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# Two weight Sobolev norm inequalities for smooth Calderón–Zygmund operators and doubling weights

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## Abstract

Let $\mu$ be a positive locally finite Borel measure on $\mathbb{R}^n$ that is doubling, and define the homogeneous $W^s(\mu)$-Sobolev norm squared $\|f\|^2_{W^s(\mu)}$ of a function $f \in L^2_{\mathrm{loc}}(\mu)$ by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{f(x)-f(y)}{|x-y|^s}\right)^2 \frac{d\mu(x)\,d\mu(y)}{\left|B\left(\frac{x+y}{2},\frac{|x-y|}{2}\right)\right|_\mu},$$

and denote by $W^s(\mu)$ the corresponding Hilbert space completion (when $\mu$ is Lebesgue measure, this is the familiar Sobolev space on $\mathbb{R}^n$). We prove in particular that for $0 \le \alpha < n$, and $\sigma$ and $\omega$ doubling measures on $\mathbb{R}^n$, there is a positive constant $\theta$ such for $0 < s < \theta$, any smooth $\alpha$-fractional convolution singular integral $T^\alpha$ with homogeneous kernel that is nonvanishing in some coordinate direction, is bounded from $W^s(\sigma)$ to $W^s(\omega)$ *if and only if* the classical fractional Muckenhoupt condition on the measure pair holds,

$$A_2^\alpha \equiv \sup_{Q\in\mathcal{Q}^n}\frac{|Q|_\omega |Q|_\sigma}{|Q|^{2\left(1-\frac{\alpha}{n}\right)}} < \infty,$$

as well as the Sobolev $\mathbf{1}$-testing and $\mathbf{1}^*$-testing conditions for the operator $T^\alpha$,

$$\left\|T^\alpha_\sigma \mathbf{1}_I\right\|_{W^s(\omega)} \le \mathfrak{T}_{T^\alpha}(\sigma,\omega)\sqrt{|I|_\sigma}\,\ell(I)^{-s}, \quad I\in\mathcal{Q}^n,$$
$$\left\|T^{\alpha,*}_\omega \mathbf{1}_I\right\|_{W^s(\sigma)^*} \le \mathfrak{T}_{T^{\alpha,*}}(\omega,\sigma)\sqrt{|I|_\omega}\,\ell(I)^{s}, \quad I\in\mathcal{Q}^n,$$


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taken over the family of indicator test functions $\{\mathbf{1}_I\}_{I\in\mathcal{P}^n}$. Here $\mathcal{Q}^n$ is the collection of all cubes with sides parallel to the coordinate axes, and $W^s(\mu)^*$ denotes the dual of $W^s(\mu)$ determined by the usual $L^2(\mu)$ bilinear pairing, which we identify with a dyadic Sobolev space $W^{-s}_{\mathrm{dyad}}(\mu)$ of negative order. The sufficiency assertion persists for more general singular integral operators $T^\alpha$.

## Contents



## 1 Introduction

The Nazarov–Treil–Volberg $T1$ conjecture on the boundedness of the Hilbert transform from one weighted space $L^2(\sigma)$ to another $L^2(\omega)$, affirmatively in the two part paper [9, 10], with the case of common point masses included in [7]. Since then there have been a number of generalizations of boundedness of Calderón–Zygmund operators from one weighted $L^2$ space to another, both to higher dimensional Euclidean spaces (see e.g. [10, 11, 17]), and also to spaces of homogeneous type (see e.g. [4]). In addition there have been some generalizations to Sobolev spaces in place of $L^2$ spaces, but only in the setting of a single weight (see e.g. [3, 7]).

The purpose of this paper is to prove a *two weight* $T1$ theorem on *weighted Sobolev* spaces for general smooth $\alpha$-fractional singular integrals on $\mathbb{R}^n$, but with *doubling* measures.[1] In order to state our theorem, we need a number of definitions, some of which are recalled and explained in detail further below. Let $\mu$ be a positive locally finite Borel measure on $\mathbb{R}^n$ that is doubling, let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$, let $\kappa \in \mathbb{N}$ and let $\left\{\triangle^{\mu}_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ be the set of weighted Alpert projections on $L^2(\mu)$ (see [14]). When $\kappa = 1$, these are the familiar weighted Haar projections $\triangle^{\mu}_{Q} = \triangle^{\mu}_{Q;1}$.

**Definition 1** Let $\mu$ be a doubling measure on $\mathbb{R}^n$. Given $s \in \mathbb{R}$, we define the $\mathcal{D}$-dyadic homogeneous $W^s_{\mathcal{D};\kappa}(\mu)$-Sobolev norm of a function $f \in L^2_{\text{loc}}(\mu)$ by

$$\|f\|^2_{W^s_{\mathcal{D};\kappa}(\mu)} \equiv \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\|\triangle^{\mu}_{Q;\kappa} f\right\|^2_{L^2(\mu)}$$

and we denote by $W^s_{\mathcal{D};\kappa}(\mu)$ the corresponding Hilbert space completion[2] of $f \in L^2_{\text{loc}}(\mu)$ with

$$\|f\|_{W^s_{\mathcal{D};\kappa}(\mu)} < \infty.$$

Note that $W^0_{\mathcal{D};\kappa}(\mu) = L^2(\mu)$. We will show below that $W^s_{\mathcal{D};\kappa}(\mu) = W^s_{\mathcal{D}';\kappa'}(\mu)$ for all $s \in \mathbb{R}$ with $|s|$ sufficiently small, for all $\kappa, \kappa' \geq 1$, and for all dyadic grids $\mathcal{D}$ and $\mathcal{D}'$. Thus for a sufficiently small real $s$ depending on the doubling measure $\mu$, there is essentially just one weighted 'dyadic' Sobolev space of order $s$, which we will denote by $W^s_{\text{dyad}}(\mu)$. Moreover, for $s > 0$ and small enough and $\mu$ doubling, there is a more familiar equivalent 'continuous' norm,

$$\|f\|_{W^s(\mu)} = \sqrt{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{f(x)-f(y)}{|x-y|^s}\right)^2 \frac{d\mu(x)\,d\mu(y)}{\left|B\left(\frac{x+y}{2},\frac{|x-y|}{2}\right)\right|_{\mu}}}. \tag{1.1}$$

We also show that the dual spaces $W^s_{\mathcal{D};\kappa}(\mu)^*$ under the $L^2(\mu)$ pairing are given by $W^{-s}_{\mathcal{D};\kappa}(\mu)$ for all grids $\mathcal{D}$ and integers $\kappa$. Thus when $\mu$ is doubling, we can identify $W^s(\mu)^*$ with any of the spaces $W^{-s}_{\mathcal{D};\kappa}(\mu)$ for $|s|$ sufficiently small, and it will be convenient to denote $W^s(\mu)^*$ by $W^{-s}(\mu)$, even though the above formula for $\|f\|_{W^{-s}(\mu)}$ diverges when $s \geq 0$.

Finally, we note that *without* the doubling hypothesis on $\mu$, we in general need to include additional Haar projections $\left\{\mathbb{E}^{\mu}_{T}\right\}_{T\in\mathcal{T}}$ onto one-dimensional spaces of constant functions on certain 'tops' $T$ of the grid $\mathcal{D}$, where a top is the union of a maximal tower in $\mathcal{D}$ (see [2]). Without these additional projections, we may not recover all of $L^2(\mu)$ in general, and moreover, the spaces $W^s_{\mathcal{D};\kappa}(\mu)$ defined above may actually depend on the dyadic grid $\mathcal{D}$. For example, if $d\mu(x) = \mathbf{1}_{[-1,1)}(x)\,dx$ and $f(x) = \mathbf{1}_{[0,1)}(x)$, the reader can easily check

1 Weighted Sobolev spaces are not canonically defined for general weights, and doubling is a convenient hypothesis that gives equivalence of the various definitions.

2 For general measures, the functional $\|\,\|_{W^s_{\mathcal{D};\kappa}(\mu)}$ may only be a seminorm, but this is avoided for doubling measures.

that the functional $\|f\|_{W^s_{\mathcal{D};1}(\mu)}$ vanishes when $\mathcal{D}$ is the standard dyadic grid, but is positive when $\mathcal{D}$ is any grid containing $[-1,1)$.[3]

Note that we will not use the Hilbert space duality (that identifies the dual of a Hilbert space with itself) to analyze the two weight boundedness of $T^\alpha_\sigma : W^s_{\text{dyad}}(\sigma) \to W^s_{\text{dyad}}(\omega)$, but rather we will use the $L^2(\omega)$ and $L^2(\sigma)$ pairings in which case the dual of $W^s_{\text{dyad}}(\mu)$ is identified with $W^{-s}_{\text{dyad}}(\mu)$ as above. The reason for this is that the weighted Alpert projections $\left\{\triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ satisfy telescoping identities, while the orthogonal projections $\left\{\ell(Q)^s \triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ do not.

Denote by $\Omega_{\text{dyad}}$ the collection of all dyadic grids in $\mathbb{R}^n$, and let $\mathcal{Q}^n$ denote the collection of all cubes in $\mathbb{R}^n$ having sides parallel to the coordinate axes. A positive locally finite Borel measure $\mu$ on $\mathbb{R}^n$ is said to be doubling if there is a constant $C_{\text{doub}}$, called the doubling constant, such that

$$|2Q|_\mu \leq C_{\text{doub}} |Q|_\mu \qquad \text{for all cubes } Q \in \mathcal{Q}^n.$$

Finally, for $0 \leq \alpha < n$ we define a smooth $\alpha$-fractional Calder ón–Zygmund kernel $K^\alpha(x,y)$ to be a function $K^\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following fractional size and smoothness conditions

$$\left|\nabla^j_x K^\alpha(x,y)\right| + \left|\nabla^j_y K^\alpha(x,y)\right| \leq C_{\alpha,j} |x-y|^{\alpha-j-n}, \quad 0 \leq j < \infty, \tag{1.2}$$

and we denote by $T^\alpha$ the associated $\alpha$-fractional singular integral on $\mathbb{R}^n$. We say that $T^\alpha_\sigma$, where $T^\alpha_\sigma f \equiv T^\alpha(f\sigma)$, is bounded from $W^s_{\text{dyad}}(\sigma)$ to $W^s_{\text{dyad}}(\omega)$ if for all admissible truncations $\widetilde{T^\alpha}$ we have

$$\left\|\widetilde{T^\alpha_\sigma} f\right\|_{W^s_{\text{dyad}}(\omega)} \leq \mathfrak{N}_{T^\alpha}(\sigma,\omega) \|f\|_{W^s_{\text{dyad}}(\sigma)}, \quad \text{for all } f \in W^s_{\text{dyad}}(\sigma).$$

Here $\mathfrak{N}_{T^\alpha}(\sigma,\omega)$ denotes the best constant in these inequalities uniformly over all admissible truncations of $T^\alpha$. See below for a precise definition of admissible truncations, as well as the interpretation of the testing conditions appearing in the next theorem. The case $s=0$ of Theorem 2 is in [1].

**Theorem 2** ($T1$ for doubling measures) *Let $0 \leq \alpha < n$, and let $T^\alpha$ denote a smooth $\alpha$-fractional singular integral on $\mathbb{R}^n$. Let $\sigma$ and $\omega$ be doubling Borel measures on $\mathbb{R}^n$. Then there is a positive constant $\theta$, depending only on the doubling constants of $\sigma$ and $\omega$, such that if $0 < s < \theta$, then $T^\alpha_\sigma$, where $T^\alpha_\sigma f \equiv T^\alpha(f\sigma)$, is bounded from $W^s(\sigma)$ to $W^s(\omega)$, i.e.*

$$\left\|T^\alpha_\sigma f\right\|_{W^s(\omega)} \leq \mathfrak{N}_{T^\alpha} \|f\|_{W^s(\sigma)}, \tag{1.3}$$

*provided the classical fractional Muckenhoupt condition on the measure pair holds,*

$$A^\alpha_2 \equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\omega |Q|_\sigma}{|Q|^{2\left(1-\frac{\alpha}{n}\right)}} < \infty,$$

*as well as the Sobolev $\mathbf{1}$-testing and $\mathbf{1}^*$-testing conditions for the operator $T^\alpha$,*

$$\left\|T^\alpha_\sigma \mathbf{1}_I\right\|_{W^s(\omega)} \leq \mathfrak{T}_{T^\alpha}(\sigma,\omega)\sqrt{|I|_\sigma}\,\ell(I)^{-s}, \quad I \in \mathcal{Q}^n,$$
$$\left\|T^{\alpha,*}_\omega \mathbf{1}_I\right\|_{W^{-s}(\sigma)} \leq \mathfrak{T}_{T^{\alpha,*}}(\omega,\sigma)\sqrt{|I|_\omega}\,\ell(I)^{s}, \quad I \in \mathcal{Q}^n,$$

[3] Moreover, even taking into account the behaviour at infinity, one can show that $\|f\|_{W^s_{\mathcal{D};1}(\mu)} = 0$ when $\mathcal{D}$ is the standard grid and $s > 0$.

*taken over the family of indicator test functions* $\{\mathbf{1}_I\}_{I\in\mathcal{Q}^n}$.
*Conversely, the testing conditions are necessary for* (1.3)*, and if in addtion* $T^\alpha$ *is a smooth convolution operator with homogeneous kernel that is nonvanishing in some coordinate direction, then* $A_2^\alpha < \infty$ *whenever the two weight norm inequality* (1.3) *holds for some* $s > 0$.

***Remark 3*** One can weaken the smoothness assumption on the kernel $K$ depending on the doubling constants of the measures $\sigma$ and $\omega$, but we will not pursue this here. See [16] for sharper assumptions in the $L^2$ case.

**Problem 4** *$T1$ theorems for Sobolev norms involving general measures, even for the Hilbert transform on the line, remain open at this time.*

The proof of Theorem 2 expands on that for $L^2$ spaces with doubling measures using weighted Alpert wavelets [1], but with a number of differences. For example:

(1) The map $f \to |f|$ fails to be bounded on $W_{\text{dyad}}^{-s}(\omega)$ for $s > 0$[4], which gives rise to significant obstacles in dealing with bilinear inequalities (using the $L^2$ inner product as a duality pairing) that require control of both $T : W_{\text{dyad}}^{s}(\sigma) \to W_{\text{dyad}}^{s}(\omega)$ and $T^* : W_{\text{dyad}}^{-s}(\omega) \to W_{\text{dyad}}^{-s}(\sigma)$.
(2) As a consequence, we are no longer able to use Calderón–Zygmund decompositions and Carleson embedding theorems that require use of the modulus $|f|$ of a Sobolev function $f$. Instead, we derive a stronger form of the pivotal condition, that permits a new Carleson condition to circumvent these hurdles.
(3) The estimation of Sobolev norms in the paraproduct form requires the use both of Alpert and Haar wavelets, in connection with the new Carleson condition. This in turn requires the identification of different wavelet spaces.
(4) In Proposition 44, we extend the Intertwining Proposition from [16] to Sobolev spaces using a new stronger form of the $\kappa$ -pivotal condition, which also results in an overall simplification of this proof.
(5) A power decay of doubling measures near zero sets of polynomials is needed to estimate Sobolev norms of moduli of Alpert wavelets in Lemma 26 when $s < 0$, where the logarithmic decay obtained in [16] is insufficient.
(6) Finally, we prove the comparability of the various Sobolev space norms for a fixed $s$ and doubling measure (the case $s = 0$ being trivial), including the familiar continuous norm in (1.1) when $s > 0$. This equivalence is needed in particular to implement the good/bad technology of Nazarov, Treil and Volberg.

## 2 Preliminaries: Sobolev spaces and doubling measures

Denote by $\mathcal{Q}^n$ the collection of cubes in $\mathbb{R}^n$ having sides parallel to the coordinate axes. A positive locally finite Borel measure $\mu$ on $\mathbb{R}^n$ is said to satisfy the*doubling condition* if there is a pair of constants $(\beta, \gamma) \in (0,1)^2$, called doubling parameters, such that

$$|\beta Q|_\mu \geq \gamma\, |Q|_\mu \quad \text{for all cubes } Q \in \mathcal{Q}^n, \tag{2.1}$$

[4] For example, if $d\mu = dx$ and $f = \sum_{k=1}^{2N} (-1)^k\, 1_{[k-1,k)}$, then $\|f\|^2_{W_{\text{dyad}}^{-s}} \approx N$, while $\||f|\|^2_{W_{\text{dyad}}^{-s}} \approx N^{1+2s}$.

and the *reverse doubling condition* if there is a pair of constants $(\beta, \gamma) \in (0,1)^2$, called reverse doubling parameters, such that

$$|\beta Q|_\mu \leq \gamma \, |Q|_\mu \quad \text{for all cubes } Q \in \mathcal{Q}^n. \tag{2.2}$$

Note that the inequality in (2.2) has been reversed from that in the definition of the doubling condition in (2.1). A familiar equivalent reformulation of (2.1) is that there is a positive constant $C_{\text{doub}}$, called the doubling constant, such that $|2Q|_\mu \leq C_{\text{doub}} \, |Q|_\mu$ for all cubes $Q \in \mathcal{Q}^n$. There is also a positive constant $\theta_\mu^{\text{doub}}$, called a*doubling exponent*, such that

$$\sup_{Q \in \mathcal{Q}^n} \frac{|sQ|_\mu}{|Q|_\mu} \leq s^{\theta_\mu^{\text{doub}}}, \qquad \text{for all sufficiently large } s > 0.$$

It is well known (see e.g. the introduction in [15]) that doubling implies reverse doubling, and that $\mu$ is reverse doubling if and only if there exists a positive constant $\theta_\mu^{\text{rev}}$, called a *reverse doubling exponent*, such that

$$\sup_{Q \in \mathcal{Q}^n} \frac{|sQ|_\mu}{|Q|_\mu} \leq s^{\theta_\mu^{\text{rev}}}, \qquad \text{for all sufficiently small } s > 0.$$

### 2.1 Decay of doubling measures near zero sets of polynomials

In order to deal with Sobolev norms and doubling measures, we will need the following estimate on doubling measures of 'halos' of zero sets of normalized polynomials, which follows the same plan of proof as in the case of boundaries of cubes proved in [16, Lemma 24]. We first recall a slight variant of a remark from [16].

For any polynomial $P$ and cube $Q$, we say that $P$ is $Q$*-normalized* if $\|P\|_{L^\infty(Q)} = 1$.

