Mathematische Zeitschrift (2023) 305:26
https://doi.org/10.1007/s00209-023-03349-9

Mathematische Zeitschrift

Algebraic reverse Khovanskii–Teissier inequality via Okounkov bodies

Chen Jiang[1] · Zhiyuan Li[1]

Received: 31 March 2022 / Accepted: 17 August 2023 / Published online: 25 September 2023

Abstract
Let X be a projective variety of dimension n over an algebraically closed field of arbitrary characteristic and let A, B, C be nef divisors on X. We show that for any integer $1 \leq k \leq n-1$,

$$(B^k \cdot A^{n-k}) \cdot (A^k \cdot C^{n-k}) \geq \frac{k!(n-k)!}{n!}(A^n) \cdot (B^k \cdot C^{n-k}).$$

The same inequality in the analytic setting was obtained by Lehmann and Xiao for compact Kähler manifolds using the Calabi–Yau theorem, while our approach is purely algebraic using (multipoint) Okounkov bodies. We also discuss applications of this inequality to Bézout-type inequalities and inequalities on degrees of dominant rational self-maps.

Keywords Reverse Khovanskii–Teissier inequality · Okounkov bodies · Intersection numbers · Volumes

Mathematics Subject Classification 14C20 · 14M25 · 14C17

1 Introduction

In [23], Lehmann and Xiao have proved the so called reverse Khovanskii–Teissier inequality on compact Kähler manifolds. Namely, for nef (1, 1)-classes α, β and γ on a compact Kähler manifold of dimension n, we have

$$(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geq \frac{k!(n-k)!}{n!}(\alpha^n) \cdot (\beta^k \cdot \gamma^{n-k}). \tag{1.1}$$

In fact such an inequality was first observed by Xiao [30] with a weaker constant $\frac{k!(n-k)!}{4n!}$ which can be improved to $\frac{k!(n-k)!}{n!}$ by the technique of Popovici [26] (cf. [30, Remark 3.1]). Such

✉ Chen Jiang
chenjiang@fudan.edu.cn

Zhiyuan Li
zhiyuan_li@fudan.edu.cn

1 Shanghai Center for Mathematical Sciences, Fudan University, Jiangwan Campus, Shanghai 200438, China

an inequality between intersection numbers plays an important role in the proof of Morse type inequalities (see also [26, 30]) and it also has interesting applications to dynamical degrees of rational maps [1, 7]. Besides, it has an interesting analogue in convex geometry [23, Theorem 5.9] and applications in the study of convolution of convex valuations [8, Theorem 2.9].

The proof of Popovici and Lehmann–Xiao depends on solving Monge–Ampére equations which deeply relies on the Calabi–Yau theorem, and it has been asked in [21, Remark 9.3] (the arXiv version of [22]) whether there is an algebraic approach working for projective varieties defined over arbitrary fields. In [7, Theorem 3.4.3], by studying intersections of numerical cycles, Dang has proved a weaker form of (1.1) in the algebraic geometry setting with the constant $\frac{k!(n-k)!}{n!}$ replacing by $\frac{1}{(n-k+1)^k}$.

In this paper, we use the (multipoint) Okounkov bodies to prove this optimal inequality in the algebraic geometry setting. The main result is

Theorem 1.1 *Let X be a projective variety of dimension n over an algebraically closed field of arbitrary characteristic. Let A, B, C be nef divisors on X. Then for any integer $1 \leq k \leq n-1$,*

$$(B^k \cdot A^{n-k}) \cdot (A^k \cdot C^{n-k}) \geq \frac{k!(n-k)!}{n!}(A^n) \cdot (B^k \cdot C^{n-k}). \tag{1.2}$$

Moreover, if A, B, C are ample, then this inequality is strict.

We shall mention that the constant $\frac{k!(n-k)!}{n!}$ in (1.2) is optimal. For instance, if $X = \mathbb{P}^k \times \mathbb{P}^{n-k}$, one can take A, B, C to be the divisors on X of type $(1, 1)$, $(1, 0)$, $(0, 1)$, respectively, then the equality holds in (1.2). It is interesting and natural to ask for the characterization of the equality case of (1.2), but the answer might be very complicated. One naive guess is that if the equality holds while both sides are non-zero, then B and C have numerical dimensions k and $n-k$ respectively and A lies on the plane spanned by B and C in the Néron–Severi group $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We will give such a characterization when X is a surface in Proposition 5.1.

Our proof makes use of the connection between volumes of big divisors and volumes of (multipoint) Okounkov bodies discovered in [15, 20, 29]. The desired inequality follows from a dedicated comparison between the Okounkov bodies associated to an admissible flag and the convex bodies associated to sub-flags obtained by cutting the admissible flag with very general hyperplanes. In this way, we actually obtain a more general result for restricted volumes, see Theorems 3.4, 3.5, and 3.8 for details. Another advantage of this method is that we only need to deal with the intersection theory of divisors, which is much easier to handle than that of algebraic cycles (cf. [7]). Very recently, Hu and Xiao [12] give a purely combinatorial proof of Theorem 1.1.

As a consequence, we can get a Bézout-type inequality which was proved by Xiao [31, Theorem 1.6] when the base field is $\mathbb{C}$, generalizing the classical Bézout theorem for hypersurfaces. Here we get a better constant than [31, Theorem 1.6].

Corollary 1.2 (cf. [31, Theorem 1.6]) *Let X be a projective variety with an ample divisor H as polarization and let $A_1, \ldots, A_r$ be nef divisor classes on X. Assume that $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$ and $|a| := \sum_{i=1}^r a_i \leq n$. Let $Y_1, \ldots, Y_r$ be subvarieties of cycle classes $A_1^{a_1}, \ldots, A_r^{a_r}$, and assume that they have proper intersection, then*

$$\deg_H(Y_1 \cap \cdots \cap Y_r) \leq \min_k \left\{ \frac{\prod_{i=1}^r \binom{|a|}{a_i}}{\binom{|a|}{a_k}(H^n)^{r-1}} \right\} \prod_{i=1}^r \deg_H(Y_i).$$

There is another application on degrees of the iterations of rational self-maps on projective varieties. Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety X of dimension n defined over an algebraically closed field of arbitrary characteristic and let H be a nef and big divisor on X. For any integer $0 \le i \le n$, the *i-th degree* of f with respect to H is defined by

$$\deg_{i,H}(f) = (\pi_1^* H^{n-i} \cdot \pi_2^* H^i),$$

where π_1 and π_2 are the projections from the normalization of the graph of f in $X \times X$ onto the first and the second factor, respectively. In [7], Dang proved a weaker form of (1.2) in order to study the sequence of intermediate degrees of the iterations of a dominant rational self-map and to recover the results in [1, 9, 28]. As an application of Theorem 1.1, we give a new and simple proof of [7, Theorem 1] with better constants.

