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Normal forms for Dirac–Jacobi bundles and splitting theorems for Jacobi structures

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Abstract

The aim of this paper is to prove a normal form Theorem for Dirac–Jacobi bundles using a recent techniques of Bursztyn, Lima and Meinrenken. As the most important consequence, we can prove the splitting theorems of Jacobi pairs which was proposed by Dazord, Lichnerowicz and Marle. As another application we provide an alternative proof of the splitting theorem of homogeneous Poisson structures.

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1 Introduction

Since the work of Weinstein [17], in which he proved his famous local splitting theorem for Poisson manifolds, many works appeared concerning different viewpoints on the proof and even giving more general statements, namely normal form theorems. Frejlich and Marcut

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proved a normal form theorem around Poisson (cosymplectic) transversals of Poisson manifolds in [7]. In [6] they used the techniques of Dual Pairs to prove a similar statement for Dirac structures. Finally, there is a unified approach by Bursztyn, Lima and Meinrenken in [3] to prove normal forms for Poisson related structures.

Jacobi geometry was introduced by Kirillov in [10] as local Lie algebras and independently by Lichnerowicz [11]. They have a deep connection to Poisson geometry, since every Poisson structure defines a Jacobi bracket. Moreover, every Jacobi structure induces a Poisson structure on a manifold of one dimension higher, this is known as the symplectization or homogenization, see [2] and its references for a detailed discussion. In Jacobi geometry there is also a local splitting theorem available, which was proven by Dazord, Lichnerowicz and Marle in [5]. Nevertheless, after this work, the parallels in the work of Poisson and Jacobi geometry stopped, at least in the context of local structure. The aim of this paper is to fill these gaps, prove normal form theorems for Jacobi bundles and give a more intrinsic proof of the splitting theorems. To do so, we will choose the approach of [3] and start with so-called Dirac–Jacobi bundles which generalize the notion of Jacobi structures.

Dirac–Jacobi bundles were introduced in [14] by Vitagliano and are a slight generalization of Wade's $\mathcal{E}^1(M)$-Dirac structures (see [16]). Moreover, these bundles are a Dirac theoretic generalizations of Jacobi bundles, as usual Dirac structures are for Poisson manifolds.

We want to stress that the methods used in this note are also suitable for proving splittings for involutive fat anchored vector bundles $(E, L \to M, \rho)$, i.e. a vector bundle $E \to M$, a line bundle $L \to M$ and a bundle map $\rho\colon E \to DL$ where DL is the Atiyah algebroid of L, such that $\Gamma^\infty(\rho(E))$ is closed with respect to the bracket, as well as Jacobi-algebroids (see [13]). We do not want to treat that in detail since every involutive fat anchored vector bundle is in particular, by composing ρ with the anchor of DL, an involutive anchored vector bundle and can be treated with the methods in [3]. The same holds true for Jacobi-algebroids.

This note is organized as follows: we recall the necessary structures in order to define the setting for Dirac–Jacobi structures, the omni-Lie algebroid of a line bundle (see [4]) in Sect. 2. Afterwards, we introduce the notion of Euler-like derivations, which are the crucial ingredient for the proofs of the main theorems. After this we are able to provide a normal form theorem for Dirac–Jacobi bundles, which is the main part of Sect. 4. In the following section, we want to apply this normal form theorem to the special case of Jacobi bundles, which allows us to state and prove two normal form theorems for Jacobi bundles, which allow us to give a different proof of the splitting theorems of Jacobi pairs, first provided in [5]. Moreover, we can apply these theorems to provide a splitting theorem for homogeneous Poisson structures around points where the homogeneity does not vanish, which was also done in [5]. Note that in [5] the proof works exactly the other way around: they prove a local splitting of homogeneous Poisson structures and use it to prove the splitting of Jacobi structures.

2 Preliminaries and notation

This introductory section is divided into two parts: first we recall the Atiyah algebroid of a vector bundle and the corresponding Der-complex with applications to contact and Jacobi geometry. Afterwards, we introduce the arena for the so-called Dirac–Jacobi bundles, the omni-Lie algebroids, and give a quick reminder of Dirac–Jacobi bundles together with the properties we will need afterwards.

2.1 Notation and a brief reminder on Jacobi geometry

The notions of Atiyah algebroid of a vector bundle and the associated *Der*-complex are known and are used in many other situations. This section is basically meant to fix notation. A more complete introduction to this can be found in [14] and its references. Nevertheless, the notion of omni-Lie algebroids was first defined in [4], in order to study Lie algebroids and local Lie algebra structures on vector bundles.

For a vector bundle $E \to M$, we denote its *gauge* or *Atiyah* algebroid by $DE \to M$ and by $\sigma\colon DE \to TM$ its anchor. Note that D is a functor from the category of vector bundles with regular, i.e. fiberwise invertible, vector bundle morphisms to Lie algebroids. Hence, for a regular $\Phi\colon E \to E'$, we denote by

$$D\Phi\colon DE \to DE'$$

the corresponding Lie algebroid morphism. We are mostly dealing with line bundles $L \to M$ for which we have the identity $DL = (J^1L)^* \otimes L$, where J^1L is the first jet bundle and sections of DL are canonically identified with the first order differential operators $\mathrm{DiffOp}^1(L, L)$ and are called derivations of L. The philosophy is that DL is supposed to replace the tangent bundle in the category of line bundles, and hence sections of it are playing the role of vector fields. The (local) flow of a derivation $\Delta \in \Gamma^\infty(DL)$ is defined as a one-parameter group of automorphisms $\Phi_t \in \mathrm{Aut}(L)$ covering $\phi_t \in \mathrm{Diffeo}(M)$ given as the unique solution of the ODE

$$(\Phi_t^*\Delta)(\lambda) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^*\lambda$$

for all $\lambda \in \Gamma^\infty(L)$, where $\Phi_t^*\Delta = D\Phi_t^{-1}(\Delta) \circ \phi_t$ and $\Phi_t^*\lambda(p) = \Phi_t^{-1}\lambda(\phi_t(p))$.

The gauge algebroid $DL \to M$ has a (tautological) Lie algebroid representation on L. The corresponding Lie algebroid complex with values in L is denoted by

$$\left(\Omega_L^\bullet(M) = \Gamma^\infty(\Lambda^\bullet(DL)^* \otimes L), \mathrm{d}_L\right).$$

Elements of $\Omega_L^\bullet(M)$ are referred to as *Atiyah* forms. Since there is an insertion

$$\iota\colon \Gamma^\infty(DL) \times \Omega_L^\bullet(M) \ni (\Delta, \alpha) \mapsto \iota_\Delta\alpha \in \Omega_L^{\bullet-1}(M),$$

we can also define a Lie derivative in the direction of $\Delta \in \Gamma^\infty(DL)$ by

$$\mathscr{L}_\Delta\colon \Omega_L^\bullet(M) \ni \alpha \mapsto [\iota_\Delta, \mathrm{d}_L]\alpha = \iota_\Delta\,\mathrm{d}_L\alpha + \mathrm{d}_L\iota_\Delta\alpha \in \Omega_L^\bullet(M).$$

Note that $\mathscr{L}_{\mathbb{1}} = \mathrm{id}_{\Omega_L^\bullet(M)}$, which can be computed directly. This means nothing else but $\iota_{\mathbb{1}}$ is a contracting homotopy of the differential d_L.

We briefly discuss Jacobi brackets in this setting. A Jacobi bracket is a local Lie algebra structure on the smooth sections of a line bundle $L \to M$, i.e. a Lie bracket $\{-,-\}\colon \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$, such that

$$\{\lambda, -\} \in \Gamma^\infty(DL)$$

for all $\lambda \in \Gamma^\infty(L)$.

Remark 2.1 Let $\{-,-\}$ be a Jacobi bracket on a line bundle $L \to M$. Then there is a unique tensor, called the Jacobi tensor, $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$, such that

$$\{\lambda, \mu\} = J(j^1\lambda, j^1\mu)$$

for $\lambda, \mu \in \Gamma^\infty(L)$. Conversely, every L-valued 2-form J on J^1L defines a skew-symmetric bilinear bracket $\{-,-\}$, but the latter needs not to be a Jacobi bracket. Specifically, it does not need to fulfill the Jacobi identity. However, there is the notion of a Gerstenhaber-Jacobi bracket

$$[-,-]\colon \Gamma^\infty(\Lambda^i(J^1L)^*\otimes L)\times\Gamma^\infty(\Lambda^j(J^1L)^*\otimes L)\to\Gamma^\infty(\Lambda^{i+j-1}(J^1L)^*\otimes L),$$

such that the Jacobi identity of $\{-,-\}$ is equivalent to $[J,J]=0$ see [13, Chapter 1.3] for a detailed discussion. Finally, a Jacobi tensor defines a map $J^\sharp\colon J^1L\to(J^1L)^*\otimes L=DL$.

When L is the trivial line bundle, than the notion of Jacobi bracket boils down to that of a *Jacobi pair*.

Remark 2.2 (Trivial Line bundle) Let $\mathbb{R}_M\to M$ be the trivial line bundle and let J be a Jacobi tensor on it. Let us denote by $1_M\in\Gamma^\infty(\mathbb{R}_M)$ the canonical global section. Using the canonical connection

$$\nabla\colon TM\ni v\mapsto(f\cdot 1_M\mapsto v(f)1_M)\in D\mathbb{R}_M,$$

we can see that $DL\cong TM\oplus\mathbb{R}_M$ and hence

$$J^1\mathbb{R}_M=(D\mathbb{R}_M)^*\otimes\mathbb{R}_M=T^*M\oplus\mathbb{R}_M.$$

With this splitting, we see that

$$J=\Lambda+\mathbb{1}\wedge E$$

for some $(\Lambda,E)\in\Gamma^\infty(\Lambda^2TM\oplus TM)$. The Jacobi identity is equivalent to $[\![\Lambda,\Lambda]\!]_s+2E\wedge\Lambda=0$ and $\mathscr{L}_E\Lambda=0$, where $[\![\cdot,\cdot]\!]_s$ is the Schouten bracket. The pair (Λ,E) is often referred to as *Jacobi pair* and in fact, Jacobi structures have been introduced in [11] as Jacobi pairs and the splitting theorem in [5] is proven for Jacobi pairs. Moreover, if we denote by $\mathbb{1}^*\in\Gamma^\infty(J^1\mathbb{R}_M)$ the canonical section then we can write any $\psi\in J^1\mathbb{R}_M$ as $\psi=\alpha+r\mathbb{1}^*\in\Gamma^\infty(J^1\mathbb{R}_M)$, for some $\alpha\in T^*M$ and $r\in\mathbb{R}$. We obtain

$$J^\sharp(\alpha+r\mathbb{1}^*)=\Lambda^\sharp(\alpha)+rE-\alpha(E)\mathbb{1}.$$

A more detailed discussion about Jacobi structures on trivial line bundles can be found in [13, Chapter 2]. In a similar way, we can see that $\Omega_L(M)^\bullet=\Gamma^\infty(\Lambda^\bullet(T^*M\oplus\mathbb{R}_M))=\Gamma^\infty(\Lambda^\bullet T^*M\oplus\mathbb{1}^*\wedge\Lambda^{\bullet-1}T^*M)$. Here $\mathbb{1}^*$ is the canonical section of $\mathbb{R}_M$, moreover the differential $\mathrm{d}_{\mathbb{R}_M}$ is defined by the relations

$$\mathrm{d}_{\mathbb{R}_M}(\mathbb{1}^*)=0\ \text{ and }\ \mathrm{d}_{\mathbb{R}_M}=\mathrm{d}_{dR}+\mathbb{1}^*\wedge.$$

2.2 The Omni-Lie algebroid of a line bundle and its automorphisms

The omni-Lie algebroid plays the same role in Dirac–Jacobi geometry as the generalized tangent bundle does in Dirac geometry. In fact, the parallels are evidently enormous. The following definitions and Lemmas are obvious adaptations of the case of Dirac structures, this is why we omit proofs. The following definitions and results in Dirac–Jacobi geometry can be found in [14].

Definition 2.3 Let $L\to M$ be a line bundle. The vector bundle $\mathbb{D}L:=DL\oplus J^1L$ together with

1. the (Dorfman-like) bracket
$$[\![(\Delta_1, \psi_1), (\Delta_2, \psi_2)]\!] = ([\Delta_1, \Delta_2], \mathscr{L}_{\Delta_1}\psi_2 - \iota_{\Delta_2} \mathrm{d}_L \psi_1)$$
2. the non-degenerate L-valued pairing
$$\langle\!\langle (\Delta_1, \psi_1), (\Delta_2, \psi_2) \rangle\!\rangle := \psi_1(\Delta_2) + \psi_2(\Delta_1)$$
3. the canonical projection $\mathrm{pr}_D : \mathbb{D}L \to DL$

is called the omni-Lie algebroid of $L \to M$.

Remark 2.4 In principle, one can consider the omni-Lie algebroid together with a bracket twisted by an Atiyah 3-form, as it is done in Dirac geometry. But in the case of line bundles the cohomology of the Der-complex is trivial and hence we prefer not to include it since anyway, we can find an isomorphism of the two brackets.

We shall now introduce automorphisms of the omni-Lie algebroid, which mirrors the definition of automorphisms of the generalized tangent bundle.

Definition 2.5 Let $L \to M$ be a line bundle. A pair $(F, \Phi) \in \mathrm{Aut}(\mathbb{D}L) \times \mathrm{Aut}(L)$ is called Courant-Jacobi automorphism, if

1. $D\Phi \circ \mathrm{pr}_D = \mathrm{pr}_D \circ F$,
2. $\Phi^* \langle\!\langle -, - \rangle\!\rangle = \langle\!\langle F-, F- \rangle\!\rangle$ and
3. $F^*[\![-, -]\!] = [\![F^*-, F^*-]\!]$.