***Remark 5*** Since all norms on a finite dimensional vector space are equivalent, we have upon rescaling the cube $Q$ to the unit cube,

$$\|P\|_{L^\infty(Q)} \approx |P(0)| + \sqrt{n}\ell(Q) \, \|\nabla P\|_{L^\infty(Q)}, \quad \deg P < \kappa, \tag{2.3}$$

with implicit constants depending only on $n$ and $\kappa$. In particular there is a positive constant $K_{n,\kappa}$ such that $\sqrt{n}\ell(Q) \, \|\nabla P\|_{L^\infty(Q)} \leq K_{n,\kappa}$ for all $Q$-normalized polynomials $P$. Then for every $Q$ -normalized polynomial $P$ of degree less than $\kappa$, there is a ball $B\left(y, \frac{\sqrt{n}}{2K_{n,\kappa}} \ell(Q)\right) \subset Q$ on which $P$ is nonvanishing. Indeed, if there is no such ball, then

$$1 = \|P\|_{L^\infty(Q)} \leq \frac{\sqrt{n}}{2K_{n,\kappa}} \ell(Q) \, \|\nabla P\|_{L^\infty(Q)} \leq \frac{1}{2}$$

is a contradiction.

Here is the result proved in [16, Lemma 24].

**Lemma 6** *Suppose $\mu$ is a doubling measure on $\mathbb{R}^n$ and that $Q \in \mathcal{Q}^n$. Then for $0 < \delta < 1$ we have*

$$|Q \setminus (1-\delta) Q|_\mu \leq \frac{C}{\ln \frac{1}{\delta}} \, |Q|_\mu \, .$$

We will need to improve significantly on this as follows. Without loss of generality, suppose that $Q = [0,1] \times [0,1]$ in the plane and $d\mu(x,y) = w(x,y)\,dxdy$. Define $W$ to be even on $[-1,1]$ by

$$W(y) \equiv \int_0^1 w(x,y)\,dx, \quad 0 \le y \le 1.$$

and note that $W(y)\,dy$ is a doubling measure on $[0,1]$, hence also reverse doubling with exponent $\theta^{\text{rev}}$. Thus from the reverse doubling property applied to the subinterval $[0,t]$ of $[-1,1]$ we have that

$$\int_0^t W(y)\,dy \le C t^{\theta^{\text{rev}}} \int_{-1}^1 W(y)\,dy \le C' t^{\theta^{\text{rev}}} \int_0^1 W(y)\,dy,$$

which says that

$$|[0,1] \times [0,t]|_\mu = \int_0^t W(y)\,dy \le C' t^{\theta^{\text{rev}}} \int_0^1 W(y)\,dy = C' t^{\theta^{\text{rev}}} |[0,1] \times [0,1]|_\mu . \tag{2.4}$$

This gives power decay instead of logarithmic decay, which will prove crucial below. The next lemma is a generalization of [16, Lemma 24].

**Lemma 7** *Let $\kappa \in \mathbb{N}$. Suppose $\mu$ is a doubling measure on $\mathbb{R}^n$ and that $Q \in \mathcal{Q}^n$. Let $Z$ denote the zero set of a $Q$-normalized polynomial $P$ of degree less than $\kappa$, and for $0 < \delta < 1$, let*

$$Z_\delta = \left\{ y \in \mathbb{R}^n : |y - z| < \delta \text{ for some } z \in Z \right\}$$

*denote the $\delta$-halo of $Z$. Then for a positive constant $C_{n,\kappa}$ depending only on $n$ and $\kappa$, and not on $P$ itself, we have*

$$|Q \cap Z_\delta|_\mu \le \frac{C_{n,\kappa}}{\ln \frac{1}{\delta}} |Q|_\mu .$$

***Proof*** Let $\delta = 2^{-m}$. Denote by $\mathfrak{C}^{(m)}(Q)$ the set of $m^{th}$ generation dyadic children of $Q$, so that each $I \in \mathfrak{C}^{(m)}(Q)$ has side length $\ell(I) = 2^{-m}\ell(Q)$, and define the collections

$$\mathfrak{G}^{(m)}(Q) \equiv \left\{ I \in \mathfrak{C}^{(m)}(Q) : I \subset Q \text{ and } \partial I \cap Z \neq \emptyset \right\},$$
$$\mathfrak{H}^{(m)}(Q) \equiv \left\{ I \in \mathfrak{C}^{(m)}(Q) : 3I \subset Q \text{ and } \partial(3I) \cap Z \neq \emptyset \right\}.$$

Then

$$Q \cap Z_\delta = \bigcup_{I \in \mathfrak{G}^{(m)}(Q)} I \text{ and } Q \setminus Z_\delta = \dot{\bigcup}_{k=2}^{m} \bigcup_{I \in \mathfrak{H}^{(k)}(Q)} I.$$

From Remark 5, we obtain that the the union $\bigcup_{I \in \mathfrak{H}^{(k)}(Q)} RI$ contains $Q \cap Z_\delta$ for $k \ge cm$ for some $c = c_{n,\kappa} \in (0,1)$ depending only on $n$ and $\kappa$, and in particular independent of $m$. Then from the doubling condition we have $|RI|_\mu \le D_R |I|_\mu$ for all cubes $I$ and some constant $D_R$, and so for $k \ge cm$,

$$\left|\mathfrak{H}^{(k)}(Q)\right|_\mu = \sum_{I \in \mathfrak{H}^{(k)}(Q)} |I|_\mu \ge \sum_{I \in \mathfrak{H}^{(k)}(Q)} \frac{1}{D_R} |RI|_\mu = \frac{1}{D_R} \int \left( \sum_{I \in \mathfrak{H}^{(k)}(Q)} \mathbf{1}_{RI} \right) d\mu$$

$$\geq \frac{1}{D_R} \int \left( \sum_{I \in \mathfrak{G}^{(k)}(Q)} \mathbf{1}_I \right) d\mu = \frac{1}{D_R} \left| \mathfrak{G}^{(k)}(Q) \right|_\mu \geq \frac{1}{D_R} \left| \mathfrak{G}^{(m)}(Q) \right|_\mu$$
$$= \frac{1}{D_R} |Q \cap Z_\delta|_\mu .$$

Thus we have

$$|Q|_\mu \geq \sum_{k=cm}^{m} \left| \mathfrak{H}^{(k)}(Q) \right|_\mu \geq \frac{m(1-c)}{D_R} |Q \cap Z_\delta|_\mu$$

which proves the lemma. □

We can apply the method used in (2.4) to obtain a power decay instead of a logarithmic decay.

**Corollary 8** *Let $\kappa \in \mathbb{N}$. Suppose $\mu$ is a doubling measure on $\mathbb{R}^n$ and that $Q \in \mathcal{Q}^n$. Let $Z$ denote the zero set of a $Q$-normalized polynomial $P$ of degree less than $\kappa$, and for $0 < \delta < 1$, let $Z_\delta$ denote the $\delta$-halo of $Z$. Then for a positive constant $C_{n,\kappa}$ depending only on $n$ and $\kappa$, and not on $P$ itself, and for some $\theta > 0$, we have*

$$|Q \cap Z_\delta|_\mu \leq C_{n,\kappa} \delta^\theta |Q|_\mu .$$

*In particular this holds for $Z = \partial Q$, which is a finite union of zero sets of linear functions.*

***Proof*** Without loss of generality $Q$ is the unit cube $[0, 1]^n$. Define an even function $w(t)$ on $[-1, 1]$, that is increasing on $[0, 1]$, by the formula

$$w(t) \equiv |Z_t|_\mu \qquad 0 \leq t \leq 1.$$

Since $P$ is a $Q$-normalized polynomial of degree less than $\kappa$, there are positive constants $t_0,\ c_0, C_0, A$ such that for every $0 < t < t_0$, there is a collection of cubes $\left\{Q_i^t\right\}_i$ with

$$\begin{aligned} \ell\left(Q_i^t\right) &= c_0 t, \\ \mathbf{1}_{Z_t}(x) &\leq \sum_i \mathbf{1}_{Q_i^t}(x) \leq A \mathbf{1}_{Z_t}(x) \\ \mathbf{1}_{Z_{2t}}(x) &\leq \sum_i \mathbf{1}_{C_0 Q_i^t}(x) \leq A. \end{aligned}$$

Thus we have

$$w(2t) \leq \sum_i \left|C_0 Q_i^t\right|_\mu \leq \sum_i C_{\mathrm{doub}} \left|Q_i^t\right|_\mu \leq C_{\mathrm{doub}} M |Z_t|_\mu = C_{\mathrm{doub}} A w(t), \quad 0 < t < t_0,$$

and hence there is a doubling exponent $\theta_w^{\mathrm{doub}}$ such that

$$\frac{w(st)}{w(t)} \leq s^{\theta_w^{\mathrm{doub}}}, \quad \text{for all sufficiently large } s.$$

We claim $w(t)$ also satisfies the reverse doubling condition

$$w(\delta) \leq C \delta^\varepsilon w(1), \qquad 0 < \delta < t_0.$$

Indeed, let $d\mu \equiv \frac{dw}{dt}$. Then assuming $s \geq 5$ in the definition of $\theta_w^{\text{doub}}$, we obtain for $Q = [0, t]$ that

$$\begin{aligned} |3Q \setminus Q|_\mu &= \sum_{I\in\mathcal{D}:\ I\subset 3Q\setminus Q, \ell(I)=\ell(Q)} |I|_\mu \geq \sum_{I\in\mathcal{D}:\ I\subset 3Q\setminus Q, \ell(I)=\ell(Q)} 5^{-\theta_\mu^{\text{doub}}} |5I|_\mu \\ &\geq \left(3^n - 1\right) 5^{-\theta_\mu^{\text{doub}}} |Q|_\mu \\ &\Longrightarrow |Q|_\mu = |3Q|_\mu - |3Q \setminus Q|_\mu \leq \left(1 - \frac{3^n - 1}{5^{\theta_\mu^{\text{doub}}}}\right) |3Q|_\mu \end{aligned}$$

which gives reverse doubling, and hence

$$|Q \cap Z_\delta|_\mu = \int_0^\delta w(t)\, dt \leq C\delta^\varepsilon |Q|_\mu, \quad \text{for } 0 < \delta < t_0,$$

and trivially this is extended to $\delta < 1$ by possibly increasing the constant $C$. □

### 2.2 Weighted Alpert bases for $L^2(\mu)$ and $L^\infty$ control of projections

The following theorem was proved in [14], which establishes the existence of Alpert wavelets, for $L^2(\mu)$ in all dimensions, having the three important properties of orthogonality, telescoping and moment vanishing. Since the statement is simplified for doubling measures, and this is the only case considered in our main theorem, we restrict ourselves to this case here.

We first recall the basic construction of weighted Alpert wavelets in [14] restricted to doubling measures. Let $\mu$ be a doubling measure on $\mathbb{R}^n$, and fix $\kappa \in \mathbb{N}$. For $Q \in \mathcal{Q}^n$, the collection of cubes with sides parallel to the coordinate axes, denote by $L^2_{Q;\kappa}(\mu)$ the finite dimensional subspace of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of $Q$ multiplied by polynomials of degree less than $\kappa$, and such that the linear combinations have vanishing $\mu$-moments on the cube $Q$ up to order $\kappa - 1$:

$$L^2_{Q;\kappa}(\mu) \equiv \left\{ f = \sum_{Q'\in\mathfrak{C}(Q)} \mathbf{1}_{Q'} p_{Q';\kappa}(x) : \int_Q f(x)\, x^\beta d\mu(x) = 0, \quad \text{for } 0 \leq |\beta| < \kappa \right\},$$

where $p_{Q';\kappa}(x) = \sum_{\beta\in\mathbb{Z}^n_+:|\beta|\leq\kappa-1} a_{Q';\beta} x^\beta$ is a polynomial in $\mathbb{R}^n$ of degree less than $\kappa$. Here $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$. Let $d_{Q;\kappa} \equiv \dim L^2_{Q;\kappa}(\mu)$ be the dimension of the finite dimensional linear space $L^2_{Q;\kappa}(\mu)$.

Let $\mathcal{D}$ denote a dyadic grid on $\mathbb{R}^n$ and for $Q \in \mathcal{D}$, let $\triangle^\mu_{Q;\kappa}$ denote orthogonal projection onto the finite dimensional subspace $L^2_{Q;\kappa}(\mu)$, and let $\mathbb{E}^\mu_{Q;\kappa}$ denote orthogonal projection onto the finite dimensional subspace

$$\mathcal{P}^n_{Q;\kappa}(\sigma) \equiv \text{Span}\{\mathbf{1}_Q x^\beta : 0 \leq |\beta| < \kappa\}.$$

**Theorem 9** *(Weighted Alpert Bases) Let $\mu$ be a doubling measure on $\mathbb{R}^n$, fix $\kappa \in \mathbb{N}$, and fix a dyadic grid $\mathcal{D}$ in $\mathbb{R}^n$.*

(1) *Then $\left\{\triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ is a complete set of orthogonal projections in $L^2(\mu)$ and*

$$f = \sum_{Q\in\mathcal{D}} \triangle^\mu_{Q;\kappa} f, \quad f \in L^2(\mu), \tag{2.5}$$

$$\left\langle \triangle_{P;\kappa}^{\mu} f, \triangle_{Q;\kappa}^{\mu} f \right\rangle_{L^2(\mu)} = 0 \, for \ P \neq Q,$$

*where convergence in the first line holds both in $L^2(\mu)$ norm and pointwise $\mu$-almost everywhere.*

(2) *Moreover we have the telescoping identities*

$$\mathbf{1}_Q \sum_{I:\, Q \subsetneqq I \subset P} \triangle_{I;\kappa}^{\mu} = \mathbb{E}_{Q;\kappa}^{\mu} - \mathbf{1}_Q \mathbb{E}_{P;\kappa}^{\mu} \quad for \ \ P, Q \in \mathcal{D} \ with \ Q \subsetneqq P, \tag{2.6}$$

(3) *and the moment vanishing conditions*

$$\int_{\mathbb{R}^n} \triangle_{Q;\kappa}^{\mu} f(x) \ x^{\beta} d\mu(x) = 0, \quad for \ Q \in \mathcal{D}, \ \beta \in \mathbb{Z}_+^n, \ 0 \leq |\beta| < \kappa. \tag{2.7}$$

We can fix an orthonormal basis $\left\{ h_{Q;\kappa}^{\mu,a} \right\}_{a \in \Gamma_{Q,n,\kappa}}$ of $L_{Q;\kappa}^2(\mu)$ where $\Gamma_{Q,n,\kappa}$ is a convenient finite index set. Then

$$\left\{ h_{Q;\kappa}^{\mu,a} \right\}_{a \in \Gamma_{Q,n,\kappa} \text{ and } Q \in \mathcal{D}}$$

is an orthonormal basis for $L^2(\mu)$. In particular we have

$$\|f\|_{L^2(\mu)}^2 = \sum_{Q \in \mathcal{D}} \left\| \triangle_{Q;\kappa}^{\mu} f \right\|_{L^2(\mu)}^2 = \sum_{Q \in \mathcal{D}} \left| \widehat{f}(Q) \right|^2,$$

$$\text{where } \widehat{f}(Q) = \left\{ \left\langle f, h_{Q;\kappa}^{\mu,a} \right\rangle_{\mu} \right\}_{a \in \Gamma_{Q,n,\kappa}} \quad \text{and } \left| \widehat{f}(Q) \right|^2 \equiv \sum_{a \in \Gamma_{Q,n,\kappa}} \left| \left\langle f, h_{Q;\kappa}^{\mu,a} \right\rangle_{\mu} \right|^2.$$

In terms of the Alpert coefficient vectors $\widehat{f_\kappa}(I) \equiv \left\{ \left\langle f, a_{I;\kappa,j}^{\mu} \right\rangle \right\}_{j=1}^{\kappa}$, we have for the special case of a doubling measure $\mu$ (see [16, (4.7) on page 14]),

$$\left| \widehat{f_\kappa}(I) \right| = \left\| \triangle_{I;\kappa}^{\mu} f \right\|_{L^2(\mu)} \leq \left\| \triangle_{I;\kappa}^{\mu} f \right\|_{L^\infty(\mu)} \sqrt{|I|_\mu} \leq C \left\| \triangle_{I;\kappa}^{\mu} f \right\|_{L^2(\mu)} = C \left| \widehat{f_\kappa}(I) \right|, \tag{2.8}$$

and in particular,

$$\left\| h_{Q;\kappa}^{\mu,a} \right\|_{L^\infty(\mu)} \approx \frac{1}{\sqrt{|I|_\mu}}, \tag{2.9}$$

since $\triangle_{I;\kappa}^{\mu} h_{Q;\kappa}^{\mu,a} = h_{Q;\kappa}^{\mu,a}$.

**Notation 10** *For doubling measures $\mu$, the cardinality of $\Gamma_{Q,n,\kappa}$ depends only on $n$ and $\kappa$, which are usually known from context, and so we will simply write $\Gamma$ when $\mu$ is doubling.*

From now on, all measures considered will be assumed to doubling, and often without explicit mention. We now inroduce Sobolev spaces defined by weighted Alpert projections instead of weighted Haar projections. We show below that these spaces are actually equivalent. For convenience we repeat the definition of weighted Sobolev space used in this paper.

**Definition 11** Given $\kappa \in \mathbb{N}$ and $s \in \mathbb{R}$ and $\mathcal{D} \in \Omega_{\text{dyad}}$ a dyadic grid, we define the $\kappa$-dyadic homogeneous $W_{\mathcal{D};\kappa}^{s}(\mu)$-Sobolev norm of a function $f \in L^2(\mu)$ by

$$\|f\|_{W_{\mathcal{D};\kappa}^{s}(\mu)}^2 \equiv \sum_{Q \in \mathcal{D}} \ell(Q)^{-2s} \left\| \triangle_{Q;\kappa}^{\mu} f \right\|_{L^2(\mu)}^2$$

and we denote by $W^s_{\mathcal{D};\kappa}(\mu)$ the corresponding Hilbert space completion.

