{β} and $\widetilde{T}^{(n)} := \{x^n \mid x \in T\}$. So once we know $\tilde{\pi}|_{Z(\widetilde{G}_r)}$, the Harish-Chandra character of $\tilde{\pi}$, or even $\tilde{\pi}$ itself, is determined by π via (3.2).

We write $\text{Sqrt}_{\tilde{\omega}}(\widetilde{G}_{\beta})$ and $\text{Sqrt}_{\omega}(G_{\beta})$ as in §2.1. We denote by $\text{Sqrt}_{\omega}^{(n)}(G_{\beta})$ the subset of $\text{Sqrt}_{\omega}(G_{\beta})$ consisting of representations whose central character restricted to $\prod_{i=1}^k \mu_n$ is trivial. Here $\prod_{i=1}^k \mu_n$ is realized as a subgroup of $Z(G_{\beta}) = \text{diag}(F^{\times} I_{r_1}, \dots, F^{\times} I_{r_k})$ via the block-diagonal embedding.

Remark 3.3 Consider $x = \text{diag}(x_1, \dots, x_k) \in G_{\beta}$ such that x^n is elliptic, then the t in (3.2) ranges over $\text{diag}(x_1 \zeta_1, \dots, x_k \zeta_k)$ with $\zeta_i \in \mu_n$. In this case for $\pi \in \text{Sqrt}_{\omega}^{(n)}(G_{\beta})$ the right hand side of (3.2) becomes

$$\frac{n^k}{d_F(d_r) |n|_F^{r/2}} \cdot \Delta^{G_{\beta}}(x) \cdot \theta_{\pi}(x).$$

Since such x 's form a dense subset of the locus of elliptic elements in G_{β} , such π is uniquely determined by $\tilde{\pi}$ via (3.2).

Let γ_i be a composition of r_i for each i and let $\gamma = \gamma_1 + \dots + \gamma_k$ be a refinement of β as before.

Proposition 3.4 Assume $\tilde{\pi}_i \in \text{Irr}_{\epsilon}(\widetilde{G}_{\gamma_i})$ and $\pi_i \in \text{Irr}(G_{\gamma_i})$ to be a lift of $\tilde{\pi}_i$ for each $i = 1, \dots, k$. Let $\pi = \pi_1 \otimes \dots \otimes \pi_k \in \text{Irr}(G_{\gamma})$ and let $\tilde{\pi} = (\tilde{\pi}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\pi}_k) \tilde{\omega} \in \text{Irr}_{\epsilon}(\widetilde{G}_{\gamma})$ with a compatible character $\tilde{\omega}$ of $Z(\widetilde{G}_r)$ satisfying (2.5), then $\tilde{\pi}$ lifts to π .

Proof First by Corollary 2.4, we have

$$d_F(d_r) \cdot \tilde{\pi}^{(n)} \simeq \prod_{i=1}^k d_F(d_{r_i}) \cdot (\tilde{\pi}_1^{(n)} \tilde{\otimes} \dots \tilde{\otimes} \tilde{\pi}_k^{(n)}).$$

In addition for $x = \text{diag}(x_1, \dots, x_k), t = \text{diag}(t_1, \dots, t_k) \in G_\beta = G_{\gamma_1} \times \dots \times G_{\gamma_k}$, we have

$$D^{G_\gamma}(x^n) = \prod_{i=1}^k D^{G_{\gamma_i}}(x_i^n) \quad \text{and} \quad D^{G_\gamma}(t) = \prod_{i=1}^k D^{G_{\gamma_i}}(t_i).$$

By definition we have $x^*, t^* \in \widetilde{G_{\gamma, \beta}^{(n)}}$ and $x_i^*, t_i^* \in \widetilde{G_{\gamma_i}^{(n)}}$ for each i . We also regard each x_i^* and t_i^* as elements in $\widetilde{G_\gamma}$ via the corresponding block-diagonal embedding. □

Lemma 3.5 *We have $x^* = x_1^* x_2^* \dots x_k^*$ and $t^* = t_1^* t_2^* \dots t_k^*$.*

Proof We only need to prove the first equation and to consider the $k = 2$ and $\gamma = \beta$ case. By definition and (2.1) we have

$$s(x) = s(\text{diag}(x_1, x_2)) = s(x_2)s(x_1) = s(x_1)s(x_2)(\det(x_1), \det(x_2))_n^{-1}.$$

Thus by direct calculation

$$\begin{aligned} x^* &= s(\text{diag}(x_1, x_2))^n u(x) = s(x_1)^n s(x_2)^n (\det(x_1), \det(x_2))_n^{-n(n+1)/2} u(x) \\ &= x_1^* x_2^* (\det(x_1), \det(x_2))_n^{-n(n+1)/2} u(x) / u(x_1) u(x_2) = x_1^* x_2^*. \end{aligned}$$

where for the last step we use the formula

$$u(x)/u(x_1)u(x_2) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ (\det(x_1), \det(x_2))_2 & \text{if } n \text{ is even.} \end{cases}$$

given in [18, Sect. 4].

Using the facts above, we have

$$\begin{aligned} \Delta^{G_\gamma}(x^n) \cdot \theta_{\tilde{\pi}}(x^*) &= \frac{\prod_{i=1}^k d_F(d_{r_i})}{d_F(d_r)} \prod_{i=1}^k \Delta^{G_{\gamma_i}}(x_i^n) \cdot \theta_{\tilde{\pi}_i}(x_i^*) \\ &= \frac{\prod_{i=1}^k d_F(d_{r_i})}{d_F(d_r)} \prod_{i=1}^k \frac{1}{d_F(d_{r_i}) |n|_F^{r_i/2}} \sum_{\{t_i \in G_{\gamma_i} | t_i^n = x_i^n\}} \Delta^{G_{\gamma_i}}(t_i) \cdot \theta_{\tilde{\pi}_i}(t_i) \cdot \epsilon(x_i^*/t_i^*) \\ &= \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t \in G_\gamma | t^n = x^n\}} \Delta^{G_\gamma}(t) \cdot \theta_{\tilde{\pi}}(t) \cdot \epsilon(x^*/t^*). \end{aligned}$$

□

Using the above proposition, we may consider the metaplectic correspondence for essentially square integrable representations of $\widetilde{G_\beta}$ and G_β .

Proposition 3.6 *For any $\pi \in \text{Sqrt}_\omega^{(n)}(G_\beta)$, there exists a unique $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\widetilde{G_\beta})$ which lifts to π , and conversely any $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\widetilde{G_\beta})$ lifts to a unique $\pi \in \text{Sqrt}_\omega^{(n)}(G_\beta)$.*

Proof The $k = 1$ case is studied by Flicker-Kazhdan based on the simple trace formula ([18, Theorem 26.1]), which will be our starting point. Now we consider the general case.

For $\pi \in \text{Sqrt}_\omega^{(n)}(G_\beta)$, there exists a unique $\pi_i \in \text{Sqrt}(G_{r_i})$ for each i such that $\pi = \pi_1 \otimes \dots \otimes \pi_k$. Similarly for $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\widetilde{G_\beta})$, there exists $\tilde{\pi}_i \in \text{Sqrt}_\epsilon(\widetilde{G_{r_i}})$ for each i such that $\tilde{\pi} = (\tilde{\pi}_1 \otimes \dots \otimes \tilde{\pi}_k)_\omega$, where $\tilde{\omega}$ is a character of $Z(\widetilde{G_r})$ satisfying (2.5).

So if $\pi = \pi_1 \otimes \cdots \otimes \pi_k \in \text{Sqrt}_\omega^{(n)}(G_\beta)$ is given, then for each i we choose $\tilde{\pi}_i \in \text{Sqrt}_\epsilon(\widetilde{G_{r_i}})$ that lifts to π_i . Using Proposition 3.4 the metaplectic tensor product $\tilde{\pi} = (\tilde{\pi}_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \tilde{\pi}_k)_{\widetilde{\omega}}$ lifts to π . Conversely for any given $\tilde{\pi} = (\tilde{\pi}_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \tilde{\pi}_k)_{\widetilde{\omega}}$, we let π_i be the lift of $\tilde{\pi}_i$. Still by Proposition 3.4 $\pi = \pi_1 \otimes \cdots \otimes \pi_k$ is a lift of $\tilde{\pi}$. Finally the uniqueness follows from the formula (3.2) and the linear independence of Harish-Chandra characters, see Remark 3.2 and Remark 3.3. □

Definition 3.7 By the above proposition, we define the *metaplectic correspondence*

$$\text{MC} : \text{Sqrt}_{\widetilde{\omega}}(\widetilde{G}_\beta) \longrightarrow \text{Sqrt}_\omega(G_\beta)$$

as an injection mapping each representation to its lift, whose image is $\text{Sqrt}_\omega^{(n)}(G_\beta)$.

Remark 3.8 We note that the generalization of metaplectic correspondence to Levi subgroups could have been done in [18, §26.2]. However there exists a mistake in their study of metaplectic tensor product during the construction, as already been pointed out by other authors. In [38] and [39], Mezo somehow managed to fix the mistake of Flicker and Kazhdan, however at least for general r and n he didn't write down all the details, especially it is unclear to the author why the scalar (i.e. $1/(d_F(d_r) |n|_F^{r/2})$) in the formula (3.2) of Harish-Chandra characters should be the correct one. Thus, our contribution here is to use the refined “metaplectic tensor product” studied in [31] to fix the mistake above, which in particular gives the correct scalar in (3.2).

3.2 The case of a maximal split torus

We digress for a moment to study an important example. Hopefully it provides a better understanding of (3.2) and Proposition 3.6. This subsection is independent, so the readers are free to skip it.

In this subsection we assume $\beta = (1, 1, \dots, 1)$, or in other words, G_β is the diagonal torus of G_r . We notice that in this special case we have $Z(G_\beta) = G_\beta$ and $G_{\beta,\beta}^{(n)} = F^{\times n} \times \cdots \times F^{\times n}$.

In particular, both G_β and $\widetilde{G_{\beta,\beta}^{(n)}}$ are abelian.

By definition every irreducible representation χ of G_β is indeed a character. Correspondingly, let $\tilde{\chi}$ be an irreducible genuine representation of \widetilde{G}_β . By Proposition 2.1.(2), we have

$$\tilde{\chi}^{(n)} := \tilde{\chi}|_{\widetilde{G_{\beta,\beta}^{(n)}}} = n_\beta \cdot \tilde{\tau}, \text{ where } \tilde{\tau} \text{ is a genuine character of } \widetilde{G_{\beta,\beta}^{(n)}}.$$

Proposition 3.9 *Let $\chi \in \text{Irr}(G_\beta)$ and let $\tilde{\chi} \in \text{Irr}_\epsilon(\widetilde{G}_\beta)$. Then $\tilde{\chi}$ lifts to χ if and only if*

1. χ is trivial on $\prod_{i=1}^r \mu_n \subset G_\beta$ (via the diagonal embedding).
2. $\tilde{\tau}$ is the genuine character satisfying $\tilde{\tau}(x^*) = \chi(x)$ for $x \in G_\beta$.

Proof Taking $\tilde{\pi} = \tilde{\chi}$ and $\pi = \chi$, the left-hand side of (3.2) becomes

$$n_\beta \cdot \tilde{\tau}(x^*).$$

Taking $\pi = \chi$, the right-hand side of (3.2) becomes

$$\frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t \in G_\beta | t^n = x^*\}} \chi(t) \cdot \epsilon(x^*/t^*) = \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\zeta \in \prod_{i=1}^r \mu_n} \chi(x\zeta).$$

So it equals 0 unless χ is trivial on $\prod_{i=1}^r \mu_n$, and in this case it equals

$$\frac{d_F(n^r)}{d_F(d_r)} \cdot \chi(x).$$

Finally using Proposition 2.3.(2) we have $n_\beta = d_F(n^r)/d_F(d_r)$, which finishes the proof. □

Thus for $\chi \in \text{Irr}(G_\beta)$ trivial on $\prod_{i=1}^r \mu_n$ and $\tilde{\chi} \in \text{Irr}_\epsilon(\widetilde{G}_\beta)$, we have $\tilde{\chi}$ lifts to χ if and only if $\tilde{\chi}^{(n)}$ contains $\tilde{\tau}$ as in the proposition. If we also fix $\tilde{\omega}$ as the central character of $\tilde{\chi}$ restricted to $Z(\widetilde{G}_r)$, then $\tilde{\chi}$ itself is uniquely determined by $\tilde{\tau}$ and $\tilde{\omega}$ (cf. Theorem 2.1). This justifies Proposition 3.6 in this special case.

3.3 Compatibility with parabolic induction and Jacquet module

As before we let β, γ be compositions of r , such that γ is a refinement of β . Let $P_{\beta,\gamma}$ be a parabolic subgroup of G_β having the Levi subgroup G_γ , and let $N_{\beta,\gamma}$ be the corresponding unipotent radical. By [40, Appendix I] there exists a canonical splitting $s_{N_{\beta,\gamma}} : N_{\beta,\gamma} \rightarrow \widetilde{G}_\beta$ as a group homomorphism that is $P_{\beta,\gamma}$ -equivariant. Using this splitting we also realize $N_{\beta,\gamma}$ as a subgroup of \widetilde{G}_β . We consider the following normalized parabolic induction and normalized Jacquet module with respect to the decomposition $P_{\beta,\gamma} = G_\gamma N_{\beta,\gamma}$ and $\widetilde{P}_{\beta,\gamma} = \widetilde{G}_\gamma N_{\beta,\gamma}$:

$$\begin{aligned} i_{\gamma,\beta} &:= i_{N_{\beta,\gamma},1} : \text{Rep}(G_\gamma) \rightarrow \text{Rep}(G_\beta) \quad (\text{or } \text{Rep}(\widetilde{G}_\gamma) \rightarrow \text{Rep}(\widetilde{G}_\beta)) \\ r_{\beta,\gamma} &:= r_{N_{\beta,\gamma},1} : \text{Rep}(G_\beta) \rightarrow \text{Rep}(G_\gamma) \quad (\text{or } \text{Rep}(\widetilde{G}_\beta) \rightarrow \text{Rep}(\widetilde{G}_\gamma)) \end{aligned}$$

in the sense of [5, §2.3].