The group of Courant-Jacobi automorphisms is denoted by $\mathrm{Aut}_{CJ}(L)$.

For a line bundle $L \to M$ and $\Phi \in \mathrm{Aut}(L)$, we define
$$\mathbb{D}\Phi : \mathbb{D}L \ni (\Delta, \alpha) \mapsto (D\Phi(\Delta), (D\Phi^{-1})^*\alpha) \in \mathbb{D}L,$$
which gives canonically an automorphism $\mathbb{D}\Phi \in \mathrm{Aut}(\mathbb{D}L)$, moreover $(\mathbb{D}\Phi, \Phi)$ is a Courant-Jacobi automorphism. For a closed 2-form $B \in \Omega^2_L(M)$, we define
$$\exp(B) : \mathbb{D}L \ni (\Delta, \alpha) \mapsto (\Delta, \alpha + \iota_\Delta B) \in \mathbb{D}L,$$
and see that $(\exp(B), \mathrm{id}) \in \mathrm{Aut}_{CJ}(L)$. We can combine these two special kinds of morphisms together with the action of $\mathrm{Aut}(L)$ on $\mathbb{D}L$ and find the following

Lemma 2.6 *Let $L \to M$ be a line bundle. If we denote by $Z^2_L(M)$ the closed 2-forms, then*
$$\mathfrak{I} : Z^2_L(M) \rtimes \mathrm{Aut}(L) \ni (B, \Phi) \mapsto (\exp(B) \circ \mathbb{D}\Phi, \Phi) \in \mathrm{Aut}_{CJ}(L)$$
is an ismorphism of groups.

The group structure of the semi-direct product $Z^2_L(M) \rtimes \mathrm{Aut}(L)$ is given by
$$(\Omega_1, \Phi_1) \cdot (\Omega_2, \Phi_2) = (\Omega_1 + (\Phi_1)_* \Omega_2, \Phi_1 \circ \Phi_2)$$
for $(\Omega_i, \Phi_i) \in Z^2_L(M) \rtimes \mathrm{Aut}(L)$ for $i = 1, 2$. In a similar way, we can define infinitesimal automorphisms of the omni-Lie algebroid.

Definition 2.7 Let $L \to M$ be line bundle. A pair $(D, \Delta) \in \Gamma^\infty(D\mathbb{D}L) \times \Gamma^\infty(DL)$ is called infinitesimal Courant-Jacobi automorphism, if

1. $[\Delta, \mathrm{pr}_D(\varepsilon)] = \mathrm{pr}_D(D(\varepsilon))$,

2. $\Delta\langle\!\langle \varepsilon, \chi \rangle\!\rangle = \langle\!\langle D(\varepsilon), \xi \rangle\!\rangle + \langle\!\langle \epsilon, D(\chi) \rangle\!\rangle$ and
3. $D([\![\varepsilon, \chi]\!]_H) = [\![D(\epsilon), \chi]\!] + [\![\epsilon, D(\chi)]\!]$

for all $\varepsilon, \chi \in \Gamma^\infty(\mathbb{D}L)$. The Lie algebra of infinitesimal Courant-Jacobi automorphisms is denoted by $\mathfrak{aut}_{CJ}(L)$.

Note that the flow of an infinitesimal Courant-Jacobi automorphism gives a (local) Courant-Jacobi automorphism, in this sense, we can see $\mathfrak{aut}_{CJ}(L)$ as the Lie algebra of $\mathrm{Aut}_{CJ}(L)$. Similarly to the autmorphism case, we have

Lemma 2.8 *Let $L \to M$ be a line bundle. Then*

$$\mathfrak{i}\colon Z^2_L(M) \rtimes \Gamma^\infty(DL) \ni (B, \Delta) \to ((\Box, \beta) \mapsto ([\Delta, \Box], \mathscr{L}_\Delta \beta + \iota_\Box(B))) \in \mathfrak{aut}_{CJ}(L)$$

is an isomorphism of Lie algebras.

The Lie algebra structure on the semi-direct product is given by

$$[(\Omega_1, \Delta_1), (\Omega_2, \Delta_2)] = (\mathscr{L}_{\Delta_1}\Omega_2 - \mathscr{L}_{\Delta_2}\Omega_1, [\Delta_1, \Delta_2])$$

for $(\Omega_i, \Delta_i) \in Z^2_L(M) \rtimes \Gamma^\infty(DL)$ and $i = 1, 2$.

For every section $(\Delta, \alpha) \in \Gamma^\infty(\mathbb{D}L)$ the map $[\![(\Delta, \alpha), -]\!]$ is an infinitesimal Courant-Jacobi automorphism, in fact it is realized in $Z^2_L(M) \rtimes \Gamma^\infty(DL)$ by

$$\mathfrak{i}(-\,\mathrm{d}_L\alpha, \Delta) = [\![(\Delta, \alpha), -]\!].$$

For later use, we want to talk about the flow of infinitesimal Courant-Jacobi automorphisms and want to compute them as explicit as possible.

Lemma 2.9 *Let $L \to M$ be line bundle. Let additionally $(B, \Delta) \in Z^2_L(M) \rtimes \Gamma^\infty(DL)$. The flow of $\mathfrak{i}(B, \Delta)$ is given by*

$$\begin{aligned}\mathfrak{I}(\gamma_t, \Phi^\Delta_t) &= \mathfrak{I}\Big(- \int_0^t (\Phi^\Delta_{-\tau})^* B \,\mathrm{d}\tau, \Phi^\Delta_t \Big) \\ &= \exp\big(- \int_0^t (\Phi^\Delta_{-\tau})^*(B)\,\mathrm{d}\tau\big) \circ \mathbb{D}\Phi^\Delta_t.\end{aligned}$$

Corollary 2.10 *Let $L \to M$ be a line bundle. For every $(\Delta, \alpha) \in \Gamma^\infty(\mathbb{D}L)$ the flow of $[\![(\Delta, \alpha), -]\!]$ is given by*

$$\exp\big(\int_0^t (\Phi^\Delta_{-\tau})^* \,\mathrm{d}_L\alpha\,\mathrm{d}\tau\big) \circ \mathbb{D}\Phi^\Delta_t.$$

2.3 Dirac–Jacobi bundles

After having discussed the arena, we want to introduce the subbundles of interest: so-called *Dirac–Jacobi Bundles*. As the name suggests, they are the analogue of Dirac structures in the generalized tangent bundle. In fact, the definition is (up to some obvious replacements) the same.

Definition 2.11 Let $L \to M$ be a line bundle. A subbundle $\mathfrak{L} \subseteq \mathbb{D}L$ is called a Dirac–Jacobi structure if

1. $\mathfrak{L}$ is involutive with respect to $[\![-, -]\!]$,

2. $\mathfrak{L}$ is maximally isotropic with respect to $\langle\!\langle -, - \rangle\!\rangle$.

Remark 2.12 Let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi structure on a line bundle $L \to M$ and let (F, Φ) be a Courant-Jacobi automorphism, then

$$F(\mathfrak{L}) \subseteq \mathbb{D}L$$

is a Dirac–Jacobi structure on $L \to M$. Moreover, we denote a transformation with a closed 2-form $B \in \Omega^2_L(M)$ by

$$\mathfrak{L}^B := \exp(B)(\mathfrak{L}).$$

Proposition 2.13 *Let $L \to M$ be a line bundle and let $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi structure, then*

$$\mathfrak{L}_J := \{(J^\sharp(\psi), \psi) \in \mathbb{D}L \mid \psi \in J^1 L\}$$

is a Dirac–Jacobi structure. If a Dirac–Jacobi structure $\mathfrak{L} \subseteq \mathbb{D}L$ fulfills

$$DL \cap \mathfrak{L} = \{0\},$$

then there is a unique Jacobi structure $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$, such that $\mathfrak{L}_J = \mathfrak{L}$

Proof The result follows the same lines as the well-known fact in Poisson geometry. □

Another interesting example of Dirac–Jacobi bundles, which also plops up in Jacobi geometry, is

Definition 2.14 Let $L \to M$ be a line bundle. A Dirac–Jacobi structure $\mathfrak{L} \subseteq \mathbb{D}L$ is called of homogeneous Poisson type, if

$$\operatorname{rank}(\mathfrak{L} \cap DL) = 1.$$

The name of these objects is justified by the following

Lemma 2.15 *Let $L \to M$ be a line bundle and let $\mathfrak{L} \subseteq \mathbb{D}L$ a Dirac–Jacobi structure of homogeneous Poisson type, then for every point $p \in M$ there exists a local trivialization $L_U = U \times \mathbb{R}$, a flat connection $\nabla\colon TU \to DL_U \cong TU \oplus \mathbb{R}_U$ and a homogeneous Poisson structure $\pi \in \Gamma^\infty(\Lambda^2 TU)$ with homogeneity $Z \in \Gamma^\infty(TM)$, i.e. $\mathscr{L}_Z\pi = -\pi$, such that*

$$\mathfrak{L}\Big|_U = \{(r(\mathbb{1} - \nabla_Z) + \nabla_{\pi^\sharp(\alpha)}, \alpha + \alpha(Z)\mathbb{1}^*) \in \mathbb{D}L\Big|_U \mid r \in \mathbb{R},\ \alpha \in T^*M\},$$

where we use the inclusion $T^*M \to J^1 L$ by $\alpha(\nabla_X) = \alpha(X)$ and $\alpha(\mathbb{1}) = 0$.

Proof Let $p \in M$ and $U \subseteq M$ be an open subset containing p, such that $L_U \cong U \times \mathbb{R}$ with corresponding trivialization of the gauge algebroid $DL_U = TU \oplus \mathbb{R}_U$ together with the trivialization $J^1 L_U = T^*U \oplus \mathbb{R}_U$. Let us denote by $\nabla^{\mathrm{can}}\colon TU \ni X \mapsto (X, 0) \in TU \oplus \mathbb{R}_U$ the canonical flat connection. In a possibly smaller neighborhood, denoted also by U, we find a non-vanishing section $\Delta = (-X, f) \in \Gamma^\infty(\mathfrak{L} \cap DL)$. We can distinguish two cases: the first is that $f(p) \neq 0$, then we find a (possibly smaller) neighbourhood of p, such that f is non-vanishing, hence $(-\frac{X}{f}, 1) =: (-Z, 1)$ spans $\mathfrak{L} \cap DL$ in that neighbourhood and hence $DL_U = \langle(-Z, 1)\rangle \oplus \langle(X, 0)\rangle_{X \in TU}$. Moreover, we have the short exact sequence

$$0 \to DL \cap \mathfrak{L} \to \mathfrak{L} \to \mathrm{pr}_{J^1 L}\,\mathfrak{L} \to 0$$

and hence $\operatorname{rank}(\operatorname{pr}_{J^1L}\mathfrak{L}) = \dim(M)$, with the canonical inclusion $T^*U \ni \alpha \to (\alpha, \alpha(Z)) \in \operatorname{pr}_{J^1L}\mathfrak{L}$. Summarizing, for all $\alpha \in T^*U$ there exists a unique $X \in TU$, such that

$$\big((X,0),(\alpha,\alpha(Z))\big) \in \mathfrak{L}\big|_U$$

Because of the maximal isotropy of $\mathfrak{L}$, this assignment comes from a bivector field $\pi \in \Gamma^\infty(\Lambda^2 TU)$, i.e. $X = \pi^\sharp(\alpha)$. Finally, we can write

$$\{\big(h(-Z,1) + (\pi^\sharp(\alpha),0),(\alpha,\alpha(Z))\big) \in \mathbb{D}L_U \mid h \in \mathbb{R},\ \alpha \in T^*U\},$$

which is nothing else but

$$\{\big(h(\mathbb{1} - \nabla^{\mathrm{can}}_Z) + \nabla^{\mathrm{can}}_{\pi^\sharp(\alpha)}, \alpha + \alpha(Z)\mathbb{1}^*)\big) \in \mathbb{D}L_U \mid h \in \mathbb{R},\ \alpha \in T^*U\}$$

The claim follows by using the flatness of ∇^{can} and the involutivity of $\mathfrak{L}$.

Now we have to treat the case $f(p) = 0$. Since $\Delta = (-X, f)$ is non-vanishing, we conclude that $X(p) \neq 0$, hence there is a closed one form $\beta \in \Gamma^\infty(T^*U)$ such that $\beta(X) = -1$ around p. We define the flat connection

$$\nabla\colon TU \ni Y \mapsto (Y, \beta(Y)) \in DL_U.$$

With this connection we see that $\Delta = (f-1)\mathbb{1} - \nabla_X$ and since $f(p) = 0$, we have that $f - 1 \neq 0$ in a whole neighbourhood of p and hence we choose $\Delta' = \frac{1}{f-1}\Delta$ as a generating section of $\mathfrak{L} \cap DL$ around p. We can now repeat the same argument as for the case $f(p) \neq 0$ by using the connection ∇ instead of ∇^{can}, since $\Delta' = \mathbb{1} - \nabla_Z$ for $Z = \frac{1}{f-1}X$. □

In the category of Dirac–Jacobi bundles there are not just automorphisms of the omni-Lie algebroid as morphisms, one of the possibilities is to include so-called *backwards transformations* as in the Dirac geometry case.