**Lemma 12** *The set* $\left\{\ell(Q)^s h^{\mu,a}_{Q;\kappa}\right\}_{(Q,a)\in\mathcal{D}\times\Gamma}$ *is an orthonormal basis for* $W^s_{\mathcal{D};\kappa}(\mu)$*, and thus for any subset* $\mathcal{H}$ *of the dyadic grid* $\mathcal{D}$*, we have,*

$$\left\|\sum_{(I,a)\in\mathcal{H}\times\Gamma} c_{I,a}\ell(I)^s h^{\mu,a}_{I;\kappa}\right\|^2_{W^s_{\mathcal{D};\kappa}(\mu)} = \sum_{(I,a)\in\mathcal{H}\times\Gamma}\left|c_{I,a}\right|^2. \tag{2.10}$$

***Proof*** We have

$$\begin{aligned}\left\langle \ell(I)^s h^{\mu,a}_{I;\kappa}, \ell(J)^s h^{\mu,b}_{J;\kappa}\right\rangle_{W^s_{\mathcal{D};\kappa}(\mu)} &= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s}\left\langle \triangle^\mu_{Q;\kappa}\ell(I)^s h^{\mu,a}_{I;\kappa}, \triangle^\mu_{Q;\kappa}\ell(J)^s h^{\mu,b}_{J;\kappa}\right\rangle_{L^2(\mu)}\\ &= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s}\ell(I)^s\ell(J)^s\left\langle \triangle^\mu_{Q;\kappa}h^{\mu,a}_{I;\kappa}, \triangle^\mu_{Q;\kappa}h^{\mu,b}_{J;\kappa}\right\rangle_{L^2(\mu)}\\ &= \begin{cases}1 & \text{if } I=J \text{ and } a=b\\ 0 & \text{if } I\neq J \text{ or } a\neq b\end{cases},\end{aligned}$$

since $\triangle^\mu_{Q;\kappa}h^{\mu,a}_{K;\kappa}$ vanishes if $Q\neq K$, and equals $h^{\mu,a}_{K;\kappa}$ if $Q=K$. Thus $\left\{\ell(Q)^s h^{\mu,a}_{Q;\kappa}\right\}_{(Q,a)\in\mathcal{D}\times\Gamma}$ is an orthonormal basis and we conclude that (2.10) holds. □

### 2.3 Equivalence of Sobolev spaces

In [23], Triebel defines the usual homogeneous unweighted Sobolev space $W^s$ with norm $\|f\|_{W^s}$ given by

$$\|f\|^2_{W^s} \equiv \int_{\mathbb{R}^n}\left|(-\triangle)^{\frac{s}{2}} f(x)\right|^2 dx = \int_{\mathbb{R}^n}\left|\left(|\xi|^2\right)^{\frac{s}{2}}\widehat{f}(\xi)\right|^2 d\xi,$$

and the corresponding inhomogeneous version with norm squared $\int_{\mathbb{R}^n}\left|(I-\triangle)^{\frac{s}{2}} f(x)\right|^2 dx$, which we will not consider here. Combining results of Triebel [23] with those of Seeger and Ullrich [20] shows that

**Lemma 13** ([23], [20]) *The following three statements are equivalent for* $\mu$ *equal to Lebesgue measure and* $s\in\mathbb{R}$.

(1) $W^s_{\text{dyad}} = W^s$,
(2) $\left\{\ell(I)^s h_I\right\}_{I\in\mathcal{D}}$ *is an orthonormal basis for* $W^s$,
(3) $-\frac{1}{2} < s < \frac{1}{2}$.

Here is the first step toward proving the equivalence of the different dyadic Sobolev spaces over all grids $\mathcal{D}$ and integers $\kappa\in\mathbb{N}$, which in particular is used to implement the good/bad cube technology of Nazarov, Treil and Volberg. The reader can notice that the doubling property of the measure $\mu$ is not explicitly used in this argument, rather only in the definition of the Sobolev spaces.

**Lemma 14** *Let* $\mu$ *be a doubling measure on* $\mathbb{R}^n$ *and let* $\mathcal{D}$ *be a dyadic grid on* $\mathbb{R}^n$*. Then for* $\kappa_1,\kappa_2\in\mathbb{N}$ *and* $s\in\mathbb{R}$*, we have,*

$$W^s_{\mathcal{D};\kappa_1}(\mu) = W^s_{\mathcal{D};\kappa_2}(\mu),$$

*with equivalence of norms.*

***Proof*** We first claim that for $s<0$,

$$W^s_{\mathcal{D};\kappa_2}(\mu)\subset W^s_{\mathcal{D};\kappa_1}(\mu)\quad \text{for } 1\leq\kappa_1\leq\kappa_2, \tag{2.11}$$

which by duality gives for $s>0$,

$$W^s_{\mathcal{D};\kappa_1}(\mu)\subset W^s_{\mathcal{D};\kappa_2}(\mu)\quad \text{for } s>0 \text{ and } 1\leq\kappa_1\leq\kappa_2. \tag{2.12}$$

Indeed, for any subset $\mathcal{H}\subset\mathcal{D}$, we have

$$\begin{aligned}
&\left\|\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a}h^{\mu,a}_{I;\kappa_2}\right\|^2_{W^s_{\mathcal{D};\kappa_1}(\mu)} = \sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\left\|\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}:\ Q\subset I} c_{I,a}\,\triangle^\mu_{Q;\kappa_1}h^{\mu,a}_{I;\kappa_2}\right\|^2_{L^2(\mu)}\\
&= \sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}\left(c_{I,a}\right)^2\left\|\triangle^\mu_{Q;\kappa_1}h^{\mu,a}_{I;\kappa_2}\right\|^2_{L^2(\mu)}\\
&\quad+\sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\sum_{I,I'\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa},a'\in\Gamma_{I',n,\kappa}:\ Q\subset I\cap I'}\int_{\mathbb{R}^n}c_{I,a}\left(\triangle^\mu_{Q;\kappa_1}h^{\mu,a}_{I;\kappa_2}\right)c_{I',a'}\left(\triangle^\mu_{Q;\kappa_1}h^{\mu,a'}_{I';\kappa_2}\right)d\mu\\
&\equiv A+B,
\end{aligned}$$

where the first term satisfies

$$\begin{aligned}
A&=\sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}\left(c_{I,a}\right)^2\left\|\triangle^\mu_{Q;\kappa_1}h^{\mu,a}_{I;\kappa_2}\right\|^2_{L^2(\mu)}\\
&=\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}\left(c_{I,a}\right)^2\sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\left\|\triangle^\mu_{Q;\kappa_1}h^{\mu,a}_{I;\kappa_2}\right\|^2_{L^2(\mu)}\\
&=\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}\left(c_{I,a}\right)^2\left\|h^{\mu,a}_{I;\kappa_2}\right\|^2_{W^s_{\mathcal{D};\kappa_1}(\mu)}\leq C\sum_{I\in\mathcal{H}}\left(c_{I,a}\right)^2\left\|h^{\mu,a}_{I;\kappa_2}\right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)}\\
&=C\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}\left(c_{I,a}\right)^2\ell(I)^{-2s}=C\left\|\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}}c_{I,a}h^{\mu,a}_{I;\kappa_2}\right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)},
\end{aligned}$$

and where the final equality follows from Lemma 12.

To handle the second term, we write

$$\begin{aligned}
B&=\sum_{I\neq I'\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa},a'\in\Gamma_{I',n,\kappa}}c_{I,a}c_{I',a'}\int_{\mathbb{R}^n}\left\{\sum_{Q\in\mathcal{D}:\ Q\subset I\cap I'}\ell(Q)^{-2s}\,\triangle^\mu_{Q;1}h^{\mu,a}_{I;\kappa_2}\right\}h^{\mu,a'}_{I';\kappa_2}d\mu\\
&=\left\{\sum_{I\subsetneqq I'\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa},a'\in\Gamma_{I',n,\kappa}}+\sum_{I'\subsetneqq I\in\mathcal{H};\ a,a'\in\Gamma}\right\}\\
&\quad\times c_{I,a}c_{I',a'}\int_{\mathbb{R}^n}\left\{\sum_{Q\in\mathcal{D}:\ Q\subset I}\ell(Q)^{-2s}\,\triangle^\mu_{Q;1}h^{\mu,a}_{I;\kappa_2}\right\}h^{\mu,a'}_{I';\kappa_2}d\mu\\
&\equiv B_1+B_2
\end{aligned}$$

where $B_1$ and $B_2$ are symmetric. So it suffices to estimate

$$B_1 = \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} \int_{\mathbb{R}^n} \left\{ \sum_{Q\in\mathcal{D}:\ Q\subset I} \ell(Q)^{-2s} \bigtriangleup^{\mu}_{Q;1} h^{\mu,a}_{I;\kappa_2} \right\} \left\{ \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I'5,n,\kappa}:\ I\subsetneqq I'} c_{I',a'} h^{\mu,a'}_{I';\kappa_2} \right\} d\mu,$$

where

$$\left\| \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} c_{I',a'} h^{\mu,a}_{I';\kappa_2} \right\|^2_{L^2(\mu)} = \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2,$$

$$\left\| \sum_{Q\in\mathcal{D}:\ Q\subset I} \ell(Q)^{-2s} \bigtriangleup^{\mu}_{Q;1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)} = \sum_{Q\in\mathcal{D}:\ Q\subset I} \ell(Q)^{-4s} \left\| \bigtriangleup^{\mu}_{Q;1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)}.$$

So for $s<0$ we have the estimate,

$$\begin{aligned} B_1 &\le \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left| c_{I,a} \right| \ell(I)^{-2s} \left( \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2 \right)^{\frac{1}{2}} \\ &\le \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left| c_{I,a} \right|^2 \ell(I)^{-2s}} \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left( \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2 \right)}, \end{aligned}$$

where the second factor squared is

$$\begin{aligned} &\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left( \sum_{I'\in\mathcal{H};\ a'\in\Gamma:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2 \right) \\ &\qquad = \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2 \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}:\ I\subsetneqq I'} \ell(I)^{-2s} \\ &\qquad \approx \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I\subsetneqq I'} \left| c_{I',a'} \right|^2 \ell\left(I'\right)^{-2s} \end{aligned}$$

since $s<0$. Altogether we have

$$B_1 \lesssim \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left| c_{I,a} \right|^2 \ell(I)^{-2s} = C \left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)},$$

which together with the estimate for term $I$ proves our claim (2.11).

Now we claim that for $s>0$ we have

$$W^s_{\mathcal{D};\kappa_1}(\mu) \subset W^s_{\mathcal{D};\kappa_2}(\mu) \quad \text{for all } \kappa_1 \ge \kappa_2 \ge 1. \tag{2.13}$$

Indeed, for any subset $\mathcal{H}\subset\mathcal{D}$, we have

$$\left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_1}(\mu)} = \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}:\ Q\subset I} c_{I,a} \bigtriangleup^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)}$$

$$
\begin{aligned}
&= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \left\| \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)} \\
&\quad + \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \sum_{I,I'\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa},a'\in\Gamma_{I',n,\kappa}:\ Q\supset I\vee I'} \int_{\mathbb{R}^n} c_{I,a}\left(\triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2}\right) c_{I',a'}\left(\triangle^{\mu}_{Q;\kappa_1} h^{\mu,a'}_{I';\kappa_2}\right) d\mu \\
&\equiv A + B,
\end{aligned}
$$

where $I \vee I'$ denotes the smallest dyadic cube containing both $I$ and $I'$ if it exists; otherwise the sum over $Q \supset I \vee I'$ is empty. Just as before, the first term satisfies

$$
\begin{aligned}
A &= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \left\| \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)} \\
&= \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\| \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)} \\
&= \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \left\| h^{\mu,a}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_1}(\mu)} \leq C \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \left\| h^{\mu,a}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)} \\
&= C \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left(c_{I,a}\right)^2 \ell(I)^{-2s} = C \left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)},
\end{aligned}
$$

and where the final equality follows from Lemma 12.

To handle the second term $B$, we only need to consider the two cases $I' \subset I$ and $I' \cap I = \emptyset$, $\ell(I') \leq \ell(I)$. For the first case, we have the estimate,

$$
\begin{aligned}
B_{\text{case1}} &\equiv \left| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} \int_{\mathbb{R}^n} \left\{ \sum_{Q\in\mathcal{D}:\ Q\supset I} \ell(Q)^{-2s} \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\} \left\{ \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} c_{I',a'} h^{\mu,a'}_{I';\kappa_2} \right\} d\mu \right| \\
&\leq \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left|c_{I,a}\right| \left\| \sum_{Q\in\mathcal{D}:\ Q\supset I} \ell(Q)^{-2s} \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|_{L^2(\mu)} \left\| \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} c_{I',a'} h^{\mu,a'}_{I';\kappa_2} \right\|_{L^2(\mu)},
\end{aligned}
$$

where for $s > 0$,

$$
\left\| \sum_{Q\in\mathcal{D}:\ Q\supset I} \ell(Q)^{-2s} \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|_{L^2(\mu)} \leq \ell(I)^{-2s} \left\| \sum_{Q\in\mathcal{D}:\ Q\supset I} \triangle^{\mu}_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2} \right\|_{L^2(\mu)} = \ell(I)^{-2s},
$$

and

$$
\left\| \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} c_{I',a'} h^{\mu,a}_{I';\kappa_2} \right\|_{L^2(\mu)} = \sqrt{\sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} \left|c_{I',a'}\right|^2}.
$$

Thus we obtain the estimate

$$
\begin{aligned}
B_{\text{case1}} &\leq \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left|c_{I,a}\right| \ell(I)^{-2s} \sqrt{\sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} \left|c_{I',a'}\right|^2} \\
&\leq \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left|c_{I,a}\right|^2 \ell(I)^{-2s}} \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} \left|c_{I',a'}\right|^2}
\end{aligned}
$$

where

$$\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ I'\subsetneqq I} \left|c_{I',a'}\right|^2$$
$$= \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}} \left|c_{I',a'}\right|^2 \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}:\ I'\subsetneqq I} \ell(I)^{-2s}$$
$$\approx \sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}} \left|c_{I',a'}\right|^2 \ell\left(I'\right)^{-2s}.$$

Altogether we have

$$B_{\text{case1}} \lesssim \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \left|c_{I,a}\right|^2 \ell(I)^{-2s} \approx \left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)},$$

which is the desired estimate for $B_{\text{case1}}$.

Turning finally to the second case $I'\cap I=\emptyset$, $\ell(I)\le\ell\left(I'\right)$, we have

$$B_{\text{case2}} \equiv \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \sum_{\substack{I,I'\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa},a'\in\Gamma_{I',n,\kappa}:\ Q\supset I\vee I' \\ I'\cap I=\emptyset,\ell(I')\le\ell(I)}} \int_{\mathbb{R}^n} c_{I,a}\left(\triangle^\mu_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2}\right) c_{I',a'}\left(\triangle^\mu_{Q;\kappa_1} h^{\mu,a'}_{I';\kappa_2}\right) d\mu$$
$$= \sum_{K\in\mathcal{D}} \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-2s} \sum_{m=1}^{\infty} \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\le\ell(I)}}$$
$$\times \int_{\mathbb{R}^n} c_{I,a}\left(\triangle^\mu_{Q;\kappa_1} h^{\mu,a}_{I;\kappa_2}\right) c_{I',a'}\left(\triangle^\mu_{Q;\kappa_1} h^{\mu,a'}_{I';\kappa_2}\right) d\mu$$
$$= \sum_{K\in\mathcal{D}} \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-2s}\, \triangle^\mu_{Q;\kappa_1}\left( \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2}\right)$$
$$\times \triangle^\mu_{Q;\kappa_1}\left( \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\le\ell(I)}} c_{I',a'} h^{\mu,a'}_{I';\kappa_2}\right) d\mu.$$

Now we compute that for each $K\in\mathcal{D}$,

$$\left\| \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\le\ell(I)}} c_{I',a'} h^{\mu,a}_{I';\kappa_2} \right\|^2_{L^2(\mu)} = \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\le\ell(I)}} \left|c_{I',a'}\right|^2,$$

and

$$\left\| \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-2s}\, \triangle^\mu_{Q;\kappa_1}\left[ \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2}\right] \right\|^2_{L^2(\mu)}$$
$$= \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-4s} \left\| \triangle^\mu_{Q;\kappa_1}\left[ \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2}\right] \right\|^2_{L^2(\mu)}$$

$$\leq \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-4s} \left\| \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2} \right\|^2_{L^2(\mu)}$$
$$= C\ell(K)^{-4s} \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} \left|c_{I,a}\right|^2 = C2^{-4sm} \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-4s} \left|c_{I,a}\right|^2,$$

and hence

$$B_{\text{case2}} = \sum_{K\in\mathcal{D}} \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{D}:\ K\subset Q} \ell(Q)^{-2s} \triangle^{\mu}_{Q;\kappa_1} \left( \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu,a}_{I;\kappa_2} \right)$$
$$\times \triangle^{\mu}_{Q;\kappa_1} \left( \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\leq\ell(I)}} c_{I',a'} h^{\mu,a}_{I';\kappa_2} \right) d\mu$$
$$\leq \sum_{K\in\mathcal{D}} \sum_{m=1}^{\infty} C2^{-sm} \sqrt{\sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma} \ell(I)^{-2s} \left|c_{I,a}\right|^2}$$
$$\times \sqrt{\ell(K)^{-2s} \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\leq\ell(I)}} \left|c_{I',a'}\right|^2}$$
$$\leq \sum_{m=1}^{\infty} C2^{-sm} \sqrt{\sum_{K\in\mathcal{D}} \sum_{I\in\mathcal{H}\cap\mathfrak{C}^{(m)}(K);\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left|c_{I,a}\right|^2}$$
$$\times \sqrt{\sum_{K\in\mathcal{D}} \ell(K)^{-2s} \sum_{\substack{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}:\ \pi^{(m)}(I')\subset K \\ \ell(I')\leq\ell(I)}} \left|c_{I',a'}\right|^2},$$

and regrouping we obtain

$$B_{\text{case2}} \leq \sum_{m=1}^{\infty} C2^{-sm} \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left|c_{I,a}\right|^2} \sqrt{\sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}} \left|c_{I',a'}\right|^2 \sum_{K\in\mathcal{D}:\ \pi^{(m)}(I')\subset K} \ell(K)^{-2s}}$$
$$\leq \sum_{m=1}^{\infty} C2^{-2sm} \sqrt{\sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left|c_{I,a}\right|^2} \sqrt{\sum_{I'\in\mathcal{H};\ a'\in\Gamma_{I',n,\kappa}} \ell(I')^{-2s} \left|c_{I',a'}\right|^2}$$
$$\lesssim \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} \ell(I)^{-2s} \left|c_{I,a}\right|^2 \approx \left\| \sum_{I\in\mathcal{H};\ a\in\Gamma_{I,n,\kappa}} c_{I,a} h^{\mu}_{I;\kappa_2} \right\|^2_{W^s_{\mathcal{D};\kappa_2}(\mu)},$$

which is the desired estimate for $B_{\text{case2}}$.