The metaplectic correspondence is compatible with the parabolic induction and Jacquet functor in the following sense.

- Proposition 3.10** 1. For $\rho \in \text{Irr}(G_\gamma)$ as a lift of $\tilde{\rho} \in \text{Irr}_\epsilon(\widetilde{G}_\gamma)$, we have that $i_{\gamma,\beta}(\tilde{\rho})$ and $i_{\gamma,\beta}(\rho)$ satisfy the Eq. (3.2). In other words, $i_{\gamma,\beta}(\tilde{\rho})$ lifts to $i_{\gamma,\beta}(\rho)$ in the general sense (without sticking to irreducible representations).
2. Let $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\widetilde{G}_\beta)$ and let $\pi \in \text{Sqrt}_\omega^{(n)}(G_\beta)$ be a lift of $\tilde{\pi}$. If $r_{\beta,\gamma}(\pi) \in \text{Sqrt}_\omega^{(n)}(G_\gamma)$, then $r_{\beta,\gamma}(\tilde{\pi}) \in \text{Sqrt}(G_\gamma)$ (thus in particular it is irreducible) which lifts to $r_{\beta,\gamma}(\pi)$; otherwise $r_{\beta,\gamma}(\tilde{\pi}) = 0$.

Proof The statement (1) has already been used in [18, Proposition 26.2] with a one-sentence explanation, and here we give a rather detailed proof for completeness. For $x \in G_\beta$ such that x^n is semi-simple regular, we have

$$\begin{aligned} &\Delta^{G_\beta}(x^n) \cdot \theta_{i_{\beta,\gamma}(\tilde{\rho})}(x^*) \\ &= \sum_{\{\tilde{g} \in \widetilde{G}_\beta / \widetilde{P}_{\beta,\gamma} \mid \tilde{g}^{-1} x^* \tilde{g} \in \widetilde{G}_\gamma\}} \Delta^{G_\gamma}(g^{-1} x^n g) \cdot \theta_{\tilde{\rho}}(\tilde{g}^{-1} x^* \tilde{g}) \\ &= \sum_{\{g \in G_\beta / P_{\beta,\gamma} \mid g^{-1} x^n g \in G_\gamma\}} \Delta^{G_\gamma}(g^{-1} x^n g) \cdot \theta_{\tilde{\rho}}((g^{-1} x g)^*) \\ &= \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{g \in G_\beta / P_{\beta,\gamma} \mid g^{-1} x^n g \in G_\gamma\}} \sum_{\{t \mid t^n = x^n\}} \Delta^{G_\gamma}(g^{-1} t g) \cdot \theta_{\rho}(g^{-1} t g) \cdot \epsilon((g^{-1} x g)^* / (g^{-1} t g)^*) \\ &= \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t \mid t^n = x^n\}} \sum_{\{g \in G_\beta / P_{\beta,\gamma} \mid g^{-1} t g \in G_\gamma\}} \Delta^{G_\gamma}(g^{-1} t g) \cdot \theta_{\rho}(g^{-1} t g) \cdot \epsilon(x^* / t^*) \end{aligned}$$

$$= \frac{1}{d_F(d_r) |n|_F^{r'/2}} \sum_{\{t|t^n=x^n\}} \Delta^{G_\beta}(t) \cdot \theta_{t_{\gamma,\beta}(\rho)}(t) \cdot \epsilon(x^*/t^*).$$

Here we use [50, Theorem 3], which also works for covering groups with the same argument, for the first and final equations. We use the fact $s(g)^{-1}x^*s(g) = (g^{-1}xg)^*$ for the second equation and (3.2) for the third equation. Finally $g^{-1}tg \in G_\beta$ is equivalent to $g^{-1}t^n g \in G_\beta$, since $t^n = x^n$ is semi-simple regular. We use this statement and the fact that $(g^{-1}xg)^*/(g^{-1}tg)^* = s(g)^{-1}(x^*/t^*)s(g) = x^*/t^*$ for the fourth equation.

For the statement (2), we write $\gamma = (r'_1, \dots, r'_l)$. We first recall a result of Casselman. Let $t = \text{diag}(t_1, \dots, t_l) \in G_\gamma$, such that t_j is an element in $G_{r'_j}$ with t_j^n being elliptic for each j , and moreover

$$|\det(t_j)|_F^{1/r'_j} / |\det(t_{j+1})|_F^{1/r'_{j+1}} > 1 \quad \text{for } j = 1, \dots, l - 1. \tag{3.3}$$

Then we have (cf. [14, Theorem 5.2] and [18, Theorem 14])

$$\Delta^{G_\beta}(t^n) \cdot \theta_{\tilde{\pi}}(t^*) = \Delta^{G_\gamma}(t^n) \cdot \theta_{r_{\beta,\gamma}(\tilde{\pi})}(t^*) \quad \text{and} \quad \Delta^{G_\beta}(t) \cdot \theta_\pi(t) = \Delta^{G_\gamma}(t) \cdot \theta_{r_{\beta,\gamma}(\pi)}(t). \tag{3.4}$$

To continue, we need the following technical lemma.

Lemma 3.11 *Let $\gamma = (r'_1, \dots, r'_l)$ be a composition of r as above and let $\tilde{\tau}_1, \dots, \tilde{\tau}_{s'}$ $\in \text{Sqrt}_\epsilon(\widetilde{G_{\gamma,(r)}^{(n)}})$ with $G_{\gamma,(r)}^{(n)} := G_{r'_1}^{(n)} \times \dots \times G_{r'_l}^{(n)}$. If for certain $c_i \in \mathbb{C}$ the equation*

$$\sum_{i=1}^{s'} c_i \cdot \theta_{\tilde{\tau}_i}(t^*) = 0 \tag{3.5}$$

holds for any t as above, then we have

$$\sum_{i=1}^{s'} c_i \cdot \theta_{\tilde{\tau}_i} = 0.$$

Proof First we need the following basic lemma.

Lemma 3.12 *Let $\tilde{\sigma}_1, \dots, \tilde{\sigma}_u \in \text{Sqrt}_\epsilon(\widetilde{G_{r'}^{(n)}})$ be pairwise non-isomorphic representations and let $N > 0$, then there exist elliptic elements $g_1, \dots, g_u \in G_{r'}^{(n)}$ satisfying $|\det(g_j)|_F < 1/N$ for each $j = 1, \dots, u$, such that the matrix $(\theta_{\tilde{\sigma}_k}(\zeta_j s(g_j)))_{1 \leq j, k \leq u}$ is invertible for any $\zeta_j \in \mu_n$.*

Proof The lemma follows from the linear independence of $\theta_{\tilde{\sigma}_k}$ on elliptic locus if we don't impose any restriction on $|\det(g_j)|_F$. In general, we just need to replace g_j by $g_j \varpi_F^m$ for m large enough. □

We remark that each $\tilde{\tau}_i$, as an essentially square integrable representation, is determined by the restriction of its Harish-Chandra character to the elliptic locus. We write $\tilde{\tau}_i = \tilde{\tau}_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \tilde{\tau}_{i_l}$, where $\tilde{\tau}_{i_j} \in \text{Sqrt}_\epsilon(\widetilde{G_{r'_j}^{(n)}})$ for $j = 1, \dots, l$.

We prove Lemma 3.11 by induction on l . The $l = 1$ case is clear. Assume the lemma holds for $l - 1$. We consider the division $\{1, \dots, s'\} = I_1 \sqcup \dots \sqcup I_u$, such that $\tilde{\tau}_{i_1 l} \simeq \tilde{\tau}_{i_2 l}$ if and only if $i_1, i_2 \in I_k$ for some $k = 1, \dots, u$. And we denote by $\tilde{\tau}_{I_k l}$ the isomorphism class

of $\tilde{\tau}_{il}$ for $i \in I_k$. We write $\tilde{\tau}'_i = \tilde{\tau}_i \tilde{\otimes} \cdots \tilde{\otimes} \tilde{\tau}_{il-1}$ and $\tilde{\tau}_i = \tilde{\tau}'_i \tilde{\otimes} \tilde{\tau}_{il}$ for each $i = 1, \dots, s'$. For $t = \text{diag}(t_1, \dots, t_l)$ as above, we write $t' = \text{diag}(t_1, \dots, t_{l-1})$. By (3.5) we have

$$\sum_{k=1}^u \theta_{\tilde{\tau}_{I_k}}(t_l^*) \sum_{i \in I_k} c_i \cdot \theta_{\tilde{\tau}'_i}(t'^*) = 0. \tag{3.6}$$

Using Lemma 3.12 for $\tilde{\sigma}_k = \tilde{\tau}_{I_k}$, we may choose u 's different t_l with the corresponding t_l^* denoted by $\tilde{g}_1, \dots, \tilde{g}_u$, such that the matrix $(\theta_{\tilde{\tau}_{I_k}}(\tilde{g}_j))_{1 \leq j, k \leq u}$ is invertible. Plugging $t_l^* = \tilde{g}_j$ into (3.6) for $j = 1, \dots, u$ we get

$$\sum_{i \in I_k} c_i \cdot \theta_{\tilde{\tau}'_i}(t'^*) = 0, \quad k = 1, \dots, u.$$

Using the induction hypothesis and the definition of I_k , we have

$$\sum_{i \in I_k} c_i \cdot \theta_{\tilde{\tau}'_i} = 0 \quad \text{and} \quad \sum_{i \in I_k} c_i \cdot \theta_{\tilde{\tau}_i} = \left(\sum_{i \in I_k} c_i \cdot \theta_{\tilde{\tau}'_i} \right) \tilde{\otimes} \theta_{\tilde{\tau}_{I_k}} = 0, \quad k = 1, \dots, u.$$

Summing over k we finish the proof. □

We finish the proof of Proposition 3.10.(2). We consider $\pi, \tilde{\pi}$ as in the statement.

If $r_{\beta, \gamma}(\pi) \in \text{Sqrt}_{\omega}^{(n)}(G_{\gamma})$, then we let $\tilde{\pi}'$ be the unique essentially square integrable representation of \widetilde{G}_{γ} satisfying $\text{MC}(\tilde{\pi}') = r_{\beta, \gamma}(\pi)$ and having the central character $\tilde{\omega}$ restricted to $Z(\widetilde{G}_{\gamma})$. Then using (3.2) and (3.4) with a similar argument to that in (1), for t as above we have

$$\theta_{\tilde{\pi}'}(t^*) = \theta_{r_{\beta, \gamma}(\tilde{\pi})}(t^*) = \sum_{i=1}^s \theta_{\tilde{\pi}'_i}(t^*), \tag{3.7}$$

where $\tilde{\pi}'_1, \dots, \tilde{\pi}'_s$ are all the irreducible subquotients of $r_{\beta, \gamma}(\tilde{\pi})$. Considering the restriction of each $\tilde{\pi}'_i$ and $\tilde{\pi}'$ to $\widetilde{G}_{\gamma, (r)}^{(n)}$, using Theorem 2.1.(2) and Lemma 3.11, and comparing the number of irreducible components (notice that the total multiplicities of each $\tilde{\pi}'_i|_{\widetilde{G}_{\gamma, (r)}^{(n)}}$ and $\tilde{\pi}'|_{\widetilde{G}_{\gamma, (r)}^{(n)}}$ are the same), we must have $s = 1$ and $\tilde{\pi}'|_{\widetilde{G}_{\gamma, (r)}^{(n)}} \simeq \tilde{\pi}'_1|_{\widetilde{G}_{\gamma, (r)}^{(n)}}$. Comparing the central character we also have $\omega_{\tilde{\pi}'}|_{Z(\widetilde{G}_{\gamma})} = \omega_{\tilde{\pi}'_1}|_{Z(\widetilde{G}_{\gamma})}$. By Remark 3.2 or Theorem 2.1 we have $\tilde{\pi}' \simeq \tilde{\pi}'_1 = r_{\beta, \gamma}(\tilde{\pi})$, so $\text{MC}(r_{\beta, \gamma}(\tilde{\pi})) = r_{\beta, \gamma}(\pi)$.

Finally we consider the case $r_{\beta, \gamma}(\pi) \notin \text{Sqrt}_{\omega}^{(n)}(G_{\gamma})$.

Lemma 3.13 *For $\pi' = \pi'_1 \otimes \cdots \otimes \pi'_l \in \text{Sqrt}_{\omega}(G_{\gamma}) - \text{Sqrt}_{\omega}^{(n)}(G_{\gamma})$ and $x = \text{diag}(x_1, \dots, x_l) \in G_{\gamma}$ such that x_i^n is elliptic for any $i = 1, \dots, l$, we have*

$$\prod_{i=1}^l \sum_{\{t_i | t_i^n = x_i^n\}} \Delta^{G_{\gamma}}(t_i) \cdot \theta_{\pi'_i}(t_i) \cdot \epsilon(x_i^*/t_i^*) = 0.$$

As a result $\tilde{\pi}' = 0$ may be regard as the “null representation” that lifts to π' .