Definition 2.16 Let $L_i \to M_i$ for $i = 1, 2$ be two line bundles and let $\Phi\colon L_1 \to L_2$ be a regular line bundle morphism covering $\phi\colon M_1 \to M_2$. Let $\mathfrak{L} \subseteq \mathbb{D}L_2$ be a Dirac–Jacobi bundle. The (not necessarily smooth) family of vector spaces

$$\mathfrak{B}_\Phi(\mathfrak{L}) := \{(\Delta_p, (D\Phi)^*\alpha_{\phi(p)}) \in \mathbb{D}L_1 \mid (D\Phi(\Delta_p), \alpha_{\phi(p)}) \in \mathfrak{L}\}$$

is called backwards transformation of $\mathfrak{L}$.

Remark 2.17 Let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle on a line bundle $L \to M$ and let $\Phi \in \operatorname{Aut}(L)$ an automorphism, then

$$\mathfrak{B}_\Phi(\mathfrak{L}) = \mathbb{D}\Phi^{-1}(\mathfrak{L}).$$

The backwards transformation of a Dirac–Jacobi bundle need not to be Dirac–Jacobi any more, but there are sufficient conditions on the subbundle $\mathfrak{L}$ and the line bundle morphism Φ which can be seen, e.g. in [14]:

Theorem 2.18 *Let $\Phi\colon L_1 \to L_2$ be a regular line bundle morphism over $\phi : M_1 \to M_2$ and let $\mathfrak{L} \in \mathbb{D}L_2$ be a Dirac–Jacobi bundle. If* $\ker D\Phi^* \cap \phi^*\mathfrak{L}$ *has constant rank, then $\mathfrak{B}_\Phi(\mathfrak{L})$ is a Dirac–Jacobi bundle, where $\phi^*\mathfrak{L}$ defines the pull-back bundle of $\mathfrak{L} \to M_2$ seen as a vector bundle.*

Proof The proof can be found in [14, Proposition 8.4]. □

A Dirac–Jacobi bundle $\mathfrak{L} \subseteq \mathbb{D}L$ has, as Dirac structures, a canonical involutive and integrable (singular) distribution obtained by

$$\sigma(\mathrm{pr}_D(\mathfrak{L})) \subseteq TM.$$

This distribution generally has two different kind of leaves, whereas in Dirac geometry all of the leaves are pre-symplectic. This mirrors the fact that Jacobi bundles may have two types of leaves: contact and locally conformal symplectic.

The details about this facts can be found in [14], we just want to briefly recall some properties.

Lemma 2.19 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle and let $S \hookrightarrow M$ be a leaf of the characteristic distribution $\sigma(\mathrm{pr}_D(\mathfrak{L})) \subseteq TM$. Then the two cases*

1. $\mathbb{1}_p \in \mathrm{pr}_D \mathfrak{L}$ *for some point $p \in S$ and*
2. $\mathbb{1}_p \notin \mathrm{pr}_D \mathfrak{L}$ *for some point $p \in S$*

are mutually exclusive.

This motivates the following definition.

Definition 2.20 Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle and let $\iota\colon S \hookrightarrow M$ be a leaf. Then

1. S is called pre-contact, if $\mathbb{1} \in \Gamma^\infty(\mathrm{pr}_D \mathfrak{L}\big|_S)$ and
2. S is called locally conformal pre-symplectic, if it is not pre-contact.

To justify this we have the following

Corollary 2.21 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq DL$ be a Dirac–Jacobi bundle and let $\iota\colon S \hookrightarrow M$ be a leaf of its characteristic foliation and denote by $I\colon L_S \to L$ the embedding of L restricted to S. If S is a*

1. *pre-contact leaf, then there exists a $\omega \in \Omega^2_{L_S}(S)$, such that*
$$\mathfrak{B}_I(\mathfrak{L}) = \{(\Delta, \iota_\Delta\omega) \in \mathbb{D}L_S \mid \Delta \in DL_S\}$$
and $\mathrm{d}_{L_S}\omega = 0$.
2. *locally conformal pre-symplectic leaf, then there exists a flat connection $\nabla\colon TS \to DL_S$ and an L_S-valued 2-form $\omega \in \Gamma^\infty(\Lambda^2 T^*S \otimes L_S)$, such that*
$$\mathfrak{B}_I(\mathfrak{L}) = \{(\nabla_X, \sigma^*(\iota_X\omega) + \alpha) \in \mathbb{D}L_S \mid X \in TS \textit{ and } \alpha \in \mathrm{Ann}(\mathrm{im}(\nabla))\}$$
and $\mathrm{d}^\nabla\omega = 0$.

3 Submanifolds and Euler-like vector fields

In this subsection we want to discuss Euler-like vector fields. These vector fields, in particular, induce a homogeneity structure on the manifold, which is equivalent, under some additional conditions which are in our case always fulfilled, to the manifold being the total space of a vector bundle, see e.g. [8]. This total space turns out to be the normal bundle for some submanifold, which is an input datum for an Euler-like vector field. Nevertheless, we will not go more in details with these features, since we work directly with tubular neighbourhoods. We will begin collecting facts about tubular neighbourhoods, submanifolds and corresponding maps, which can be found in [3] and describe afterwards the notion of Euler-like vector fields and extend this notion to derivations of a line bundle. As a final remark, we want to stress that all of the used submanifolds are actually *embedded* submanifolds.

3.1 Normal bundles and tubular neighborhoods

For a pair of manifolds (M, N), i.e. a submanifold $N \hookrightarrow M$, we denote

$$\nu(M, N) = \frac{TM\big|_N}{TN}$$

the normal bundle. If the ambient space is clear, we will just write ν_N instead. Given a map of pairs

$$\Phi\colon (M, N) \to (M', N'),$$

i.e. a map $\Phi\colon M \to M$, such that $\Phi(N) \subseteq N'$, we denote by

$$\nu(\Phi)\colon \nu(M, N) \to \nu(M', N')$$

the induced map on the normal bundle. For a vector field X on M tangent to N, we have that the flow Φ_t^X is a map of pairs from (M, N) to itself. Hence we define

$$T\nu(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\nu(\Phi_t^X) \in \Gamma^\infty(T\nu_N).$$

Moreover, for a vector bundle $E \to M$ and $\sigma \in \Gamma^\infty(E)$, such that $\sigma\big|_N = 0$ for a submanifold $N \hookrightarrow M$, we denote by

$$\mathrm{d}^N\sigma\colon \nu_N \to E\big|_N$$

the map which is $\nu(\sigma)$, for σ seen as a map $\sigma\colon (M, N) \to (E, M)$, followed by the canonical identification $\nu(E, M) = E$, given by

$$C_E\colon E \ni v_p \to [\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} t v_p]_{TM} \in \nu(E, M).$$

Before we prove the next results, we want to find a useful description of C_E^{-1}. Let us therefore consider a curve $\gamma\colon I \to E$ for an open interval I containing 0, such that $\gamma(0) = 0_p$ for $p \in M$, then one can prove in local coordinates

$$C_E^{-1}([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma(t)]) = \lim_{t\to 0}\frac{\gamma(t)}{t}. \tag{3.1}$$

Proposition 3.1 *Let $E_i \to M_i$ be vector bundles for $i = 1, 2$ and let $\Phi\colon E_1 \to E_2$ be a vector bundle morphism. Then, for $\Phi\colon (E_1, M_1) \to (E_2, M_2)$,*

$$C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} = \Phi$$

Proof Let $v_p \in E_1$, then

$$\begin{aligned}
(C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1})(v_p) &= (C_{E_2}^{-1} \circ \nu(\Phi))([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} t v_p]_{TM_1}) \\
&= C_{E_2}^{-1}([T\Phi\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} t v_p]_{TM_2}) \\
&= C_{E_2}^{-1}([\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} t\Phi(v_p)]_{TM_2}) \\
&= \Phi(v_p)
\end{aligned}$$

□

Proposition 3.2 *Let $E_i \to M$ be vector bundles for $i = 1, 2$ and let $\Phi\colon E_1 \to E_2$ be a vector bundle morphism covering the identity. Then, for every section $\sigma \in \Gamma^\infty(E_1)$, such that $\sigma\big|_N = 0$ for some submanifold $N \hookrightarrow M$,*

$$\mathrm{d}^N \Phi(\sigma) = \Phi(\mathrm{d}^N \sigma)$$

holds.

Proof We consider the map $\Phi(\sigma)\colon (M, N) \to (E_2, M)$, then we have

$$\begin{aligned} C_{E_2}^{-1} \circ \nu(\Phi(\sigma)) &= C_{E_2}^{-1} \circ \nu(\Phi) \circ \nu(\sigma) \\ &= C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} \circ C_{E_1}^{-1} \circ \nu(\sigma) \\ &= \Phi \circ C_{E_1}^{-1} \circ \nu(\sigma) \end{aligned}$$

and the claim follows if we restrict these maps. □

Proposition 3.3 *Let (M, N) be a pair of manifolds and let $X \in \Gamma^\infty(TM)$ be such that $X\big|_N = 0$. Then*

$$T\Phi_t^X\big|_N = \mathrm{e}^{tD_X}$$

for a unique $D_X \in \Gamma^\infty(\mathrm{End}(TM\big|_N))$, moreover $TN \subseteq \ker(D_X)$ and

$$\begin{tikzcd} TM\big|_N \arrow[r, "D_X"] \arrow[d] & TM\big|_N \\ \nu_N \arrow[ur, "\mathrm{d}^N X"'] & \end{tikzcd}$$

commutes.

Proof Since $X\big|_N = 0$ its flow fixes all elements of N. This means that

$$T\Phi_t^X\colon T_pM \to T_pM$$

for all $t \in \mathbb{R}$ and $p \in N$. Moreover, it fulfills the property,

$$T\Phi_t^X \circ T\Phi_s^X = T\Phi_{t+s}^X$$

and $T\Phi_0^X = \mathrm{id}$ and hence the claim follows. □

Definition 3.4 Let (M, N) be a pair of manifolds. A tubular neighbourhood of N is an open subset $U \subseteq M$ containing N together with a diffeomorphism

$$\psi\colon \nu_N \to U,$$

such that $\psi\big|_N\colon N \to N$ is the identity and for $\psi\colon (\nu_N, N) \to (M, N)$ the map

$$\nu(\psi)\colon \nu(\nu_N, N) \to \nu_N$$

is inverse of $C_{\nu_N}\colon \nu_N \to \nu(\nu_N, N)$.

3.2 Euler-like vector fields and derivations

In this part, we basically recall just the notion of Euler-like vector fields from [3] and extend this notion to derivations of a line bundle.

Definition 3.5 Let (M, N) be a pair of manifolds. A vector field $X \in \Gamma^\infty(TM)$ is called Euler-like, if

1. $X\big|_N = 0$,
2. X has complete flow,
3. $T\nu(X) = \mathcal{E}$,

where $\mathcal{E}$ is the Euler vector field on $\nu_N \to N$.

Proposition 3.6 *Let (M, N) be a pair of manifolds, then there exists an Euler-like vector field.*

Proof The proof can be found in [3]. □

Lemma 3.7 *Let M be a manifold, $N \hookrightarrow M$ a submanifold and $X \in \Gamma^\infty(TM)$ be a Euler-like vector field. Then there exists a unique tubular neighbourhood embedding*

$$\psi \colon \nu_N \to U,$$

such that $\psi^* X = \mathcal{E}$.

Proof The proof can be found in [3]. □

Proposition 3.8 *Let (M, N) be a pair of manifolds and let $X \in \Gamma^\infty(TM)$ be a complete vector field, such that $X\big|_N = 0$. Then X is Euler-like if and only if $\mathrm{d}^N X$ followed by the projection $TM\big|_N \to \nu_N$ is identity.*

Proof The proof can be found in [3]. □

Note that for a pair of manifolds (M, N) and an Euler like vector field $X \in \Gamma^\infty(TM)$, the set

$$\{p \in M \mid \lim_{t \to -\infty} \Phi_t^X(p) \text{ exists and lies in } N\}$$

is an open subset in M containing N, such that that the action of Φ_t^X restricts to this set. Moreover, for a tubular neighbourhood $\psi \colon \nu_N \to U$, such that $\psi^* X = \mathcal{E}$, we have that

$$U = \{p \in M \mid \lim_{t \to -\infty} \Phi_t^X(p) \text{ exists and lies in } N\}.$$

The proof of this statement and a more detailed discussion can be found in [3]. Let us denote by $\lambda_s = \Phi^X_{\log(s)}\big|_U$. We obtain, that λ_s is smooth for all $s \in \mathbb{R}_0^+$. Moreover, we have that

$$\psi \circ \kappa_s = \lambda_s \circ \psi, \tag{3.2}$$

where we denote by $\kappa_s \colon \nu_N \to \nu_N$ the map $[X_p] \mapsto [sX_p]$. Note that $\kappa_0 \colon \nu_N \to N$ coincides with the bundle projection, to be more precise $\kappa_0 = j \circ \mathrm{pr}_\nu$, where pr_ν is the bundle projection and $j \colon N \to \nu_N$ the canonical inclusion.

Let us add now the line bundle case

Definition 3.9 Let $L \to M$ be a line bundle and $N \hookrightarrow M$ be a submanifold. A derivation $\Delta \in \Gamma^\infty(DL)$ is called Euler-like, if

1. $\Delta\big|_N = 0$,
2. $\sigma(\Delta)$ is an Euler-like vector field.

This definition turns out to be the correct one for our purposes, since we can prove basically all results, which are available for Euler-like vector fields. Let us start collecting them.