Thus from (2.12) and (2.11) we obtain $W^s_{\mathcal{D};\kappa_1}(\mu) = W^s_{\mathcal{D};\kappa_2}(\mu)$ for all $s > 0$, all grids $\mathcal{D}$ and all integers $\kappa_1, \kappa_2 \in \mathbb{N}$. Duality now establishes these equalities for $s < 0$ as well, and the case $s = 0$ is automatic. □

Following Peetre [13] and Stein [21], we define the homogeneous *difference* Sobolev space $W^s_{\mathrm{diff};\kappa}(\mu)$ by

$$W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu) \equiv \left\{ f \in L^2(\mu) : \|f\|_{W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}}(\mu) < \infty \right\}, \qquad s \in \mathbb{R} \text{ and } \kappa \in \mathbb{N},$$

where

$$\|f\|^2_{W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu)} \equiv \sum_{Q\in\mathcal{D}} \int_Q \left| \frac{f(x) - \mathbb{E}^\mu_{Q;\kappa} f(x)}{\ell(Q)^s} \right|^2 d\mu(x),$$

$$\text{and } \mathbb{E}^\mu_{Q;\kappa} f(x) \equiv \left(E^\mu_Q f\right) \mathbf{1}_Q(x) = \left( \frac{1}{|Q|_\mu} \int_Q f d\mu \right) \mathbf{1}_Q(x).$$

The proof of the next lemma does not explicitly use the doubling property of $\mu$ either.

**Lemma 15** *Suppose $\mu$ is a doubling measure on $\mathbb{R}^n$ and $\mathcal{D}$ is a dyadic grid on $\mathbb{R}^n$. Then for $s > 0$ and $\kappa \in \mathbb{N}$, we have*

$$W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu) = W^s_{\mathcal{D};\kappa}(\mu),$$

*with equivalence of norms.*

***Proof*** We expand the function

$$f(x) - \mathbb{E}^\mu_{Q;\kappa} f(x) = \sum_{I\subset Q} \triangle^\mu_{I;\kappa} f(x),$$

and so obtain for $s > 0$ that

$$\begin{aligned} \|f\|^2_{W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}} &= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \sum_{I\subset Q} \left\| \triangle^\mu_{I;\kappa} f \right\|^2_{L^2(\mu)} \\ &= \sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left\| \triangle^\mu_{I;\kappa} f \right\|^2_{L^2(\mu)} \sum_{\substack{Q\in\mathcal{D}\\ Q\supset I}} \left( \frac{\ell(Q)}{\ell(I)} \right)^{-2s} \\ &\approx \sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left\| \triangle^\mu_{I;\kappa} f \right\|^2_{L^2(\mu)} = \|f\|^2_{W^s_{\mathcal{D};\kappa}(\mu)}. \end{aligned}$$

□

We just showed in Lemma 14 above that the weighted Alpert Sobolev spaces $W^s_{\mathcal{D};\kappa}(\mu)$ coincide for $s \in \mathbb{R}$ and $\kappa \geq 1$, and so we simply write $W^s_{\mathcal{D}}(\mu)$ for these spaces, and for specificity we use the norm of $W^s_{\mathcal{D};1}(\mu)$. Now we will show that the spaces $W^s_{\mathcal{D};\kappa}(\mu)$ are independent of the dyadic grid $\mathcal{D}$ for $|s|$ sufficiently small provided the measure $\mu$ is *doubling*, and extend this to include the difference spaces $W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu)$ as well.

**Theorem 16** *Suppose $\mu$ is a doubling measure on $\mathbb{R}^n$ and $\mathcal{D}$ and $\mathcal{E}$ are dyadic grids on $\mathbb{R}^n$. Then for $\kappa \in \mathbb{N}$, and $|s|$ sufficiently small, we have*

$$W^s_{\mathcal{D};\kappa}(\mu) = W^s_{\mathcal{E};\kappa}(\mu).$$

***Proof*** Note that for $J \in \mathcal{E}$ and $0 < \delta = 2^{-m} \leq 1$, and $F$ is any finite linear combination of Alpert wavelets,

$$\sum_{I\in\mathcal{D}:\ \ell(I)=\delta\ell(J)} \int_{\mathbb{R}^n} \left| \triangle^\mu_{I;\kappa} \triangle^\mu_{J;\kappa} F \right|^2 d\mu$$

$$= \sum_{I\in\mathcal{D}:\ \ell(I)=\delta\ell(J)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{I;\kappa}\left(\mathbf{1}_{H_\delta(J)}\,\triangle^\mu_{J;\kappa}\,F\right)\right|^2 d\mu \le \int_{H_\delta(J)} \left|\triangle^\mu_{J;\kappa}F\right|^2 d\mu$$

$$\le \int_{H_\delta(J)} \left\|\triangle^\mu_{J;\kappa}F\right\|^2_\infty d\mu = \left\|\triangle^\mu_{J;\kappa}F\right\|^2_\infty \left|H_\delta(J)\right|_\mu \le \left\|\triangle^\mu_{J;\kappa}F\right\|^2_\infty C\delta^\varepsilon \left|J\right|_\mu$$

$$\le C\delta^\varepsilon \int_{\mathbb{R}^n} \left|\triangle^\mu_{J;\kappa}F\right|^2 d\mu,$$

which gives for $\eta > 0$

$$\sum_{I\in\mathcal{D}:\ \ell(I)\le\ell(J)} \ell(I)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{I;\kappa}\,\triangle^\mu_{J;\kappa}\,F\right|^2 d\mu$$

$$= \sum_{m=1}^\infty \sum_{I\in\mathcal{D}:\ \ell(I)=2^{-m}\ell(J)} \ell(I)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{I;\kappa}\,\triangle^\mu_{J;\kappa}\,F\right|^2 d\mu$$

$$\le \sum_{m=1}^\infty C2^{m(2s+\eta-\varepsilon)}\ell(J)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{J;\kappa}F\right|^2 d\mu \le C_{\varepsilon,\eta,s}\ell(J)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{J;\kappa}F\right|^2 d\mu,$$

provided $s < \frac{\varepsilon-\eta}{2}$.

On the other hand, for $J \in \mathcal{E}$ and $\delta = 2^m > 1$, there are at most $2^n$ cubes $I$ such that $\ell(I) = \delta\ell(J)$ and $I \cap J \ne \emptyset$, and then following the line of reasoning in (2.14) we have,

$$\sum_{I\in\mathcal{D}:\ \ell(I)>\ell(J)} \ell(I)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{I;\kappa}\,\triangle^\mu_{J;\kappa}\,F\right|^2 d\mu$$

$$= \sum_{I\in\mathcal{D}:\ \ell(I)>\ell(J)\text{ and }I\cap J\ne\emptyset} \ell(I)^{-(2s+\eta)} \int_{\mathbb{R}^n} \left|\triangle^\mu_{I;\kappa}\,\triangle^\mu_{J;\kappa}\,F\right|^2 d\mu$$

$$\approx \sum_{I\in\mathcal{D}:\ \ell(I)>\ell(J)\text{ and }I\cap J\ne\emptyset} \ell(I)^{-(2s+\eta)} \left|I\right|_\mu \left\|\triangle^\mu_{I;\kappa}\,\triangle^\mu_{J;\kappa}\,F\right\|^2_\infty$$

$$\approx \left|J\right|_\mu \sum_{I\in\mathcal{D}:\ \ell(I)>\ell(J)\text{ and }I\cap J\ne\emptyset} \ell(I)^{-(2s+\eta)} \frac{\left|J\right|_\mu}{\left|I\right|_\mu} \left\|\triangle^\mu_{J;\kappa}F\right\|^2_\infty$$

$$\lesssim \left|J\right|_\mu \left\|\triangle^\mu_{J;\kappa}F\right\|^2_\infty C_\eta \ell(J)^\eta \sum_{m=1}^\infty 2^{-m(2s+\eta)}\ell(J)^{-(2s+\eta)} \frac{\left|J\right|_\mu}{\left|\pi^n J\right|_\mu}$$

$$\le \ell(J)^{-2s} \int_{\mathbb{R}^n} \left|\triangle^\mu_{J;\kappa}F\right|^2 d\mu \sum_{m=1}^\infty 2^{-m(2s+\eta)} c2^{-m\theta^{\mathrm{rev}}_\mu} \le C\ell(J)^{-2s} \int_{\mathbb{R}^n} \left|\triangle^\mu_{J;\kappa}F\right|^2,$$

provided $s > \frac{\eta-\theta^{\mathrm{rev}}_\mu}{2}$ where $\theta^{\mathrm{rev}}_\mu > 0$ is the reverse doubling exponent of $\mu$, i.e.

$$\left|\pi^n J\right|_\mu \ge c2^{n\theta^{\mathrm{rev}}_\mu} \left|J\right|_\mu .$$

Now we compute

$$\|f\|^2_{W^s_{\mathcal{D};\kappa}(\mu)} = \sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left\|\triangle^\mu_{I;\kappa}f\right\|^2_{L^2(\mu)} = \sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left\|\triangle^\mu_{I;\kappa}\left(\sum_{J\in\mathcal{E}}\triangle^\mu_{J;\kappa}\right)f\right\|^2_{L^2(\mu)}$$

$$\leq 2\sum_{I\in\mathcal{D}}\ell(I)^{-2s}\left\|\sum_{J\in\mathcal{E}:\ \ell(I)\leq\ell(J)}\triangle_{I;\kappa}^{\mu}\triangle_{J;\kappa}^{\mu}f\right\|_{L^2(\mu)}^2$$

$$+2\sum_{I\in\mathcal{D}}\ell(I)^{-2s}\left\|\sum_{J\in\mathcal{E}:\ \ell(I)>\ell(J)}\triangle_{I;\kappa}^{\mu}\triangle_{J;\kappa}^{\mu}f\right\|_{L^2(\mu)}^2.$$

We bound the first sum by

$$C_\eta\sum_{I\in\mathcal{D}}\ell(I)^{-2s}\sum_{J\in\mathcal{E}:\ \ell(I)\leq\ell(J)}\left(\frac{\ell(J)}{\ell(I)}\right)^{\eta}\left\|\triangle_{I;\kappa}^{\mu}\triangle_{J;\kappa}^{\mu}f\right\|_{L^2(\mu)}^2$$

$$\leq C_\eta\sum_{J\in\mathcal{E}}\ell(J)^{\eta}\sum_{I\in\mathcal{D}:\ \ell(I)\leq\ell(J)}\ell(I)^{-(2s+\eta)}\left\|\triangle_{I;\kappa}^{\mu}\triangle_{J;\kappa}^{\mu}f\right\|_{L^2(\mu)}^2$$

$$\leq C_\eta\sum_{J\in\mathcal{E}}\ell(J)^{\eta}C_{\varepsilon,\eta,s}\ell(J)^{-(2s+\eta)}\int_{\mathbb{R}^n}\left|\triangle_{J;\kappa}^{\mu}f\right|^2d\mu$$

$$\leq C_{\varepsilon,\eta,s}\sum_{J\in\mathcal{E}}\ell(J)^{-2s}\int_{\mathbb{R}^n}\left|\triangle_{J;\kappa}^{\mu}f\right|^2d\mu=C_{\varepsilon,\eta,s}\left\|f\right\|_{W_{\mathcal{E};\kappa}^{s}(\mu)}^2,$$

and the second sum by

$$\sum_{I\in\mathcal{D}}\ell(I)^{-2s}\sum_{J\in\mathcal{E}:\ \ell(I)>\ell(J)}\left(\frac{\ell(I)}{\ell(J)}\right)^{\eta}\left\|\triangle_{I;\kappa}^{\mu}\triangle_{J;\kappa}^{\mu}f\right\|_{L^2(\mu)}^2\leq C_\eta\left\|f\right\|_{W_{\mathcal{E};\kappa}^{s}(\mu)}^2$$

provided $s>\frac{\eta-\theta_\mu^{\mathrm{rev}}}{2}$. Altogether we obtain

$$\left\|f\right\|_{W_{\mathcal{D};\kappa}^{s}(\mu)}\leq C_\eta\left\|f\right\|_{W_{\mathcal{E};\kappa}^{s}(\mu)}$$

provided

$$\frac{\eta-\theta_\mu^{\mathrm{rev}}}{2}<s<\frac{\varepsilon-\eta}{2},$$

and interchanging the roles of the dyadic grids $\mathcal{D}$ and $\mathcal{E}$ completes the proof. □

Finally, we will explicitly compute the norm $\left\|f\right\|_{W_{\mathcal{D}_{\mathrm{diff}};1}^{s}(\mu)}$ by starting with

$$\frac{1}{|Q|_\mu}\int_Q\int_Q(f(x)-f(y))^2d\mu(x)\,d\mu(y)$$

$$=\frac{1}{|Q|_\mu}\int_Q\int_Q\left\{f(x)^2-2f(x)f(y)+f(y)^2\right\}d\mu(x)\,d\mu(y)$$

$$=2\int_Q f^2d\mu-2|Q|_\mu\left(E_Q^\mu f\right)^2=2\int_Q\left\{f(x)^2-\left(E_Q^\mu f\right)^2\right\}d\mu,$$

to obtain the representation

$$\left\|f\right\|_{W_{\mathcal{D}_{\mathrm{diff}};1}^{s}(\mu)}^2=\sum_{Q\in\mathcal{D}}\int_Q\left|\frac{f(x)-E_Q^\mu f}{\ell(Q)^s}\right|^2d\mu(x)$$

$$=\sum_{Q\in\mathcal{D}}\ell(Q)^{-2s}\int_Q\left\{f(x)^2-2\left(E_Q^\mu f\right)f(x)+\left(E_Q^\mu f\right)^2\right\}d\mu(x)$$

$$= \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \int_Q \left\{ f(x)^2 - \left(E_Q^\mu f\right)^2 \right\} d\mu(x)$$
$$= \frac{1}{2} \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \frac{1}{|Q|_\mu} \int_Q \int_Q (f(x) - f(y))^2 \, d\mu(x) \, d\mu(y).$$

We will next show that this last expression is comparable to the expression

$$\|f\|^2_{W^s(\mu)} \equiv \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{f(x) - f(y)}{|x-y|^s} \right)^2 \frac{d\mu(x) \, d\mu(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right) \right|_\mu}.$$

**Theorem 17** *Suppose that $\mu$ is doubling on $\mathbb{R}^n$. For $s > 0$ sufficiently small, we have*

$$\|f\|^2_{W^s_{\mathcal{D}_{\mathrm{diff}};1}(\mu)} \approx \|f\|^2_{W^s(\mu)}.$$

***Proof*** From the formula above we have

$$\|f\|^2_{W^s_{\mathcal{D}_{\mathrm{diff}};1}(\mu)} = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \sum_{Q\in\mathcal{D}} \frac{\mathbf{1}_{Q\times Q}(x,y)}{\ell(Q)^{2s} |Q|_\mu} \right\} (f(x) - f(y))^2 \, d\mu(x) \, d\mu(y)$$
$$\leq \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{C}{|x-y|^{2s} \left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right) \right|_\mu} (f(x) - f(y))^2 \, d\mu(x) \, d\mu(y) = \frac{C}{2} \|f\|^2_{W^s(\mu)},$$

since $|Q|_\mu \gtrsim \left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right) \right|_\mu$ whenever $(x, y) \in Q \times Q$. Conversely we use the one third trick for dyadic grids. Namely that there is a finite collection of dyadic grids $\{\mathcal{D}_m\}_{m=1}^{3^n}$ so that for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, there is some $m$ and some $Q \in \mathcal{D}_m$ such that

$$B(x, C|x-y|), B(y, C|x-y|) \subset Q \quad \text{and} \quad \ell(Q) \leq C|x-y|,$$

where $C$ is a large constant that will be fixed below. In particular this gives

$$|Q|_\mu \approx |B(x, c|x-y|)|_\mu \approx |B(y, c|x-y|)|_\mu, \quad \text{for } \frac{1}{100} < c < \frac{1}{2}.$$

Then we cover the product space $\mathbb{R}^n \times \mathbb{R}^n$ with a collection of product balls

$$\{B(x_k, c|x_k - y_k|) \times B(y_k, c|x_k - y_k|)\}_{k=1}^\infty$$

where $E \equiv \{(x_k, y_k)\}_{k=1}^\infty$ is a discrete subset of $\mathbb{R}^n \times \mathbb{R}^n$, and provided $c$ is chosen sufficiently small, this collection of product balls has bounded overlap. Now denote by $Q_k$ the cube chosen above by the point $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have

$$|Q_k|_\mu \approx |B(x, c|x-y|)|_\mu \approx |B(y, c|x-y|)|_\mu$$
$$\text{for all } (x, y) \text{ in the product ball } B(x_k, c|x_k - y_k|) \times B(y_k, c|x_k - y_k|),$$

and so

$$\|f\|^2_{W^s(\mu)} \leq \sum_{k=1}^\infty \int_{B(x_k, c|x_k - y_k|)} \int_{B(y_k, c|x_k - y_k|)} \left( \frac{f(x) - f(y)}{|x-y|^s} \right)^2 \frac{d\mu(x) \, d\mu(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right) \right|_\mu}$$
$$\lesssim \sum_{k=1}^\infty \int_{Q_k} \int_{Q_k} \left( \frac{f(x) - f(y)}{\ell(Q_k)^s} \right)^2 \frac{d\mu(x) \, d\mu(y)}{|Q_k|_\mu}$$

$$\leq \sum_{m=1}^{3^n} \sum_{Q\in\mathcal{D}_m} \ell(Q)^{-2s} \frac{1}{|Q|_\mu} \int_Q \int_Q (f(x) - f(y))^2 \, d\mu(x) \, d\mu(y)$$

$$= 2 \sum_{m=1}^{3^n} \|f\|^2_{W^s_{\mathcal{D}_m,\mathrm{diff};1}(\mu)} \leq C \|f\|^2_{W^s_{\mathcal{D}_{\mathrm{diff}};1}(\mu)}$$

since $\|f\|^2_{W^s_{\mathcal{D}_m,\mathrm{diff};1}(\mu)}$ is independent of the dyadic grid $\mathcal{D}_m$ by Theorem 16. □

In particular, we have thus obtained one of the main results of this subsection.