Corollary 1.3 (cf. [7, Theorem 1]) *Let X be a projective variety of dimension n and let H be a nef and big divisor on X. Fix an integer $0 \le i \le n$.*

(1) *For any dominant rational self-maps f, g on X,*

$$\deg_{i,H}(f \circ g) \le \frac{\binom{n}{i}}{(H^n)} \deg_{i,H}(f) \cdot \deg_{i,H}(g).$$

(2) *For any nef and big divisor L on X and any dominant rational self-maps f on X,*

$$\deg_{i,H}(f) \le \frac{\binom{n}{i}^2 (H^{n-i} \cdot L^i) \cdot (L^{n-i} \cdot H^i)}{(L^n)^2} \deg_{i,L}(f).$$

This result is essential in the definition of the *dynamical degree* of a self-rational map (cf. [1, 7, 9, 29]).

This paper is organized as the following. In Sect. 2, we introduce definitions and basic knowledge on (multipoint) Okounkov bodies. In Sect. 3, we develop properties of Okounkov bodies and prove Theorem 1.1 along with several general statements. In Sect. 4, we give the applications of the main theorem. In Sect. 5, we give a characterization of the equality case of Theorem 1.1 on surfaces.

2 Preliminaries

2.1 Notation and conventions

We work over an algebraically closed field of arbitrary characteristic. We adopt the standard notation and definitions in [18–20]. A *variety* is reduced and irreducible. A *divisor* on a projective variety always means a Cartier divisor. When the base field is uncountable, a property holds for a *very general* choice of data if it is satisfied away from a countable union of proper closed subvarieties of the relevant parameter space.

2.2 Volumes

Let X be a projective variety of dimension n and let D be a divisor on X. The *volume* of D is the real number

$$\mathrm{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

We say that D is *big* if $\mathrm{vol}_X(D) > 0$. For more details and properties of volumes, we refer to [18, 2.2.C] and [19, 11.4.A]. By the homogeneity of volumes, this definition can be extended to $\mathbb{Q}$-divisors. Note that if D is a nef divisor, then $\mathrm{vol}_X(D) = (D^n)$.

2.3 Restricted volumes

We recall the notation in [20, §2.4]. Let V be a projective variety and let D be a big divisor on V. The *augmented base locus* $\mathbf{B}_+(D) \subset V$ is defined to be $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ for any sufficiently small ample $\mathbb{Q}$-divisor A, where $\mathbf{B}(D - A)$ is the *stable base locus* of $D - A$. Let X be an irreducible closed subvariety of V of dimension n, then the *restricted volume* of D from V to X is defined by

$$\mathrm{vol}_{V|X}(D) = \lim_{m\to\infty} \frac{\dim\left(\mathrm{Im}\left(H^0(V, mD) \xrightarrow{\text{restr}} H^0(X, mD|_X)\right)\right)}{m^n/n!}.$$

For a sufficiently divisible integer $m > 0$, consider $\pi_m : V_m \to V$ to be the blowing-up of V along the base ideal of $|mD|$, then we have

$$\pi_m^*|mD| = |M_m| + E_m$$

where M_m is free, and E_m is the fixed part. If $X \not\subset \mathbf{B}(D)$, the *asymptotic intersection number* of D and X is defined to be

$$\|D^n \cdot X\| = \limsup_{m\to\infty} \frac{(M_m^n \cdot X_m)}{m^n},$$

where X_m is the strict transform of X on V_m (cf. [10, Definition 2.6]). Recall that we have the following Fujita's approximation theorem of restricted volumes.

Theorem 2.1 ([10, Theorem 2.13], [20, Remark 3.6]) *Let V be a projective variety and let D be a big divisor on V. Let X be an irreducible closed subvariety of V of dimension n such that $X \not\subset \mathbf{B}_+(D)$. Then*

$$\mathrm{vol}_{V|X}(D) = \|D^n \cdot X\|.$$

2.4 Okounkov bodies

Okounkov bodies of big divisors were introduced in [15, 20] motivated by earlier works of Okounkov [24, 25]. There have been many interesting applications of Okounkov bodies in the study of geometric properties of divisors, for example, [2–6, 13, 14, 16, 17, 27].

We recall the definition of Okounkov bodies from [20]. Let X be a projective variety of dimension n. Consider an *admissible flag* $X_\bullet$ on X

$$X_\bullet : X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{n-1} \supseteq X_n = \{x\}$$

where each X_i is an irreducible closed subvariety of X which is non-singular at the point x and $\mathrm{codim}\, X_i = i$. For a big divisor D on X, we consider the $\mathbb{Q}$-linear system

$$|D|_{\mathbb{Q}} = \{D' \mid D \sim_{\mathbb{Q}} D' \geq 0\}$$

and a valuation-like function

$$\begin{aligned} \nu_{X_\bullet} : |D|_{\mathbb{Q}} &\to \mathbb{R}^n_{\geq 0}, \\ D' &\mapsto \nu_{X_\bullet}(D') = (\nu_1, \nu_2, \ldots, \nu_n), \end{aligned}$$

where ν_i are defined inductively as follows:

(1) define $\nu_1 := \mathrm{mult}_{X_1} D' \in \mathbb{Q}_{\geq 0}$ and $D'_1 := D' - \nu_1 X_1$ on X_0, and inductively,
(2) assuming that $\nu_i \in \mathbb{Q}_{\geq 0}$ and D'_i on X_{i-1} are defined, then define $\nu_{i+1} := \mathrm{mult}_{X_{i+1}}(D'_i|_{X_i}) \in \mathbb{Q}_{\geq 0}$ and $D'_{i+1} = D'_i|_{X_i} - \nu_{i+1} X_{i+1}$ on X_i.