Proof Since x_i is elliptic for $i = 1, \dots, l$ for each t_i in the sum there exists $\zeta_i \in \mu_n$ such that $x_i = t_i \zeta_i$. Thus $D^{G_{r'_i}}(x_i) = D^{G_{r'_i}}(t_i)$ and $\epsilon(x_i^*/t_i^*) = 1$, and we only need to prove that

$$\prod_{i=1}^l \left(\sum_{\zeta_i \in \mu_n} \omega_{\pi'_i}(\zeta_i) \right) \cdot \theta_{\pi'_i}(x_i) = 0.$$

The lemma follows from the fact that $\omega_{\pi'_i}|_{\mu_n}$ is not trivial for some i . □

Using this lemma, (3.2) and (3.4) and a similar argument to the previous case, we have

$$\theta_{r_{\beta,\gamma}(\tilde{\pi})}(t^*) = \sum_{i=1}^s \theta_{\tilde{\pi}'_i}(t^*) = 0, \tag{3.8}$$

where $\tilde{\pi}'_1, \dots, \tilde{\pi}'_s$ are all the irreducible subquotients of $r_{\beta,\gamma}(\tilde{\pi})$ and t is considered as above.

Still considering the restriction of each $\tilde{\pi}'_i$ to $G_{\gamma,(r)}^{(n)}$, and using Theorem 2.1.(2) and Lemma 3.11, we have $s = 0$ and $r_{\beta,\gamma}(\tilde{\pi}) = 0$. □

3.4 Classification of essentially square integrable representations

In this part we classify $\text{Sqrt}_\epsilon(\widetilde{G}_r)$ as well as its image under the metaplectic correspondence. We let $\nu = |\det(\cdot)|_F$ be a character of $G_{r'}$ for each positive integer r' . For a composition $\beta = (r_1, \dots, r_k)$ of r , representations $\tilde{\pi}_i \in \text{Irr}_\epsilon(\widetilde{G}_{r_i})$ for $i = 1, \dots, k$ and a genuine character $\tilde{\omega}$ of $Z(\widetilde{G}_r)$ satisfying (2.5), we define the following Bernstein-Zelevinsky product

$$(\tilde{\pi}_1 \tilde{\times} \dots \tilde{\times} \tilde{\pi}_k)_{\tilde{\omega}} := i_{\beta,(r)}((\tilde{\pi}_1 \tilde{\otimes} \dots \tilde{\otimes} \tilde{\pi}_k)_{\tilde{\omega}}).$$

First we focus on a result which was previously studied in [18, Lemma 27.1]. But unfortunately both the statement and proof in *loc. cit.* are flawed. So we give a correct one.

Proposition 3.14 1. Let $\tilde{\rho}_1 \in \text{Cusp}_\epsilon(\widetilde{G}_{r_1})$ and $\tilde{\rho}_2 \in \text{Cusp}_\epsilon(\widetilde{G}_{r_2})$. Then $(\tilde{\rho}_1 \tilde{\times} \tilde{\rho}_2)_{\tilde{\omega}}$ is reducible only if $r_1 = r_2$, and $(\tilde{\rho}_2 \tilde{\times} \tilde{\rho}_1)_{\tilde{\omega}}$ is isomorphic to $(\tilde{\rho}_1 \tilde{\times} \tilde{\rho}_2)_{\tilde{\omega}}$ twisted by an unramified character of $G_{r_1+r_2}$.

2. Let $\tilde{\rho} \in \text{Cusp}_\epsilon(\widetilde{G}_{r_0})$. Then there exists a unique positive real number $s(\tilde{\rho})$ such that $(\tilde{\rho} \tilde{\times} \tilde{\rho} \nu^{s(\tilde{\rho})})_{\tilde{\omega}}$ is not irreducible. Moreover, $(\tilde{\rho} \tilde{\times} \tilde{\rho} \nu^s)_{\tilde{\omega}}$ is not irreducible for some $s \in \mathbb{R}$ if and only if $s = \pm s(\tilde{\rho})$.

3. In (2), we let $\{\rho \nu^a, \rho \nu^{a+1}, \dots, \rho \nu^b\}$ be the cuspidal support of $\text{MC}(\tilde{\rho})$, where $m = b - a + 1$ is a positive integer and $\rho \in \text{Cusp}(G_{r_0/m})$. Then $s(\tilde{\rho}) = m/n$.

In (1) and (2), $\tilde{\omega}$ are compatible genuine characters of $Z(\widetilde{G}_{r_1+r_2})$ and $Z(\widetilde{G}_{2r_0})$ respectively, such that the corresponding metaplectic tensor products make sense.

Proof The statement (1) and (2) have been proved in [31, Proposition 6.10]. So we focus on statement (3). We let $\beta_0 = (r_0, r_0)$ be a composition of $2r_0$. The representation $\tilde{\pi} = (\tilde{\rho} \tilde{\times} \tilde{\rho} \nu^{s(\tilde{\rho})})_{\tilde{\omega}}$ is reducible and of length 2, and moreover its Jacquet module $r_{(2r_0),\beta_0}(\tilde{\pi})$ consists of $(\tilde{\rho} \tilde{\otimes} \tilde{\rho} \nu^{s(\tilde{\rho})})_{\tilde{\omega}}$ and $(\tilde{\rho} \nu^{s(\tilde{\rho})} \tilde{\otimes} \tilde{\rho})_{\tilde{\omega}}$ as its subquotient. Using Casselman’s criterion for discrete series (cf. [1, Theorem 3.4]), there exists an essentially square integrable subquotient $\tilde{\pi}'$ of $\tilde{\pi}$. Thus $r_{(2r_0),\beta_0}(\tilde{\pi}')$ is isomorphic to $(\tilde{\rho} \tilde{\otimes} \tilde{\rho} \nu^{s(\tilde{\rho})})_{\tilde{\omega}}$ or $(\tilde{\rho} \nu^{s(\tilde{\rho})} \tilde{\otimes} \tilde{\rho})_{\tilde{\omega}}$. We let $\pi' = \text{MC}(\tilde{\pi}') \in \text{Sqrt}^{(n)}(G_{r_0})$. Using Proposition 3.10.(2), we have

$$r_{(2r_0),\beta_0}(\pi') \simeq r_{(2r_0),\beta_0}(\text{MC}(\tilde{\pi}')) \simeq \text{MC}(r_{(2r_0),\beta_0}(\tilde{\pi}')),$$

which is isomorphic to $\text{MC}(\tilde{\rho}) \otimes \text{MC}(\tilde{\rho}v^{s(\tilde{\rho})})$ or $\text{MC}(\tilde{\rho}v^{s(\tilde{\rho})}) \otimes \text{MC}(\tilde{\rho})$ by Proposition 3.4. Thus the cuspidal support of π' equals the union of that of $\text{MC}(\tilde{\rho})$ and $\text{MC}(\tilde{\rho}v^{s(\tilde{\rho})})$ (cf. [53]). Since $\text{MC}(\tilde{\rho}v^{s(\tilde{\rho})}) \simeq \text{MC}(\tilde{\rho})v^{ns(\tilde{\rho})}$, the union above is

$$\{\rho v^a, \rho v^{a+1}, \dots, \rho v^b, \rho v^{a+ns(\tilde{\rho})}, \rho v^{a+ns(\tilde{\rho})+1}, \dots, \rho v^{b+ns(\tilde{\rho})}\}.$$

By [53, Sect. 9] the only possibility is $a + ns(\tilde{\rho}) = b + 1$, meaning that $s(\tilde{\rho}) = m/n$. □

Let m be a positive integer dividing r , let a, b two real numbers such that $b - a + 1 = m$, and let $\rho \in \text{Cusp}(G_{r/m})$. Let $\beta_m = (r/m, \dots, r/m)$ be a composition of r , and we consider G_{β_m} as a Levi subgroup of G_r . We let $L(\rho, [a, b])$ be the unique irreducible quotient of the parabolic induction $\rho v^a \times \rho v^{a+1} \times \dots \times \rho v^b := i_{\beta_m, (r)}(\rho v^a \otimes \rho v^{a+1} \otimes \dots \otimes \rho v^b)$ as an essentially square integrable representation of G_r as in [53]. We have a similar theory for \widetilde{G}_r

Lemma 3.15 [31, Proposition 7.2] *Let $\tilde{\rho} \in \text{Cusp}_\epsilon(\widetilde{G}_r)$ and let $\tilde{\omega}$ be a compatible character of $Z(\widetilde{G}_r)$ such that $(\tilde{\rho}v^{as(\tilde{\rho})} \tilde{\omega} \tilde{\rho}v^{(a+1)s(\tilde{\rho})} \tilde{\omega} \dots \tilde{\omega} \tilde{\rho}v^{bs(\tilde{\rho})} \tilde{\omega})_{\tilde{\omega}}$ makes sense,*

1. *There exists a unique irreducible subrepresentation (resp. quotient) of*

$$(\tilde{\rho}v^{as(\tilde{\rho})} \tilde{\omega} \tilde{\rho}v^{(a+1)s(\tilde{\rho})} \tilde{\omega} \dots \tilde{\omega} \tilde{\rho}v^{bs(\tilde{\rho})} \tilde{\omega})_{\tilde{\omega}}$$

denoted by $Z(\tilde{\rho}, [a, b])_{\tilde{\omega}}$, (resp. $L(\tilde{\rho}, [a, b])_{\tilde{\omega}}$). Moreover, $L(\tilde{\rho}, [a, b])_{\tilde{\omega}}$ is an essentially square integrable representation of \widetilde{G}_r ;

2. *For an integer $0 \leq l \leq r$, we have*

$$\begin{aligned} & r_{(r), (l, r-l)}(Z(\tilde{\rho}, [a, b])_{\tilde{\omega}}) \\ &= \begin{cases} (Z(\tilde{\rho}, [a, a+l'])_{\tilde{\omega}_1} \tilde{\omega}' \otimes Z(\tilde{\rho}, [a+l'+1, b])_{\tilde{\omega}_2})_{\tilde{\omega}} & \text{if } l' = ml/r \in \mathbb{Z}, \\ 0 & \text{if } r \nmid ml. \end{cases} \end{aligned}$$

and

$$\begin{aligned} & r_{(r), (l, r-l)}(L(\tilde{\rho}, [a, b])_{\tilde{\omega}}) \\ &= \begin{cases} (L(\tilde{\rho}, [b-l'+1, b])_{\tilde{\omega}'_1} \tilde{\omega}' \otimes L(\tilde{\rho}, [a, b-l'])_{\tilde{\omega}'_2})_{\tilde{\omega}} & \text{if } l' = ml/r \in \mathbb{Z}, \\ 0 & \text{if } r \nmid ml. \end{cases} \end{aligned}$$

where $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}'_1, \tilde{\omega}'_2$ are compatible genuine characters.

The following proposition classifies the image of the metaplectic correspondence of cuspidal and essentially square integrable representations.

Proposition 3.16 *Let m be a positive integer dividing r , let $\rho \in \text{Cusp}(G_{r/m})$ and let $\pi = L(\rho, [a, b])$ with $m = b - a + 1$ satisfying $\omega_\pi|_{\mu_n} = 1$. Let s be the order of $\omega_\rho|_{\mu_n}$ which divides m .*

1. *If π is the lift a certain $\tilde{\pi} \in \text{Cusp}_\epsilon(\widetilde{G}_r)$, then we have $s = m$. Conversely if $s = m$, then any $\tilde{\pi} \in \text{Sqrt}_\epsilon(\widetilde{G}_r)$ lifting to π is cuspidal.*
2. *Let $m' = m/s$ and let $\tilde{\rho} \in \text{Cusp}_\epsilon(\widetilde{G}_{r/m'})$ such that $\text{MC}(\tilde{\rho}) = L(\rho, [a, a+s-1])$. Then for $\omega = \omega_\pi$ and a genuine character $\tilde{\omega}$ of $Z(\widetilde{G}_r)$ satisfying (3.1), we have $\text{MC}(L(\tilde{\rho}, [0, m'-1])_{\tilde{\omega}}) = \pi$. Conversely, any $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\widetilde{G}_r)$ is of the form $L(\tilde{\rho}, [0, m'-1])_{\tilde{\omega}}$ for some m' dividing r and some $\tilde{\rho} \in \text{Cusp}_\epsilon(\widetilde{G}_{r/m'})$.*

Proof We first prove (1). We choose $\tilde{\pi} \in \text{Sqrt}_\epsilon(\widetilde{G}_r)$ such that $\text{MC}(\tilde{\pi}) = \pi$. First we assume $\tilde{\pi}$ to be cuspidal. Then for any $l' = 1, \dots, m - 1$ and $l = l'/m$, we have $r_{(r), (l, r-l)}(\tilde{\pi}) = 0$. Using a similar argument to the proof of Proposition 3.10.(2) (more precisely, the combination of (3.2) and (3.4)) we may prove that for any $x = \text{diag}(x_1, x_2) \in G_l \times G_{r-l}$ such that x_1^n and x_2^n are elliptic and $|\det(x_1)|_F^{1/l} > |\det(x_2)|_F^{1/(r-l)}$, we have

$$0 = \left(\sum_{\{t_1 | t_1^n = x_1^n\}} \theta_{L(\rho, [b-l'+1, b])}(t_1) \cdot \epsilon(x_1^*/t_1^*) \right) \cdot \sum_{\{t_2 | t_2^n = x_2^n\}} \theta_{L(\rho, [a, b-l'])}(t_2) \cdot \epsilon(x_2^*/t_2^*)$$

$$= \theta_{L(\rho, [b-l'+1, b])}(x_1) \cdot \theta_{L(\rho, [a, b-l'])}(x_2) \cdot \left(\sum_{\zeta_1 \in \mu_n} \omega_\rho(\zeta_1)^{l'} \right) \cdot \left(\sum_{\zeta_2 \in \mu_n} \omega_\rho(\zeta_2)^{m-l'} \right),$$

where for the second equation we use the fact that x_i is elliptic, and thus $x_i = t_i \zeta_i$ for a certain $\zeta_i \in \mu_n$ and $x_i^* = t_i^*$. Since l' and x are arbitrary, $\omega_\rho|_{\mu_n}$ must be of order m . Conversely if $\omega_\rho|_{\mu_n}$ is of order m , then for any $1 \leq l \leq r - 1$ the representation $r_{(r), (l, r-l)}(L(\rho, [a, b]))$ is not in $\text{Sqrt}_\omega^{(n)}(G_l \times G_{r-l})$, thus using Proposition 3.10.(2) $r_{(r), (l, r-l)}(\tilde{\pi}) = 0$. So $\tilde{\pi}$ is cuspidal.