**Theorem 18** *For all grids $\mathcal{D}$ on $\mathbb{R}^n$, all positive integers $\kappa$, and all sufficiently small $s > 0$ depending only on the doubling constant of $\mu$, we have*

$$W^s(\mu) = W^s_{\mathcal{D};\kappa}(\mu) = W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu),$$

*with equivalence of norms.*

As a consequence of this theorem, there is essentially just one notion of a weighted Sobolev space for a doubling measure $\mu$ provided $|s|$ is sufficiently small, namely $W^s(\mu)$ when $s > 0$, and any of the dyadic spaces $W^s_{\mathcal{D};\kappa}(\mu)$ when $s < 0$. For specificity we will use the norm of $W^s_{\mathcal{D}_0;1}(\mu)$ on these spaces, where $\mathcal{D}_0$ is the standard dyadic grid on $\mathbb{R}^n$.

**Definition 19** Define $W^s_{\mathrm{dyad}}(\mu) = W^s_{\mathcal{D};\kappa}(\mu) = W^s_{\mathcal{D}_{\mathrm{diff}};\kappa}(\mu)$ for $|s|$ sufficiently small, and norm $W^s_{\mathrm{dyad}}(\mu)$ with the norm of $W^s_{\mathcal{D};1}(\mu)$.

Note that $W^s_{\mathrm{dyad}}(\mu) = W^s(\mu)$ for $s > 0$ sufficiently small.

***Remark 20*** The Sobolev space $W^s(\mu)$ used here is different from the Sobolev space introduced on a space of homogeneous type in [5], since one can show that the norm squared used in [5] is comparable to

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{f(x) - f(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right)\right|_\mu^\alpha} \right)^2 \frac{d\mu(x) \, d\mu(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right)\right|_\mu}.$$

It seems likely that our proof extends to the analogous $T1$ theorem for these weighted Sobolev spaces using the doubling measure inequalities $\frac{|Q|_\mu}{|2^m Q|_\mu} \lesssim \left(\frac{\ell(Q)}{\ell(2^m Q)}\right)^{\theta^{\mathrm{rev}}_\mu}$ and $\frac{|2^m Q|_\mu}{|Q|_\mu} \lesssim \left(\frac{\ell(2^m Q)}{\ell(Q)}\right)^{\theta^{\mathrm{doub}}_\mu}$.

**Problem 21** *Does a $T1$ theorem hold in the context of weighted Sobolev spaces with doubling measures and norm squared given by*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{f(x) - f(y)}{\varphi(|x-y|)} \right)^2 \frac{d\mu(x) \, d\mu(y)}{\left| B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right)\right|_\mu},$$

*where $\varphi : (0, \infty) \to (0, \infty)$ satisfies*

$$\left(\frac{s}{t}\right)^{\theta_1} \lesssim \frac{\varphi(s)}{\varphi(t)} \lesssim \left(\frac{s}{t}\right)^{\theta_2} \text{ for } 0 < s \leq t < \infty\,?$$

***Remark 22*** The inner product for the Hilbert space $W^s_{\text{dyad}}(\mu)$ is given by

$$\langle f, g\rangle_{W^s_{\text{dyad}}(\mu)} \equiv \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\langle \triangle^\mu_{Q;\kappa} f, \triangle^\mu_{Q;\kappa} g\right\rangle_{L^2(\mu)}$$

where the inner product for $L^2(\mu)$ is given by

$$\langle f, g\rangle_{L^2(\mu)} \equiv \int_{\mathbb{R}^n} f(x)\, g(x)\, d\mu(x).$$

In Lemma 12 we showed that $\left\{\ell(Q)^s \triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ is a complete set of orthogonal projections on $W^s_{\text{dyad}}(\mu)$, nevertheless we will not use the Hilbert space duality that identifies the dual of a Hilbert space with the conjugate of itself under the inner product $\langle f, g\rangle_{W^s_{\text{dyad}}(\mu)}$, but rather the $L^2(\mu)$ inner product which identifies the dual of $W^s_{\text{dyad}}(\mu)$ with $W^{-s}_{\text{dyad}}(\mu)$. As mentioned in the introduction, the reason for this is that the weighted Alpert projections $\left\{\triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ satisfy telescoping identities, while the orthogonal projections $\left\{\ell(Q)^s \triangle^\mu_{Q;\kappa}\right\}_{Q\in\mathcal{D}}$ do not.

## 2.4 Haar, Alpert and indicator functions

The Alpert projections $\left\{\triangle^\mu_{I;\kappa}\right\}_{I\in\mathcal{D}}$ form a complete family of orthogonal projections on $L^2(\mu)$, where

$$\triangle^\mu_{I;\kappa} f \equiv \mathbb{E}^\mu_{I;\kappa} f - \sum_{I'\in\mathfrak{C}_{\mathcal{D}}(I)} \mathbb{E}^\mu_{I';\kappa} f = \sum_{a\in\Gamma} \left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)} h^{\mu,a}_{I;\kappa}.$$

Thus we have

$$\begin{aligned}
\sum_{a\in\Gamma} \left|\left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)}\right|^2 \left\|h^{\mu,a}_{I;\kappa}\right\|^2_{W^s_{\text{dyad}}(\mu)} &= \sum_{a\in\Gamma} \left\|\left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)} h^{\mu,a}_{I;\kappa}\right\|^2_{W^s_{\text{dyad}}(\mu)} \\
= \left\|\triangle^\mu_{I;\kappa} f\right\|^2_{W^s_{\text{dyad}}(\mu)} = \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\|\triangle^\mu_{Q;\kappa} \triangle^\mu_{I;\kappa} f\right\|^2_{L^2(\mu)} &= \ell(I)^{-2s} \left\|\triangle^\mu_{I;\kappa} f\right\|^2_{L^2(\mu)} \\
= \ell(I)^{-2s} \sum_{a\in\Gamma} \left|\left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)}\right|^2 &
\end{aligned}$$

for all $f \in W^s_{\text{dyad}}(\mu)$, which implies upon taking $f = h^{\mu,a}_{I;\kappa}$, that

$$\left\|h^{\mu,a}_{I;\kappa}\right\|^2_{W^s_{\text{dyad}}(\mu)} = \ell(I)^{-2s}, \quad I \in \mathcal{D}.$$

Now we compute the weighted Alpert Sobolev norms of indicators. By independence of $\kappa$, we may assume $\kappa = 1$. Using (2.8 ) and (2.9), we then note that

$$\left\|\triangle^\mu_Q \mathbf{1}_I\right\|^2_\infty \approx \frac{\left\|\triangle^\mu_Q \mathbf{1}_I\right\|^2_{L^2(\mu)}}{|Q|_\mu} \approx \frac{\frac{|I|^2_\mu}{|Q|_\mu}}{|Q|_\mu} = \left(\frac{|I|_\mu}{|Q|_\mu}\right)^2 \quad \text{for } I \subset Q' \in \mathfrak{C}_{\mathcal{D}}(Q),$$

and hence from (2.8) we obtain

$$\|\mathbf{1}_I\|^2_{W^s_{\mathrm{dyad}}(\mu)} = \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\|\triangle^\mu_{Q;1}\mathbf{1}_I\right\|^2_{L^2(\mu)} = \sum_{Q\in\mathcal{D}:\ Q\supsetneqq I} \ell(Q)^{-2s} \left\|\triangle^\mu_{Q;1}\mathbf{1}_I\right\|^2_{L^2(\mu)} \tag{2.14}$$

$$\approx \sum_{Q\in\mathcal{D}:\ Q\supsetneqq I} \ell(Q)^{-2s} |Q|_\mu \left\|\triangle^\mu_{Q;1}\mathbf{1}_I\right\|^2_\infty \approx |I|_\mu \sum_{Q\in\mathcal{D}:\ Q\supsetneqq I} \ell(Q)^{-2s} \frac{|I|_\mu}{|Q|_\mu}$$

$$= |I|_\mu \sum_{n=1}^\infty 2^{-2ns}\ell(I)^{-2s} \frac{|I|_\mu}{|\pi^n I|_\mu} = \ell(I)^{-2s} |I|_\mu \sum_{n=1}^\infty 2^{-2ns} \frac{|I|_\mu}{|\pi^n I|_\mu}.$$

Since $\mu$ is doubling, it also satisfies a *dyadic reverse doubling* condition with reverse doubling exponent $\theta^{\mathrm{rev}}_\mu > 0$ depending on the doubling constant, i.e.

$$\left|\pi^n I\right|_\mu \geq c2^{n\theta^{\mathrm{rev}}_\mu} |I|_\mu .$$

Then for $s > -\theta^{\mathrm{rev}}_\mu$ we have $\sum_{n=1}^\infty 2^{-2ns} \frac{|I|_\mu}{|\pi^n I|_\mu} \leq \sum_{n=1}^\infty 2^{-2n\left(s+\theta^{\mathrm{rev}}_\mu\right)} < \infty$, and so

$$\|\mathbf{1}_I\|^2_{W^s_{\mathrm{dyad}}(\mu)} \approx \ell(I)^{-2s} |I|_\mu .$$

Altogether we have proved the following lemma.

**Lemma 23** *Suppose $\mu$ is a locally finite positive Borel measure on $\mathbb{R}^n$. Then*

$$\left\|h^{\mu,a}_{I;\kappa}\right\|_{W^s_{\mathrm{dyad}}(\mu)} = \ell(I)^{-s}, \quad \textit{for all } I\in\mathcal{D}, a\in\Gamma_{I,n,\kappa}, \kappa\geq 1 \textit{ and } s\in\mathbb{R},$$

*and if $\mu$ is a doubling measure,*

$$\|\mathbf{1}_I\|_{W^s_{\mathrm{dyad}}(\mu)} \approx \ell(I)^{-s}\sqrt{|I|_\mu}, \quad \textit{for all } I\in\mathcal{D}, \kappa\geq 1 \textit{ and } s > -\theta^{\mathrm{rev}}_\mu .$$

### 2.4.1 Sharpness

Here we construct measures for which $\|\mathbf{1}_I\|^2_{W^s_{\mathrm{dyad}}(\mu)} = \infty$ for all intervals $I$ and $s<0$, and thus are *not* dyadic reverse doubling. A trivial example is any finite measure $\mu$, and an infinite example is $d\mu(x) = \mathbf{1}_{[e,\infty)}(x)\frac{1}{x\ln x}$. In fact we have the following lemma.

**Lemma 24** *If there is $s<0$ such that*

$$\|\mathbf{1}_I\|^2_{W^s_{\mathrm{dyad}}(\mu)} \leq C\ell(I)^{-2s} |I|_\mu$$

*for all dyadic intervals $I$, then $\mu$ is a dyadic reverse doubling measure with exponent $|s|$.*

***Proof*** We have

$$\sum_{n=0}^\infty 2^{(n+1)|s|}\frac{1}{|\pi^n I|_\mu} \lesssim |I|^{-2}_\mu \ell(I)^{2s} \|\mathbf{1}_I\|^2_{W^s_{\mathrm{dyad}}(\mu)} \leq C|I|^{-1}_\mu, \quad \text{for all intervals } I,$$

which shows that $|\pi^n I|_\mu \geq \frac{1}{C}2^{(n+1)|s|}|I|_\mu$ for all intervals $I$, which is the dyadic reverse doubling condition with $t=|s|$. □

***Remark 25*** There is an asymmetry inherent in the homogeneous Sobolev two weight inequality (3.1) for general measures. If we wish to use *cube testing* to characterize the Sobolev inequality (3.1) for some $s>0$ *assuming* the estimate $\|\mathbf{1}_I\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)} \leq C\ell(I)^{2s}|I|_\omega$, then from the equivalence with the bilinear inequality (3.2) and the discussion and lemma above, we see that $\omega$ needs to be restricted, in fact by reverse dyadic doubling with exponent essentially greater than $2|s|$, while no restriction needs to be made on $\sigma$.

### 2.4.2 Norms of moduli of Alpert wavelets

Let $\mathbf{h}^\omega_{J;\kappa} = \left(h^{\omega,a}_{J;\kappa}\right)_{a\in\Gamma}$ be the vector of Alpert wavelets associated with the cube $J$. Note that for $\omega$ doubling and $0\leq s<1$, we have

$$\frac{\left\|\mathbf{h}^\omega_{J;\kappa}\right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}{\ell(J)^{2s}} = 1,$$

and we now show that the same sort of Sobolev estimate holds for the absolute value $\left|\mathbf{h}^\omega_{J;\kappa}\right|$ of the vector Alpert wavelet (which remains trivial in the case $\kappa=1$ since $\mathbf{h}^\omega_{J;\kappa}$ is then constant on dyadic children of $J$).

**Lemma 26** *Let $\mu$ be a doubling measure on $\mathbb{R}^n$. Then the modulus of a vector of Alpert wavelets $\mathbf{h}^\mu_{J;\kappa} = \left\{h^{\mu,a}_{J;\kappa}\right\}_{a\in\Gamma_{J,n,\kappa}}$ satisfies*

$$\frac{\left\|\left|\mathbf{h}^\mu_{J;\kappa}\right|\right\|^2_{W^{-s}_{\mathrm{dyad}}(\mu)}}{\ell(J)^{2s}} \lesssim C, \quad \textit{for } J\in\mathcal{D}.$$

***Proof*** We expand

$$\left\|\left|h^{\mu,a}_{J;\kappa}\right|\right\|^2_{W^{-s}_{\mathrm{dyad}}(\mu)} = \sum_Q \ell(Q)^{2s}\left\|\triangle^\mu_{Q;\kappa}\left|h^{\mu,a}_{J;\kappa}\right|\right\|^2_{L^2(\mu)}.$$

For a cube $Q$ contained in a child $J'$ of $J$ that is disjoint from the zero set of the polynomial $\mathbf{1}_{J'}h^{\mu,a}_{J;\kappa}$ on the child $J'$, the absolute values on the Alpert wavelet can be removed, and we obtain that $\triangle^\mu_{Q;\kappa}\left|h^{\mu,a}_{J;\kappa}\right| = \pm\triangle^\mu_{Q;\kappa}h^{\mu,a}_{J;\kappa}$ vanishes. On the other hand, if $Q\subset J'$ intersects the zero set $Z$ of the polynomial $\mathbf{1}_{J'}h^{\mu,a}_{J;\kappa}$, then we use $\left\|\triangle^\mu_{Q;\kappa}f\right\|_{L^2(\mu)} \leq C\frac{|\widehat{f}(Q)|}{\sqrt{|Q|_\mu}}$ to obtain the crude estimate,

$$\begin{aligned}\left\|\triangle^\mu_{Q;\kappa}\left|h^{\mu,a}_{J;\kappa}\right|\right\|^2_{L^2(\mu)} &= \left|\left\langle h^{\mu,a}_{Q;\kappa},\left|h^{\mu,a}_{J;\kappa}\right|\right\rangle_{L^2(\mu)}\right|^2\int_{\mathbb{R}^n}\left|h^{\mu,a}_{J;\kappa}\right|^2 d\mu = \left|\left\langle h^{\mu,a}_{Q;\kappa},\left|h^{\mu,a}_{J;\kappa}\right|\right\rangle_{L^2(\mu)}\right|^2\\ &\lesssim \left(\int_Q\left|h^{\mu,a}_{J;\kappa}\right|^2 d\mu\right)\left\|h^{\mu,a}_{J;\kappa}\right\|^2_\infty |Q|_\mu \lesssim \frac{|Q|_\mu}{|J|_\mu},\end{aligned}$$

and together with Corollary 8, we obtain

$$\sum_{Q\subset J':\ Q\cap Z\neq\emptyset}\ell\quad(Q)^{2s}\left\|\triangle^\mu_{Q;\kappa}\left|h^{\mu,a}_{J;\kappa}\right|\right\|^2_{L^2(\mu)} \lesssim \sum_{Q\subset J':\ Q\cap Z\neq\emptyset}\ell(Q)^{2s}\frac{|Q|_\mu}{|J|_\mu}$$

$$= \sum_{m=1}^{\infty} \left(2^{-m}\ell(J)\right)^{2s} \sum_{\substack{Q\subset J':\ Q\cap Z\neq\emptyset \\ \ell(Q)=2^{-m}\ell(J)}} \frac{|Q|_\mu}{|J|_\mu}$$
$$\lesssim \left( \sum_{m=1}^{\infty} 2^{-2ms} \sum_{\substack{Q\subset J':\ Q\cap Z\neq\emptyset \\ \ell(Q)=2^{-m}\ell(J)}} \frac{|Q|_\mu}{|J|_\mu} \right) \ell(J)^{2s}$$
$$\lesssim \left( \sum_{m=1}^{\infty} 2^{-2ms} 2^{-\varepsilon m} \right) \ell(J)^{2s}$$

which is the estimate we want when $s > 0$ or $-\frac{\varepsilon}{2} < s \le 0$, where $\varepsilon = \varepsilon(\mu) > 0$.