Here we remark that as X_{i-1} is non-singular at x, $D'_i|_{X_i}$ is a well-defined $\mathbb{Q}$-divisor in a neighborhood of x and $\mathrm{mult}_{X_{i+1}}(D'_i|_{X_i})$ can be well-defined. The *Okounkov body* of D with respect to $X_\bullet$ is defined as

$$\Delta_{X_\bullet}(D) := \text{the convex closure of } \nu_{X_\bullet}(|D|_{\mathbb{Q}}) \text{ in } \mathbb{R}^n_{\geq 0}.$$

Here the definition is equivalent to [20, Definition 1.8] but we use $\mathbb{Q}$-linear systems instead of global sections as in [20] to make the notation simpler. Such a formulation appears for example in [2]. By [20, Theorem A], we have

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{X_\bullet}(D)) = \frac{1}{n!}\mathrm{vol}_X(D) \tag{2.1}$$

for every admissible flag $X_\bullet$ on X.

2.5 Multipoint Okounkov bodies

In this paper, we will study the property of the Okounkov body of a given admissible flag cutting by very general hyperplanes (see Proposition 3.2). For example, given an admissible flag $X_\bullet$ and a very general hyperplane H on X, in order to naturally define an admissible flag $H_\bullet$, we need to pick a closed point in $X_{n-1} \cap H$. But in this way we lost the information of other points in $X_{n-1} \cap H$. So the natural idea is to consider the whole set of points $X_{n-1} \cap H$ instead of just picking one. To this end, we need to extend the definition of admissible flags and Okounkov bodies to multipoint admissible flags and multipoint Okounkov bodies.

Now we recall the definition of multipoint Okounkov bodies in [29]. For our purpose, we only introduce a special case. Let Z be a projective variety of dimension k. Consider a *multipoint admissible flag* $Z_\bullet$ on Z

$$Z_\bullet : Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{k-1} \supseteq Z_k = \{p_1, \ldots, p_N\}$$

where Z_k consists of N distinct points and for $0 \leq i < k$, each Z_i is an irreducible closed subvariety of Z which is non-singular at the points $p_1, \ldots, p_N$ and $\mathrm{codim}\, Z_i = i$. For $1 \leq j \leq N$, denote by $Z_\bullet(p_j)$ the admissible flag

$$Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{k-1} \supseteq \{p_j\}.$$

Then for a big divisor D on Z, we can consider the functions $\nu_{Z_\bullet(p_j)}(D')\,(1 \leq j \leq N)$ for $D' \in |D|_{\mathbb{Q}}$. Note that $\nu_{Z_\bullet(p_j)}(D')$ only differs on the last coordinate in $\mathbb{R}^k$. So the lexicographical order of $\{\nu_{Z_\bullet(p_j)}(D') \mid 1 \leq j \leq N\}$ is just the order of the last coordinates. We define the subset $V_j(D) \subset |D|_{\mathbb{Q}}$ by

$$V_j(D) = \left\{ D' \in |D|_{\mathbb{Q}} \;\middle|\; \nu_{Z_\bullet(p_j)}(D') < \nu_{Z_\bullet(p_i)}(D') \text{ for all } i \neq j \right\}. \tag{2.2}$$

The *multipoint Okounkov body* of D with respect to $Z_\bullet$ and p_j is defined in [29, Definition 3.4] as

$$\Delta_{Z_\bullet, j}(D) := \text{the convex closure of} \nu_{Z_\bullet(p_j)}(V_j(D)) \text{ in } \mathbb{R}^k_{\geq 0}.$$

By [29, Theorem 1.2], we have

$$\sum_{j=1}^{N} \mathrm{vol}_{\mathbb{R}^k}(\Delta_{Z_\bullet,j}(D)) = \frac{1}{k!}\mathrm{vol}_Z(D) \tag{2.3}$$

for every multipoint admissible flag $Z_\bullet$ on Z.

By the homogeneity (cf. [29, Proposition 3.10]), the definition of (multipoint) Okounkov bodies can be extended to big $\mathbb{Q}$-divisors. In particular, (2.1) and (2.3) hold for big $\mathbb{Q}$-divisors.

Remark 2.2 In [29], it has been assumed that the base field is $\mathbb{C}$ and Z is smooth. But as the proof of [29, Theorem 1.2] follows the line of [20, Theorem A], it is not hard to check that [29, Theorem 1.2] holds for any projective variety over any algebraically closed field. Here as in [20, Remark 3.7], we do not need to assume that the base field is uncountable.

2.6 Cutting a divisor by general hyperplanes

We will use the following lemma which is a direct consequence of Bertini's theorem.

Lemma 2.3 *Let X be a projective variety. Let D be a $\mathbb{Q}$-divisor on X and let P be a prime divisor whose generic point lies in the smooth locus of X. Then for a general very ample divisor H on X, the following statements hold:*

(1) *if* $\dim X > 2$, *then $P|_H$ is a prime divisor whose generic point lies in the smooth locus of H;*
(2) *if* $\dim X = 2$, *then $P|_H$ is a reduced divisor (consisting of points) lies in the smooth locus of H;*
(3) $\mathrm{mult}_P D = \mathrm{mult}_{P'} D|_H$ *for each irreducible component P' of $P|_H$.*

Proof (1) and (2) follows from Bertini's theorem [11, Theorem II.8.18]. To get (3), we just need to choose H general so that

- for each irreducible component P_1 of D, $P_1|_H$ is reduced, and
- for irreducible components P_1 and P_2 of D, $P_1|_H$ and $P_2|_H$ has no common irreducible component.

This is again by Bertini's theorem. □

3 Inequalities between volumes via Okounkov bodies

3.1 A comparision result on Okounkov bodies

We define $\mathrm{pr}_{>k} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ to be the projection of the last $n-k$ coordinates and $\mathrm{pr}_{\leq k} : \mathbb{R}^n \to \mathbb{R}^k$ to be the projection of the first k coordinates.