Now we prove (2). We have that $\text{MC}(L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}})$ is in $\text{Sqrt}(G_r)$ and by Proposition 3.14, $s(\tilde{\rho}) = s/n$. Using Proposition 3.6, Proposition 3.10.(2) and Lemma 3.15.(2), for the composition $\beta_{m'} = (r/m', \dots, r/m')$ of r , we have

$$r_{(r), \beta_{m'}}(\text{MC}(L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}})) = \text{MC}(r_{(r), \beta_{m'}}(L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}}))$$

$$= \text{MC}((\tilde{\rho} v^{s(\tilde{\rho})(m'-1)} \tilde{\omega} \dots \tilde{\omega} \tilde{\rho} v^{s(\tilde{\rho})} \tilde{\omega} \tilde{\rho})_{\tilde{\omega}})$$

$$= L(\rho, [a + (m' - 1)s, b]) \otimes \dots \otimes L(\rho, [a + s, a + 2s - 1]) \otimes L(\rho, [a, a + s - 1]).$$

Thus the cuspidal support of $\text{MC}(L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}})$ is $\{\rho v^a, \rho v^{a+1}, \dots, \rho v^b\}$, implying that $\pi = \text{MC}(L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}})$. Finally any $\tilde{\pi} \in \text{Sqrt}_\omega(\widetilde{G}_r)$ lifts to a certain $\pi = L(\rho, [a, b])$ as above. By Proposition 3.6 we must have $\tilde{\pi} = L(\tilde{\rho}, [0, m' - 1])_{\tilde{\omega}}$ for $m' = m/s$ and $\tilde{\rho}$ satisfying $\text{MC}(\tilde{\rho}) = L(\rho, [a, a + s - 1])$. \square

3.5 Classification of essentially tempered representations

We further study essentially tempered representations.

- Proposition 3.17** 1. Let $\tilde{\pi}_i$ be a genuine irreducible square integrable representation of \widetilde{G}_{r_i} for each i with $r_1 + \dots + r_k = r$ and let $\tilde{\omega}$ be a genuine character of $Z(\widetilde{G}_r)$ satisfying (2.5), then $(\tilde{\pi}_1 \tilde{\omega} \dots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$ is an irreducible and tempered representation of \widetilde{G}_r .
2. Conversely every genuine irreducible tempered representation $\tilde{\pi}$ of \widetilde{G}_r is of the form $(\tilde{\pi}_1 \tilde{\omega} \dots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$ as above. Such $\tilde{\pi}$ correspond in bijection with \widetilde{G}_r -conjugacy classes of the metaplectic tensor product $(\tilde{\pi}_1 \tilde{\omega} \dots \tilde{\omega} \tilde{\pi}_k)_{\tilde{\omega}}$ with $r_i, \tilde{\pi}_i, \tilde{\omega}$ as in (1).

Proof This proposition was first noted in [18, Proposition 27, p98], with an essential usage of the metaplectic correspondence (cf. [18, Lemma 27, p98]) as well as some non-trivial results in the book of Silberger that are not explicitly written down for a covering group. Here we give a more elementary proof instead.

Using Proposition 3.16, for each i we may find a certain positive integer m_i dividing r_i , a genuine unitary cuspidal representation $\tilde{\rho}_i$ of \widetilde{G}_{r_i/m_i} and a genuine character $\tilde{\omega}_i$ of $Z(\widetilde{G}_{r_i})$ such that $\tilde{\pi}_i = L(\tilde{\rho}_i, [(-m_i + 1)/2, (m_i - 1)/2])_{\tilde{\omega}_i}$.

Lemma 3.18 *For a compatible genuine character $\tilde{\omega}$ of $Z(\tilde{G}_r)$, the representation*

$$L(\tilde{\rho}_1, [(-m_1 + 1)/2, (m_1 - 1)/2])_{\tilde{\omega}_1} \tilde{\times} \cdots \tilde{\times} L(\tilde{\rho}_k, [(-m_k + 1)/2, (m_k - 1)/2])_{\tilde{\omega}_k} \tilde{\omega}$$

is irreducible.

Proof Write Δ_i for the segment $\{\tilde{\rho}_i v^{(-m_i+1)s(\tilde{\rho}_i)/2}, \tilde{\rho}_i v^{(-m_i+3)s(\tilde{\rho}_i)/2}, \dots, \tilde{\rho}_i v^{(m_i-1)s(\tilde{\rho}_i)/2}\}$, then the lemma follows from the fact that Δ_i are pairwise weakly unlinked. See [31, Theorem 7.5]. □

Using this lemma and the fact that the parabolic induction of a square integrable representation is (not necessarily irreducible) tempered (cf. [51, Lemma III.2.3]), $(\tilde{\pi}_1 \tilde{\times} \cdots \tilde{\times} \tilde{\pi}_k)_{\tilde{\omega}}$ is tempered. The second statement follows readily from the following lemma.

Lemma 3.19 [51, Proposition III.4.1] *Any irreducible tempered representation $\tilde{\pi}$ is a direct summand of a certain parabolic induction $i_{\beta(r)}(\tilde{\rho})$, where $\tilde{\rho}$ is an irreducible square integrable representation of a Levi subgroup G_β of G_r . Such a pair $(G_\beta, \tilde{\rho})$ is unique up to G_r -conjugacy.*

We remark that although the results we cited above are not for covering groups, the same argument indeed works for a general finite central extension of a p -adic reductive group. Note that the argument is based on the definition of square integrable and tempered representations via exponents, which is known in general [1, Theorem 3.4 and Theorem 3.5]. So the original proof can also be generalized without difficulty. □

Finally we study the metaplectic correspondence for essentially tempered representations. Let $r' = r'_1 + \cdots + r'_l$ and $\pi' = \rho_1 \times \cdots \times \rho_l \in \text{Temp}(G_{r'})$ with $\rho_i \in \text{Sqrt}(G_{r'_i})$. The representation π' is called *metic*³ if each central character ω_{ρ_i} restricted to μ_n is trivial. In general, a representation $\pi = \pi_1 \otimes \cdots \otimes \pi_k \in \text{Temp}(G_\beta)$ with each $\pi_i \in \text{Temp}(G_{r_i})$ is called *metic* if each π_i is metic, where $\beta = (r_1, \dots, r_k)$ is a composition of r as before. We denote by $\text{Temp}_\omega(G_\beta)$ the set of equivalence classes of irreducible essentially tempered representations whose central character restricted to $Z(G_r)$ is ω , and by $\text{Temp}_\omega^{(n)}(G_\beta)$ its subset consisting of metic ones.

Proposition 3.20 *For any $\pi \in \text{Temp}_\omega^{(n)}(G_\beta)$, there exists a unique $\tilde{\pi} \in \text{Temp}_{\tilde{\omega}}(\tilde{G}_\beta)$ that lifts to π , and conversely any such $\tilde{\pi}$ lifts to some π .*

Proof We first consider $\beta = (r)$ and $G_\beta = G_r$. We write $\tilde{\pi} = (\tilde{\rho}_1 \tilde{\times} \cdots \tilde{\times} \tilde{\rho}_l)_{\tilde{\omega}}$ with $\tilde{\rho}_i$ essentially square integrable. We let $\rho_i = \text{MC}(\tilde{\rho}_i)$ for each i . By Proposition 3.4 and Proposition 3.10.(1), $\tilde{\pi}$ lifts to $\rho_1 \times \cdots \times \rho_l$ that is essentially tempered and metic. Conversely let $\pi = \rho_1 \times \cdots \times \rho_l$ with each ρ_i essentially square integrable, such that the restriction of ω_{ρ_i} to μ_n is trivial. Using Proposition 3.6 we choose $\tilde{\rho}_i$ to be an essentially square integrable representation such that $\text{MC}(\tilde{\rho}_i) = \rho_i$, then $(\tilde{\rho}_1 \tilde{\times} \cdots \tilde{\times} \tilde{\rho}_l)_{\tilde{\omega}}$ lifts to π . For general β using Proposition 3.4 and the above case, similarly we may prove that every $\tilde{\pi} \in \text{Temp}_\omega^{(n)}(G_\beta)$ lifts to a $\pi \in \text{Temp}_\omega(G_\beta)$, and conversely every $\pi \in \text{Temp}_\omega(G_\beta)$ is a lift of a certain $\tilde{\pi} \in \text{Temp}_\omega^{(n)}(G_\beta)$. Finally such $\tilde{\pi}$ lifting to π is unique by Remark 3.2. □

However in the above proposition, it is unclear to the author if $\pi \in \text{Temp}_\omega^{(n)}(G_\beta)$, as a lift of a certain $\tilde{\pi} \in \text{Temp}_{\tilde{\omega}}(\tilde{G}_\beta)$, is unique or not. So until now we cannot claim the existence of a map $\text{MC} : \text{Temp}_{\tilde{\omega}}(\tilde{G}_\beta) \rightarrow \text{Temp}_\omega^{(n)}(G_\beta), \tilde{\pi} \mapsto \pi$.

³ It is an abbreviation of metaplectic, a notation introduced by Flicker-Kazhdan.

4 Calculation of Whittaker dimension

In this section, we consider one of the main important applications of metaplectic correspondence: calculating the Whittaker dimension of a representation $\tilde{\pi}$ in $\text{Irr}_\epsilon(\tilde{G}_r)$.

4.1 A general introduction of the Harish–Chandra germ expansion

In this subsection, let G be a connected reductive group over F , let \mathfrak{g} be the Lie algebra of G . We won't recall concrete definition of some notation in this part, but indeed the only case we are interested in is G being a Levi subgroup G_β of G_r . So the readers may just imagine what happens in general and refer to the reference we provide.

Let π be an irreducible representation of G . First we briefly recall the definition of the Harish-Chandra germ function (cf. [26]). Let θ_π be the Harish-Chandra character of π , which has the following germ expansion

$$\theta_\pi(x \exp(X)) = \sum_{\mathcal{O} \in \text{Nil}(\mathfrak{g}_x)} c_{\pi, \mathcal{O}}(x) \hat{j}(X, \mathcal{O}),$$

where

- x is a semi-simple element of G ;
- X is a semi-simple regular element in a small neighborhood of 0 in \mathfrak{g} ;
- $\exp : \mathfrak{g} \rightarrow G$ denotes the exponential map;
- \mathfrak{g}_x denotes the Lie algebra of centralizer of x in G ;
- $\text{Nil}(\mathfrak{g}_x)$ denotes the set of nilpotent orbits of \mathfrak{g}_x ;
- $\hat{j}(\cdot, \mathcal{O})$ denotes the Fourier transform of the orbital integral along \mathcal{O} , which is a smooth function on semi-simple regular elements of \mathfrak{g} ;
- $c_{\pi, \mathcal{O}}(x)$ is a complex number depending on π , \mathcal{O} and x .

We write $\text{Nil}_{\text{reg}}(\mathfrak{g}_x)$ for the set of regular nilpotent orbits of \mathfrak{g}_x , and we define

$$c_\pi(x) := |\text{Nil}_{\text{reg}}(\mathfrak{g}_x)|^{-1} \sum_{\mathcal{O}_0 \in \text{Nil}_{\text{reg}}(\mathfrak{g}_x)} c_{\pi, \mathcal{O}_0}(x)$$

as a complex function on the set of semi-simple elements in G , called the *Harish-Chandra germ function* of π . The following formula can be used to calculate this germ function ([6, Proposition 4.5.1]):

$$\begin{aligned} &\Delta^G(x) c_\pi(x) \\ &= \begin{cases} |W(G_x, T_x)|^{-1} \lim_{x' \rightarrow x} \Delta^G(x') \theta_\pi(x'), & \text{if } G_x \text{ is quasi-split,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

where

- $D^G(x)$ denotes the Weyl discriminant of G and $\Delta^G(x) = D^G(x)^{1/2}$;
- G_x denotes the neutral connected component of the centralizer of x in G ;
- when G_x is quasi-split, T_x denotes the maximal split torus contained in a Borel subgroup of G_x ;
- $W(G_x, T_x)$ denotes the corresponding Weyl group;
- x' in the limit are semi-simple regular elements in T_x .