Finally, for big cubes $Q$ containing $J$ there is only the tower above $J$ to consider, and trivial estimates work:

$$\sum_{Q\in\mathcal{D}:\ Q\supset J} \ell(Q)^{2s} \left\| \triangle^{\mu}_{Q;\kappa} \left| h^{\mu,a}_{J;\kappa} \right| \right\|^2_{L^2(\mu)}$$
$$= \sum_{Q\in\mathcal{D}:\ Q\supset J} \ell(Q)^{2s} \left| \left\langle h^{\mu,a}_{Q;\kappa}, \left| h^{\mu}_{J;\kappa} \right| \right\rangle_{L^2(\mu)} \right|^2 \int_{\mathbb{R}^n} \left| h^{\mu,a}_{J;\kappa} \right|^2 d\mu$$
$$\le \sum_{m=1}^{\infty} 2^{2ms} \left| \left\langle h^{\mu,a}_{\pi^{(m)}J;\kappa}, \left| h^{\mu,a}_{J;\kappa} \right| \right\rangle_{L^2(\mu)} \right|^2 \ell(J)^{2s} \lesssim \sum_{m=1}^{\infty} 2^{2ms} \left( \int_J \left| h^{\mu,a}_{\pi^{(m)}J;\kappa} \right|^2 d\mu \right) \ell(J)^{2s}$$
$$\lesssim \sum_{m=1}^{\infty} 2^{2ms} \frac{|J|_\mu}{\left|\pi^{(m)}J\right|_\mu} \ell(J)^{2s} \lesssim \ell(J)^{2s},$$

provided $s < 0$ or $s$ is small enough depending on the doubling constant of $\mu$. □

### 2.5 Duality

Here we compute the dual space of $W^s_{\mathrm{dyad}}(\mu)$ under the $L^2(\mu)$ pairing

$$\langle f, g\rangle_{L^2(\mu)} = \int_{\mathbb{R}^n} f(x)\, g(x)\, d\mu(x) = \sum_{I\in\mathcal{D}, J\in\mathcal{D}} \int_{\mathbb{R}^n} \triangle^\mu_I f\ \triangle^\mu_J g d\mu = \sum_{I\in\mathcal{D}} \int_{\mathbb{R}^n} \triangle^\mu_I f\ \triangle^\mu_I g d\mu.$$

**Lemma 27** *Let $-1 < s < 1$. Then*

$$\left(W^s_{\mathrm{dyad}}(\mu)\right)^* = W^{-s}_{\mathrm{dyad}}(\mu),$$

*holds in the sense that if $g \in W^{-s}_{\mathrm{dyad}}(\mu)$ then $f \to \langle f, g\rangle_{L^2(\mu)}$ defines a bounded linear functional on $W^s_{\mathrm{dyad}}(\mu)$, and conversely that every bounded linear functional on $W^s_{\mathrm{dyad}}(\mu)$ arises in this way.*

***Proof*** For $\kappa$ sufficiently large, Cauchy–Schwarz gives

$$\left|\langle f, g\rangle_{L^2(\mu)}\right| = \left| \left\langle \sum_{I\in\mathcal{D}} \triangle^\mu_{I;\kappa} f, \sum_{J\in\mathcal{D}} \triangle^\mu_{J;\kappa} g \right\rangle_{L^2(\mu)} \right|$$

$$
\begin{aligned}
&= \left| \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{D}} \int_{\mathbb{R}^n} \left(\triangle^\mu_{I;\kappa} f\right) \left(\triangle^\mu_{J;\kappa} g\right) d\mu \right| \\
&= \left| \sum_{I\in\mathcal{D}} \int_{\mathbb{R}^n} \left(\triangle^\mu_{I;\kappa} f\right) \left(\triangle^\mu_{I;\kappa} g\right) d\mu \right| \\
&= \left| \sum_{I\in\mathcal{D}} \int_{\mathbb{R}^n} \ell(I)^{-s} \left(\triangle^\mu_{I;\kappa} f\right) \ell(I)^{s} \left(\triangle^\mu_{I;\kappa} g\right) d\mu \right| \\
&\leq \sqrt{\int_{\mathbb{R}^n} \sum_{I\in\mathcal{D}} \left| \ell(I)^{-s} \left(\triangle^\mu_{I;\kappa} f\right)\right|^2 d\mu} \sqrt{\int_{\mathbb{R}^n} \sum_{I\in\mathcal{D}} \left| \ell(I)^{s} \left(\triangle^\mu_{I;\kappa} g\right)\right|^2 d\mu} \\
&= \|f\|_{W^s_{\mathrm{dyad}}(\mu)} \|g\|_{W^{-s}_{\mathrm{dyad}}(\mu)} .
\end{aligned}
$$

Conversely, if $\Lambda \in W^s_{\mathrm{dyad}}(\mu)^*$ is a continuous linear functional on $W^s_{\mathrm{dyad}}(\mu)$, then for $\kappa$ sufficiently large

$$
\begin{aligned}
&\left| \sum_{I\in\mathcal{D}} \ell(I)^{-s} \left\langle f, h^\mu_{I;\kappa} \right\rangle_{L^2(\mu)} \ell(I)^s \Lambda h^\mu_{I;\kappa} \right| \\
&\quad = \left| \sum_{I\in\mathcal{D}} \left\langle f, h^\mu_{I;\kappa} \right\rangle_{L^2(\mu)} \Lambda h^\mu_{I;\kappa} \right| = \left| \sum_{I\in\mathcal{D}} \Lambda \left(\triangle^\mu_{I;\kappa} f\right) \right| = |\Lambda f| \\
&\quad \leq \|\Lambda\| \|f\|_{W^s_{\mathrm{dyad}}(\mu)} = \|\Lambda\| \sqrt{\int_{\mathbb{R}^n} \sum_{I\in\mathcal{D}} \left| \ell(I)^{-s} \left(\triangle^\mu_{I;\kappa} f\right)\right|^2 d\mu} \\
&\quad = \|\Lambda\| \sqrt{\sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left| \left\langle f, h^\mu_{I;\kappa} \right\rangle_{L^2(\mu)} \right|^2}
\end{aligned}
$$

for all choices of coefficients $\left\{ \ell(I)^{-s} \left\langle f, h^\mu_{I;\kappa} \right\rangle_{L^2(\mu)} \right\}_{I\in\mathcal{D}} \in \ell^2(\mathcal{D})$, and so we have $\ell(I)^s \Lambda h^\mu_{I;\kappa} \in \ell^2(\mathcal{D})$, i.e.

$$
\sqrt{\sum_{I\in\mathcal{D}} \ell(I)^{2s} \left| \Lambda h^\mu_{I;\kappa} \right|^2} \leq \|\Lambda\| .
$$

Thus if we define $g$ to have Alpert coefficients $\Lambda h^{\mu,a}_{I;\kappa}$, i.e.

$$
g = \sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \left\langle g, h^{\mu,a}_{I;\kappa} \right\rangle_{L^2(\mu)} h^{\mu,a}_{I;\kappa} \equiv \sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \left(\Lambda h^{\mu,a}_{I;\kappa}\right) h^{\mu,a}_{I;\kappa}
$$

then $g \in W^{-s}_{\mathrm{dyad}}(\mu)$ since

$$
\begin{aligned}
\|g\|_{W^{-s}_{\mathrm{dyad}}(\mu)} &= \sqrt{\int_{\mathbb{R}^n} \sum_{I\in\mathcal{D}} \left| \ell(I)^{s} \left(\triangle^\mu_{I;\kappa} g\right)\right|^2 d\mu} = \sqrt{\sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \ell(I)^{2s} \left| \left\langle g, h^{\mu,a}_{I;\kappa} \right\rangle_{L^2(\mu)} \right|^2} \\
&= \sqrt{\sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \ell(I)^{2s} \left| \Lambda h^{\mu,a}_{I;\kappa} \right|^2} \leq \|\Lambda\| < \infty,
\end{aligned}
$$

and finally we have

$$\Lambda f = \sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)} \Lambda h^{\mu,a}_{I;\kappa} = \sum_{\substack{I\in\mathcal{D}\\ a\in\Gamma}} \left\langle f, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)} \left\langle g, h^{\mu,a}_{I;\kappa}\right\rangle_{L^2(\mu)}$$
$$= \int_{\mathbb{R}^n} f(x)\, g(x)\, d\mu(x) = \langle f, g\rangle_{L^2(\mu)} .$$

□

### 2.6 Quasiorthogonality in weighted Sobolev spaces

Let $\triangle^{\mu,s}_{I;\kappa_1} = \ell(I)^{-s}\,\triangle^{\mu}_{I;\kappa_1}$ and $\mathbb{E}^{\mu,s}_{I;\kappa_1} = \ell(I)^{-s}\,\mathbb{E}^{\mu}_{I;\kappa_1}$. Since $\left\{\triangle^{\mu,s}_{I;\kappa_1}\right\}_{I\in\mathcal{D}}$ is a complete set of orthogonal projections on $W^s_{\mathrm{dyad}}(\mathbb{R}^n)$, we have

$$\sum_{I\in\mathcal{D}} \ell(I)^{-2s} \left\|\triangle^{\mu}_{I;\kappa_1} f\right\|^2_{L^2(\mu)} = \sum_{I\in\mathcal{D}} \left\|\triangle^{\mu,s}_{I;\kappa_1} f\right\|^2_{W^s_{\mathrm{dyad}}(\mu)} = \left\|\sum_{I\in\mathcal{D}} \triangle^{\mu,s}_{I;\kappa_1} f\right\|^2_{W^2_{\mathrm{dyad}}(\mu)}$$
$$= \|f\|^2_{W^s_{\mathrm{dyad}}(\mu)}$$

and then if $\left\{\mathbb{E}^{\mu,s}_{F;\kappa_1}\right\}_{F\in\mathcal{F}}$ is a collection of projections, indexed by a subgrid $\mathcal{F}$ of $\mathcal{D}$ satisfying an appropriate Carleson condition, we expect to have

$$\sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\|\mathbb{E}^{\mu}_{F;\kappa_1} f\right\|^2_{L^2(\mu)} = \sum_{F\in\mathcal{F}} \left\|\mathbb{E}^{\mu,s}_{F;\kappa_1} f\right\|^2_{L^2(\mu)} \lesssim \|f\|^2_{W^s_{\mathrm{dyad}}(\mu)} .$$

Here is the quasiorthogonality lemma appropriate for Sobolev spaces, in which $|f|$ does not appear, and which can be viewed as a Sobolev space version of the Carleson Embedding Theorem.

**Lemma 28** (Quasiorthogonality Lemma) *Let $\mu$ be a doubling measure on $\mathbb{R}^n$. Suppose that for some $\varepsilon > 0$, the subgrid $\mathcal{F}\subset\mathcal{D}$ satisfies the $\varepsilon$-strong $\mu$-Carleson condition,*

$$\sum_{\substack{F\in\mathcal{F}\\ F\subset F'}} \left(\frac{\ell(F')}{\ell(F)}\right)^{\varepsilon} |F|_\mu \le C\left|F'\right|_\mu \qquad F'\in\mathcal{F}. \tag{2.15}$$

*Then for $s < \frac{\varepsilon}{2}$ we have*

$$\sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\|\mathbb{E}^{\mu}_{F;\kappa} f\right\|^2_{L^2(\mu)} \lesssim \|f\|^2_{W^s_{\mathrm{dyad}}(\sigma)} .$$

***Proof*** Since $\frac{\triangle^{\mu}_{\pi^{(m)}F;\kappa} f}{\left\|\triangle^{\mu}_{\pi^{(m)}F;\kappa} f\right\|_\infty}$ is a normalized polynomial of degree less than $\kappa$ on the $\mathcal{D}$-child $\left(\pi^{(m)}F\right)_F$ of $\pi^{(m)}F$ that contains $F$, we have by (2.8) and (2.9),

$$|F|_\mu \left\|\mathbb{E}^{\sigma}_{F;\kappa}\,\triangle^{\mu}_{\pi^{(m)}F;\kappa} f\right\|^2_\infty \approx \int_F \left|\mathbb{E}^{\sigma}_{F;\kappa}\,\triangle^{\mu}_{\pi^{(m)}F;\kappa} f\right|^2 d\mu \le \int_F \left|\triangle^{\mu}_{\pi^{(m)}F;\kappa} f\right|^2 d\mu$$
$$\approx |F|_\mu \left\|\triangle^{\mu}_{\pi^{(m)}F;\kappa} f\right\|^2_\infty$$

and so for any $t < 0$ we have

$$\sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\| \mathbb{E}^{\mu}_{F;\kappa} f \right\|^2_{L^2(\mu)} = \sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\| \mathbb{E}^{\mu}_{F;\kappa_1} \left( \sum_{I\in\mathcal{D}} \triangle^{\mu}_{I;\kappa} f \right) \right\|^2_{L^2(\mu)}$$

$$= \sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\| \sum_{m=1}^{\infty} \mathbb{E}^{\mu}_{F;\kappa_1} \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|^2_{L^2(\mu)}$$

$$\lesssim \sum_{F\in\mathcal{F}} \ell(F)^{-2s} |F|_{\mu} \left( \sum_{m=1}^{\infty} \left\| \mathbb{E}^{\mu}_{F;\kappa} \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|_{\infty} \right)^2$$

$$\leq \sum_{F\in\mathcal{F}} \ell(F)^{-2s} |F|_{\mu} \left( \sum_{m=1}^{\infty} \left\| \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|_{\infty} \right)^2$$

$$\leq \sum_{F\in\mathcal{F}} \ell(F)^{-2s} |F|_{\mu} \left( \sum_{m=0}^{\infty} \ell\left(\pi^{(m)}F\right)^{2t} \right) \left( \sum_{m=0}^{\infty} \ell\left(\pi^{(m)}F\right)^{-2t} \left\| \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|^2_{\infty} \right)$$

$$\approx \sum_{F\in\mathcal{F}} \ell(F)^{2t-2s} |F|_{\mu} \sum_{m=0}^{\infty} \ell\left(\pi^{(m)}F\right)^{-2t} \left\| \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|^2_{\infty}.$$

Substituting $F'$ for $\pi^{(m)}F$, and letting $t = s - \frac{\varepsilon}{2} < 0$, we obtain from the $\varepsilon$-strong Carleson condition (2.15) that

$$\sum_{F\in\mathcal{F}} \ell(F)^{-2s} \left\| \mathbb{E}^{\mu}_{F;\kappa} f \right\|^2_{L^2(\mu)}$$

$$\lesssim \sum_{F\in\mathcal{F}} \ell(F)^{2t-2s} \sum_{m=0}^{\infty} \frac{|F|_{\mu}}{\left|\left(\pi^{(m)}F\right)_F\right|_{\mu}} \ell\left(\pi^{(m)}F\right)^{-2t} \int_{\left(\pi^{(m)}F\right)_F} \left\| \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right\|^2_{\infty} d\mu$$

$$\approx \sum_{F\in\mathcal{F}} \ell(F)^{2t-2s} \sum_{m=0}^{\infty} \frac{|F|_{\mu}}{\left|\left(\pi^{(m)}F\right)_F\right|_{\mu}} \ell\left(\pi^{(m)}F\right)^{-2t} \int_{\left(\pi^{(m)}F\right)_F} \left| \triangle^{\mu}_{\pi^{(m)}F;\kappa} f \right|^2 d\mu$$

$$= \sum_{F'\in\mathcal{F}} \left( \sum_{\substack{F\in\mathcal{F}\\ F\subset F'}} \ell(F)^{-\varepsilon} \frac{|F|_{\mu}}{|(F')_F|_{\mu}} \right) \ell\left(F'\right)^{\varepsilon-2s} \left\| \triangle^{\mu}_{F';\kappa} f \right\|^2_{L^2(\mu)}$$

$$\leq \sum_{F'\in\mathcal{F}} \left( C\ell\left(F'\right)^{-\varepsilon} \frac{\left|F'\right|_{\mu}}{|(F')_F|_{\mu}} \right) \ell\left(F'\right)^{\varepsilon-2s} \left\| \triangle^{\mu}_{F';\kappa} f \right\|^2_{L^2(\mu)}$$

$$\lesssim \sum_{F'\in\mathcal{F}} \ell\left(F'\right)^{-2s} \left\| \triangle^{\mu}_{F';\kappa} f \right\|^2_{L^2(\mu)} \lesssim \|f\|^2_{W^s_{\mathrm{dyad}}(\mu)}.$$

□

***Remark 29*** We can replace $f$ by its modulus $|f|$ in the above lemma when $\kappa = 1$ and $s > 0$ is sufficiently small. Indeed, by the reverse triangle inequality we have

$$|||f|||^2_{W^s_{\mathrm{diff};1}(\mu)} = \sum_{Q\in\mathcal{D}} \int_Q \left| \frac{f(x) - \mathbb{E}^{\mu}_{Q;1} f(x)}{\ell(Q)^s} \right|^2 d\mu(x)$$

$$= \sum_{Q\in\mathcal{D}} \int_Q \left| \frac{f(x) - \left(\frac{1}{|Q|_\mu}\int_Q f d\mu\right) \mathbf{1}_Q(x)}{\ell(Q)^s} \right|^2 d\mu(x)$$

$$= 2\sum_{Q\in\mathbb{D}} \frac{1}{|Q|_\mu^2} \int_Q \int_Q \left| \frac{|f(x)| - |f(y)|}{\ell(Q)^s} \right|^2 d\mu(x)\, d\mu(y)$$

$$\leq 2\sum_{Q\in\mathbb{D}} \frac{1}{|Q|_\mu^2} \int_Q \int_Q \left| \frac{f(x) - f(y)}{\ell(Q)^s} \right|^2 d\mu(x)\, d\mu(y) = \|f\|^2_{W^s_{\mathrm{diff};1}(\mu)},$$

and now we use the equivalence $\|\cdot\|^2_{W^s_{\mathrm{diff};1}(\mu)} \approx \|\cdot\|^2_{W^s_{\mathrm{dyad};1}(\mu)}$ for $|s|$ sufficiently small.

## 3 Preliminaries: weighted Sobolev norm inequalities

Duality shows the equivalence of weighted norm inequalities with bilinear inequalities.

**Lemma 30** *The Sobolev norm inequality*

$$\left\|T^\alpha_\sigma f\right\|_{W^s_{\mathrm{dyad}}(\omega)} \leq \left\|T^\alpha\right\|_{\mathrm{op}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \tag{3.1}$$

*is equivalent to the bilinear inequality*

$$\left| \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{D}} \int_{\mathbb{R}^n} \left(T^\alpha_\sigma \triangle^\sigma_I f\right) \triangle^\omega_J g d\omega \right| \leq \left\|T^\alpha\right\|_{\mathrm{bil}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \quad \text{for } f, g \in L^2(\mu). \tag{3.2}$$

***Proof*** Indeed, if the bilinear inequality holds and $f = \sum_{I\in\mathcal{D}} \triangle^\sigma_I f$ and $g = \sum_{J\in\mathcal{D}} \triangle^\omega_J g$, then

$$\left|\int_{\mathbb{R}^n} \left(T^\alpha_\sigma f\right) g d\omega\right| = \left| \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{D}} \int_{\mathbb{R}^n} \left(T^\alpha_\sigma \triangle^\sigma_I f\right) \triangle^\omega_J g d\omega \right| \leq \left\|T^\alpha\right\|_{\mathrm{bil}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

shows that

$$\begin{aligned} \left\|T^\alpha_\sigma f\right\|_{W^s_{\mathrm{dyad}}(\omega)} &= \sup_{\|g\|_{W^s_{\mathrm{dyad}}(\omega)^*}\leq 1} \left|\int_{\mathbb{R}^n} \left(T^\alpha_\sigma f\right) g d\omega\right| = \sup_{\|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\leq 1} \left|\int_{\mathbb{R}^n} \left(T^\alpha_\sigma f\right) g d\omega\right| \\ &\leq \left\|T^\alpha\right\|_{\mathrm{bil}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \\ &\Longrightarrow \left\|T^\alpha\right\|_{\mathrm{op}} \leq \left\|T^\alpha\right\|_{\mathrm{bil}} \text{ since } L^2(\mu) \text{ is dense in } W^s_{\mathrm{dyad}}(\mu). \end{aligned}$$

Conversely, if the norm inequality holds, then

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \left(T^\alpha_\sigma f\right) g d\omega\right| &\leq \left\|T^\alpha_\sigma f\right\|_{W^s_{\mathrm{dyad}}(\omega)} \|g\|_{W^s_{\mathrm{dyad}}(\omega)^*} \leq \left\|T^\alpha\right\|_{\mathrm{op}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \|g\|_{W^s_{\mathrm{dyad}}(\omega)^*} \\ &= \left\|T^\alpha\right\|_{\mathrm{op}} \|f\|_{W^s_{\mathrm{dyad}}(\sigma)} \|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \\ &\Longrightarrow \left\|T^\alpha\right\|_{\mathrm{bil}} \leq \left\|T^\alpha\right\|_{\mathrm{op}}. \end{aligned}$$

□

### 3.1 The good–bad decomposition

Here we follow the random grid idea of Nazarov, Treil and Volberg. Denote by $\Omega_{\text{dyad}}$ the collection of all dyadic grids $\mathcal{D}$. For a weight $\mu$, we consider a random choice of dyadic grid $\mathcal{D}$ on the natural probability space $\Omega_{\text{dyad}}$.