The following lemma gives a comparison of the Okounkov bodies of a certain admissible flag and its sub-flag via the natural projections.

Lemma 3.1 *Let X be a projective variety of dimension n, let D be a big divisor on X, and let $X_\bullet$ be an admissible flag. Fix an integer $1 \leq k \leq n$. Denote $Y = X_k$ and $Y_\bullet = X_{k+\bullet}$. Suppose that $D|_Y$ is big and $\mathcal{O}_{X_{i-1}}(X_i)|_Y$ is a semiample line bundle on Y for $1 \leq i \leq k$. Then*

$$\mathrm{pr}_{>k}(\Delta_{X_\bullet}(D)) \subset \Delta_{Y_\bullet}(D|_Y).$$

Proof Fix any $D' \in |D|_{\mathbb{Q}}$. Suppose that $\nu_{X_\bullet}(D') = (\nu_1, \ldots, \nu_n)$. Recall that for $0 \le i \le n-1$ we define by induction that $\nu_{i+1} = \mathrm{mult}_{X_{i+1}} D'_i|_{X_i}$ and $D'_{i+1} = D'_i|_{X_i} - \nu_{i+1} X_{i+1}$. Then by definition $\nu_{Y_\bullet}(D'_k|_Y) = (\nu_{k+1}, \ldots, \nu_n)$. By construction,

$$D'_k|_Y \sim_{\mathbb{Q}} D|_Y - \sum_{i=1}^{k} \nu_i X_i|_Y,$$

where we view $X_i|_Y$ as a divisor defined by the line bundle $\mathcal{O}_{X_{i-1}}(X_i)|_Y$. So by our assumption, $D|_Y - D'_k|_Y$ is semiample. Therefore, we can find $\tilde{D} \in |D|_Y|_{\mathbb{Q}}$ such that $\nu_{Y_\bullet}(\tilde{D}) = \nu_{Y_\bullet}(D'_k|_Y) = (\nu_{k+1}, \ldots, \nu_n)$. This concludes the desired inclusion. □

The following proposition describes the behavior of the Okounkov body of an admissible flag when cutting by very general hyperplanes.

Proposition 3.2 *Let X be a projective variety of dimension n over an uncountable algebraically closed field, let D be a big divisor on X, and let $X_\bullet$ be an admissible flag. Fix an integer $1 \le k \le n-1$. For very general very ample divisors $H_1, \ldots, H_{n-k}$ on X, denote $Z = H_1 \cap \cdots \cap H_{n-k}$, then there is a natural multipoint admissible flag $Z_\bullet$ on Z given by $Z_i = X_i \cap Z$ for $0 \le i \le k-1$, and $Z_k = X_k \cap Z = \{p_1, \ldots, p_N\}$. Then*

$$\mathrm{pr}_{\le k}(\Delta_{X_\bullet}(D)) \subset \bigcap_{m \in \mathbb{Z}_{>0}} \Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m} A_Z\right)$$

for all $1 \le j \le N$ and any ample divisor A_Z on Z.

Remark 3.3 By [29, Proof of Theorem 1.2], if $\Delta_{Z_\bullet, j}(D|_Z)^\circ \ne \emptyset$, then

$$\bigcap_{m \in \mathbb{Z}_{>0}} \Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m} A_Z\right) = \Delta_{Z_\bullet, j}(D|_Z).$$

But we do not need this fact in this paper.

Proof By definition, the set $\nu_{X_\bullet}(|D|_{\mathbb{Q}})$ lies in $\mathbb{Q}^n_{\ge 0}$, which is a countable set. So there exists a countable set $S \subset |D|_{\mathbb{Q}}$ such that $\Delta_{X_\bullet}(D)$ is the convex closure of $\{\nu_{X_\bullet}(D') \mid D' \in S\}$. It suffices to show that for all $D' \in S \subset |D|_{\mathbb{Q}}$ and for very general very ample divisors $H_1, \ldots, H_{n-k}$ on X, we have

$$\mathrm{pr}_{\le k}(\nu_{X_\bullet}(D')) \in \Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m} A_Z\right) \tag{3.1}$$

for all $m \in \mathbb{Z}_{>0}$.

In fact, it suffices to show that (3.1) holds for a single $D' \in S$ and for very general very ample divisors $H_1, \ldots, H_{n-k}$ on X. More precisely, for fixed very ample linear systems $\mathcal{L}_1, \ldots, \mathcal{L}_{n-k}$ on X, if (3.1) holds for every $D' \in S$ and every $H_i \in \mathcal{L}_i \backslash Z_{i,D'}$ where $Z_{i,D'} \subset \mathcal{L}_i$ is a countable union of proper closed subset (depending on D') for each i, then (3.1) holds for every $D' \in S$ and every $H_i \in \mathcal{L}_i \backslash \bigcup_{D' \in S} Z_{i,D'}$. Here $\bigcup_{D' \in S} Z_{i,D'}$ is again a countable union of proper closed subset of $\mathcal{L}_i$ as S is countable.

Now we consider a fixed $D' \in S$. Suppose that $\nu_{X_\bullet}(D') = (\nu_1, \ldots, \nu_n)$. Recall that for $0 \le i \le n-1$ we define by induction that $\nu_{i+1} = \mathrm{mult}_{X_{i+1}} D'_i|_{X_i}$ and $D'_{i+1} = D'_i|_{X_i} - \nu_{i+1} X_{i+1}$. By applying Lemma 2.3 inductively, by taking $H_1, \ldots, H_{n-k}$ general, we get that

- $\nu_{i+1} = \mathrm{mult}_{Z_{i+1}} D'_i|_{Z_i}$ and $D'_{i+1}|_{Z_i} = D'_i|_{Z_i} - \nu_{i+1} Z_{i+1}$ for $0 \le i \le k-2$;

- $\nu_k = \text{mult}_{p_s} D'_{k-1}|_{Z_{k-1}}$ for all $1 \leq s \leq N$.