Similarly we consider the corresponding theory for a covering group. Let \tilde{G} be a central extension of G by μ_n and let $\tilde{\pi}$ be an irreducible representation of \tilde{G} . Let $x^* \in \tilde{G}$ such that

$\mathfrak{p}(x^*)$ is semi-simple. To simplify our discussion, we also assume x^* to be *good*, meaning that for any $y \in G$ commuting with $\mathfrak{p}(x^*)$, we have that x^* and $s(y)$ commute. Indeed, when x^* is semi-simple and good, the character $[x^*, \cdot] : \widetilde{G}_{\mathfrak{p}(x^*)} \rightarrow \mu_n$ is trivial, where $[\cdot, \cdot]$ denotes the commutator in \widetilde{G} . Thus in this special case the results in [36, Sect. 4] are largely simplified.

For the Harish-Chandra character $\theta_{\widetilde{\pi}}$ we still have the following germ expansion ([36, Théorème 4.3.2]):

$$\theta_{\widetilde{\pi}}(x^* \exp(X)) = \sum_{\mathcal{O} \in \text{Nil}(\mathfrak{g}_{\mathfrak{p}(x^*)})} c_{\widetilde{\pi}, \mathcal{O}}(x^*) \hat{j}(X, \mathcal{O}),$$

where

- X is a semi-simple regular element in a small neighborhood of 0 in \mathfrak{g} ;
- $\exp(X)$ is realized as an element in \widetilde{G} via the splitting s ;
- $c_{\widetilde{\pi}, \mathcal{O}}(x^*)$ is a complex number depending on $\widetilde{\pi}$, \mathcal{O} and x^* .

We similarly define

$$c_{\widetilde{\pi}}(x^*) := |\text{Nil}_{\text{reg}}(\mathfrak{g}_{\mathfrak{p}(x^*)})|^{-1} \sum_{\mathcal{O}_0 \in \text{Nil}_{\text{reg}}(\mathfrak{g}_{\mathfrak{p}(x^*)})} c_{\widetilde{\pi}, \mathcal{O}_0}(x^*)$$

as a complex function on the set of good semi-simple elements in G , called the *Harish-Chandra germ function* of $\widetilde{\pi}$. An identical argument of [6, Proposition 4.5.1] gives us

$$\begin{aligned} &\Delta^G(\mathfrak{p}(x^*))c_{\widetilde{\pi}}(x^*) \\ &= \begin{cases} |W(G_{\mathfrak{p}(x^*)}, T_{\mathfrak{p}(x^*)})|^{-1} \lim_{x'^* \rightarrow x^*} \Delta^G(\mathfrak{p}(x'^*))\theta_{\widetilde{\pi}}(x'^*), & \text{if } G_{\mathfrak{p}(x^*)} \text{ is quasi-split,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.2}$$

where x'^* in the limit are good semi-simple regular elements in $\widetilde{T}_{\mathfrak{p}(x^*)}$.

4.2 Whittaker space and dimension

We come back to our study of Kazhdan-Patterson covering groups. Let $\beta = (r_1, \dots, r_k)$ be a composition of r , and we write $G = G_{\beta}$ to simplify our notation. Let N be the unipotent subgroup of a Borel subgroup of G . Let ψ be a generic character of N , which means that for any simple root α related to N and the corresponding subgroup N_{α} of N , the restriction $\psi|_{N_{\alpha}}$ is non-trivial. It is easy to verify that such kind of pairs (N, ψ) form a single G -conjugacy class.

For a finite length genuine representation $\widetilde{\pi}$ of \widetilde{G} , we define

$$\text{Wh}(\widetilde{\pi}) = \text{Hom}_N(\widetilde{\pi}, \psi)$$

the *Whittaker space* of $\widetilde{\pi}$, where as before we regard N as a subgroup of \widetilde{G} . As we explained, this vector space essentially does not depend on the choice of the pair (N, ψ) .

We note that Wh , as a functor from $\text{Rep}_{\epsilon}(\widetilde{G})$ to the category of complex vector spaces, is exact. Since when $\beta = (r)$, using the argument in [32, Theorem I.5.3] the functor Wh is realized by taking the r -th Bernstein-Zelevinsky derivative, which is an exact functor. In general, for each i we may similarly define r_i -th ‘‘partial Bernstein-Zelevinsky functor’’ with respect to the i -th block in G , which is also exact. The functor Wh is realized by taking the r_i -th partial Bernstein-Zelevinsky functor for each i .

We define $d_{\tilde{\pi}} := \dim_{\mathbb{C}}(\text{Wh}(\tilde{\pi}))$, which is finite because of the following proposition.

Proposition 4.1 [32, Theorem I.5.3] or [42, Theorem 2] *For $\tilde{\pi}$ irreducible we have $d_{\tilde{\pi}} = c_{\tilde{\pi}}(I_r)$.*

Let $\tilde{\pi}_i \in \text{Irr}_{\epsilon}(\tilde{G}_{r_i})$ for each $i = 1, \dots, k$ and let $\tilde{\omega}$ be a genuine character of $Z(\tilde{G}_r)$ satisfying (2.5).

Proposition 4.2 *We have $d_{(\tilde{\pi}_1 \tilde{\omega} \dots \tilde{\pi}_k)_{\tilde{\omega}}} = d_{(\tilde{\pi}_1 \tilde{\omega} \dots \tilde{\pi}_k)_{\tilde{\omega}}} = d_F(d_r)^{-1} \cdot \prod_{i=1}^k d_F(d_{r_i}) \cdot d_{\tilde{\pi}_i}$.*

Proof The first equation follows from [2, Theorem]. The second equation follows from Corollary 2.4 and the fact that the unipotent radical N is contained in $G_{\beta, \beta}^{(n)}$. \square

Here, $d_r = \gcd(n, 2rc - r + 1)$ is defined as in §2.3.

Let $\tilde{\omega}$ and ω be characters satisfying (3.1). Let $\tilde{\pi} \in \text{Sqrt}_{\tilde{\omega}}(\tilde{G})$ and let $\pi \in \text{Sqrt}_{\omega}(G)$ be the metaplectic lift of $\tilde{\pi}$. We study the relation between the Harish-Chandra germ functions $c_{\tilde{\pi}}$ and c_{π} .

Proposition 4.3 *Let x be a semi-simple element in G such that x^n is semi-simple and the centralizers G_x and G_{x^n} are equal. Let T be a maximal split torus contained in G_x that contains x . Then*

$$\begin{aligned} & \Delta^G(x^n) \cdot |W(G_x, T)| \cdot c_{\tilde{\pi}}(x^*) \\ &= \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t \in T | t^n = x^n\}} \Delta^G(t) \cdot |W(G_t, T)| \cdot c_{\pi}(t) \cdot \epsilon(x^*/t^*). \end{aligned}$$

Proof We choose x' to be an element in T sufficiently close to x , such that x'^n is semi-simple regular. Using (3.2) we have

$$\Delta^G(x'^n) \cdot \theta_{\tilde{\pi}}(x'^*) = \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{\{t' \in T | t'^n = x'^n\}} \Delta^G(t') \cdot \theta_{\pi}(t') \cdot \epsilon(x'^*/t'^*).$$

Taking the limit $x' \rightarrow x$ and using (4.1) and (4.2), we finish the proof. \square

Remark 4.4 In Proposition 4.3 the assumption $G_x = G_{x^n}$ is important, which guarantees that T is a maximal split torus of each G_t and G_{x^n} . We don't know what happens if this condition is not satisfied.

Taking $x = I_r$ and using Propositions 4.1 and 4.3, we have the following important corollary.

Corollary 4.5 *We have*

$$d_{\tilde{\pi}} = \frac{1}{d_F(d_r) |n|_F^{r/2} r!} \sum_t \Delta^G(t) \cdot |W(G_t, T)| \cdot c_{\pi}(t), \tag{4.3}$$

where t in the sum ranges over $\text{diag}(\zeta_1, \dots, \zeta_r)$ with $\zeta_1, \dots, \zeta_r \in \mu_n$, and T denotes the torus of diagonal matrices.

4.3 The main Whittaker dimensional formula for essentially square integrable representations

In this subsection we focus on the case where $G = G_r$. We let T be the diagonal torus of G . In general we let $T_{r'}$ be the diagonal torus of $G_{r'}$ for any $r' \geq 1$.

First we propose the following conjecture related to a so-called ‘‘Zelevinsky standard module’’. Assume $r = r_0m$ for positive integers r_0 and m . For $\rho \in \text{Cusp}(G_{r_0})$ and two real numbers a, b such that $b - a + 1 = m$, we define $Z(\rho, [a, b])$ to be the unique irreducible subrepresentation of

$$\rho v^a \times \rho v^{a+1} \times \dots \times \rho v^b.$$

Conjecture 4.6 For $x = \text{diag}(\zeta_1, \dots, \zeta_r)$ with $\zeta_i \in \mu_n$, we have

$$c_{Z(\rho, [a, b])}(x) = \begin{cases} \prod_{i=1}^m \omega_\rho(\zeta'_i) & \text{if } x \text{ is } G\text{-conjugate to } \text{diag}(\zeta'_1 I_{r_0}, \dots, \zeta'_m I_{r_0}), \\ & \zeta'_i \in \mu_n \text{ pairwise different for } i = 1, \dots, m; \\ 0 & \text{otherwise.} \end{cases}$$

This conjecture helps us to calculate the Whittaker dimension of a genuine representation of \tilde{G} . We state the following main theorem.

Theorem 4.7 Let $\tilde{\pi} \in \text{Sqrt}_\epsilon(\tilde{G})$. Let m be the positive integer dividing r and let $\rho \in \text{Cusp}(G_{r/m})$, such that $\text{MC}(\tilde{\pi}) = L(\rho, [0, m - 1])$ (cf. Proposition 3.16). Let $r_0 = r/m$ and let s be the order of $\omega_\rho|_{\mu_n}$ that divides m . When $\text{gcd}(n, p) = 1$, we have

$$d_{\tilde{\pi}} = \frac{1}{d_r} \cdot \binom{m/s + n/s - 1}{m/s}.$$

Proof We prove Theorem 4.7 on assuming Conjecture 4.6. First we prove a general lemma concerning the Harish–Chandra germ function. For a finite length representation π of G having irreducible subquotients π_1, \dots, π_l (counting the multiplicity), we define $\theta_\pi := \sum_{i=1}^l \theta_{\pi_i}$ and $c_\pi := \sum_{i=1}^l c_{\pi_i}$.

Lemma 4.8 Let $\beta = (r_1, \dots, r_k)$ be a composition of r , let $\pi_i \in \text{Irr}(G_{r_i})$, let T be the diagonal torus of G and G_β , and let $x = \text{diag}(\zeta_1, \dots, \zeta_r) \in T$ such that $\zeta_i \in F^\times$. Then

$$|W(G_x, T)| \cdot c_{\pi_1 \times \dots \times \pi_k}(x) = \frac{1}{r_1! \dots r_k!} \sum_{w \in W(G, T)} \prod_{i=1}^k |W((G_{r_i})_{x_{w,i}}, T_{r_i})| \cdot c_{\pi_i}(x_{w,i}),$$

where for $x \in T$ and $w \in W(G, T)$ we write $w^{-1}xw = \text{diag}(x_{w,1}, \dots, x_{w,k})$ with $x_{w,i} \in G_{r_i}$, and $(G_{r_i})_{x_{w,i}}$ denotes the centralizer of $x_{w,i}$ in G_{r_i} .

Proof We consider $x' = \text{diag}(\zeta'_1, \dots, \zeta'_r)$ with $\zeta'_i \in F^\times$ pairwise different for $i = 1, \dots, r$. We denote by $W(G, T)$ and $W(G_\beta, T)$ the corresponding Weyl groups. Using [50, Theorem 3] we have

$$\begin{aligned}
 \Delta^{G_r}(x')\theta_\pi(x') &= \sum_{\{g \in G_r/P_{(r),\beta} | g^{-1}x'g \in G_\beta\}} \Delta^{G_\beta}(g^{-1}x'g)\theta_{\pi_1 \otimes \dots \otimes \pi_k}(g^{-1}x'g) \\
 &= \sum_{w \in W(G,T)/W(G_\beta,T)} \Delta^{G_\beta}(w^{-1}x'w)\theta_{\pi_1 \otimes \dots \otimes \pi_k}(w^{-1}x'w) \\
 &= |W(G_\beta, T)|^{-1} \sum_{w \in W(G,T)} \Delta^{G_\beta}(w^{-1}x'w)\theta_{\pi_1 \otimes \dots \otimes \pi_k}(w^{-1}x'w) \\
 &= \frac{1}{r_1! \cdots r_k!} \sum_{w \in W(G,T)} \prod_{i=1}^k \Delta^{G_{r_i}}(x'_{w,i})\theta_{\pi_i}(x'_{w,i})
 \end{aligned}$$

Taking the limit $x' \rightarrow x$ and using (4.1) we finish the proof. □

Now we focus on the proof of Theorem 4.7. We write $\pi = MC(\tilde{\pi}) = L(\rho, [0, m - 1])$. Using Tadic’s determinantal formula [35, Theorem 1], we have

$$\pi = \sum_{m_1 + \dots + m_l = m} (-1)^{m-l} Z(\rho, [0, t_1 - 1]) \times Z(\rho, [t_1, t_2 - 1]) \times \dots \times Z(\rho, [t_{l-1}, t_l - 1]),$$

where m_i in the sum are positive integers, and we define $t_0 = 0$ and $t_i = m_1 + \dots + m_i$ for each $i = 1, 2, \dots, l$, and the equation is taken in the Grothendieck group of finite length representations of G . Taking the Harish–Chandra germ function and using Lemma 4.8, we get

$$\begin{aligned}
 |W(G_x, T)| \cdot c_\pi(x) &= |W(G_x, T)| \sum_{m_1 + \dots + m_l = m} (-1)^{m-l} c_{(\times_{i=1}^l Z(\rho, [t_{i-1}, t_i - 1]))}(x) \\
 &= \sum_{m_1 + \dots + m_l = m} \frac{(-1)^{m-l}}{\prod_{i=1}^l (r_0 m_i)!} \sum_{w \in W(G,T)} \prod_{i=1}^l |W((G_{r_0 m_i})_{x_{w,i}}, T_{r_0 m_i})| \cdot c_{Z(\rho, [t_{i-1}, t_i - 1])}(x_{w,i}).
 \end{aligned}$$