**Definition 31** For a positive integer $r$ and $0 < \varepsilon < 1$, a cube $J \in \mathcal{D}$ is said to be $(r,\varepsilon)$-*bad* if there is a cube $I \in \mathcal{D}$ with $|I| \geq 2^r |J|$, and

$$\operatorname{dist}(e(I), J) \leq \tfrac{1}{2} |J|^{\varepsilon} |I|^{1-\varepsilon}.$$

Here, $e(J)$ is the union of the boundaries of the children of the cube $J$. (This contains the set of discontinuities of $h^{\mu}_{J;\kappa}$ and its derivatives less than order $\kappa$.) Otherwise, $J$ is said to be $(r,\varepsilon)$-good.

The basic proposition here is this, see e.g. [24] and e.g. [11] or [17] for higher dimensions.

**Proposition 32** *There is the conditional probability estimate*

$$\mathbb{P}^{\Omega_{\text{dyad}}}_{\text{cond}} \left( J \text{ } is \text{ } (r,\varepsilon) \text{-bad} : J \in \mathcal{D} \right) \leq C_{\varepsilon} 2^{-\varepsilon r}.$$

Define projections

$$\mathsf{P}^{\mu}_{\text{good};\mathcal{D}} f = \mathsf{P}^{\mu}_{\text{good}} f \equiv \sum_{I \text{ is } (r,\varepsilon)\text{-good } \in \mathcal{D}} \Delta^{\mu}_{I} f \text{ and } \mathsf{P}^{\mu}_{\text{bad};\mathcal{D}} f = \mathsf{P}^{\mu}_{\text{bad}} f \equiv f - \mathsf{P}^{\mu}_{\text{good}} f. \tag{3.3}$$

Recall that

$$\|f\|^2_{W^s_{\mathcal{D}}(\mu)} \equiv \sum_{Q \in \mathcal{D}} \ell(Q)^{-2s} \left\| \triangle^{\mu}_{Q} f \right\|^2_{L^2(\mu)},$$
$$\|f\|^2_{W^s_{\text{dyad}}(\mu)} \approx \|f\|^2_{W^s_{\mathcal{D}}(\mu)}, \quad \text{for all } \mathcal{D} \in \Omega_{\text{dyad}}.$$

The basic Proposition is then this.

**Proposition 33** (cf. Theorem 17.1 in [24] where the middle line below is treated) *We have the estimates*

$$\mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} \left\| P^{\mu}_{\text{bad};\mathcal{D}} f \right\|_{W^s_{\mathcal{D}}(\mu)} \leq C_{\varepsilon} 2^{-\frac{\varepsilon}{2} r} \|f\|_{W^s_{\text{dyad}}(\mu)},$$
$$\mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} \left\| P^{\mu}_{\text{bad};\mathcal{D}} f \right\|_{L^p(\mu)} \leq C_{\varepsilon} 2^{-\frac{\varepsilon r}{p}} \|f\|_{L^p(\mu)},$$
$$\mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} \left\| P^{\mu}_{\text{bad};\mathcal{D}} f \right\|_{W^{-s}_{\mathcal{D}}(\mu)} \leq C_{\varepsilon} 2^{-\frac{\varepsilon}{2} r} \|f\|_{W^{-s}_{\text{dyad}}(\mu)}.$$

***Proof*** We have

$$\mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} \left( \left\| \mathsf{P}^{\mu}_{\text{bad};\mathcal{D}} f \right\|^2_{W^s_{\text{dyad}}(\mu)} \right) = \mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} \sum_{I \in \mathcal{D} \text{ is } (r,\varepsilon)\text{-bad}} \ell(I)^{-2s} \left\langle f, h^{\mu}_{I} \right\rangle^2_{L^2(\mu)}$$
$$\leq C_{\varepsilon} \mathbb{E}^{\mathcal{D}}_{\Omega_{\text{dyad}}} 2^{-\varepsilon r} \sum_{I \in \mathcal{D}} \ell(I)^{-2s} \left\langle f, h^{\mu}_{I} \right\rangle^2_{L^2(\mu)} = C_{\varepsilon} 2^{-\varepsilon r} \|f\|^2_{W^s_{\text{dyad}}(\mu)},$$

and then

$$\mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}}\left(\left\| \mathsf{P}^{\mu}_{\mathrm{bad};\mathcal{D}} f \right\|_{W^{s}_{\mathrm{dyad}}(\mu)}\right) \leq \sqrt{\mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}}\left(\left\| \mathsf{P}^{\mu}_{\mathrm{bad};\mathcal{D}} f \right\|^{2}_{W^{s}_{\mathrm{dyad}}(\mu)}\right)} \leq C_{\varepsilon} 2^{-\frac{\varepsilon}{2} r} \left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\mu)} .$$

Similarly for $L^{p}(\mu)$ and $W^{-s}_{\mathrm{dyad}}(\mu)$ in place of $W^{s}_{\mathrm{dyad}}(\mu)$. □

From this we conclude the following: Given any $0 < \varepsilon < 1$, there is a choice of $r$, depending on $\varepsilon$, so that the following holds. Let $T : W^{s}_{\mathrm{dyad}}(\sigma) \to W^{s}_{\mathrm{dyad}}(\omega)$ be a bounded linear operator, where for specificity we take $W^{s}_{\mathrm{dyad}} = W^{s}_{\mathcal{D}_0}$, and $\mathcal{D}_0$ is the standard dyadic grid on $\mathbb{R}^{n}$. We then have

$$\left\| T \right\|_{W^{s}_{\mathrm{dyad}}(\sigma) \to W^{s}_{\mathrm{dyad}}(\omega)} \leq 2 \sup_{\left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)}=1} \sup_{\left\| \phi \right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}=1} \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T \mathsf{P}^{\sigma}_{\mathrm{good};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{good};\mathcal{D}} g \right\rangle_{\omega} \right| . \tag{3.4}$$

Indeed, we can choose $f \in W^{s}_{\mathrm{dyad}}(\sigma)$ of norm one, and $g \in W^{-s}_{\mathrm{dyad}}(\omega)$ of norm one, and we can write

$$f = \mathsf{P}^{\sigma}_{\mathrm{good};\mathcal{D}} f + \mathsf{P}^{\sigma}_{\mathrm{bad};\mathcal{D}} f$$

and similarly for $g$ and $\mathcal{E}$, so that

$$\begin{aligned}
\left\| T \right\|_{W^{s}_{\mathrm{dyad}}(\sigma) \to W^{s}_{\mathrm{dyad}}(\omega)} &= \left| \left\langle T_{\sigma} f, g \right\rangle_{\omega} \right| \\
&\leq \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T_{\sigma} \mathsf{P}^{\sigma}_{\mathrm{good};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{good};\mathcal{E}} g \right\rangle_{\omega} \right| + \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T_{\sigma} \mathsf{P}^{\sigma}_{\mathrm{bad};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{good};\mathcal{E}} g \right\rangle_{\omega} \right| \\
&\quad + \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T_{\sigma} \mathsf{P}^{\sigma}_{\mathrm{good};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{bad};\mathcal{E}} g \right\rangle_{\omega} \right| + \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T_{\sigma} \mathsf{P}^{\sigma}_{\mathrm{bad};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{bad};\mathcal{E}} g \right\rangle_{\omega} \right| \\
&\leq \mathbb{E}^{\mathcal{D}}_{\Omega_{\mathrm{dyad}}} \mathbb{E}^{\mathcal{E}}_{\Omega_{\mathrm{dyad}}} \left| \left\langle T_{\sigma} \mathsf{P}^{\sigma}_{\mathrm{good};\mathcal{D}} f, \mathsf{P}^{\omega}_{\mathrm{good};\mathcal{E}} g \right\rangle_{\omega} \right| + 3 C_{\varepsilon} 2^{-\frac{\varepsilon}{2} r} \left\| T \right\|_{W^{s}_{\mathrm{dyad}}(\sigma) \to W^{s}_{\mathrm{dyad}}(\omega)} .
\end{aligned}$$

And this proves (3.4) for $r$ sufficiently large.

This has the following implication for us: *Given any linear operator $T$ and* $0 < \varepsilon < 1$, *it suffices to consider only* $(r,\varepsilon)$-good *cubes for $r$ sufficiently large, and prove an estimate for $\left\| T \right\|_{W^{s}_{\mathrm{dyad}}(\sigma) \to W^{s}_{\mathrm{dyad}}(\omega)}$ that is independent of this assumption.* Accordingly, we will call $(r,\varepsilon)$-good cubes just good *cubes* from now on. At certain points in the arguments below, such as in the treatment of the neighbour form for $W^{s}_{\mathrm{dyad}}(\sigma)$, we will need to further restrict the parameter $\varepsilon$ (and accordingly $r$ as well).

### 3.2 Defining the norm inequality

We now turn to a precise definition of the weighted norm inequality

$$\left\| T^{\alpha}_{\sigma} f \right\|_{W^{s}_{\mathrm{dyad}}(\omega)} \leq \mathfrak{N}_{T^{\alpha}} \left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} , \quad f \in W^{s}_{\mathrm{dyad}}(\sigma) , \tag{3.5}$$

where $W^{s}_{\mathrm{dyad}}(\sigma)$ is the Hilbert space completion of the space of functions $f \in L^{2}_{\mathrm{loc}}(\sigma)$ for which

$$\left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} < \infty.$$

A similar definition holds for $W^{s}_{\mathrm{dyad}}(\omega)$. For a precise definition of (3.5), it is possible to proceed with the notion of associating operators and kernels through an identity for functions with disjoint support as in [22]. However, we choose to follow the approach in [18, see page 314]. So we suppose that $K^{\alpha}$ is a smooth $\alpha$-fractional Calderón–Zygmund kernel, and we introduce a family $\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K^{\alpha}_{\delta,R}(x,y)=\eta^{\alpha}_{\delta,R}(|x-y|)\,K^{\alpha}(x,y)$ are bounded with compact support for fixed $x$ or $y$, and uniformly satisfy (1.2). Then the truncated operators

$$T^{\alpha}_{\sigma,\delta,R}f(x)\equiv\int_{\mathbb{R}^n}K^{\alpha}_{\delta,R}(x,y)\,f(y)\,d\sigma(y)\,,\quad x\in\mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ as an $\alpha$-fractional singular integral operator, which we typically denote by $T^{\alpha}$, suppressing the dependence on the truncations.

**Definition 34** We say that an $\alpha$-fractional singular integral operator $T^{\alpha}=\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ satisfies the norm inequality (3.5) provided

$$\left\|T^{\alpha}_{\sigma,\delta,R}f\right\|_{W^s_{\mathrm{dyad}}(\omega)}\leq\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)\,\|f\|_{W^s_{\mathrm{dyad}}(\sigma)}\,,\quad f\in W^s_{\mathrm{dyad}}(\sigma)\,,0<\delta<R<\infty.$$

Independence of Truncations In the presence of the classical Muckenhoupt condition $A^{\alpha}_2$, the norm inequality ( 3.5) is essentially independent of the choice of truncations used, including *nonsmooth* truncations as well—see [9]. However, in dealing with the Monotonicity Lemma 40 below, where $\kappa^{th}$ order Taylor approximations are made on the truncated kernels, it is necessary to use sufficiently smooth truncations. Similar comments apply to the Cube Testing conditions (3.6) and (3.7) below.

#### 3.2.1 Ellipticity of kernels

Modifying slightly the definition in [21, (39) on page 210], we say that an $\alpha$-fractional Calderón–Zygmund kernel $K^{\alpha}$ is *elliptic in the sense of Stein* if there is a unit coordinate vector $\mathbf{e}_k\in\mathbb{R}^n$ for some $1\leq k\leq n$, and a positive constant $c>0$ such that

$$\left|K^{\alpha}(x,x+t\mathbf{e}_k)\right|\geq c\,|t|^{\alpha-n}\,,\quad\text{for all }t\in\mathbb{R}.$$

For example, the Beurling, Cauchy and Riesz transform kernels, as well as those for $k$-iterated Riesz transforms are elliptic in the sense of Stein for any $k\geq 1$.

#### 3.2.2 Cube testing

While the next more general testing conditions with $\kappa>1$, introduced in [14, 16], are not used in the statements of our theorems, they will be used in the course of our proof.

The *$\kappa$-cube testing conditions* associated with an $\alpha$ -fractional singular integral operator $T^{\alpha}$, introduced in [14] for $s=0$, are given by

$$\left(\mathfrak{T}^{\kappa,s}_{T^{\alpha}}(\sigma,\omega)\right)^2\equiv\sup_{Q\in\mathcal{Q}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{\ell(Q)^{-2s}\,|Q|_{\sigma}}\left\|\mathbf{1}_QT^{\alpha}_{\sigma}\left(\mathbf{1}_Qm^{\beta}_Q\right)\right\|^2_{W^s_{\mathrm{dyad}}(\omega)}<\infty,$$
$$\left(\mathfrak{T}^{\kappa,-s}_{(T^{\alpha})^*}(\omega,\sigma)\right)^2\equiv\sup_{Q\in\mathcal{Q}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{\ell(Q)^{2s}\,|Q|_{\omega}}\left\|\mathbf{1}_QT^{\alpha,*}_{\omega}\left(\mathbf{1}_Qm^{\beta}_Q\right)\right\|^2_{W^{-s}_{\mathrm{dyad}}(\sigma)}<\infty,\tag{3.6}$$

where $(T^{\alpha,*})_{\omega}=\left(T^{\alpha}_{\sigma}\right)^*$, with $m^{\beta}_Q(x)\equiv\left(\frac{x-c_Q}{\ell(Q)}\right)^{\beta}$ for any cube $Q$ and multiindex $\beta$, where $c_Q$ is the center of the cube $Q$, and where we interpret the right hand sides as holding uniformly over all sufficiently smooth truncations of $T^{\alpha}$. Equivalently, in the presence of $A_2^{\alpha}$, we can take a single suitable truncation, see Independence of Truncations in Subsubsection 3.2 above.

We also use the larger *triple $\kappa$-cube testing conditions* in which the integrals over $Q$ are extended to the triple $3Q$ of $Q$:

$$\begin{aligned}\left(\mathfrak{TR}_{T^{\alpha}}^{\kappa,s}(\sigma,\omega)\right)^2 &\equiv \sup_{Q\in\mathcal{Q}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{\ell(Q)^{-2s}|Q|_{\sigma}}\left\|\mathbf{1}_{3Q}T_{\sigma}^{\alpha}\left(\mathbf{1}_Q m_Q^{\beta}\right)\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^2<\infty,\\ \left(\mathfrak{TR}_{(T^{\alpha})^*}^{\kappa,-s}(\omega,\sigma)\right)^2 &\equiv \sup_{Q\in\mathcal{Q}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{\ell(Q)^{2s}|Q|_{\omega}}\left\|\mathbf{1}_{3Q}T_{\omega}^{\alpha,*}\left(\mathbf{1}_Q m_Q^{\beta}\right)\right\|_{W_{\mathrm{dyad}}^{-s}(\sigma)}^2<\infty.\end{aligned} \tag{3.7}$$

### 3.3 Necessity of the classical Muckenhoupt condition

Suppose that $T^{\alpha}f=K*f$ where $K^{\alpha}(x)=\frac{\Omega(x)}{|x|^{n-\alpha}}$, and $\Omega(x)$ is homogeneous of degree 0 and smooth away from the origin. Note that we do not require any cancellation properties on $\Omega$, except that when $\alpha=0$ we suppose $\int\Omega(x)\,d\sigma_{n-1}(x)=0$ where $\sigma_{n-1}$ is surface measure on the sphere (see e.g. [21] page 68 for the case $\alpha=0$). We assume $\Omega$ is nontrivial in the sense that there is a coordinate direction $\Theta\in\mathbb{S}^{n-1}$ such that $\Omega(\Theta)\neq 0$. Then there is a cone $\Gamma$ centered on $\Theta$ on which $K(x)=\frac{\Omega(x)}{|x|^{n-\alpha}}$ and $\Omega(x)\geq c>0$ for $x\in\Gamma$. Consider pairs of separated dyadic cubes in direction $\Theta$,

$$\begin{aligned}\mathcal{SP}_{\Theta}\equiv\Big\{&\left(I,I'\right):\mathrm{dist}\left(I,I'\right)\approx\ell(I)=\ell\left(I'\right),R_1\ell(I)\\ &\leq\mathrm{dist}\left(I,I'\right)\approx R\ell(I),\text{ and } I' \text{ has direction } \Theta \text{ from } I\Big\},\end{aligned}$$

where $R$ is chosen large enough that if the cone $\Gamma$ is translated to any point in $I$, then it contains any cube $I'$ for which $\left(I,I'\right)\in\mathcal{SP}_{\Theta}$.