As A_Z is ample, we can find $A'_Z \in |A_Z|_{\mathbb{Q}}$ whose support is very ample which does not contain p_j and Z_{k-1} but contains p_s for all $s \neq j$, in other words,

- $\text{mult}_{p_j} A'_Z = \text{mult}_{Z_{k-1}} A'_Z = 0$, and
- $\text{mult}_{p_s} A'_Z = \mu_s > 0$ for all $s \neq j$.

Then from the construction,

$$\nu_{Z_\bullet(p_j)}\left(D'|_Z + \frac{1}{m}A'_Z\right) = (\nu_1, \ldots, \nu_k)$$

and

$$\nu_{Z_\bullet(p_s)}\left(D'|_Z + \frac{1}{m}A'_Z\right) = \left(\nu_1, \ldots, \nu_{k-1}, \nu_k + \frac{1}{m}\mu_s\right)$$

for $s \neq j$. Then we have $D'|_Z + \frac{1}{m}A'_Z \in V_j(D|_Z + \frac{1}{m}A_Z)$ as defined in (2.2) and hence $(\nu_1, \ldots, \nu_k) \in \Delta_{Z_\bullet, j}(D|_Z + \frac{1}{m}A_Z)$. □

3.2 The generalized reverse Khovanskii–Teissier inequality

As the volume of Okounkov body can compute the volume of big divisors, the comparision result in Sect. 3.1 will yield a stronger version of the reverse Khovanskii–Teissier inequality as follows.

Theorem 3.4 *Let X be a projective variety of dimension n over an uncountable algebraically closed field. Fix an integer $1 \leq k \leq n-1$. Let D be a big divisor on X and let $B_1, \ldots, B_k, C_1, \ldots, C_{n-k}$ be very general very ample divisors on X. Denote $Y = B_1 \cap \cdots \cap B_k$ and $Z = C_1 \cap \cdots \cap C_{n-k}$. Then*

$$\text{vol}_Y(D|_Y) \cdot \text{vol}_Z(D|_Z) \geq \frac{k!(n-k)!}{n!} \text{vol}_X(D) \cdot (Y \cdot Z).$$

Proof It is easy to construct an admissible flag $X_\bullet$ such that $X_i = B_1 \cap \cdots \cap B_i$ for any $1 \leq i \leq k$. By applying Lemma 3.1, we get

$$\text{pr}_{>k}(\Delta_{X_\bullet}(D)) \subset \Delta_{Y_\bullet}(D|_Y).$$

By applying Proposition 3.2 to the case $H_i = C_i$ for $1 \leq i \leq n-k$, we get

$$\text{pr}_{\leq k}(\Delta_{X_\bullet}(D)) \subset \bigcap_{m \in \mathbb{Z}_{>0}} \Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m}A_Z\right)$$

for all $1 \leq j \leq N$, where $Z = C_1 \cap \cdots \cap C_{n-k}$, $N = (X_k \cdot Z) = (Y \cdot Z)$, and A_Z is an ample divisor on Z. So

$$\Delta_{X_\bullet}(D) \subset \Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m}A_Z\right) \times \Delta_{Y_\bullet}(D|_Y) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$$

for all $1 \le j \le N$ and all $m \in \mathbb{Z}_{>0}$. Combining with (2.1) and (2.3), this yields

$$\begin{aligned}\frac{1}{n!}(Y \cdot Z) \cdot \mathrm{vol}_X(D) &= N \,\mathrm{vol}_{\mathbb{R}^n}(\Delta_{X_\bullet}(D)) \\ &\le \sum_{j=1}^{N} \mathrm{vol}_{\mathbb{R}^k}\left(\Delta_{Z_\bullet, j}\left(D|_Z + \frac{1}{m}A_Z\right)\right) \cdot \mathrm{vol}_{\mathbb{R}^{n-k}}(\Delta_{Y_\bullet}(D|_Y)) \\ &= \frac{1}{k!}\mathrm{vol}_Z\left(D|_Z + \frac{1}{m}A_Z\right) \cdot \frac{1}{(n-k)!}\mathrm{vol}_Y(D|_Y).\end{aligned}$$

We can conclude the assertion by taking $m \to \infty$, as vol_Z is a continuous function for big $\mathbb{Q}$-divisors by [20, Corollary 4.12]. □

As a consequence, we can obtain a general form of Theorem 1.1.

Theorem 3.5 *Let X be a projective variety of dimension n. Fix an integer $1 \le k \le n-1$. Let $A, B_1, \dots, B_k, C_1, \dots, C_{n-k}$ be nef divisors on X. Then*

$$\begin{aligned}&(B_1 \cdot \dots \cdot B_k \cdot A^{n-k}) \cdot (A^k \cdot C_1 \cdot \dots \cdot C_{n-k}) \\ &\quad \ge \frac{k!(n-k)!}{n!}(A^n) \cdot (B_1 \cdot \dots \cdot B_k \cdot C_1 \cdot \dots \cdot C_{n-k}). \end{aligned} \tag{3.2}$$

Moreover, if $A, B_1, \dots, B_k, C_1, \dots, C_{n-k}$ are ample, then this inequality is strict.

Proof After base change we may assume that the base field is uncountable. As the inequality is homogeneous and nef divisors are the limits of ample $\mathbb{Q}$-divisors in the Néron–Severi group $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, it suffices to prove the inequality (3.2) for very ample divisors $A, B_1, \dots, B_k, C_1, \dots, C_{n-k}$ on X. Then the inequality (3.2) follows from Theorem 3.4 by taking $D = A$.