Using (4.3), we have

$$\begin{aligned}
 d_{\tilde{\pi}} &= \frac{1}{d_F(d_r) |n|_F^{r/2} r!} \sum_x \Delta^G(x) \cdot |W(G_x, T)| \cdot c_\pi(x) \\
 &= \frac{1}{d_F(d_r) |n|_F^{r/2} r!} \sum_{m_1 + \dots + m_l = m} \frac{(-1)^{m-l}}{\prod_{i=1}^l (r_0 m_i)!} \sum_{w \in W(G,T)} \sum_x f(m_1, \dots, m_l, w, x)
 \end{aligned}$$

where x ranges over $\text{diag}(\zeta_1, \dots, \zeta_r)$ with $\zeta_i \in \mu_n$ for each i , and

$$f(m_1, \dots, m_l, w, x) := \Delta^G(x) \cdot \prod_{i=1}^l |W((G_{r_0 m_i})_{x_{w,i}}, T_{r_0 m_i})| \cdot c_{Z(\rho, [t_{i-1}, t_i - 1])}(x_{w,i}).$$

It is clear that

$$\sum_{w \in W(G,T)} \sum_x f(m_1, \dots, m_l, w, x) = |W(G, T)| \sum_x f(m_1, \dots, m_l, 1, x),$$

so we further have

$$d_{\tilde{\pi}} = \frac{1}{d_F(d_r) |n|_F^{r/2}} \sum_{m_1 + \dots + m_l = m} \frac{(-1)^{m-l}}{\prod_{i=1}^l (r_0 m_i)!} \sum_x f(m_1, \dots, m_l, 1, x). \tag{4.4}$$

For the identity element $1 \in W(G, T)$, we write $x_i = x_{1,i}$ for short for each $i = 1, \dots, l$. Using Conjecture 4.6,

$$f(m_1, \dots, m_l, 1, x) = \Delta^G(x) \prod_{i=1}^l |W((G_{r_0 m_i})_{x_i}, T_{r_0 m_i})| \cdot c_{Z(\rho, [l_{i-1}, l_i - 1])}(x_i)$$

is 0 except the following condition holds: For every $i = 1, \dots, l$, the matrix x_i is $G_{r_0 m_i}$ -conjugate to a matrix of the form $\text{diag}(\zeta_{i1} I_{r_0}, \dots, \zeta_{i m_i} I_{r_0})$ with $\zeta_{ij} \in \mu_n$ pairwise different for $j = 1, \dots, m_i$. In this case we have $|W((G_{r_0 m_i})_{x_i}, T_{r_0 m_i})| = (r_0!)^{m_i}$ for each i and

$$f(m_1, \dots, m_l, 1, x) = \Delta^G(x) \prod_{i=1}^l (r_0!)^{m_i} \prod_{i=1}^l \omega_\rho(\zeta_{ij}).$$

From now on we assume $\text{gcd}(n, p) = 1$. The most important consequence of this assumption is that, $\Delta^G(x) = 1$ for any x of the form $\text{diag}(\zeta_1, \dots, \zeta_n)$ with $\zeta_i \in \mu_n$, which seems to be crucial in the calculation below. Moreover, we also have $\mu_n \subset \mathbb{F}_q^\times$, $d_F(d_r) = d_r$ and $|n|_F = 1$.

Then we have

$$\begin{aligned} d_{\tilde{\pi}} &= \frac{1}{d_r} \sum_{m_1 + \dots + m_l = m} \left(\prod_{i=1}^l \frac{(-1)^{m_i - 1} (r_0!)^{m_i}}{(r_0 m_i)!} \sum_{x_i} \prod_{j=1}^{m_i} \omega_\rho(\zeta_{ij}) \right) \\ &= \frac{1}{d_r} \sum_{m_1 + \dots + m_l = m} \left(\prod_{i=1}^l \frac{(-1)^{m_i - 1}}{m_i!} \sum_{\substack{\zeta_{i1}, \dots, \zeta_{i m_i} \in \mu_n \\ \text{pairwise different}}} \prod_{j=1}^{m_i} \omega_\rho(\zeta_{ij}) \right) \end{aligned}$$

where for each i , the corresponding x_i in the sum ranges over the matrices in $T_{r_0 m_i}$ that are $G_{r_0 m_i}$ -conjugate to $\text{diag}(\zeta_{i1} I_{r_0}, \dots, \zeta_{i m_i} I_{r_0})$ with $\zeta_{ij} \in \mu_n$ pairwise different.

Lemma 4.9 *For each i we have*

$$\frac{1}{m_i!} \sum_{\substack{\zeta_{i1}, \dots, \zeta_{i m_i} \in \mu_n \\ \text{pairwise different}}} \prod_{j=1}^{m_i} \omega_\rho(\zeta_{ij}) = \begin{cases} (-1)^{(s+1)m_i/s} \cdot \binom{n/s}{m_i/s} & \text{if } s \text{ divides } m_i; \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let ζ be a generator of μ_n , and let $\zeta_0 = \omega_\rho(\zeta)$ which is a generator of μ_s . Then the left-hand side becomes

$$\sum_{k_1 + \dots + k_s = m_i} \prod_{j=1}^s \binom{n/s}{k_j} \zeta_0^{j k_j},$$

where k_j in the sum are non-negative integers, meaning that for each j there are k_j 's ζ_{il} with $1 \leq l \leq m_i$ such that $\omega_\rho(\zeta_{il}) = \zeta_0^j$. Considering the generating function

$$\prod_{j=1}^s (\zeta_0^j X + 1)^{n/s} = ((-1)^{s+1} X^s + 1)^{n/s}$$

and comparing the coefficient of X^{m_i} , we finish the proof. □

Using this lemma, we further have

$$\begin{aligned}
 d_{\tilde{\pi}} &= \frac{1}{d_r} \sum_{\substack{m_1 + \dots + m_l = m \\ s \text{ divides } m_i}} \left(\prod_{i=1}^l (-1)^{m_i-1} (-1)^{(s+1)m_i/s} \cdot \binom{n/s}{m_i/s} \right) \\
 &= \frac{1}{d_r} \sum_{m'_1 + \dots + m'_l = m/s} \left(\prod_{i=1}^l (-1)^{m'_i-1} \cdot \binom{n/s}{m'_i} \right) \\
 &= \frac{1}{d_r} \cdot \binom{m/s + n/s - 1}{m/s},
 \end{aligned}$$

where for the last step we compare the coefficient of $X^{m/s}$ in the generating function

$$\sum_{i=0}^{+\infty} \left(\sum_{j=1}^{n/s} (-1)^{j-1} \binom{n/s}{j} X^j \right)^i = \frac{1}{(1-X)^{n/s}} = \sum_{i=0}^{+\infty} \binom{i + n/s - 1}{i} X^i,$$

so we finish the proof. □

Remark 4.10 By Theorem 4.7 and Proposition 4.2, it is theoretically and algorithmically possible to calculate $d_{\tilde{\pi}}$ for any $\tilde{\pi} \in \text{Irr}_{\tilde{\omega}}(\tilde{G}_r)$, since in the Grothendieck group $\tilde{\pi}$ can be written as a linear combination (with coefficients being integers that can be determined) of representations of the form

$$(L(\tilde{\rho}_1, [0, m_1 - 1])_{\tilde{\omega}_1} \tilde{\times} \dots \tilde{\times} L(\tilde{\rho}_l, [0, m_l - 1])_{\tilde{\omega}_l})_{\tilde{\omega}},$$

where $\tilde{\rho}_i \in \text{Cusp}_{\epsilon}(\tilde{G}_{r_i})$ such that $\sum_{i=1}^l r_i m_i = r$ and $\tilde{\omega}_i$ are genuine compatible characters of $Z(\tilde{G}_{r_i, m_i})$ for each i (cf. [31]).

In particular we have the following corollary.

Corollary 4.11 *Let $\tilde{\rho} \in \text{Cusp}_{\epsilon}(\tilde{G}_{r'_0})$. Assume $\text{MC}(\tilde{\rho}) = L(\rho, [0, s - 1])$ for $r'_0 = r_0 s$ and $\rho \in \text{Cusp}(G_{r_0})$, such that $\omega_{\rho|_{\mu_n}}$ is of order s (cf. Proposition 3.16). If $\text{gcd}(n, \rho) = 1$, then*

$$d_{L(\tilde{\rho}, [0, k-1])_{\tilde{\omega}}} = \frac{1}{d_r} \binom{k + n/s - 1}{k} \text{ and } d_{Z(\tilde{\rho}, [0, k-1])_{\tilde{\omega}}} = \frac{1}{d_r} \binom{n/s}{k}.$$

Proof The first statement is nothing but a reformulation of Theorem 4.7. For the second statement, recall that we have the following determinantal formula (cf. [31, 35])

$$\begin{aligned}
 &Z(\tilde{\rho}, [0, k - 1])_{\tilde{\omega}} \\
 &= \sum_{k_1 + \dots + k_l = k} (-1)^{k-l} (L(\tilde{\rho}, [t_0, t_1 - 1])_{\tilde{\omega}_1} \tilde{\times} L(\tilde{\rho}, [t_1, t_2 - 1])_{\tilde{\omega}_2} \tilde{\times} \dots \tilde{\times} L(\tilde{\rho}, [t_{l-1}, t_l - 1])_{\tilde{\omega}_l})_{\tilde{\omega}},
 \end{aligned}$$

where we write $t_0 = 0$ and $t_i = k_1 + \dots + k_i$ for each $i = 1, \dots, l$, and the above two sums are in the Grothendieck group of finite length representations of \tilde{G}_r . Taking the Whittaker dimension and using the first formula and Proposition 4.2, we get

$$d_r \cdot d_{Z(\tilde{\rho}, [0, k-1])_{\tilde{\omega}}} = \sum_{k_1 + \dots + k_l = k} \prod_{i=1}^l (-1)^{k_i-1} \binom{k_i + n/s - 1}{k_i} = \binom{n/s}{k},$$

where the last equation follows by considering the coefficient of X^k in the generating function

$$\sum_{i=0}^{+\infty} \left(\sum_{j=1}^{+\infty} (-1)^{j-1} \binom{j+n/s-1}{j} X^j \right)^i = \sum_{i=0}^{+\infty} (1 - (1+X)^{-n/s})^i = (1+X)^{n/s}.$$

□

In particular we have the following interesting special case.

Corollary 4.12 *We keep the notation of Corollary 4.11. Then*

$$d_{Z(\tilde{\rho}, [0, n/s-1])_{\tilde{\omega}}} = 1 \quad \text{and} \quad d_{Z(\tilde{\rho}, [0, k-1])_{\tilde{\omega}}} = 0 \quad \text{if } k > n/s.$$

It should be interesting to give an independent algebraic proof of the above corollary, like what Bernstein-Zelevinsky did in [5, 53] in the $n = 1$ case. Their proof, which is mysterious enough, relies on the multiplicity one theorem of the Whittaker model of a generic representation. So giving an independent proof of the about corollary includes providing a better understanding of the old but prominent theory of Bernstein-Zelevinsky.

4.4 Evidence of Conjecture 4.6

In this subsection we still let $G = G_r$. We verify Conjecture 4.6 for two special cases.

First we consider the case where $r_0 = 1$, then ρ is indeed a character of F^\times . In this case by definition we have

$$Z(\rho, [a, b]) = (\rho v^{(b+a)/2}) \circ \det$$

which is a character of G_r . Then for $x = \text{diag}(\zeta_1, \dots, \zeta_r)$ with $\zeta_i \in \mu_n$ we have

$$\Delta^G(x) \cdot c_{Z(\rho, [a, b])}(x) = \lim_{x' \rightarrow x} \Delta^G(x') \cdot (\rho v^{(b+a)/2})(\det(x'))$$

where in the limit $x' = \text{diag}(\zeta'_1, \dots, \zeta'_r)$ with $\zeta'_i \in \mu_n$ pairwise different. If there exist $i \neq j$ such that $\zeta_i = \zeta_j$, then

$$\lim_{x' \rightarrow x} \Delta^G(x') = 0 \quad \text{and thus} \quad c_{Z(\rho, [a, b])}(x) = 0;$$

if ζ_i are pairwise different for $i = 1, \dots, r$, we have

$$c_{Z(\rho, [a, b])}(x) = (\rho v^{(b+a)/2})(\det(x)) = \prod_{i=1}^r \rho(\zeta_i).$$

So in this case the conjecture is verified. In particular we have the following important corollary as a special case of Corollary 4.11.

Corollary 4.13 *Let $\tilde{\chi} \in \text{Irr}_e(\tilde{F}^\times)$. Assume $\text{gcd}(n, p) = 1$, then*

$$d_{L(\tilde{\chi}, [0, k-1])_{\tilde{\omega}}} = \frac{1}{d_r} \binom{k+n-1}{k} \quad \text{and} \quad d_{Z(\tilde{\chi}, [0, k-1])_{\tilde{\omega}}} = \frac{1}{d_r} \binom{n}{k}.$$

Remark 4.14 In the case where $\tilde{\chi}$ is an unramified representation, $Z(\tilde{\chi}, [0, k-1])_{\tilde{\omega}}$ is usually called an “exceptional representation” or a “theta representation”, whose Whittaker dimension is calculated in [32, Theorem I.3.5] by considering functional equations related to intertwining operators. So we give an independent proof of their result and also generalize it to ramified case.