Proposition 3.10 *Let $L \to M$ be a line bundle and let $\Delta \in \Gamma^\infty(DL)$ be an Euler-like derivation with respect to $N \hookrightarrow M$, then the (complete) flow $\Phi_t^\Delta \in \mathrm{Aut}(L)$ of Δ induces the map*

$$\Lambda_s = \Phi^\Delta_{\log(s)},$$

which, restricted to $U = \{p \in M \mid \lim_{t \to -\infty} \Phi_t^{\sigma(X)}(p)$ exists and lies in $N\}$, can be extended smoothly to $s = 0$. Moreover, the map

$$\Lambda_0 \colon L_U \to L_N$$

is a regular line bundle morphism.

Proof The proof is an easy verification using a tubular neighbourhood $\psi \colon \nu_N \to U$, such that $\psi^*\sigma(X) = \mathcal{E}$. □

Definition 3.11 Let $L \to M$ be a line bundle and $N \hookrightarrow M$ be a submanifold. A fat tubular neighbourhood is a regular line bundle morphism

$$\Psi \colon L_\nu \to L_U,$$

where the line bundle L_ν is given by the pull-back

$$\begin{array}{ccc} L_\nu & \longrightarrow & L_N \\ \downarrow & & \downarrow \\ \nu_N & \longrightarrow & N \end{array},$$

covering a tubular neighborhood $\psi \colon \nu_N \to U$, such that $\Psi\big|_N \colon L_N \to L_N$ is the identity.

Lemma 3.12 *Let $L \to M$ be a line bundle, let $N \hookrightarrow M$ be a submanifold and let $\psi \colon \nu_N \to U$ be a tubular neighborhood. Then there exists a fat tubular neighbourhood covering ψ.*

Proof The proof can be found in [13, Chapter 3]. □

For a line bundle $L \to N$ and a vector bundle $E \to N$ there is always a canonical Derivation $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$, such that $\sigma(\Delta_{\mathcal{E}}) = \mathcal{E}$ constructed as follows: Consider the pull-back L_E of L along $p \colon E \to M$ as in the diagram

$$\begin{array}{ccc} L_E & \xrightarrow{P} & L \\ \downarrow & & \downarrow \\ E & \xrightarrow{p} & N \end{array}$$

and the corresponding map $DP \colon L_E \to L_N$. We have that canonically $\ker(DP) \cong \mathrm{Ver}(E) := \ker(Tp) \subseteq TE$, which induces a flat (partial) connection $\nabla \colon \mathrm{Ver}(E) \to DL_E$. Since the Euler vector field is canonically vertical, we can define $\Delta_{\mathcal{E}} = \nabla_{\mathcal{E}}$.

Proposition 3.13 *Let $L \to N$ be a line bundle and let $E \to N$ be a vector bundle. Then the flow Φ_t of $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$ is given by*

$$\Phi_t(v_p, l_p) = (\mathrm{e}^t \cdot v_p, l_p)$$

for all $(v_p, l_p) \in L_E$.

Proof Note that since $\nabla_{\mathcal{E}}$ is in the kernel of DP, it is related to the 0 derivation on $L \to M$ and hence we have for its flow

$$P \circ \Phi_t = P.$$

Since $L_E = E \times_M L$, we have that

$$\Phi_t(v_p, l_p) = (\phi_t(v_p), l_p)$$

where ϕ_t is the flow of the symbol of $\nabla_{\mathcal{E}}$, which is by construction the Euler vector field and hence the claim follows. □

Note that for the flow Φ_t of the canonical Euler-like derivation $\Delta_{\mathcal{E}} \in \Gamma^\infty(DL_E)$, we have that

$$P_s = \Phi_{\log(s)} \colon L_E \to L_E$$

is defined for all $s > 0$ and can be extended smoothly to $s = 0$, moreover P_0 coincides with the canonical projection $P \colon L_E \to L$ followed by the canonical inclusion $J \colon L \to L_E$.

Lemma 3.14 *Let $L \to M$ be a line bundle, let $N \hookrightarrow M$ be a submanifold and let $\Delta \in \Gamma^\infty(DL)$ be an Euler-like derivation. Then there is a unique fat tubular neighbourhood $\Psi \colon L_\nu \to L_U$ such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$.*

Proof First, we want to prove existence. It is clear that any such Ψ has to cover the unique tubular neighbourhood $\psi \colon \nu_N \to U$, such that $\psi^*\sigma(\Delta) = \mathcal{E}$. So let us choose a fat tubular neighbourhood $\tilde{\Psi} \colon L_\nu \to L_U$ covering ψ. We consider now $\tilde{\Psi}^*\Delta \in \Gamma^\infty(DL_\nu)$. We have $\sigma(\tilde{\Psi}^*\Delta) = \psi^*\sigma(\Delta) = \mathcal{E}$. Thus $\sigma(\Delta_{\mathcal{E}}) = \sigma(\tilde{\Psi}^*\Delta)$. Consider now the derivation $\Box = \Delta_{\mathcal{E}} - \tilde{\Psi}^*\Delta$ and

$$\Box_t = -\frac{1}{t}\Phi^*_{\log(t)}\Box,$$

where Φ_t is the flow of $\Delta_{\mathcal{E}}$ and extend it smoothly to $t = 0$. Let us denote the flow of $\Box_t$ by ϕ_t. Note that it is complete, since $\sigma(\Box_t) = 0$, indeed there is even a explicit formula for it, which we do not need. Note however, that $\phi_t \in \mathrm{Aut}(L_\nu)$ and it fixes every base point for all $t \in \mathbb{R}$, since $\sigma(\Box_t) = 0$. Let us compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*(\Delta_{\mathcal{E}} + t\Box_t) &= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] + \frac{\mathrm{d}}{\mathrm{d}t}t\Box_t) \\
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] - \frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_{\log(t)}\Box) \\
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] - \frac{1}{t}[\Delta_{\mathcal{E}}, \Phi^*_{\log(t)}\Box]) \\
&= \phi_t^*([\Box_t, \Delta_{\mathcal{E}}] + [\Delta_{\mathcal{E}}, \Box_t]) \\
&= 0.
\end{aligned}$$

Hence we see that $\Delta_{\mathcal{E}} = \phi_0^*(\Delta_{\mathcal{E}}) = \phi_1^*(\Delta_{\mathcal{E}} + \Box_1) = \phi_1^*(\tilde{\Psi}^*\Delta)$. Therefore, we have that the map $\Psi = \tilde{\Psi} \circ \phi_1$ will do the job, since $\phi_1\big|_N = \mathrm{id}$, because $\Box\big|_N$ is trivial and hence also $\Box_t$.

Let us now assume that we have $\Psi_1, \Psi_2 \colon L_\nu \to L_U$, such that $\Psi_1^*\Delta = \Psi_2^*\Delta = \Delta_{\mathcal{E}}$. Note that since both have to cover the unique $\psi \colon \nu_N \to U$, the target L_U is the same for both. Let us consider $\Xi := \Psi_1^{-1} \circ \Psi_2 \colon L_\nu \to L_\nu$, which covers the identity, which implies that there is a nowhere vanishing function $f \in \mathscr{C}^\infty(\nu_N)$, such that $\Xi(l_p) = f(p)l_p$ for all $l_p \in L_\nu$. Moreover, we have that $\Xi\big|_N = \mathrm{id}_{L_\nu}\big|_N$, hence $f(0_n) = 1$ for all $n \in N$, and $\Xi^*\Delta_{\mathcal{E}} = \Delta_{\mathcal{E}}$. We consider now an arbitrary section $\lambda \in \Gamma^\infty(L_\nu)$ and compute

$$
\begin{aligned}
\Delta_{\mathcal{E}}(\lambda) &= (\Xi^*\Delta_{\mathcal{E}})(\lambda) \\
&= \Xi^*(\Delta_{\mathcal{E}}(\Xi_*\lambda)) \\
&= \frac{1}{f}(\Delta_{\mathcal{E}}(f\lambda)) \\
&= \frac{\mathcal{E}(f)}{f}\lambda + \Delta_{\mathcal{E}}(\lambda).
\end{aligned}
$$

Hence $\mathcal{E}(f) = 0$, which means that $f = \mathrm{pr}_\nu^* g$ for some function $g \in \mathscr{C}^\infty(N)$, but since $1 = f(0_n) = g(n)$ for all $n \in N$, we have that $\Xi = \mathrm{id}_{L_\nu}$. □

For a line bundle $L \to M$, a submanifold N and an Euler-like derivation $\Delta \in \Gamma^\infty(DL)$ with respect to N, we have that

$$
\Lambda_s := \Phi^\Delta_{\log(s)} \colon L_U \to L_U
$$

is well defined for $s > 0$ and can be extended smoothly to $s = 0$, where L_U is the target of the unique fat tubular neighbourhood $\Psi \colon L_\nu \to L_U$, such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$. Moreover, we have that

$$
\Lambda_s \circ \Psi = \Psi \circ P_s \tag{3.3}
$$

for all $s \geq 0$. Note that if we project this equation to the manifold level, this simply gives Eq. 3.2.

4 Normal forms of Dirac–Jacobi bundles

Using Euler-like derivations, we want to prove a normal form theorem for Dirac–Jacobi bundles. In fact, if the submanifold N is a transversal, then we can find special Euler like derivations which are, in some sense, controlling the behaviour of the Dirac–Jacobi bundles near N. The aim is now to prove the existence of this special kind of Euler-like derivations and afterwards, we are able to prove a normal form theorem and deduce some corollloraries from it.

Definition 4.1 Let $L \to M$ be a line bundle and let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle. A submanifold $N \hookrightarrow M$ is called transversal, if

$$
DL_N + \mathrm{pr}_D \mathfrak{L}\big|_N = (DL)\big|_N.
$$

Proposition 4.2 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle and let $N \hookrightarrow M$ be a transversal. Then*

$$
\mathfrak{B}_I(\mathfrak{L}) := \{(\Delta_p, (DI)^*\alpha_{\iota(p)}) \in \mathbb{D}L_1 \mid (DI(\Delta_p), \alpha_{\phi(p)}) \in \mathfrak{L}\}
$$

is a Dirac–Jacobi bundle, where $I\colon L_N \to L$ *is the canonical inclusion.*

Proof This is an easy consequence of Theorem 2.18. □

Lemma 4.3 *Let* $L \to M$ *be a line bundle, let* $\mathfrak{L} \subseteq \mathbb{D}L$ *be a Dirac–Jacobi bundle and let* $\iota\colon N \hookrightarrow M$ *be a transversal. The backwards transformation* $\mathfrak{B}_I(\mathfrak{L})$ *is canonically isomorphic (as vector bundles) to the fibred product* $I^!\mathfrak{L}$*, which is defined by the diagram*

$$\begin{array}{ccc} I^!\mathfrak{L} & \longrightarrow & \mathfrak{L} \\ \downarrow & & \downarrow \mathrm{pr}_D \\ DL_N & \xrightarrow{DI} & DL \end{array} .$$

Proof We consider the linear map

$$\Xi\colon I^!\mathfrak{L}_p \ni (\Delta_p, (\Box_{\iota(p)}, \alpha_{\iota(p)})) \mapsto (\Delta_p, DI^*\alpha_{\iota(p)}) \in \mathfrak{B}_I(\mathfrak{L}),$$

which is well-defined since $DI(\Delta_p) = \Box_{\iota(p)}$. We claim now that this map is injective, let us therefore consider $(\Delta_p, (\Box_{\iota(p)}, \alpha_{\iota(p)})) \in \ker(\Xi)$. It follows immediately, that $\Delta_p = 0$ and hence $\Box_{\iota(p)} = 0$. If $(0, \alpha_{\iota(p)}) \in \mathfrak{L}$ then $\alpha_{\iota_p} \in \mathrm{Ann}(\mathrm{pr}_D L)$. Since $DI^*\alpha_{\iota(p)} = 0$, we have that $\alpha_{\iota(p)} \in \mathrm{Ann}(DL_N)$, hence $\alpha_{\iota(p)} = 0$ and the claim follows. For dimensional reasons we have that Ξ is an isomorphism. □

Proposition 4.4 *Let* $L \to M$ *be a line bundle, let* $\mathfrak{L} \subseteq \mathbb{D}L$ *be a Dirac–Jacobi bundle and let* $N \hookrightarrow M$ *be a transversal. Then there exists* $\varepsilon \in \Gamma^\infty(\mathfrak{L})$*, such that* $\varepsilon\big|_N = 0$ *and* $\mathrm{pr}_D(\varepsilon)$ *is Euler-like.*

Proof We consider the exact sequence

$$0 \to \mathfrak{B}_I(\mathfrak{L}) \to \mathfrak{L}\big|_N \to \nu_N \to 0,$$

where the first arrow is defined by the identification $\mathfrak{B}_I(\mathfrak{L}) \cong I^!\mathfrak{L}$ from Lemma 4.3 followed by the canonical map $I^!\mathfrak{L} \to \mathfrak{L}$. The second arrow is the projection $\mathrm{pr}_D\colon \mathfrak{L}\big|_N \to DL\big|_N$ followed by the symbol map $\sigma : DL\big|_N \to TM\big|_N$and finally followed by the the projection to the normal bundle $\mathrm{pr}_{\nu_N} : TM\big|_N \to \nu_N$. This map is surjective, because N is a transversal. Let us choose a section $\varepsilon \in \Gamma^\infty(\mathfrak{L})$ with $\varepsilon\big|_N = 0$, such that $\mathrm{d}^N\varepsilon\colon \nu_N \to \mathfrak{L}\big|_N$ defines a splitting of the sequence. We consider now