For the last statement, suppose that $A, B_1, \dots, B_k, C_1, \dots, C_{n-k}$ are ample. Without loss of generality, we may assume that A is very ample and is a subvariety of X. Assume to the contrary that the equality holds. Note that for any sufficiently small $t > 0$, $B_k - tA$ is ample. So applying (3.2) to $A, B_1, \dots, B_{k-1}, B_k - tA, C_1, \dots, C_{n-k}$, we get

$$\begin{aligned}&(B_1 \cdot \dots \cdot B_{k-1} \cdot (B_k - tA) \cdot A^{n-k}) \cdot (A^k \cdot C_1 \cdot \dots \cdot C_{n-k}) \\ &\quad \ge \frac{k!(n-k)!}{n!}(A^n) \cdot (B_1 \cdot \dots \cdot B_{k-1} \cdot (B_k - tA) \cdot C_1 \cdot \dots \cdot C_{n-k}).\end{aligned}$$

By the assumption, this implies that

$$\begin{aligned}&(B_1 \cdot \dots \cdot B_{k-1} \cdot A^{n-k+1}) \cdot (A^k \cdot C_1 \cdot \dots \cdot C_{n-k}) \\ &\quad \le \frac{k!(n-k)!}{n!}(A^n) \cdot (B_1 \cdot \dots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \dots \cdot C_{n-k}). \end{aligned} \tag{3.3}$$

On the other hand, applying (3.2) to $A|_A, B_1|_A, \dots, B_{k-1}|_A, C_1|_A, \dots, C_{n-k}|_A$ on the variety A, we get

$$\begin{aligned}&(B_1 \cdot \dots \cdot B_{k-1} \cdot A^{n-k+1}) \cdot (A^k \cdot C_1 \cdot \dots \cdot C_{n-k}) \\ &\quad \ge \frac{(k-1)!(n-k)!}{(n-1)!}(A^n) \cdot (B_1 \cdot \dots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \dots \cdot C_{n-k}). \end{aligned} \tag{3.4}$$

Here we remark that we need $k > 1$ in order to apply (3.2), but inequality (3.4) holds trivially if $k = 1$. Then (3.3) and (3.4) yields

$$(A^n) \cdot (B_1 \cdot \cdots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \cdots \cdot C_{n-k}) \leq 0,$$

a contradiction. □

Remark 3.6 From the proof of Theorem 3.5, it is easy to see that the inequality (3.2) is strict as long as A, B_k are ample and $(B_1 \cdot \cdots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \cdots \cdot C_{n-k}) > 0$.

Remark 3.7 Although we deduce Theorem 3.5 from Theorem 3.4, it is not hard to see that Theorem 3.5 implies Theorem 3.4 conversely, as the volume of a big divisor can be approximated by ample divisors by Fujita's approximation theorem (cf. [20, §3.1] or Theorem 2.1). This fact was pointed out by Jian Xiao.

3.3 Proof of Theorem 1.1

It is just a special case of Theorem 3.5.

3.4 Further discussions on restricted volumes

In this subsection we will discuss the generalization of Theorem 3.4 to restricted volumes, which was suggested by Jian Xiao.

Theorem 3.8 *Let V be a projective variety over an uncountable algebraically closed field and let D be a big divisor on V. Let X be an irreducible closed subvariety of V of dimension n such that $X \not\subset \mathbf{B}_+(D)$. Fix an integer $1 \leq k \leq n-1$. Let $B_1, \ldots, B_k, C_1, \ldots, C_{n-k}$ be very general very ample divisors on V. Denote $Y = B_1 \cap \cdots \cap B_k \cap X$ and $Z = C_1 \cap \cdots \cap C_{n-k} \cap X$. Then*

$$\mathrm{vol}_{V|Y}(D) \cdot \mathrm{vol}_{V|Z}(D) \geq \frac{k!(n-k)!}{n!} \mathrm{vol}_{V|X}(D) \cdot (Y \cdot Z)_X.$$

Proof For any sufficiently divisible integer $m > 0$, consider $\pi_m : V_m \to V$ to be the blowing-up of V along the base ideal of $|mD|$, then we have

$$\pi_m^*|mD| = |M_m| + E_m$$

where M_m is free, and E_m is the fixed part. Denote by X_m the strict transform of X on V_m. By taking $B_1, \ldots, B_k, C_1, \ldots, C_{n-k}$ very general, we may assume that $Y, Z \not\subset \mathbf{B}_+(D)$ and for every sufficiently divisible integer $m > 0$, $Y_m = (\pi_m^* B_1 \cdot \cdots \cdot \pi_m^* B_k \cdot X_m)$ and $Z_m = (\pi_m^* C_1 \cdot \cdots \cdot \pi_m^* C_{n-k} \cdot X_m)$ as cycles, where Y_m, Z_m are the strict transforms of Y, Z on V_m respectively. Then by applying Theorem 3.5 to

$$M_m|_{X_m}, \pi_m^* B_1|_{X_m}, \ldots, \pi_m^* B_k|_{X_m}, \pi_m^* C_1|_{X_m}, \ldots, \pi_m^* C_{n-k}|_{X_m}$$

on X_m, we get

$$\begin{aligned}
&(\pi_m^* B_1 \cdot \cdots \cdot \pi_m^* B_k \cdot M_m^{n-k} \cdot X_m) \cdot (M_m^k \cdot \pi_m^* C_1 \cdot \cdots \cdot \pi_m^* C_{n-k} \cdot X_m) \\
&\quad \geq \frac{k!(n-k)!}{n!} (M_m^n \cdot X_m) \cdot (\pi_m^* B_1 \cdot \cdots \cdot \pi_m^* B_k \cdot \pi_m^* C_1 \cdot \cdots \cdot \pi_m^* C_{n-k} \cdot X_m).
\end{aligned}$$

By taking $m \to \infty$, we get

$$\|D^k \cdot Y\| \cdot \|D^{n-k} \cdot Z\| \geq \frac{k!(n-k)!}{n!} \|D^n \cdot X\| \cdot (Y \cdot Z)_X.$$

This concludes the desired inequality by Theorem 2.1. □

4 Applications

4.1 Proof of Corollary 1.2

It is equivalent to showing that

$$(A_1^{a_1}\cdot\dots\cdot A_r^{a_r}\cdot H^{n-|a|})\le \min_k\left\{\frac{\prod_{i=1}^r\binom{|a|}{a_i}}{\binom{|a|}{a_k}(H^n)^{r-1}}\right\}\prod_{i=1}^r(A_i^{a_i}\cdot H^{n-a_i}).$$

Without loss of generality, it suffices to show that

$$(H^n)^{r-1}\cdot(A_1^{a_1}\cdot\dots\cdot A_r^{a_r}\cdot H^{n-|a|})\le\prod_{i=2}^r\binom{|a|}{a_i}\prod_{i=1}^r(A_i^{a_i}\cdot H^{n-a_i}). \tag{4.1}$$