Now we consider the other extreme for cuspidal representations, which is summed up as the following theorem.

Theorem 4.15 *Assume $\gcd(n, p) = 1$. For $\pi \in \text{Cusp}(G)$ and $x = \text{diag}(\zeta_1, \dots, \zeta_r)$ with $\zeta_i \in \mu_n \subset \mathbb{F}_q^\times \subset F^\times$, we have*

$$c_\pi(x) = \begin{cases} \omega_\pi(\zeta) & \text{if } \zeta_1 = \dots = \zeta_r = \zeta; \\ 0 & \text{if } \zeta_i \neq \zeta_j \text{ for some } i \neq j. \end{cases}$$

Proof When $\zeta_1 = \dots = \zeta_r = 1$, by [43] and [44] $c_\pi(I_r)$ equals the Whittaker dimension of π which is 1. For $\zeta_1 = \dots = \zeta_r = \zeta \in \mu_n$ in general, by (4.1) we have $c_\pi(\zeta I_r) = \omega_\pi(\zeta)c_\pi(I_r) = \omega_\pi(\zeta)$. To prove the rest, we first give an ad-hoc introduction of the simple type theory for our use, and we leave [9], or in general [12] for more details.

A simple stratum $[\mathfrak{a}, \beta]$ in $A = M_r(F)$ consists of a hereditary order \mathfrak{a} in A and an element $\beta \in G$, such that $E = F[\beta]$ is a field of degree d over F , and E^\times normalizes \mathfrak{a} . Let $B \simeq M_r(E)$ be the centralizer of E in A , thus $r = md$. Let $\mathfrak{b} = B \cap \mathfrak{a}$ be a hereditary order in B , let $\mathfrak{p}_\mathfrak{a}$ be the Jacobson radical of \mathfrak{a} and let $\mathfrak{p}_\mathfrak{b} = \mathfrak{p}_\mathfrak{a} \cap \mathfrak{b}$ be the Jacobson radical of \mathfrak{b} . We further assume our simple stratum to be maximal, saying that \mathfrak{b} is a maximal order in B . In this case, we change \mathfrak{a} up to G -conjugacy, such that

$$\mathfrak{a} = \{(a_{ij})_{1 \leq i, j \leq e} \mid a_{ij} \in M_{mf}(\mathfrak{o}_F) \text{ for } 1 \leq i \leq j \leq e \text{ and } a_{ij} \in M_{mf}(\mathfrak{p}_F) \text{ for } 1 \leq j < i \leq e\}$$

and

$$\mathfrak{p}_\mathfrak{a} = \{(a_{ij})_{1 \leq i, j \leq e} \mid a_{ij} \in M_{mf}(\mathfrak{o}_F) \text{ for } 1 \leq i < j \leq e \text{ and } a_{ij} \in M_{mf}(\mathfrak{p}_F) \text{ for } 1 \leq j \leq i \leq e\},$$

where $d = ef$ with e the ramification index and f the residue degree of E/F , and \mathfrak{o}_F (resp. \mathfrak{o}_E) denotes the ring of integers of F (resp. E), and \mathfrak{p}_F (resp. \mathfrak{p}_E) denotes the corresponding maximal ideal. We further have the \mathbb{F}_q -algebra embedding

$$\iota = (t_1, \dots, t_e) : M_m(\mathbb{F}_{q^f}) \simeq \mathfrak{b}/\mathfrak{p}_\mathfrak{b} \hookrightarrow \mathfrak{a}/\mathfrak{p}_\mathfrak{a} \simeq \underbrace{M_{mf}(\mathbb{F}_q) \times \dots \times M_{mf}(\mathbb{F}_q)}_{e\text{-copies}}, \quad (4.5)$$

where each t_i is an \mathbb{F}_q -algebra embedding⁴ from $M_m(\mathbb{F}_{q^f})$ to the i -th $M_{mf}(\mathbb{F}_q)$ on the right-hand side, which induces the corresponding group embedding

$$\iota = (t_1, \dots, t_e) : \text{GL}_m(\mathbb{F}_{q^f}) \simeq \mathfrak{b}^\times/1 + \mathfrak{p}_\mathfrak{b} \hookrightarrow \mathfrak{a}^\times/1 + \mathfrak{p}_\mathfrak{a} \simeq \underbrace{\text{GL}_{mf}(\mathbb{F}_q) \times \dots \times \text{GL}_{mf}(\mathbb{F}_q)}_{e\text{-copies}}. \quad (4.6)$$

Associated with $[\mathfrak{a}, \beta]$, we may construct an open compact pro- p -group $J^1(\mathfrak{a}, \beta) \subset 1 + \mathfrak{p}_\mathfrak{a}$ and an open compact group $J^0(\mathfrak{a}, \beta) = \mathfrak{b}^\times J^1(\mathfrak{a}, \beta) \subset \mathfrak{a}^\times$, and we have $J^0(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \simeq \mathfrak{b}^\times/1 + \mathfrak{p}_\mathfrak{b} \simeq \text{GL}_m(\mathbb{F}_{q^f})$. We will write J^0 and J^1 instead of $J^0(\mathfrak{a}, \beta)$ and $J^1(\mathfrak{a}, \beta)$ for short.

One main result of the simple type theory predicts the existence of a pair (\mathbf{J}, Λ) with $\mathbf{J} := E^\times J^0$ and an irreducible representation Λ of \mathbf{J} , such that π is isomorphic to the compact induction $\text{ind}_\mathbf{J}^G(\Lambda)$. The restriction $\lambda = \Lambda|_{J^0}$ equals the tensor product $\kappa \otimes \rho$. Here κ is a so-called β -extension as an irreducible representation of J^0 of dimension a power of p , whose construction and properties shall be irrelevant. And ρ is the inflation of a cuspidal representation $\bar{\rho}$ of $\text{GL}_m(\mathbb{F}_{q^f}) \simeq \mathfrak{b}^\times/1 + \mathfrak{p}_\mathfrak{b}$.

⁴ Each t_i is injective since $M_m(\mathbb{F}_{q^f})$ is a simple \mathbb{F}_q -algebra.

We come back to the original proof. Let $x' = \text{diag}(\zeta'_1, \dots, \zeta'_r)$ be a semi-simple regular element of G , such that $\zeta'_i \in \zeta_i(1 + \mathfrak{p}_F)$ for each i . For a finite dimensional irreducible representation Λ of an ℓ -group \mathbf{J} , we denote by $\text{tr}(\Lambda)$ the trace of Λ . Using [10, Theorem A.14], for an open compact subgroup K_1 of G , we have

$$\theta_\pi(x') = \sum_{h \in K_1 \backslash G / \mathbf{J}} \sum_{g \in K_1 h \mathbf{J} / \mathbf{J}, g^{-1}x'g \in \mathbf{J}} \text{tr}(\Lambda)(g^{-1}x'g). \tag{4.7}$$

Lemma 4.16 *If $\zeta_i \neq \zeta_j$ for some $1 \leq i < j \leq n$, then for any $g \in G$ such that $g^{-1}x'g \in \mathbf{J}$ we have $\text{tr}(\Lambda)(g^{-1}x'g) = 0$.*

It is clear that the rest of Theorem 4.15 follows from Lemma 4.16, (4.1) and (4.7), so we focus on the proof of Lemma 4.16.

Since $\det(g^{-1}x'g) = \prod_{i=1}^r \zeta'_i \in \mathfrak{o}_F^\times$, we have $g^{-1}x'g \in J^0$. We write $g^{-1}x'g = yk$ with $y \in \mathfrak{b}^\times, k \in J^1$, and \bar{y} the image of y in $\text{GL}_m(\mathbb{F}_{q^f}) \simeq \mathfrak{b}^\times / 1 + \mathfrak{p}_\mathfrak{b}$.

Lemma 4.17 *Let $P_{\bar{y}}(X)$ be the characteristic polynomial of \bar{y} in $\text{M}_m(\mathbb{F}_{q^f})$, then*

$$[\text{N}_{\mathbb{F}_{q^f}/\mathbb{F}_q}(P_{\bar{y}}(X))]^e = \prod_{i=1}^r (X - \zeta_i),$$

where $\text{N}_{\mathbb{F}_{q^f}/\mathbb{F}_q} := \prod_{\sigma \in \text{Gal}(\mathbb{F}_{q^f}/\mathbb{F}_q)} \sigma$ denotes the norm map.

Proof By definition, $P'(X) = \prod_{i=1}^r (X - \zeta'_i) \in \mathfrak{o}_F[X]$ is the characteristic polynomial of $g^{-1}x'g = yk$, thus $P(X) = \prod_{i=1}^r (X - \zeta_i) \in \mathbb{F}_q[X]$ is the characteristic polynomial of yk in $\text{GL}_r(\mathbb{F}_q) \simeq \text{GL}_r(\mathfrak{o}_F)/1 + \text{M}_r(\mathfrak{p}_F)$. Since $yk \in \mathfrak{a}^\times$, by direct calculation $P(X)$ depends only on the image of yk in $\text{GL}_{mf}(\mathbb{F}_q)^{\oplus e} \simeq \mathfrak{a}^\times / 1 + \mathfrak{p}_\mathfrak{a}$. More precisely by (4.6), $\iota(\bar{y}) = \text{diag}(\iota_1(\bar{y}), \dots, \iota_e(\bar{y}))$ is exactly the image of yk in $\text{GL}_{mf}(\mathbb{F}_q)^{\oplus e}$. Moreover, $P(X) = \prod_{i=1}^e P_{\iota_i(\bar{y})}(X)$ with $P_{\iota_i(\bar{y})}(X)$ denoting the characteristic polynomial of $\iota_i(\bar{y})$ in $\text{M}_{mf}(\mathbb{F}_q)$. Finally we have $\text{N}_{\mathbb{F}_{q^f}/\mathbb{F}_q}(P_{\bar{y}}(X)) = P_{\iota_i(\bar{y})}(X)$ for each i , thus $[\text{N}_{\mathbb{F}_{q^f}/\mathbb{F}_q}(P_{\bar{y}}(X))]^e = \prod_{i=1}^e P_{\iota_i(\bar{y})}(X) = P(X) = \prod_{i=1}^r (X - \zeta_i)$. \square

Now we finish the proof of Lemma 4.16. By definition we have $\text{tr}(\Lambda)(g^{-1}x'g) = \text{tr}(\kappa)(g^{-1}x'g) \cdot \text{tr}(\rho)(g^{-1}x'g)$ and $\text{tr}(\rho)(g^{-1}x'g) = \text{tr}(\rho)(yk) = \text{tr}(\bar{\rho})(\bar{y})$. Using [25, Theorem 12], $\text{tr}(\bar{\rho})(\bar{y}) = 0$ unless $P_{\bar{y}}(X)$ is a power of an irreducible polynomial in $\mathbb{F}_q[X]$, which happens only if $\zeta_1 = \dots = \zeta_r$ using Lemma 4.17. Thus we must have $\text{tr}(\bar{\rho})(\bar{y}) = 0$ and $\text{tr}(\Lambda)(g^{-1}x'g) = 0$ once there exist i, j such that $\zeta_i \neq \zeta_j$, finishing the proof of Lemma 4.16 and Theorem 4.15. \square

As a special case of Theorem 4.7, we have

Corollary 4.18 *Assume $\text{gcd}(n, p) = 1$. Let $\tilde{\pi} \in \text{Cusp}_e(\tilde{G})$ such that $\pi = \text{MC}(\tilde{\pi}) \in \text{Cusp}(G)$, then*

$$d_{\tilde{\pi}} = n/d_r.$$

Remark 4.19 We notice that $\tilde{\pi}$ is cuspidal if π is so, but the converse is not true in general (cf. Proposition 3.16). So this corollary calculates the Whittaker dimension of a certain class of cuspidal representations of \tilde{G} . One may also compare our result with the result of Blondel [7, Theorem 3] for depth 0 cuspidal representations, and find out that they match well if the

number k in *loc. cit.* is 1, which is indeed expected when π is cuspidal (although we do not give a proof here). This ‘‘coincidence’’ indeed motivates us to study an ‘‘explicit’’ metaplectic correspondence as what Bushnell–Henniart [11] proposed for the Jacquet–Langlands correspondence.

5 Conjecture 14 of [30] for a Kazhdan–Patterson covering group

In this section, we explain how our results could be used to study an analogue of [30, Conjecture 14] for a Kazhdan–Patterson covering group. Our discussion here is rather heuristic.

Still, all the covering groups we consider are Kazhdan–Patterson for fixed n, c . Fix two positive integers r and c_0 . Let $\tilde{\pi} \in \text{Temp}_\epsilon(\widetilde{G}_r)$, which is necessarily generic by Proposition 3.17, Proposition 4.2 and Corollary 4.11. For a compatible genuine character $\tilde{\omega}$ of $Z(\widetilde{G}_{rc_0})$, we consider the parabolic induction

$$(\tilde{\pi} v^{-(c_0n-1)/2n} \tilde{\times} \tilde{\pi} v^{-(c_0n-3)/2n} \tilde{\times} \dots \tilde{\times} \tilde{\pi} v^{(c_0n-3)/2n} \tilde{\times} \tilde{\pi} v^{(c_0n-1)/2n})_{\tilde{\omega}}. \tag{5.1}$$

By the Langlands classification [1, Theorem 4.1], it has a unique irreducible subrepresentation which we denote by $\rho_{c_0}(\tilde{\pi})_{\tilde{\omega}}$. We state [30, Conjecture 14] in our settings.