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^!\mathfrak{L} & \longrightarrow & \mathfrak{L}\big|_N & \longrightarrow & \nu_N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & TN & \longrightarrow & TM\big|_N & \longrightarrow & \nu_N & \longrightarrow & 0 \end{array}$$

and see that if $\mathrm{d}^N\varepsilon$ splits the above sequence then $(\sigma \circ \mathrm{pr}_D)\,\mathrm{d}^N\varepsilon$ splits the lower sequence. Using Proposition 3.2, we see that $(\sigma \circ \mathrm{pr}_D)\,\mathrm{d}^N\varepsilon = \mathrm{d}^N((\sigma \circ \mathrm{pr}_D)(\varepsilon))$ and by Proposition 3.8, we see that $T\nu(\sigma \circ \mathrm{pr}_D)(\varepsilon) = \mathcal{E}$. Multiplying ε by a suitable bump function we may arrange that $(\sigma \circ \mathrm{pr}_D)(\varepsilon)$ is complete and hence an Euler-like vector field. By definition $\mathrm{pr}_D(\varepsilon)$ is hence an Euler-like derivation. □

Let us fix now a Dirac–Jacobi structure $\mathfrak{L} \subseteq \mathbb{D}L$ for a line bundle $L \to M$. Let us also consider a transversal $\iota\colon N \hookrightarrow M$ and a section $\varepsilon = (\Delta, \alpha) \in \Gamma^\infty(\mathfrak{L})$, such that $\varepsilon\big|_N = 0$ and Δ is an Euler-like derivation. Due to the Lemma 3.14, we find a unique fat tubular neighbourhood

$$\begin{array}{ccc} L_\nu & \xrightarrow{\Psi} & L_U \\ \downarrow & & \downarrow \\ \nu_N & \xrightarrow{\psi} & U \end{array}$$

such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$. With this we have now two ways to construct a Dirac–Jacobi bundle on $L_\nu \to \nu_N$, namely we can take the backwards transformation $\mathfrak{B}_\Psi(\mathfrak{L}_U)$ and, if we consider

$$\begin{array}{ccccc} L_\nu & \xrightarrow{P} & L_N & \xrightarrow{I} & L \\ \downarrow & & \downarrow & & \downarrow \\ \nu_N & \longrightarrow & N & \longrightarrow & M \end{array},$$

we can take the backwards transformation $\mathfrak{B}_{I\circ P}(\mathfrak{L}) = \mathfrak{B}_P(\mathfrak{B}_I(\mathfrak{L}))$. The aim is now to compare these two structures. Let us therefore consider the flow of $[\![(\Delta,\alpha), -]\!]$, which is given by

$$(\gamma_t, \Phi_t^\Delta) \in Z_L^2(M) \rtimes \operatorname{Aut}(L),$$

where Φ_t^Δ is the flow of Δ and $\gamma_t = \int_0^t (\Phi_{-\tau}^\Delta)^* \,\mathrm{d}_L\alpha \,\mathrm{d}\tau$ by Corollary 2.10. For sure we have that the action of $(\gamma_t, \Phi_t^\Delta)$ preserves $\mathfrak{L}$, which is explicitly

$$\exp(\gamma_t) \circ \mathbb{D}\Phi_t^\Delta(\mathfrak{L}) = \mathfrak{L}.$$

This leads us directly to the following theorem.

Theorem 4.5 (Normal form for Dirac–Jacobi bundles) *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi bundle and let $N \hookrightarrow M$ be a transversal. Then there exists an open neighbourhood $U \subseteq M$ of N and a fat tubular neighbourhood $\Psi\colon L_\nu \to L_U$, such that*

$$\mathfrak{B}_\Psi(\mathfrak{L}\big|_U) = (\mathfrak{B}_{I\circ P}(\mathfrak{L}))^\omega$$

for an $\omega \in \Omega^2_{L_\nu}(\nu_N)$.

Proof According to Proposition 4.4, we can find $(\Delta,\alpha) \in \Gamma^\infty(\mathfrak{L})$, such that Δ is Euler-like. Then there is a unique fat tubular neighbourhood $\Psi\colon L_\nu \to L_U$, such that $\Psi^*\Delta = \Delta_{\mathcal{E}}$, due to Lemma 3.14. Let us denote by $(\gamma_t, \Phi_t^\Delta) \in Z_L^2(M) \rtimes \operatorname{Aut}(L)$ the flow of $[\![(\Delta,\alpha), -]\!]$. We know that $(\gamma_t, \Phi_t^\Delta)$ preserves $\mathfrak{L}$ for all $t \in \mathbb{R}$ and so will $(\gamma_{-\log(s)}, \Phi^\Delta_{-\log(s)})$ for all $s > 0$. Let us take a closer look at

$$\begin{aligned} \gamma_{-\log(s)} &= \int_0^{-\log(s)} (\Phi_{-\tau}^\Delta)^* \,\mathrm{d}_L\alpha \,\mathrm{d}\tau \\ &= \int_{-\log(1)}^{-\log(s)} (\Phi_{-\tau}^\Delta)^* \,\mathrm{d}_L\alpha \,\mathrm{d}\tau \\ &= \int_s^1 \frac{1}{t} (\Phi^\Delta_{\log(t)})^* \,\mathrm{d}_L\alpha \,\mathrm{d}t \end{aligned}$$

and we obtain that it is smoothly extendable to $s = 0$. Let us denote its limit $s \to 0$ by ω' and $\omega = \Psi^*\omega'$. We have

$$\begin{aligned}
\mathfrak{B}_\Psi(\mathfrak{L}\big|_U) &= \mathfrak{B}_\Psi(\exp(\gamma_{-\log(s)}) \circ \mathbb{D}\Phi^\Delta_{-\log(s)}(\mathfrak{L}))\\
&= \mathfrak{B}_\Psi(\exp(\gamma_{-\log(s)})\mathfrak{B}_{\Phi^\Delta_{\log(s)}}(\mathfrak{L}))\\
&= (\mathfrak{B}_\Psi(\mathfrak{B}_{\Lambda_s}(\mathfrak{L}))^{\Psi^*\gamma_{-\log(s)}}\\
&= (\mathfrak{B}_{\Lambda_s\circ\Psi}(\mathfrak{L}))^{\Psi^*\gamma_{-\log(s)}}\\
&= (\mathfrak{B}_{\Psi\circ P_s}(\mathfrak{L}))^{\Psi^*\gamma_{-\log(s)}}.
\end{aligned}$$

which holds for all $s \geq 0$. Here we used Remark 2.17 to identify Courant-Jacobi automorphisms with backwards transformations as well as Eq. 3.3. Hence, using that for the canonical inclusion $J\colon L_N \to L_\nu$ we have that $P_0 = J \circ P$ and $\Psi \circ J = I$, we get

$$\mathfrak{B}_\Psi(\mathfrak{L}\big|_U) = (\mathfrak{B}_{I\circ P}(\mathfrak{L}))^\omega$$

for $s = 0$. □

Note that this theorem says that, up to a B-field, the Dirac–Jacobi structure is fully encoded in a given transversal, and hence the term "normal form" is justified by this fact. Moreover, it is possible to distinguish two different kinds of leaves in Dirac–Jacobi geometry, see [14], so it is also possible to distinguish two kinds of transversals, which are more interesting in the Jacobi setting, since in the general Dirac–Jacobi setting the normal forms will be the same. Nevertheless, we will introduce them here and use them more extensively in the next section.

Definition 4.6 (Cosymplectic Transversal) Let $L \to M$ be a line bundle and let $\mathfrak{L} \in \mathbb{D}L$ be a Dirac–Jacobi structure. A transversal $\iota\colon N \hookrightarrow M$ is called cosymplectic, if

$$DL_N \cap \mathfrak{B}_I(\mathfrak{L}) = \{0\}.$$

Remark 4.7 The term *cosymplectic* is already occupied in the literature: it is corank one Poisson manifold with some properties. Since there is no possible risk of confusion in this note, we use this term as short for transversal to a locally conformal pre-symplectic leaf. Note that a cosymplectic transversal always inherts a Dirac–Jacobi bundle coming from a Jacobi tensor by Proposition 2.13. So let us denote $\mathfrak{L}_{J_N} = \mathfrak{B}_I(\mathfrak{L}_J) \subseteq \mathbb{D}L_N$.

These transversals naturally appear as minimal transversals to locally conformal pre-symplectic leaves, i.e. submanifolds of minimal dimension intersecting the leaf transversally, see [14] for a more detailed discussion.

Using the normal form theorem, we get in the case of cosymplectic transversals:

Corollary 4.8 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a minimal transversal to $\mathfrak{L}$ at a point p_0 in a locally conformal pre-symplectic leaf, i.e. $\sigma(\mathrm{pr}_D(\mathfrak{L}))\big|_{p_0} \oplus T_{p_0}N = T_{p_0}M$ and let $\nu_N = V \times N$ be trivializable and trivialized. Then locally around p_0:*

$$\mathfrak{B}_\Psi(\mathfrak{L}\big|_U) = \{(v + J_N^\sharp(\psi), \alpha + \psi) \in DL_\nu \mid v \in TV, \alpha \in (\mathrm{Ann}(T^*V)) \otimes L_\nu \text{ and } \psi \in J^1L_N\}^\omega$$

where J_N is the Jacobi structure on the transversal and the canonical identification $DL_{\nu_N} = TV \oplus DL_N$.

Proof First, we note that a minimal transversal to a leaf is always a transversal as in Definition 4.1 and that for a minimal transversal N at a locally conformal pre-symplectic point p_0, i.e. a point in a locally conformal pre-symplectic leaf, we have the equation

$$DL_N \cap \mathfrak{B}_I(\mathfrak{L}) = \{0\}$$

at p_0 and hence in a whole neighborhood. The rest is an application of Theorem 4.5 and the usage of the splitting $DL_{\nu_N} = TV \oplus DL_N$, since $TV = \ker(Tp)$ using the discussion in front of Proposition 3.13. □

The other kind of leaves of a Dirac–Jacobi structure are the so-called pre-contact leaves. Their minimal transversals possess the following structure:

Definition 4.9 (Cocontact Transversal) Let $L \to M$ be a line bundle and let $\mathfrak{L} \in \mathbb{D}L$ be a Dirac–Jacobi structure. A transversal $\iota\colon N \hookrightarrow M$ is called cocontact, if

$$\operatorname{rank}(DL_N \cap \mathfrak{B}_I(\mathfrak{L})) = 1.$$

Lemma 4.10 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a minimal transversal to $\mathfrak{L}$ at a pre-contact point p_0. Then*

$$\operatorname{rank}(DL_N \cap \mathfrak{B}_I(\mathfrak{L})) = 1$$

holds in a neighbourhood of p_0.

Proof Recall that a minimal transversal at p_0 is a transversal of minimal dimension, which in particular implies that

$$\sigma(\operatorname{pr}_D(\mathfrak{L}))\big|_{p_0} \oplus T_{p_0}N = T_{p_0}M.$$

It is easy to see that

$$(DL_N \cap \mathfrak{B}_I(\mathfrak{L}))\big|_{p_0} = \langle \mathbb{1}_{p_0} \rangle,$$

which follows because N is minimal and p_0 is a pre-contact point, i.e. $\mathbb{1}_{p_0} \in \operatorname{pr}_D \mathfrak{L}$. To be more precise, by using the pre-contact property of p_0 and the minimality of N, we see $(\operatorname{pr}_D \mathfrak{L} \cap DL_N)\big|_{p_0} = \langle \mathbb{1}_{p_0} \rangle$ and hence there is $\alpha \in J^1_{p_0}L$, such that $(\mathbb{1}_{p_0}, \alpha) \in \mathfrak{L}$. Let us define $\beta \in J^1_{p_0}L$ by

$$\beta(\Delta) = 0 \text{ for } \Delta \in \operatorname{pr}_D \mathfrak{L}$$

and

$$\beta(\Delta) = \alpha(\Delta) \text{ for } \Delta \in DL_N.$$

Then β is well-defined, since $\operatorname{pr}_D \mathfrak{L} \cap DL_N = \mathbb{1}$ and $\alpha(\mathbb{1}) = 0$ and moreover $(0, \beta) \in \mathfrak{L}$, since $\langle\!\langle (0, \beta), \mathfrak{L} \rangle\!\rangle = 0$ and $\mathfrak{L}$ is maximal isotropic. We consider now the element $(\mathbb{1}_{p_0}, \alpha - \beta) \in \mathfrak{L}$, thus $(\mathbb{1}_{p_0}, DI^*(\alpha - \beta)) = (\mathbb{1}_{p_0}, 0) \in \mathfrak{B}_I(\mathfrak{L})$. Moreover, since $\operatorname{pr}_D \mathfrak{L} \cap DL_N = \mathbb{1}$, we conclude $DL_N \cap \mathfrak{B}_I(\mathfrak{L}) = \langle \mathbb{1}_{p_0} \rangle$ and hence $\operatorname{rank}(DL_N \cap \mathfrak{B}_I(\mathfrak{L}))\big|_{p_0} = 1$.