Without loss of generality, we may assume that H is very ample. When $r=1$, (4.1) is trivial. Suppose that $r\ge 2$. Take general elements $H_1,\dots,H_{n-|a|}\in|H|$ and take $Z=H_1\cap\dots\cap H_{n-|a|}$. Applying Theorem 3.5 to $A_r|_Z$ and $H|_Z$, we get

$$\begin{aligned}
&(H^n)\cdot(A_1^{a_1}\cdot\dots\cdot A_r^{a_r}\cdot H^{n-|a|})\\
&=(H^{|a|}|_Z)\cdot(A_1^{a_1}|_Z\cdot\dots\cdot A_r^{a_r}|_Z)\\
&\le\binom{|a|}{a_r}(A_1^{a_1}|_Z\cdot\dots\cdot A_{r-1}^{a_{r-1}}|_Z\cdot H^{a_r}|_Z)\cdot(A_r^{a_r}|_Z\cdot H^{|a|-a_r}|_Z)\\
&=\binom{|a|}{a_r}(A_1^{a_1}\cdot\dots\cdot A_{r-1}^{a_{r-1}}\cdot H^{n-|a|+a_r})\cdot(A_r^{a_r}\cdot H^{n-a_r}).
\end{aligned}$$

So we can prove inequality (4.1) by induction on r.

Remark 4.1 The constant we obtain is slightly better than Xiao's, which is $\min_k\left\{\frac{\prod_{i=1}^r\binom{n}{a_i}}{\binom{n}{a_k}(H^n)^{r-1}}\right\}$, but the proof is basically the same as long as one knows Theorem 3.5. It remains interesting to find the optimal constant. We refer to [30, §3.4] for some related discussions.

4.2 Proof of Corollary 1.3

(1) For any dominant rational self-maps f, g on X, take a projective normal variety W with generically finite morphisms $p_j : W\to X$ for $j=1,2,3$ such that $p_2=p_1\circ g$ and $p_3=p_2\circ f$. Then by the projection formula between W and the graphs of f, g,

1. $\deg_{i,H}(f\circ g)=\frac{1}{\deg(p_1)}\cdot(p_1^*H^{n-i}\cdot p_3^*H^i)$;
2. $\deg_{i,H}(f)=\frac{1}{\deg(p_2)}\cdot(p_2^*H^{n-i}\cdot p_3^*H^i)$;
3. $\deg_{i,H}(g)=\frac{1}{\deg(p_1)}\cdot(p_1^*H^{n-i}\cdot p_2^*H^i)$.

If $i=0$ or n the statement is trivially true. If $1\le i\le n-1$, by Theorem 1.1, we have

$$\begin{aligned}
\deg_{i,H}(f\circ g)&=\frac{1}{\deg(p_1)}\cdot(p_1^*H^{n-i}\cdot p_3^*H^i)\\
&\le\frac{1}{\deg(p_1)}\cdot\frac{\binom{n}{i}}{(p_2^*H^n)}(p_1^*H^{n-i}\cdot p_2^*H^i)\cdot(p_2^*H^{n-i}\cdot p_3^*H^i)\\
&=\frac{\binom{n}{i}}{(H^n)}\deg_{i,H}(f)\cdot\deg_{i,H}(g).
\end{aligned}$$

(2) Let π_1 and π_2 be the projections from the normalization of the graph of f in $X \times X$ onto the first and the second factor, respectively. If $i = 0$ or n the statement is trivially true. If $1 \le i \le n-1$ then by applying Theorem 1.1 repeatedly, one can get

$$\begin{aligned}\deg_{i,H}(f) &= (\pi_1^* H^{n-i} \cdot \pi_2^* H^i)\\ &\le \frac{\binom{n}{i}}{(\pi_1^* L^n)} (\pi_1^* H^{n-i} \cdot \pi_1^* L^i) \cdot (\pi_1^* L^{n-i} \cdot \pi_2^* H^i)\\ &\le \frac{\binom{n}{i}}{(\pi_1^* L^n)} (\pi_1^* H^{n-i} \cdot \pi_1^* L^i) \cdot \frac{\binom{n}{i}}{(\pi_2^* L^n)} (\pi_1^* L^{n-i} \cdot \pi_2^* L^i) \cdot (\pi_2^* L^{n-i} \cdot \pi_2^* H^i)\\ &= \frac{\binom{n}{i}^2 (H^{n-i} \cdot L^i) \cdot (L^{n-i} \cdot H^i)}{(L^n)^2} \deg_{i,L}(f).\end{aligned}$$

Remark 4.2 As we apply the optimal inequality (1.2), we obtain better constants than Dang's in [7, Theorem 1]. For example, the constant in [7, Theorem 1(i)] is $\frac{(n-i+1)^i}{(H^n)}$. It remains interesting to find the optimal constants.

5 Characterization of the equality case on surfaces

In this section, we give a characterization of the equality case of Theorem 1.1 on surfaces. The proof is motivated by [1, Proposition 1.15]. For simplicity, we state the characterization for non-singular surfaces, and the general case can be easily worked out by taking a desingularization.

Proposition 5.1 *Let X be a non-singular projective surface. Let A, B, C be nef divisors on X. Then*

$$2(B \cdot A) \cdot (A \cdot C) = (A^2) \cdot (B \cdot C) \neq 0$$

if and only if the following conditions hold

(1) $A \equiv sB + tC$ in $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for some $s, t > 0$;
(2) $B^2 = C^2 = 0$, $(B \cdot C) \neq 0$.

Proof The "if" part is obvious. We only deal with the "only if" part.

Suppose that $2(B \cdot A) \cdot (A \cdot C) = (A^2) \cdot (B \cdot C) \neq 0$. Then $(B \cdot A)$, $(A \cdot C)$, (A^2), $(B \cdot C)$ are all positive numbers. Set $\Gamma = A - \frac{(A^2)}{2(A \cdot B)} B$. Then $(\Gamma \cdot C) = 0$ and $(\Gamma^2) = \frac{(A^2)^2 (B^2)}{4(A \cdot B)^2} \ge 0$.