Conjecture 5.1 *When $c_0 = 1$, the Whittaker space of $\rho_1(\tilde{\pi})_{\tilde{\omega}}$ is one-dimensional. In general, $\rho_{c_0}(\tilde{\pi})_{\tilde{\omega}}$ is a (nr, c_0) -representation (cf. [30, Definition 11]), or, a fortiori, the highest derivative of $\rho_{c_0}(\tilde{\pi})_{\tilde{\omega}}$ is of order nr and equals $v^{-1/2} \cdot \rho_{c_0-1}(\tilde{\pi})_{\tilde{\omega}'}$ for a certain compatible genuine character $\tilde{\omega}'$ of $Z(\widetilde{G}_{rn(c_0-1)})$.*

Remark 5.2 The exponent $-1/2$ in $v^{-1/2} \cdot \rho_{c_0-1}(\tilde{\pi})_{\tilde{\omega}'}$ differs from that in [30, Conjecture 14] (which is $(n-1)/2$). It is because in *loc. cit.* the author considered non-normalized derivative functors, whereas we consider normalized derivative functors.

We expect that using Corollary 4.11, the Bernstein–Zelevinsky theory of derivatives in [4, §3–4] and the theory of generalized Speh representations studied by Tadić [47] and others, this conjecture can be verified as in the linear case [13, Theorem 4]. However, we need to generalize the above results to a Kazhdan–Patterson covering group, which, although expected to be routine, has not been fully done yet. So instead we satisfy ourselves with a special case, where $\tilde{\pi}$ is a cuspidal representation.

Proposition 5.3 *When $\tilde{\pi}$ is a cuspidal representation, the above conjecture is true.*

Proof Let $s(\tilde{\pi})$ be as in Proposition 3.14, let $s = s(\tilde{\pi})n$ be a positive integer that divides n and let $m = n/s$. Let

$$\tilde{\pi}_0 = Z(\tilde{\pi}, [-(mc_0 - 1)/2, (mc_0 - 1)/2])_{\tilde{\omega}_0}$$

be the unique irreducible subrepresentation of

$$(v^{-(mc_0-1)s/2n} \tilde{\times} \tilde{\pi} \tilde{\times} v^{-(mc_0-3)s/2n} \tilde{\times} \tilde{\pi} \tilde{\times} \dots \tilde{\times} v^{(mc_0-1)s/2n} \tilde{\times} \tilde{\pi})_{\tilde{\omega}_0}$$

of \widetilde{G}_{rmc_0} , where $\tilde{\omega}_0$ is a certain compatible genuine character of $Z(\widetilde{G}_{rmc_0})$. Using the Zelevinsky classification [31], the representation (5.1) has an irreducible subrepresentation

$$(v^{-(s-1)/2n} \tilde{\times} \tilde{\pi}_0 \tilde{\times} v^{-(s-3)/2n} \tilde{\times} \tilde{\pi}_0 \tilde{\times} \dots \tilde{\times} v^{(s-1)/2n} \tilde{\times} \tilde{\pi}_0)_{\tilde{\omega}},$$

which is exactly $\rho_{c_0}(\tilde{\pi})_{\tilde{\omega}}$. Using [30, Proposition 12] whose argument works for a Kazhdan–Patterson covering group as well, we need to prove that

- When $c_0 = 1$, the Whittaker space of $Z(\tilde{\pi}, [-(m-1)/2, (m-1)/2])_{\tilde{\omega}_0}$ is one-dimensional;
- $\tilde{\pi}_0 = Z(\tilde{\pi}, [-(mc_0-1)/2, (mc_0-1)/2])_{\tilde{\omega}_0}$ is an (mr, c_0) -representation. Or, a fortiori, the highest derivative of $Z(\tilde{\pi}, [-(mc_0-1)/2, (mc_0-1)/2])_{\tilde{\omega}_0}$ is of order mr and equals

$$\begin{aligned} & Z(\tilde{\pi}, [-(mc_0-1)/2, (mc_0-1-2m)/2])_{\tilde{\omega}'_0} \\ &= v^{-1/2} \cdot Z(\tilde{\pi}, [-(mc_0-1-m)/2, (mc_0-1-m)/2])_{\tilde{\omega}''_0}, \end{aligned}$$

where $\tilde{\omega}'_0$ and $\tilde{\omega}''_0$ are certain compatible genuine characters of $Z(\widehat{G_{r(m-1)c_0}})$.

The first claim follows from Corollary 4.12. The second claim follows from Corollary 4.12, the formula of Jacquet module of $\tilde{\pi}_0$ in Lemma 3.15.(2), and the fact that the i -th derivative functor equals the composition of the Jacquet module functor related to the composition $(rmc_0 - i, i)$ with the i -th second partial derivative functor. We omit the detail but leave [53, §3] for a similar argument. \square

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

1. Ban, D., Jantzen, C.: The Langlands quotient theorem for finite central extensions of p -adic groups. *Glas. Mat. Ser. III* **48**(2), 313–334 (2013)
2. Banks, W.D.: Heredity of Whittaker models on the metaplectic group. *Pac. J. Math.* **185**(1), 89–96 (1998)
3. Banks, W.D., Levy, J., Sepanski, M.R.: Block-compatible metaplectic cocycles. *J. Reine Angew. Math.* **507**, 131–163 (1999)
4. Bernstein, I.N., Zelevinskii, A.V.: Representations of the group $GL(n, F)$, where F is a non-Archimedean local field. *Uspëhi Mat. Nauk* **31**(3), 5–70 (1976)
5. Bernstein, I.N., Zelevinsky, A.V.: Induced representations of reductive p -adic groups I. *Ann. Sci. École Norm. Sup. (4)* **10**(4), 441–472 (1977)
6. Beuzart-Plessis, R.: A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: the Archimedean case. *Astérisque* **418**, 1–299 (2020)
7. Blondel, C.: Uniqueness of Whittaker model for some supercuspidal representations of the metaplectic group. *Compos. Math.* **83**(1), 1–18 (1992)
8. Borel, A., L-functions, Automorphic: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977). Part 2, pp. 27–61 (1979)
9. Bushnell, C.J.: Arithmetic of cuspidal representations, representations of Reductive p -adic groups. *Progr. Math.* **328**, 39–126 (2019)
10. Bushnell, C.J., Henniart, G.: Local tame lifting for $GL(N)$. I. Simple characters. *Inst. Hautes Études Sci. Publ. Math.* **83**, 105–233 (1996)
11. Bushnell, C.J., Henniart, G.: The essentially tame Jacquet–Langlands correspondence for inner forms of $GL(n)$. *Pure Appl. Math. Q.* **7**(3), 469–538 (2011)
12. Bushnell, C.J., Kutzko, P.C.: The admissible dual of $GL(N)$ via compact open subgroups. *Ann. Math. Stud.* **129**, 1 (1993)
13. Cai, Y., Friedberg, S., Gourevitch, D., Kaplan, E.: The generalized doubling method: (k, c) models. *Proc. Am. Math. Soc.* **151**(7), 2831–2845 (2023)
14. Casselman, W.: Characters and Jacquet modules. *Math. Ann.* **230**(2), 101–105 (1977)
15. Chinta, G., Offen, O.: A metaplectic Casselman–Shalika formula for GL_r . *Am. J. Math.* **135**(2), 403–441 (2013)
16. Clozel, L.: Invariant harmonic analysis on the Schwartz space of a reductive p -adic group. Harmonic analysis on reductive groups. *Progr. Math.* **101**, 101–121 (1991)
17. Flicker, Y.Z.: Automorphic forms on covering groups of $GL(2)$. *Invent. Math.* **57**(2), 119–182 (1980)
18. Flicker, Y.Z., Kazhdan, D.A.: Metaplectic correspondence. *Inst. Hautes Études Sci. Publ. Math.* **64**, 53–110 (1986)

19. Gan, W.-T., Gao, F., Weissman, M.H.: L-groups and the Langlands program for covering groups: a historical introduction. *Astérisque* **398**, 1–31 (2018)
20. Gao, F.: Distinguished theta representations for certain covering groups. *Pac. J. Math.* **290**(2), 333–379 (2017)
21. Gao, F.: R-group and Whittaker space of some genuine representations. *J. Inst. Math. Jussieu* **1**(22), 1–61 (2021)
22. Gao, F.: R-group and Whittaker space of some genuine representations, II. *Res. Numb. Theory* **8**(62), 1–15 (2022)
23. Gao, F., Weissman, M.H.: Whittaker models for depth zero representations of covering groups, 2019. *Int. Math. Res. Not. IMRN* **11**, 3580–3620 (2019)
24. Gao, F., Gurevich, N., Karasiewicz, E.: Genuine pro- p Iwahori-Hecke algebras, Gelfand-Graev representations, and some applications (2022). arXiv preprint [arXiv:2204.13053](https://arxiv.org/abs/2204.13053)
25. Green, J.A.: The characters of the finite general linear groups. *Trans. Am. Math. Soc.* **80**, 402–447 (1955)
26. Harish-Chandra: Admissible Invariant Distributions on Reductive p -adic Groups, *With a preface and notes by Stephen DeBacker and Paul J. Sally, Jr.*, American Mathematical Society, p. 16 (1999)
27. Jacquet, H., Shalika, J.A.: On Euler products and the classification of automorphic representations I. *Am. J. Math.* **103**(3), 499–558 (1981)
28. Kable, A.C.: The tensor product of exceptional representations on the general linear group. *Elsevier Ann. Sci. École Norm. Sup. (4)* **34**(5), 741–769 (2001)
29. Kaplan, E.: Doubling constructions and tensor product L -functions: coverings of the symplectic group, (2019). arXiv preprint [arXiv:1902.00880](https://arxiv.org/abs/1902.00880)
30. Kaplan, E.: Rankin-Selberg Integrals and L-Functions for Covering Groups of General Linear Groups. *Int. Math. Res. Not. IMRN* (2022)
31. Kaplan, E., Lapid, E., Zou, J.: Classification of irreducible representations of metaplectic covers of the general linear group over a non-archimedean local field, (2022). arXiv preprint [arXiv:2206.14731](https://arxiv.org/abs/2206.14731)
32. Kazhdan, D.A., Patterson, S.J.: Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.* **59**, 35–142 (1984)
33. Kazhdan, D.A., Patterson, S.J.: Towards a generalized Shimura correspondence. *Adv. Math.* **60**(2), 161–234 (1986)
34. Kubota, T.: Topological covering of $SL(2)$ over a local field. *J. Math. Soc. Jpn.* **19**(1), 114–121 (1967)
35. Lapid, E., Mínguez, A.: On a determinantal formula of Tadić. *Am. J. Math.* **136**(1), 111–142 (2014)
36. Li, W.-W.: La formule des traces pour les revêtements de groupes réductifs connexes. II. Analyse harmonique locale. *Ann. Sci. Éc. Norm. Supér. (4)* **45**(5), 787–859 (2012)
37. Matsumoto, H.: Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)* **2**(1), 1–62 (1969)
38. Mezo, P.: Comparisons of general linear groups and their metaplectic coverings I. *Can. J. Math.* **54**(1), 92–137 (2002)
39. Mezo, P.: Metaplectic tensor products for irreducible representations. *Pac. J. Math.* **215**(1), 85–96 (2004)
40. Mœglin, C., Waldspurger, J.-L.: Spectral decomposition and Eisenstein series: a paraphrase of the scriptures. Cambridge University Press, Cambridge, p. 113 (1995)
41. Mœglin, C., Waldspurger, J.-L., de Whittaker, M.: Dégénérés pour des groupes p -adiques. *Math. Z.* **196**(3), 427–452 (1987)
42. Patel, S.-P.: A theorem of Mœglin and Waldspurger for covering groups. *Pac. J. Math.* **273**(1), 225–239 (2015)
43. Rodier, F.: Modèle de Whittaker et caractères de représentations, Non-commutative harmonic analysis, *Lecture Notes in Math.*, Vol. 466. Springer, Berlin, pp. 151–171 (1975)
44. Shalika, J.A.: The multiplicity one theorem for GL_n . *Ann. Math.* **100**(2), 171–193 (1974)
45. Shimura, G.: On modular forms of half integral weight. *Ann. Math.* **97**(2), 440–481 (1973)
46. Suzuki, T.: Distinguished representations of metaplectic groups. *Am. J. Math.* **120**(4), 723–755 (1998)
47. Tadić, M.: Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). *Ann. Sci. École Norm. Sup. (4)* **19**(3), 335–382 (1986)
48. Takeda, S.: Metaplectic tensor products for automorphic representation of $\widetilde{GL}(r)$. *Can. J. Math.* **68**(1), 179–240 (2016)
49. Takeda, S.: Remarks on metaplectic tensor products for covers of GL_r . *Pac. J. Math.* **290**(1), 199–230 (2017)
50. Van Dijk, G.: Computation of certain induced characters of p -adic groups. *Math. Ann.* **199**, 229–240 (1972)
51. Waldspurger, J.-L.: La formule de Plancherel pour les groupes p -adiques. D’après Harish-Chandra. *J. Inst. Math. Jussieu* **2**(2), 235–333 (2003)
52. Weil, A.: Basic Number Theory. *Die Grundlehren der Mathematischen Wissenschaften*, 3rd edn. Springer, New York-Berlin, p. 144 (1974)

53. Zelevinsky, A.V.: Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$. *Ann. Sci. École Norm. Sup.* **13**(2), 165–210 (1980)

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