Now we want to argue why this holds in a whole neighbourhood. Let us therefore consider the sum $DL_N + \mathfrak{B}_I(\mathfrak{L}) \subseteq \mathbb{D}L$ and a (local) section $\alpha \in \Omega^1_L(M)$ such that $\alpha(\mathbb{1})\big|_{p_0} \neq 0$. Let $(0, \beta) \in \big(DL_N + \mathfrak{B}_I(\mathfrak{L})\big)\big|_{p_0} \cap \langle \alpha \rangle\big|_{p_0}$, then there exists $\Delta \in D_{p_0}L$ such that $(\Delta, \beta) \in \mathfrak{B}_I(\mathfrak{L})$, but since $(\mathbb{1}, 0) \in \mathfrak{B}_I(\mathfrak{L})$, using the isotropy of $\mathfrak{B}_I(\mathfrak{L})$, we have

$$0 = \langle\!\langle (\Delta, \beta), (\mathbb{1}, 0) \rangle\!\rangle = \beta(\mathbb{1}),$$

but $\beta = k\alpha$ for $k \in \mathbb{R}$, we conclude $k = 0$ and thus $\beta = 0$ and therefore $\big(DL_N + \mathfrak{B}_I(\mathfrak{L})\big)\big|_{p_0} \cap \langle\alpha\rangle\big|_{p_0} = \{0\}$. For dimensional reasons we conclude $\mathbb{D}L\big|_{p_0} = (DL_N + \mathfrak{B}_I(\mathfrak{L}))\big|_{p_0} \oplus \langle\alpha\rangle\big|_{p_0}$. Therefore this equality holds in a whole neighbourhood of p_0, so $\operatorname{rank}(DL_N + \mathfrak{B}_I(\mathfrak{L})) = 2n + 1$ in this neighbourhood, which implies $\operatorname{rank}(DL_N \cap \mathfrak{B}_I(\mathfrak{L})) = 1$ around p_0. □

Remark 4.11 Note that a cocontact transversal does not inherit a Jacobi structure, but the pulled-back Dirac–Jacobi is of homogeneous Poisson type (see Definition 2.14).

Definition 4.12 Let $L \to M$ be a line bundle and let $\mathfrak{L} \in \mathbb{D}L$ be a Dirac–Jacobi structure. A homogeneous cocontact transversal $\iota\colon N \hookrightarrow M$ is a cocontact transversal together with a flat connection $\nabla\colon TN \to DL_N$, such that

$$\operatorname{im}(\nabla) \oplus (DL_N \cap \mathfrak{B}_I(\mathfrak{L})) = DL_N.$$

Remark 4.13 The definition of a homogeneous cocontact transversal seems a bit strange, since it includes a flat connection. This fact can be explained quite easily using the homogenization described in [14], which turns a Dirac–Jacobi structure on a line bundle $L \to M$ into a Dirac structure on $L^\times := L^*\backslash\{0_M\}$ which is homogeneous (in the sense of [14]) with respect to the restricted Euler vector field $\mathcal{E}$ on L^*. The pre-symplectic leaves of this Dirac structure have the additional property that $\mathcal{E}$ is either tangential to it at very point or nowhere tangential to it. If $\mathcal{E}$ is tangential, then the leaf corresponds to a pre-contact leaf on the base M. Hence a minimal transversal N to it is nowhere tangential to the Euler vector field and defines therefore a horizontal bundle on $L^*_{\operatorname{pr}(N)}$ and hence a flat connection. On the other hand, given a minimal transversal to the projected leaf on M, we need a flat connection to lift it to a minimal transversal to the leaf on $L^\times$. So a homogeneous cocontact transversal to a leaf of a Dirac–Jacobi structure is equivalent to a transversal of a leaf of its homogenization. Moreover, if we (locally) choose the horizontal bundle to be integrable, we have a flat connection. But we stress that in both cases, the homogenization and the line bundle point of view, this is an additional property one has to impose on the transversal.

Proposition 4.14 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a minimal transversal to $\mathfrak{L}$ at a pre-contact point p_0. Then every flat connection ∇ locally gives to N the structure of a homogeneous cocontact transversal.*

Proof In the proof of Lemma 4.10, we have seen that

$$(DL_N \cap \mathfrak{B}_I(\mathfrak{L}))\big|_{p_0} = \langle \mathbb{1} \rangle$$

and hence for every flat connection ∇, we have that $\operatorname{im}(\nabla)\big|_{p_0} \oplus (DL_N \cap \mathfrak{B}_I(\mathfrak{L}))\big|_{p_0} = DL_N$ and hence this decomposition holds in a whole neighborhood of p_0. □

An immediate consequence is:

Corollary 4.15 *Let $L \to M$ be a line bundle, let $\mathfrak{L} \subseteq \mathbb{D}L$ be a Dirac–Jacobi structure and let $\iota\colon N \hookrightarrow M$ be a homogeneous cocontact transversal with connection ∇. Then there exists a local trivialization of ν such that*

$$\mathfrak{B}_\Psi(\mathfrak{L}\big|_U) = \{(v + r(\mathbb{1} - Z_N) + \pi^\sharp(\psi), \alpha + \psi + \psi(Z_N)\mathbb{1}^*) \mid v \in TV, \alpha \in \operatorname{Ann}(T^*V) \text{ and } \psi \in T^*N\}^\omega,$$

where we used the trivializations $DL_\nu = T\nu \oplus \mathbb{R}_M$ and $J^1L = T^*M \oplus \mathbb{R}_M$ corresponding to ∇ as well as $L\big|_U \cong L_\nu$ and the homogeneous Poisson (π_N, Z_N) structure on the transversal from Lemma 2.15.

This last two corollaries can be seen as the Jacobi-geometric analogue of the results obtained by Blohmann in [1, Theorem 3.2, Corollary 3.6].

5 Normal forms and splitting Theorems of Jacobi bundles

As explained in Proposition 2.13, Jacobi bundles are a special kind of Dirac–Jacobi structures. In addition, we have that Jacobi isomorphism induces an isomorphism of the corresponding Dirac structures (this holds even for morphisms if one considers forward maps of Dirac–Jacobi structures which we will not explain here, see [14]). The converse is unfortunately not true: if the Dirac–Jacobi structures of two Jacobi structures are isomorphic, it does not follow in general that the Jacobi structures are isomorphic. The parts which are not "allowed" in Jacobi geometry are the B-fields. Nevertheless, we can keep track of them, if we make further assumptions on the transversals, namely cosymplectic and cocontact transversals.

5.1 Cosymplectic transversals of Jacobi structures

In this part, we are using the notion of cosymplectic transversals as explained in the previous section. The difference is now that in Jacobi geometry this transversal gives us more than on arbitrary Dirac–Jacobi manifolds. In fact, the Jacobi structure induces a line bundle valued symplectic structure on the normal bundle, to be seen in the following

Lemma 5.1 *Let $L \to M$ be a line bundle, $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi tensor with corresponding Dirac–Jacobi structure $\mathfrak{L}_J \in \mathbb{D}L$ and let $\iota\colon N \hookrightarrow M$ be a cosymplectic transversal. Then*

$$J^\sharp(\operatorname{Ann}(DL_N)) \oplus DL_N = DL\big|_N.$$

Proof First we prove that $J^\sharp\big|_{\operatorname{Ann}(DL_N)}$ is injective. Let therefore $\alpha \in \operatorname{Ann}(DL_N)$ be such that $J^\sharp(\alpha) = 0$. Hence, for an arbitrary $\beta \in J^1L$ we have that

$$\alpha(J^\sharp(\beta)) = -\beta(J^\sharp(\alpha)) = 0.$$

Hence, $\alpha = \operatorname{Ann}(DL_N) \cap \operatorname{Ann}(\operatorname{im}(J^\sharp)) = \operatorname{Ann}(DL_N) \cap \operatorname{Ann}(\operatorname{pr}_D \mathfrak{L}_J) = \operatorname{Ann}(DL_N + \operatorname{pr}_D \mathfrak{L}_J) = \{0\}$, and $J^\sharp\big|_{\operatorname{Ann}(DL_N)}$ is injective. Let $\Delta \in DL_N \cap J^\sharp(\operatorname{Ann}(DL_N))$, then there exists an $\alpha \in \operatorname{Ann}(DL_N)$, such that $J^\sharp(\alpha) = \Delta$. Thus, we have that $(\Delta, \alpha) \in \mathfrak{L}_J$ and moreover $(\Delta, DI^*\alpha) \in \mathfrak{B}_I(\mathfrak{L}_J)$, but since $\alpha \in \operatorname{Ann}(DL_N)$, we have that $DI^*\alpha = 0$ and hence $\Delta = 0$, since N is cosymplectic. The claim follows counting dimensions. □

Suppose that $\iota\colon N \hookrightarrow M$ is a cosymplectic transversal, then we have that

$$\operatorname{pr}_\nu \circ \sigma \circ J^\sharp \colon \operatorname{Ann}(DL_N) \to \nu_N$$

is an isomorphism. Let us chose $\alpha \in \Gamma^\infty(J^1L)$, such that $\alpha\big|_N = 0$ and such that $\mathrm{d}^N\alpha\colon \nu_N \to \operatorname{Ann}(DL_N) \subseteq J^1 L\big|_N$ is a right-inverse to $\operatorname{pr}_\nu \circ \sigma \circ J^\sharp$. We then have

$$\operatorname{pr}_\nu(\mathrm{d}^N \sigma(J^\sharp(\alpha))) = \operatorname{pr}_\nu(\sigma(J^\sharp(\mathrm{d}^N\alpha))) = \mathrm{id}_{\nu_N}$$

and hence we have that $T\nu(\sigma(J^\sharp(\alpha))) = \mathcal{E}$. Multiplying α by a bump-function, which is 1 near N, we may arrange that $\sigma(J^\sharp(\alpha))$ is complete and hence $J^\sharp(\alpha)$ is an Euler-like derivation. By Theorem 4.5 and the definition of the Jacobi structure J_N on the transversal (see Remark 4.7), we have that

$$\mathfrak{B}_\Psi(\mathfrak{L}_J) = \mathfrak{B}_{P\circ I}(\mathfrak{L})^\omega = \mathfrak{B}_P(\mathfrak{L}_{J_N})^\omega,$$

where $\omega = \Psi^* \int_0^1 \frac{1}{t}(\Phi^\Delta_{\log(t)})^*(\mathrm{d}_L\alpha)\,\mathrm{d}t$ and $\Psi\colon L_\nu \to L_U$ is the unique tubular neighborhood, such that $\Psi^*(J^\sharp(\alpha)) = \Delta_{\mathcal{E}}$.

Proposition 5.2 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ evaluated along N has kernel DL_N.*

Proof One can show, in local coordinates, that $\mathrm{d}^N\alpha([\sigma(\Box)\big|_N]) = \mathscr{L}_\Box\alpha\big|_N$ for all $\Box \in \Gamma^\infty(DL)$. Hence we have trivially $\mathscr{L}_\Delta\alpha\big|_N = 0$ for $\Delta \in \Gamma^\infty(DL)$, such that $\Delta\big|_N \in \Gamma^\infty(DL_N)$. Let now $\Delta, \Box \in \Gamma^\infty(DL)$ be such that $\Delta\big|_N \in \Gamma^\infty(DL_N)$, then

$$\begin{aligned}
\mathrm{d}_L\alpha(\Delta, \Box)\big|_N &= -(\mathrm{d}_L\iota_\Delta\alpha)(\Box)\big|_N = -\Box(\alpha(\Delta))\big|_N \\
&= -(\mathscr{L}_\Box\alpha)(\Delta)\big|_N - \alpha([\Box, \Delta])\big|_N = -(\mathscr{L}_\Box\alpha)\big|_N(\Delta) \\
&= \mathrm{d}^N\alpha([\sigma(\Box)\big|_N])(\Delta) \\
&= 0,
\end{aligned}$$

where the last equality follows since $\mathrm{d}^N\alpha\colon \nu_N \to \operatorname{Ann}(DL_N)$. Hence we have that $\ker((\mathrm{d}_L\alpha)^\flat) \supseteq DL_N$, in particular this is true for $\frac{1}{t}(\Phi^\Delta_{\log(t)})^*(\mathrm{d}_L\alpha)$, since $\Phi_{\log(s)}\big|_N$ is a gauge transformation fixing DL_N. Thus it is true also for ω, since $D\Psi\big|_{DL_N} = \mathrm{id}$. The equality follows from the fact that $\mathrm{d}^N\alpha$ is chosen to be injective, since it has a left-inverse. □

We want to describe the structure of ω at N. Note that, for a cosymplectic transversal N, the normal bundle always comes together with a canonical symplectic (i.e. non-degenerate) L_N-valued 2-form $\Theta \in \Gamma^\infty(\Lambda^2\nu_N^* \otimes L_N)$ defined by

$$\Theta(X, Y) = (\mathrm{pr}_\nu \circ \sigma \circ J^\sharp\big|_{\operatorname{Ann}(DL_N)})^{-1}(X)(Y)$$

Lemma 5.3 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ coincides, restricted to $\nu_N \subseteq DL_{\nu_N}$, with Θ.*

Proof Note that for a cosymplectic transversal, we have

$$DL\big|_N = DL_N \oplus J^\sharp(\operatorname{Ann}(DL_N)) = DL_N \oplus \nu_N$$

with the canonical identification

$$J^\sharp(\operatorname{Ann}(DL_N)) = \frac{DL\big|_N}{DL_N} = \nu_N$$

where the last equality is the identification via the symbol $\sigma\colon DL \to TM$. Moreover, we have

$$DL_\nu\big|_N = DL_N \oplus \nu_N,$$

where we include ν_N by the following map:

$$\chi\colon \nu_N \ni v_n \mapsto \left(\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_0\lambda(\kappa_t(v_p))\right) \in D_nL_\nu.$$

It is clear that $D\Psi$ fixes DL_N, since $\Psi\big|_N\colon L_N \to L_N$ is the identity. We want to show that $D\Psi(\nu_N) \subseteq J^\sharp(\mathrm{Ann}(DL_N))$. Let $v_n \in \nu_N$ and $s \in \Gamma^\infty(L_U)$, then

$$
\begin{aligned}
D\Psi(\chi(v_n))(s) = \Psi(\chi(v_n)\Psi^* s) &= \Psi \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_0(\Psi^* s(\kappa_t(v_n))) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Psi P_0(\Psi^{-1} s(\psi(\kappa_t(v_n))) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Lambda_0(s(\psi(\kappa_t(v_n))) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Lambda_0(\Lambda_{\frac{1}{t}} s(\lambda_t(\psi(v_n))) \\
&= (\frac{\mathrm{d}}{\mathrm{d}t} \Lambda_0((\Phi^\Delta_{log(t)})^* s(\psi(v_n)))\big|_{t=0} \\
&= (\frac{1}{t} \Lambda_0((\Phi^\Delta_{log(t)})^* \Delta(s)(\psi(v_n))))\big|_{t=0} \\
&= \lim_{t\to 0} \frac{1}{t} \Lambda_0(\Lambda_{\frac{1}{t}} \Delta_{\lambda_t(\psi(v_n)}(s)) \\
&= \lim_{t\to 0} \frac{1}{t} \Lambda_0(\Delta_{\lambda_t(\psi(v_n))}(s))
\end{aligned}
$$

where we used Eqs. 3.2 and 3.3 as well as the definition of Λ_t and its relation to the flow of Δ, see Proposition 3.10. The equality

$$
D\Psi(\chi(v_n)) = \lim_{t\to 0} \frac{\Delta_{\lambda_t(\psi(v_n))}}{t}
$$

follows from the fact that $\Lambda_0\big|_N = \mathrm{id}$.