We claim that Γ and C are propotional in $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ by the Hodge index theorem. Fix any ample divisor H on X, by our assumption, C is not numerically trivial, so $(H \cdot C) \neq 0$. Then $((\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C) \cdot H) = 0$. On the other hand, $((\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C)^2) \ge 0$. So the classical Hodge index theorem (see [11, Theorem V.1.9]) implies that $\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C \equiv 0$. As they are propotional, we can also get $\Gamma \equiv \frac{(\Gamma \cdot A)}{(C \cdot A)} C$, which implies that

$$A \equiv \frac{(A^2)}{2(B \cdot A)} B + \frac{(A^2)}{2(C \cdot A)} C \tag{5.1}$$

by the construction of Γ. The fact that $(B^2) = (C^2) = 0$ can be obtained by intersecting (5.1) with B, C respectively. □

Acknowledgements We are grateful to Jian Xiao for drawing our attention to this topic and many helpful comments. We would like to thank Mingchen Xia for discussions on Okounkov bodies. We would like to thank the referee for useful suggestions. The authors were supported by NSFC for Innovative Research Groups (Grant No. 12121001) and partially supported by National Key Research and Development Program of China (Grant No. 2020YFA0713200). The second author was also supported by Shanghai Pilot Program for Basic Research (No. 21TQ00). The authors are members of LMNS, Fudan University.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Boucksom, S., Favre, C., Jonsson, M.: Degree growth of meromorphic surface maps. Duke Math. J. **141**(3), 519–538 (2008)
2. Choi, S.R., Hyun, Y., Park, J., Won, J.: Asymptotic base loci via Okounkov bodies. Adv. Math. **323**, 784–810 (2018)
3. Choi, S.R., Hyun, Y., Park, J., Won, J.: Okounkov bodies associated to pseudoeffective divisors. J. Lond. Math. Soc. (2) **97**(2), 170–195 (2018)
4. Choi, S.R., Jung, S.-J., Park, J., Won, J.: A product formula for volumes of divisors via Okounkov bodies. Int. Math. Res. Not. IMRN (22), 7118–7137 (2019)
5. Choi, S.R., Park, J., Won, J.: Okounkov bodies and Zariski decompositions on surfaces. Bull. Korean Math. Soc. **54**(5), 1677–1697 (2017)
6. Choi, S.R., Park, J., Won, J.: Okounkov bodies associated to pseudoeffective divisors II. Taiwan. J. Math. **21**(3), 601–620 (2017)
7. Dang, N.-B.: Degrees of iterates of rational maps on normal projective varieties. Proc. Lond. Math. Soc. (3) **121**(5), 1268–1310 (2020)
8. Dang, N.-B., Xiao, J.: Positivity of valuations on convex bodies and invariant valuations by linear actions. J. Geom. Anal. **31**(11), 10718–10777 (2021)
9. Dinh, T.-C., Sibony, N.: Regularization of currents and entropy. Ann. Sci. École Norm. Sup. (4) **37**(6), 959–971 (2004)
10. Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., Popa, M.: Restricted volumes and base loci of linear series. Am. J. Math. **131**(3), 607–651 (2009)
11. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, No. 52. Springer, New York (1977)
12. Hu, J., Xiao, J.: Intersection theoretic inequalities via Lorentzian polynomials (2023). arXiv:2304.04191
13. Ito, A.: Okounkov bodies and Seshadri constants. Adv. Math. **241**, 246–262 (2013)
14. Jow, S.-Y.: Okounkov bodies and restricted volumes along very general curves. Adv. Math. **223**(4), 1356–1371 (2010)
15. Kaveh, K., Khovanskii, A.G.: Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Ann. Math. (2) **176**(2), 925–978 (2012)
16. Küronya, A., Lozovanu, V.: Infinitesimal Newton–Okounkov bodies and jet separation. Duke Math. J. **166**(7), 1349–1376 (2017)
17. Küronya, A., Lozovanu, V.: Positivity of line bundles and Newton–Okounkov bodies. Doc. Math. **22**, 1285–1302 (2017)
18. Lazarsfeld, R.: Positivity in Algebraic Geometry. I, volume 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Berlin, Classical Setting: Line Bundles and Linear Series (2004)
19. Lazarsfeld, R.: Positivity in Algebraic Geometry. II, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Berlin. Positivity for Vector Bundles, and Multiplier Ideals (2004)
20. Lazarsfeld, R., Mustaţă, M.: Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4) **42**(5), 783–835 (2009)
21. Lehmann, B., Xiao, J.: Convexity and Zariski decomposition structure (2015). arXiv:1507.04316v2
22. Lehmann, B., Xiao, J.: Convexity and Zariski decomposition structure. Geom. Funct. Anal. **26**(4), 1135–1189 (2016)
23. Lehmann, B., Xiao, J.: Correspondences between convex geometry and complex geometry. Épijournal Géom. Algébrique 1:Art. 6, 29 (2017)

24. Okounkov, A.: Brunn–Minkowski inequality for multiplicities. Invent. Math. **125**(3), 405–411 (1996)
25. Okounkov, A.: Why would multiplicities be log-concave? In The Orbit Method in Geometry and Physics (Marseille, 2000), volume 213 of Progress in Mathematics, pp. 329–347. Birkhäuser Boston, Boston (2003)
26. Popovici, D.: Sufficient bigness criterion for differences of two nef classes. Math. Ann. **364**(1–2), 649–655 (2016)
27. Roé, J.: Local positivity in terms of Newton–Okounkov bodies. Adv. Math. **301**, 486–498 (2016)
28. Truong, T.T.: Relative dynamical degrees of correspondences over a field of arbitrary characteristic. J. Reine Angew. Math. **758**, 139–182 (2020)
29. Trusiani, A. Multipoint Okounkov bodies. Ann. Inst. Fourier (Grenoble) **71**(6), 2595–2646 (2021)
30. Xiao, J.: Weak transcendental holomorphic Morse inequalities on compact Kähler manifolds. Ann. Inst. Fourier (Grenoble) **65**(3), 1367–1379 (2015)
31. Xiao, J.: Bézout-type inequality in convex geometry. Int. Math. Res. Not. IMRN **16**, 4950–4965 (2019)