But, by definition, we have that

$$
\mathrm{d}^N \Delta(v_n) = \lim_{t\to 0} \frac{\Delta_{\lambda_t(\psi(v_n))}}{t}
$$

hence $D\Psi \circ \chi = \mathrm{d}^N\Delta = J^\sharp \circ \mathrm{d}^N\alpha$, but α was chosen in such a way that $\mathrm{d}^N\alpha$ takes values in $\mathrm{Ann}(DL_N)$. Thus $D\Psi\big|_N$ respects the splitting. Using this,

$$
\mathfrak{B}_\Psi(\mathfrak{L}_J) = \mathfrak{B}_P(\mathfrak{L}_{J_N})^\omega,
$$

$\ker(\omega^\flat)\big|_N = DL_N$ and the definition of Θ, we see that they have to coincide at N. □

This leads us to the normal form theorem for Jacobi manifolds.

Theorem 5.4 (Normal Form for Jacobi bundles I) *Let $L \to M$ be a line bundle, let J be a Jacobi structure and let $N \to M$ be a cosymplectic transversal. For a closed 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$, such that* $\ker(\omega^\flat)\big|_N = DL_N$ *and ω coincides with Θ at $\nu_N \subseteq DL_\nu$, the Dirac–Jacobi structure*

$$
\mathfrak{B}_P(\mathfrak{L}_{J_N})^\omega
$$

is the graph of a Jacobi structure near the zero section and there exists a fat tubular neighbourhood $\Psi\colon L_\nu \to L_U$ which is a Jacobi map near the zero section.

Proof The theorem is true for

$$
\omega = \int_0^1 \frac{1}{t} (\Phi^\Delta_{\log(t)})^* \,\mathrm{d}_L\alpha \,\mathrm{d}t
$$

due to Theorem 4.5 with $\alpha \in \Gamma^\infty(J^1 L)$ chosen as in the discussion before, which ensures that $\ker(\omega^\flat)\big|_N = DL_N$ and ω coincides with Θ at $\nu_N \subseteq DL_\nu$. Let ω' be a second 2-form fulfilling these requirements, then

$$\sigma_t := t(\omega' - \omega)$$

is a (time-dependent) 2-form such that $\sigma_0 = 0$ and moreover $\sigma_t\big|_N = 0$. Thus,

$$(\mathfrak{B}_P(\mathfrak{L}_{J_N})^\omega)^{\sigma_t} = \mathfrak{B}_P(\mathfrak{L}_{J_N})^{\omega+\sigma_t}$$

is a Jacobi structure near N, since $\sigma_t\big|_N = 0$ and hence the condition $DL_\nu \cap \mathfrak{B}_P(\mathfrak{L}_{J_N})^{\omega+\sigma_t} = \{0\}$ is fulfilled along N and thus in a open neighbourhood of N. This is equivalent to $\mathfrak{B}_P(\mathfrak{L}_{J_N})^{\omega+\sigma_t}$ being a Jacobi structure by Proposition 2.13. Now we can apply Appendix A to get the result. □

An immediate consequence of this theorem is the Splitting Theorem for Jacobi manifolds around a locally conformal symplectic leaf, proven by Dazord, Lichnerowicz and Marle in [5].

Theorem 5.5 *Let $L \to M$ be a line bundle, let $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi tensor and let $p_0 \in M$ be a locally conformal symplectic point. Then there are a line bundle trivialization $L_U \cong U \times \mathbb{R}$ around p_0 and a minimal cosymplectic transversal $N \hookrightarrow U$, such that $U \cong U_{2q} \times N$ for an open subset $0 \in U_{2q} \subseteq \mathbb{R}^{2q}$ and the corresponding Jacobi pair (Λ, E) is transformed (via this isomorphism) to*

$$(\Lambda, E) = (\pi_{\mathrm{can}} + \Lambda_N + E_N \wedge Z_{\mathrm{can}}, E_N),$$

where (Λ_N, E_N) is the induced Jacobi structure on the transversal N and the canonical stuctures on the fiber are given by $(\pi_{\mathrm{can}}, Z_{\mathrm{can}}) = (\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, p_i \frac{\partial}{\partial p_i})$.

Proof We can assume from the beginning that the line bundle is trivial, since otherwise we can trivialize around p_0 and and restrict the line bundle to this open neighbourhood. Let us choose an minimal transversal N to the leaf S at p_0 (in the sense, that $S \times N = M$ holds locally around p_0). It is easy to see that

$$(DL_N \cap \mathfrak{B}_I(\mathfrak{L}_J))\big|_p = \{0\},$$

and hence we can restrict to an open neighburhood of p_0, where this equality holds. This means that every minimal transversal to a leaf is a cosymplectic transversal near the intersection point. Let us from now on denote $p_0 = (s_0, n_0)$, hence $\nu_N \cong T_{s_0}S \times N \cong \mathbb{R}^{2k} \times N$. Since the line bundle is trivial, we can identify ν_N together with Θ as a symplectic vector bundle, hence we find a possible smaller N and a vector bundle automorphism of ν_N, such that Θ is the constant symplectic form. We can now choose

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i - \mathbb{1}^* \wedge p_i \,\mathrm{d}q^i \in \Omega_{L_\nu}(\nu_N)$$

where (q, p) are the symplectic coordinates on $\nu_N \to N$. This 2-from is d_L-closed and coincides with Θ on N, moreover $\ker(\omega^\flat)\big|_N = DL_N$. Hence the requirements of Theorem 5.4 are fulfilled and the claim follows by an easy computation. □

5.2 Cocontact transversals of Jacobi structures

The second kind of transversals we want to discuss in the context of Jacobi geometry are cocontact transversals, which were also introduced before in Definition 4.9. In fact this notion is not enough for our purposes and we need to assume more information on the structure of the transversal, which is precisely the notion of homogeneous cocontact transversal from Definition 4.9.

Lemma 5.6 *Let $L \to M$ be a line bundle, $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi tensor with corresponding Dirac–Jacobi structure $\mathfrak{L}_J \in \mathbb{D}L$ and let $\iota\colon N \hookrightarrow M$ be a homogeneous cocontact transversal with connection $\nabla\colon TN \to DL_N$. Then*

$$J^\sharp(\mathrm{Ann}(\mathrm{im}(\nabla))) \oplus \mathrm{im}(\nabla) = DL\big|_N.$$

Proof The proof follows the same lines as Lemma 5.1. □

Now, as in the cosymplectic case, we pick an $\alpha \in \Gamma^\infty(J^1 L)$, such that $\alpha\big|_N = 0$ and

$$\mathrm{d}^N\alpha\colon \nu_N \to \mathrm{Ann}(\mathrm{im}(\nabla)) \subseteq J^1 L\big|_N$$

defines a splitting of $I^!\mathfrak{L} \to \mathfrak{L}\big|_N \to \nu_N$, i.e. $\mathrm{pr}_\nu \circ \sigma \circ J^\sharp \circ \mathrm{d}^N\alpha = \mathrm{id}_{\nu_N}$. Hence we have that $J^\sharp(\alpha)$, multiplied by a suitable bump function which is 1 close to N, is an Euler-like derivation. By Theorem 4.5, we have that

$$\mathfrak{B}_\Psi(\mathfrak{L}_J) = \mathfrak{B}_P(\mathfrak{B}_I(\mathfrak{L}))^\omega,$$

where $\omega = \Psi^* \int_0^1 \frac{1}{t}(\Phi^\Delta_{\log(t)})^*(\mathrm{d}_L\alpha)\,\mathrm{d}t$ and $\Psi\colon L_\nu \to L_U$ is the unique tubular neighbourhood, such that $\Psi^*(J^\sharp(\alpha)) = \Delta_{\mathcal{E}}$. We can prove, as before, the following

Proposition 5.7 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ restricted to N has kernel* $\mathrm{im}(\nabla)$.

Proof This proof follows the same lines as the proof of Proposition 5.2. □

As in the cosymplectic transversal case, we can define a skew symmetric 2-form

$$\Theta \in \Gamma^\infty(\Lambda^2 J^\sharp(\mathrm{Ann}(\mathrm{im}(\nabla)) \otimes L_N)$$

by

$$\Theta(X, Y) = (J^\sharp\big|_{\mathrm{Ann}(\mathrm{im}(\nabla))})^{-1}(X)(Y).$$

It is easy to see that Θ is non-degenerate. Moreover, we have

Lemma 5.8 *The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ coincides, restricted to $\nu_N \oplus K \subseteq DL_{\nu_N}$, with Θ, where we denote $K := (DL_N \cap \mathfrak{B}_I(\mathfrak{L}_J))$.*

Proof Using the ideas of the proof of Lemma 5.3, we can show that the fat tubular neighbourhood transports $J^\sharp(\mathrm{Ann}(\mathrm{im}(\nabla))$ to $\nu_N \oplus K$, hence the proof is copy and paste of this Lemma. □

Theorem 5.9 (Normal Form for Jacobi bundles II) *Let $L \to M$ be a line bundle, let J be a Jacobi structure and let $N \to M$ be a homogenous cocontact transversal with connection $\nabla\colon TN \to DL_N$. For a closed 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$, such that* $\ker(\omega^\flat)\big|_N = \mathrm{im}(\nabla)$ *and ω coincides with Θ at $\nu_N \oplus (\mathfrak{B}_I(\mathfrak{L}_J) \cap DL_N) \subseteq DL_\nu$, the Dirac–Jacobi structure*

$$\mathfrak{B}_P(\mathfrak{L}_N)^\omega$$

is the graph of a Jacobi structure near the zero section and there exists a fat tubular neighbourhood $\Psi \colon L_\nu \to L_U$ which is a Jacobi map near the zero section.

Proof The proof follows the lines of Theorem 5.4 with the obvious adaptions. □

The next step is to prove the second splitting Theorem of Dazord and Lichnerowicz and Marle in [5], namely the splitting of Jacobi manifolds around contact leaves.

Theorem 5.10 *Let $L \to M$ be a line bundle, let $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi tensor and let $p_0 \in M$ be a contact point. Then there are a line bundle trivialization $L_U \cong U \times \mathbb{R}$ around p_0 and a minimal homogeneous cocontact transversal $N \hookrightarrow U$, such that $U \cong U_{2q+1} \times N$ for an open subset $0 \in U_{2q+1} \subseteq \mathbb{R}^{2q+1}$ and the corresponding Jacobi pair (Λ, E) is transformed (via this isomorphism) to*

$$(\Lambda, E) = (\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N, E_{\mathrm{can}}),$$

where (π_N, Z_N) is the induced homogeneous Poisson structure on the transversal N and the contact structure on the fiber is given by $(\Lambda_{\mathrm{can}}, E_{\mathrm{can}}) = ((p_i \frac{\partial}{\partial u} + \frac{\partial}{\partial q^i}) \wedge \frac{\partial}{\partial q_i}, \frac{\partial}{\partial u})$.

Proof Let $p_0 \in M$ be a contact point and let $N \subseteq M$ be a minimal transversal, such that

$$\sigma(\operatorname{im} J^\sharp)\big|_{p_0} \oplus T_{p_0}N = T_{p_0}M.$$

Note using Proposition 4.14,this means in particular, that N is a homogeneous cocontact transversal for a any flat connection, so we assume that the line bundle $L \to M$ is trivial and choose the flat connection ∇ induced by this trivialization. In a possibly smaller neighbourhood, we can assume that also the normal bundle $\nu_N = V \times N \to N$ is trivial. We want to show that there is a trivialization of ν_N, such that Θ looks trivial, where we specialize on the way through the proof what we mean by trivial. Let us therefore denote by λ the local trivializing section of L_N, thus we can write

$$\Theta(\Delta, \Box) = \Omega(\Delta, \Box) \cdot \lambda$$

for $\Delta, \Box \in \nu_N \oplus K$ with $K = DL_N \cap \mathfrak{B}_I(\mathfrak{L}_J) \subseteq DL_N$. Hence, we can find a (local) nowhere vanishing section of K of the form $\mathbb{1} - Z$ for a unique Z. Let us now restrict

$$\Theta\big|_{\nu_N} \colon \nu_N \times \nu_N \to L_N,$$

since ν_N is odd dimensional and Θ is a skew-symmetric pairing, we can find a local non-vanishing $X \in \Gamma^\infty(\nu_N)$, such that $\Theta(X, \cdot) = 0$, moreover, since Θ is non-degenerate, we can modify X by multiplying by a non-vanishing section in such a way that

$$\Omega(\mathbb{1} - Z, X) = 1.$$

It is now easy to see that the symplectic complement $S := \langle \mathbb{1} - Z, X \rangle^\perp \subseteq \nu_N$. Finally, we find a trivialization of S such that $\Omega\big|_S$ is the trivial symplectic form with Darboux frame $\{e_2, e_{k+2}, \dots\}$. Hence, by extending this trivialization to $\nu_N = V \times N$ by using the coordinate X as e_0, we find that $\{e_0, \mathbb{1} - Z, e_1, e_{k+1}, e_2, e_{k+2}, \dots\}$ is a Darboux frame of Ω in this trivialization. with the decomposition $DL_\nu = TV \oplus TN \oplus \mathbb{R}_{\nu_N}$ we can choose

$$\omega = \sum_{i=1}^{k} dx^i \wedge dx^{i+k} + \mathbb{1}^* \wedge (dx^0 - \sum_{i=1}^{k} x^{i+k} dx^i)$$

which coincides with Θ on $\nu_N \oplus K$ and is d_L-closed. By applying Theorem 5.9, since N together with ∇ is a homogeneous cocontact transversal, we find a Jacobi morphism

$$\mathfrak{B}_P(\mathfrak{L}_N)^{\omega} \cong \mathfrak{L}_J.$$

An easy computation shows that $\mathfrak{B}_P(\mathfrak{L}_N)^{\omega}$ is the graph of the Jacobi structure of the form in the theorem. $\square$

6 Application: splitting theorem for homogeneous Poisson structures

Using the homogenization scheme from [2], one can see that Jacobi bundles are nothing else but special kinds of homogeneous Poisson manifolds. Moreover, the two most important examples of Poisson manifolds are of this kind: the cotangent bundle and the dual of a Lie algebra. Using this insight, it is easy to see that proving something for Jacobi structures gives a proof for something in homogeneous Poisson Geometry. We want to apply this philosophy to give a splitting theorem for homogeneous Poisson manifolds. The first appearance of such a theorem was [5, Theorem 5.5] in order to prove the local splitting of Jacobi pairs. Here we want to attack the problem from the other side: we use the splitting of Jacobi manifolds to prove the splitting of homogeneous Poisson structures.

Theorem 6.1 *Let (π, Z) be a homogeneous Poisson structure on a manifold M and let $p_0 \in M$ be a point such that $Z_{p_0} \neq 0$ such that $\operatorname{rank}(\pi^\sharp) = 2k$. Then there exist an open neighbourhood U of p_0, an open neighbourhood U_{2k} of $0 \in \mathbb{R}^{2k}$, a manifold N with a homogeneous Poisson structure (π_N, Z_N) and a diffeomorphism $\psi\colon U \to U_{2k} \times N$, such that*

$$\psi_*\pi = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \pi_N.$$

Additionally,

1. *if $Z \in \operatorname{im}(\pi^\sharp)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + \frac{\partial}{\partial p_k} + Z_N$.*
2. *if $Z \notin \operatorname{im}(\pi^\sharp)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + Z_N$.*

Proof Note that since $Z_{p_0} \neq 0$, we find coordinates $\{u, x^1, \dots, x^q\}$ with $p_0 = (1, 0, \dots, 0)$, such that $Z = u\frac{\partial}{\partial u}$. In this chart, using $\mathscr{L}_Z\pi = -\pi$, we have

$$\pi = \frac{1}{u}(\Lambda + u\frac{\partial}{\partial u} \wedge E)$$

for unique $\Lambda \in \Gamma^\infty(\Lambda^2 TM)$ and $E \in \Gamma^\infty(TM)$ which do not depend on u. It is easy to see, that we have

$$[\![\Lambda, \Lambda]\!]_{\mathrm{s}} = -2E \wedge \Lambda \text{ and } \mathscr{L}_E \Lambda = 0,$$

which means that (Λ, E) is a Jacobi pair. This allows us to use Theorem 5.5 and Theorem 5.10 to prove the result. We will do it just for the case where p_0 is a contact point, which means, translated to Jacobi pairs, that E_{p_0} is transversal to $\operatorname{im}(\Lambda^\sharp)\big|_{p_0}$ and thus $Z \in \operatorname{im}(\pi^\sharp)$, since the other case is exactly the same. Note that, we can apply Theorem 5.10: there exist coordinates $\{x, q^i, p_i, y^j\}$ and a local non-vanishing function a (which is basically the line bundle trivialization), such that

$$\Lambda = \frac{1}{a}(\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N) \text{ and } E = \frac{1}{a}(E_{\mathrm{can}} + \Lambda^\sharp(\mathrm{d}a)),$$

where Λ_{can} and E_{can} are just depending on $\{x, q^i, p_i\}$ and (ϕ_N, Z_N) is a homogeneous Poisson structure just depending on y^j-coordinates.

If we apply the diffeomorphism $(u, x^1, \dots, x^q) \mapsto (a \cdot u, x^1, \dots, x^q)$, we have

$$\pi = \frac{1}{u}(\Lambda_{\mathrm{can}} + \pi_N + E_{\mathrm{can}} \wedge Z_N + u\frac{\partial}{\partial u} \wedge E_{\mathrm{can}}).$$

A (quite) long and not very insightful computation shows that the diffeomorphism

$$\Phi(u, x^1, \dots, x^q) = (u, \Phi^{Z_N}_{\log(u)}(\Phi^{E_{\mathrm{can}}}_{-log(u)}(x^1, \dots, x^q))),$$

where $\Phi^{Z_N}_t$ (resp. $\Phi^{E_{\mathrm{can}}}_t$) is the flow uf Z_N (resp. E_{can}), gives us

$$\pi = \frac{1}{u}(\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}) + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} + \pi_N \text{ and } Z = u\frac{\partial}{\partial u} + p_i\frac{\partial}{\partial p_i} + Z_N$$

and with some obvious variations and renaming the coordinates we get the result. □

This Application shows us that, even though we can see Poisson structures as Jacobi manifolds, which suggests that they are more general objects than Poisson structures, the splitting theorems (of Jacobi pairs) are a refinement of the known splitting theorems for Poisson structures.

7 Generalized contact bundles

In this last section, we want to drop a word about generalized contact bundles. They were introduced recently in [15] and they are modelled to be the odd dimensional analogue to generalized complex structures. In the same way Dirac–Jacobi bundles are a generalization of Wade's $\mathcal{E}^1(M)$-Dirac structures, generalized contact bundles are a generalization of integrable *generalized almost contact structure*, which were defined in [9].

Definition 7.1 Let $L \to M$ be a line bundle. A subbundle $\mathfrak{L} \subseteq \mathbb{D}_{\mathbb{C}}L := \mathbb{D}L \otimes \mathbb{C}$ is called generalized contact structure on L, if

1. $\mathfrak{L}$ is a (complex) Dirac–Jacobi structure
2. $\mathfrak{L} \cap \overline{\mathfrak{L}} = \{0\}$

A generalized contact structure can be also seen as an endomorphism of $\mathbb{D}L$ of the form

$$\mathbb{K} = \begin{pmatrix} \phi & J^\sharp \\ \alpha^\flat & -\phi^* \end{pmatrix},$$

where $\phi \in \mathrm{End}(DL)$, $J \in \Gamma^\infty((J^1 L)^* \otimes L)$ and $\alpha \in \Omega^2_L(M)$ (see [12] and [15]). This endomorphism has to fulfill certain properties: it has to be almost complex, compatible with the pairing and integrable, which we do not explain what it means here and refer the reader to [15]. The +i-Eigenbundle produces a generalized contact structure in the sense of Definition 7.1. Moreover, we have that among many more conditions that J is a Jacobi structure. Let us now pick a (cosymplectic or cocontact) transversal to J together with an Euler-like derivation $\Delta = J^\sharp(\alpha)$, then $(\Delta, \mathrm{i}\alpha - \phi^*(\alpha)) = \mathrm{i}(0, \alpha) + \mathbb{K}(0, \alpha) \in \Gamma^\infty(\mathfrak{L})$. With the techniques from Sect. 4 and Sect. 5, one can show that

$$\mathfrak{B}_\Psi(\mathfrak{L}) = \mathfrak{B}_{I \circ P}(\mathfrak{L})^{\mathrm{i}\omega + \beta},$$

where $\omega = \int_0^1 \frac{1}{t} (\Phi^{\Delta}_{\log(t)})^* \, \mathrm{d}_L \alpha \, \mathrm{d}t$ and $\beta = -\int_0^1 \frac{1}{t} (\Phi^{\Delta}_{\log(t)})^* \, \mathrm{d}_L \phi^*(\alpha) \, \mathrm{d}t$. This is nothing else but a normal form for generalized contact bundles. This can be pushed more forward to prove a local splitting of generalized bundles, but this has already been done in [12] with similar techniques.

8 Final remarks

In [14] the Dirac-ization trick is explained, which introduces a one-to-one correspondence between Dirac–Jacobi bundles and so-called homogeneous Dirac structures, i.e. Dirac structures $\mathcal{D} \subseteq \mathbb{T}P$ on the total space of a $\mathbb{R}^\times$ principal bundle $P \to M$, such that

$$(X, \alpha) \in \Gamma^\infty(\mathcal{D}) \implies ([E, X] + X, \mathscr{L}_E \alpha) \in \Gamma^\infty(\mathcal{D})$$

for the fundamental vector field $E \in \Gamma^\infty(TP)$ of $1 \in \mathbb{R} = \mathsf{Lie}(\mathbb{R}^\times)$. Knowing this one may wonder, if one can apply the normal form theorem from [3] to the homogeneous Dirac-structure in order to obtain the normal form theorems for Dirac–Jacobi structures. This is not directly possible, at least not to the author's knowledge. The main problems are the following:

1. Homogeneous Dirac structures are not invariant with respect to the lifted $\mathbb{R}^\times$-action on $\mathbb{T}M$, so the invariant versions of the normal forms from [3] are not applicable.
2. The homogenization of a submanifold gives always a homogeneous submanifold, i.e. invariant under the principal action, but minimal transversals to contact points (points such that the leaf through the point is tangential to the homogeneity E) are not homogeneous.
3. Even though there is a notion of product in the category of $\mathbb{R}^\times$-principal bundles (the cartesian product of the total spaces modulo the diagonal action), the product is not a principal bundle over the cartesian product of the bases of the factors, which makes a splitting in this category very difficult.

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A the Moser trick for Jacobi manifolds

Let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi structure on a line bundle $L \to M$. Moreover, we assume having smooth family of closed 2-forms σ_t, such that $\sigma_0 = 0$ and $\mathcal{L}_J^{\sigma_t}$ is a Jacobi structure for all t, denoted by J_t. For

$$\alpha_t := -\frac{\partial}{\partial t}\iota_{\mathbb{1}}\sigma_t$$

the equation

$$\frac{\partial}{\partial t}\sigma_t = -\,\mathrm{d}_L\alpha_t$$

holds. We define the Moser-derivation by

$$\Delta_t := -J_t^\sharp(\alpha_t)$$

and its flow by $\Phi_t \in \mathrm{Aut}(L)$, where we assume it exists for on open subset containing $[0, 1]$. Let us compute

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* J_t &= \Phi_t^*([\Delta_t, J_t] + \tfrac{\mathrm{d}}{\mathrm{d}t} J_t) \\ &= \Phi_t^*(-[J_t^\sharp(\alpha_t), J_t] + \tfrac{\mathrm{d}}{\mathrm{d}t} J_t) \\ &= \Phi_t^*(J_t^\sharp(-\,\mathrm{d}_L\alpha_t) + \tfrac{\mathrm{d}}{\mathrm{d}t} J_t). \end{aligned} \tag{A.1}$$

It is easy to see that

$$J_t^\sharp = J^\sharp \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1}$$

by the equality $\mathcal{L}_{J_t} = \mathcal{L}_J^{\sigma_t}$. We can compute

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} J_t^\sharp &= \tfrac{\mathrm{d}}{\mathrm{d}t} J^\sharp \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \\ &= -J^\sharp \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \circ (\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{id} + \sigma_t^\flat \circ J^\sharp)) \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \\ &= -J^\sharp \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \circ ((\frac{\partial}{\partial t}\sigma_t)^\flat \circ J^\sharp) \circ (\mathrm{id} + \sigma_t^\flat \circ J^\sharp)^{-1} \\ &= -J_t^\sharp \circ (\frac{\partial}{\partial t}\sigma_t)^\flat \circ J_t^\sharp \\ &= (-J_t^\sharp(\frac{\partial}{\partial t}\sigma_t))^\sharp \\ &= \big(J_t^\sharp(\mathrm{d}_L\alpha_t)\big)^\sharp, \end{aligned}$$

and hence $\frac{\mathrm{d}}{\mathrm{d}t} J_t = J_t^\sharp(\mathrm{d}_L\alpha_t)$. If we use this equality in Eq. A.1, we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t^* J_t = 0,$$

so we finally have $J = \Phi_0^* J_0 = \Phi_1^* J_1$ and hence the two Jacobi structures are isomorphic.

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