We first derive the 'separated' Muckenhoupt condition from the full testing condition for $T^{\alpha}$, i.e.

$$\left(\frac{1}{|I'|}\int_{I'}d\omega\right)\left(\frac{1}{|I|}\int_I d\sigma\right)\leq\mathfrak{FT}_{T^{\alpha}}(s;\sigma,\omega)^2,\qquad \left(I,I'\right)\in\mathcal{SP}_{\Theta}.$$

We may assume without loss of generality that $\Theta=e_1$, the unit vector in the direction of the positive $x_1$-axis. Now we choose a special unit Haar function $h_{I'}^{\omega}$, i.e. $\triangle_{I'}^{\omega}h_{I'}^{\omega}=h_{I'}^{\omega}$ and $\left\|h_{I'}^{\omega}\right\|_{L^2(\omega)}=1$, satisfying

$$h_{I'}^{\omega}(x)=\sum_{K\in\mathfrak{C}(I)}a_K\mathbf{1}_K(x),\text{ where }\begin{cases}a_K>0 & \text{if } K \text{ lies to the right of center}\\ a_K<0 & \text{if } K \text{ lies to the left of center}\end{cases},$$

where for a cube $Q$ centered at the origin, we say a child $K$ lies to the right of center if $K$ is contained in the half space where $x_1\geq 0$. We now compute

$$\begin{aligned}\left\|\mathbf{1}_{I'}T^{\alpha}(\mathbf{1}_I\sigma)\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^2 &= \sum_{J\in\mathcal{D}}\ell(J)^{-2s}\left\|\triangle_J^{\omega}\left(\mathbf{1}_{I'}T^{\alpha}\mathbf{1}_I\sigma\right)\right\|_{L^2(\omega)}^2\\ &\geq \ell\left(I'\right)^{-2s}\left\|\triangle_{I'}^{\omega}\left(T^{\alpha}\mathbf{1}_I\sigma\right)\right\|_{L^2(\omega)}^2\\ &\geq \ell\left(I'\right)^{-2s}\left|\left\langle T^{\alpha}\mathbf{1}_I\sigma,h_{I'}^{\omega}\right\rangle_{\omega}\right|^2=\ell\left(I'\right)^{-2s}\left|\int_{I'}\left(\int_I K^{\alpha}(x,y)\,d\sigma(y)\right)h_{I'}^{\omega}(x)\,d\omega(x)\right|^2\\ &= \ell\left(I'\right)^{-2s}\left|\int_{I'}\left(\int_I\left[K^{\alpha}(x-y)-K^{\alpha}(c_{I'}-y)\right]d\sigma(y)\right)h_{I'}^{\omega}(x)\,d\omega(x)\right|^2\end{aligned}$$

$$\begin{aligned}
&= \ell\left(I'\right)^{-2s}\left|\int_{I'}\int_{I}\left[K^{\alpha}\left(x-y\right)-K^{\alpha}\left(c_{I'}-y\right)\right]h_{I'}^{\omega}\left(x\right)d\sigma\left(y\right)d\omega\left(x\right)\right|^{2}\\
&\gtrsim c\ell\left(I'\right)^{-2s}\left(\int_{I'}\int_{I}\left|\frac{1}{\sqrt{|I|_{\omega}}}\frac{\ell\left(I'\right)}{\operatorname{dist}\left(I,I'\right)^{n+1-\alpha}}\right|d\sigma\left(y\right)d\omega\left(x\right)\right)^{2}\\
&= c\ell\left(I'\right)^{-2s}\frac{|I|_{\sigma}^{2}\left|I'\right|_{\omega}}{\operatorname{dist}\left(I,I'\right)^{2n-2\alpha}},
\end{aligned}$$

since $\omega$ is doubling. Thus we have

$$\begin{aligned}
\ell\left(I'\right)^{-2s}\frac{|I|_{\sigma}^{2}\left|I'\right|_{\omega}}{\operatorname{dist}\left(I,I'\right)^{2n-2\alpha}} &\lesssim \left\|\mathbf{1}_{I'}T^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2}\\
&\leq \mathfrak{F}\mathfrak{T}_{T^{\alpha}}\left(s;\sigma,\omega\right)^{2}\left\|\mathbf{1}_{I}\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)}^{2}\approx \mathfrak{F}\mathfrak{T}_{T^{\alpha}}\left(s;\sigma,\omega\right)^{2}\ell\left(I\right)^{-2s}|I|_{\sigma}
\end{aligned}$$

which gives the desired inequality,

$$\frac{|I|_{\sigma}}{|I|^{n-\alpha}}\frac{\left|I'\right|_{\omega}}{|I'|^{n-\alpha}}\lesssim \mathfrak{F}\mathfrak{T}_{T^{\alpha}}\left(s;\sigma,\omega\right)^{2}.$$

Since the measures are doubling, we obtain the full Muckenhoupt inequality,

$$A_{2}^{\alpha}\left(\sigma,\omega\right)=\sup_{I}\frac{|I|_{\sigma}|I|_{\omega}}{|I|^{2(n-\alpha)}}\lesssim \mathfrak{F}\mathfrak{T}_{T^{\alpha}}\left(s;\sigma,\omega\right)^{2}.$$

Thus we have proved the following lemma.

**Lemma 35** *If $T^{\alpha}f=K^{\alpha}*f$ where $K^{\alpha}\left(x\right)=\frac{\Omega(x)}{|x|^{n-\alpha}}$, and $\Omega\left(x\right)$ is homogeneous of degree $0$ and smooth away from the origin and $\Omega\left(\mathbf{e}_{k}\right)\neq 0$ for some $1\leq k\leq n$, then boundedness of $T^{\alpha}$ from $W_{\mathrm{dyad}}^{s}\left(\sigma\right)$ to $W_{\mathrm{dyad}}^{s}\left(\omega\right)$, implies the $A_{2}^{\alpha}\left(\sigma,\omega\right)$ condition, more precisely,*

$$\sqrt{A_{2}^{\alpha}\left(\sigma,\omega\right)}\lesssim \mathfrak{F}\mathfrak{T}_{T^{\alpha}}\left(s;\sigma,\omega\right)\leq \mathfrak{N}_{T^{\alpha}}\left(s;\sigma,\omega\right).$$

### 3.3.1 Necessity of the strong $\kappa^{th}$ order pivotal condition for doubling weights

The smaller fractional Poisson integrals $\mathrm{P}_{\kappa}^{\alpha}\left(Q,\mu\right)$ used here, in [14] and elsewhere, are given by

$$\mathrm{P}_{\kappa}^{\alpha}\left(Q,\mu\right)=\int_{\mathbb{R}^{n}}\frac{\ell\left(Q\right)^{\kappa}}{\left(\ell\left(Q\right)+\left|y-c_{Q}\right|\right)^{n+\kappa-\alpha}}d\mu\left(y\right),\quad \kappa\geq 1,\tag{3.8}$$

and the $\kappa^{th}$-order fractional pivotal constants $\mathcal{V}_{2}^{\alpha,\kappa},\mathcal{V}_{2}^{\alpha,\kappa,*}<\infty$, $\kappa\geq 1$, are given by

$$\begin{aligned}
\left(\mathcal{V}_{2}^{\alpha,\kappa}\left(\sigma,\omega\right)\right)^{2} &= \sup_{Q\supset\dot{\cup}Q_{r}}\frac{1}{|Q|_{\sigma}}\sum_{r=1}^{\infty}\mathrm{P}_{\kappa}^{\alpha}\left(Q_{r},\mathbf{1}_{Q}\sigma\right)^{2}|Q_{r}|_{\omega} &(3.9)\\
\left(\mathcal{V}_{2}^{\alpha,\kappa,*}\left(\sigma,\omega\right)\right)^{2} &= \sup_{Q\supset\dot{\cup}Q_{r}}\frac{1}{|Q|_{\omega}}\sum_{r=1}^{\infty}\mathrm{P}_{\kappa}^{\alpha}\left(Q_{r},\mathbf{1}_{Q}\omega\right)^{2}|Q_{r}|_{\sigma}=\left(\mathcal{V}_{2}^{\alpha,\kappa}\left(\omega,\sigma\right)\right)^{2}
\end{aligned}$$

and the $\varepsilon$-strong $\kappa$-pivotal constants $\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa},\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa,*}<\infty$, $\kappa\geq 1$, $\varepsilon>0$, are given by

$$\left(\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}\left(\sigma,\omega\right)\right)^{2}=\sup_{Q\supset\dot{\cup}Q_{r}}\frac{1}{|Q|_{\sigma}}\sum_{r=1}^{\infty}\mathrm{P}_{\kappa}^{\alpha}\left(Q_{r},\mathbf{1}_{Q}\sigma\right)^{2}\left(\frac{\ell\left(Q\right)}{\ell\left(Q_{r}\right)}\right)^{\varepsilon}|Q_{r}|_{\omega}$$

$$\left(\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa,*}(\sigma,\omega)\right)^2 = \sup_{Q\supset\dot{\cup}Q_r} \frac{1}{|Q|_\omega}\sum_{r=1}^{\infty}\mathrm{P}_\kappa^\alpha\left(Q_r,\mathbf{1}_Q\omega\right)^2\left(\frac{\ell(Q)}{\ell(Q_r)}\right)^\varepsilon |Q_r|_\sigma = \left(\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}(\omega,\sigma)\right)^2 \tag{3.10}$$

and where the suprema are taken over all subdecompositions of a cube $Q \in \mathcal{Q}^n$ into pairwise disjoint dyadic subcubes $Q_r$. The case $\varepsilon = 0$ of the following lemma was obtained in [16, Subsection 4.1 on pages 12–13, especially Remark 15], where it was the point of departure for freeing the theory from reliance on energy conditions when the measures are doubling.

**Lemma 36** *Let $0 \le \alpha < n$. If $\sigma$ is a doubling measure, then for $\kappa > \theta_\sigma^{\mathrm{doub}} + \alpha - n$ and $0 < \varepsilon \le \theta_\sigma^{\mathrm{rev}}$, we have*

$$\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}(\sigma,\omega) \le C_{\kappa,\varepsilon} A_2^\alpha(\sigma,\omega).$$

***Proof*** A doubling measure $\sigma$ with doubling parameters $0 < \beta, \gamma < 1$ as in (2.1), has a 'doubling exponent' $\theta_\sigma^{\mathrm{doub}} > 0$ and a positive constant $c$ depending on $\beta,\gamma$ that satisfy the condition, see e.g. [16],

$$\left|2^{-j}Q\right|_\sigma \ge c2^{-j\theta_\sigma^{\mathrm{doub}}}|Q|_\sigma \quad \text{for all } j \in \mathbb{N}.$$

We can then exploit the doubling exponents $\theta_\sigma^{\mathrm{doub}}$ and reverse doubling exponents $\theta_\sigma^{\mathrm{rev}}$ of the doubling measure $\sigma$ in order to derive certain $\kappa^{th}$ order pivotal conditions $\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa} < \infty$. Indeed, if $\sigma$ has doubling exponent $\theta_\sigma^{\mathrm{doub}}$ and $\kappa > \theta_\sigma^{\mathrm{doub}} + \alpha - n$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus I}\frac{\ell(I)^\kappa}{(\ell(I)+|x-c_I|)^{n+\kappa-\alpha}}d\sigma(x)\\
&=\sum_{j=1}^{\infty}\ell(I)^{\alpha-n}\int_{2^jI\setminus 2^{j-1}I}\frac{1}{\left(1+\frac{|x-c_I|}{\ell(I)}\right)^{n+\kappa-\alpha}}d\sigma(x)\\
&\lesssim |I|^{\frac{\alpha}{n}-1}\sum_{j=1}^{\infty}2^{-j(n+\kappa-\alpha)}\left|2^jI\right|_\sigma \lesssim |I|^{\frac{\alpha}{n}-1}\sum_{j=1}^{\infty}2^{-j(n+\kappa-\alpha)}\frac{1}{c2^{-j\theta_\sigma^{\mathrm{doub}}}}|I|_\omega\\
&\le C_{n,\kappa,\alpha,\beta,\gamma}|I|^{\frac{\alpha}{n}-1}|I|_\sigma
\end{aligned} \tag{3.11}$$

provided $n+\kappa-\alpha-\theta_\sigma^{\mathrm{doub}} > 0$, i.e. $\kappa > \theta_\sigma^{\mathrm{doub}} + \alpha - n$. It follows that if $I \supset \dot{\bigcup}_{r=1}^{\infty} I_r$ is a subdecomposition of $I$ into pairwise disjoint cubes $I_r$, and $\kappa > \theta_\sigma^{\mathrm{doub}} + \alpha - n$, then

$$\begin{aligned}
&\sum_{r=1}^{\infty}\mathrm{P}_\kappa^\alpha(I_r,\mathbf{1}_I\sigma)^2\left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon|I_r|_\omega\\
&\lesssim\sum_{r=1}^{\infty}\left(|I_r|^{\frac{\alpha}{n}-1}|I_r|_\sigma\right)^2\left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon|I_r|_\omega = \sum_{r=1}^{\infty}\left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon\frac{|I_r|_\sigma|I_r|_\omega}{|I_r|^{2\left(1-\frac{\alpha}{n}\right)}}|I_r|_\sigma\\
&\lesssim A_2^\alpha(\sigma,\omega)\sum_{r=1}^{\infty}\left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon|I_r|_\sigma \le C_{\sigma,\varepsilon}A_2^\alpha(\sigma,\omega)|I|_\sigma
\end{aligned}$$

provided $0 < \varepsilon \le \theta_\sigma^{\mathrm{rev}}$, indeed,

$$\left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon|I_r|_\sigma \le \left(\frac{\ell(I)}{\ell(I_r)}\right)^\varepsilon C_{\sigma,\varepsilon}\left(\frac{\ell(I_r)}{\ell(I)}\right)^{\theta_\sigma^{\mathrm{rev}}}|I|_\sigma \le C_{\sigma,\varepsilon}|I|_\sigma.$$

This then gives

$$\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa} \leq C_{\kappa,\varepsilon} A_2^{\alpha}(\sigma,\omega) \quad \kappa > \theta_{\sigma}^{\mathrm{doub}} + \alpha - n \text{ and } 0 < \varepsilon \leq \theta_{\sigma}^{\mathrm{rev}} \tag{3.12}$$

where the constant $C_{\kappa,\varepsilon}$ depends on $\kappa$, $\varepsilon$ and the doubling constant of $\sigma$. A similar result holds for $\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa,*}$ if $\kappa + n - \alpha > \theta_{\omega}^{\mathrm{doub}}$ and $0 < \varepsilon \leq \theta_{\omega}^{\mathrm{rev}}$ hold for the doubling and reverse doubling exponents $\theta_{\omega}^{\mathrm{doub}}, \theta_{\omega}^{\mathrm{rev}}$ of $\omega$. □

### 3.4 The energy lemma

For $0 \leq \alpha < n$ and $m \in \mathbb{R}_+$, we recall from (3.8) the $m^{th}$-order fractional Poisson integral

$$\mathrm{P}_m^{\alpha}(J,\mu) \equiv \int_{\mathbb{R}^n} \frac{\ell(J)^m}{(\ell(J) + |y - c_J|)^{m+n-\alpha}} d\mu(y),$$

where $\mathrm{P}_1^{\alpha}(J,\mu) = \mathrm{P}^{\alpha}(J,\mu)$ is the standard Poisson integral. The case $s = 0$ of the following extension of the 'energy lemma' is due to Rahm, Sawyer and Wick [14], and is proved in detail in [16, Lemmas 28 and 29 on pages 27–30].

**Definition 37** Given a subset $\mathcal{J} \subset \mathcal{D}$, define the projection $\mathsf{P}_{\mathcal{J}}^{\omega} \equiv \sum_{J' \in \mathcal{J}} \triangle_{J';\kappa}^{\omega}$, and given a cube $J \in \mathcal{D}$, define the projection $\mathsf{P}_J^{\omega} \equiv \sum_{J' \in \mathcal{D}:\, J' \subset J} \triangle_{J';\kappa}^{\omega}$.

**Lemma 38** (Energy Lemma) *Fix $\kappa \geq 1$. Let $J$ be a cube in $\mathcal{D}$, and let $\Psi_J \in W_{\mathrm{dyad}}^{-s}(\omega)$ be supported in $J$ with vanishing $\omega$-means up to order less than $\kappa$. Let $\nu$ be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma > 1$, and let $T^{\alpha}$ be a smooth $\alpha$-fractional singular integral operator with $0 \leq \alpha < n$. Then for $|s|$ sufficiently small, we have the 'pivotal' bound*

$$\left| \left\langle T^{\alpha}(\varphi\nu), \Psi_J \right\rangle_{L^2(\omega)} \right| \lesssim C_{\gamma} \mathrm{P}_{\kappa}^{\alpha}(J,\nu)\, \ell(J)^{-s} \sqrt{|J|_{\omega}}\, \|\Psi_J\|_{W_{\mathrm{dyad}}^{-s}(\omega)} \tag{3.13}$$

*for any function $\varphi$ with $|\varphi| \leq 1$.*

We also recall from [16, Lemma 33] the following Poisson estimate, that is a straightforward extension of the case $m = 1$ due to Nazarov, Treil and Volberg in [12].

**Lemma 39** *Fix $m \geq 1$. Suppose that $J \subset I \subset K$ and that* $\mathrm{dist}(J, \partial I) > 2\sqrt{n}\ell(J)^{\varepsilon} \ell(I)^{1-\varepsilon}$. *Then*

$$\mathrm{P}_m^{\alpha}(J, \sigma \mathbf{1}_{K \setminus I}) \lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{m - \varepsilon(n+m-\alpha)} \mathrm{P}_m^{\alpha}(I, \sigma \mathbf{1}_{K \setminus I}). \tag{3.14}$$

We now give Sobolev modifications to several known arguments. The next lemma was proved in [14] for $s = 0$.

**Lemma 40** *Let $0 \leq \alpha < n$, $\kappa \in \mathbb{N}$ and $0 < \delta < 1$. Suppose that $I$ and $J$ are cubes in $\mathbb{R}^n$ such that $J \subset 2J \subset I$, and that $\mu$ is a signed measure on $\mathbb{R}^n$ supported outside $I$. Finally suppose that $T^{\alpha}$ is a smooth fractional singular integral on $\mathbb{R}^n$ with kernel $K^{\alpha}(x,y) = K_y^{\alpha}(x)$, and that $\omega$ is a locally finite positive Borel measure on $\mathbb{R}^n$. Then*

$$\left\| \triangle_{J;\kappa}^{\omega} T^{\alpha} \mu \right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^2 \lesssim \Phi_{\kappa,s}^{\alpha}(J,\mu)^2 + \Psi_{\kappa,s}^{\alpha}(J,|\mu|)^2, \tag{3.15}$$

*where for a measure $\nu$,*

$$\Phi_{\kappa,s}^{\alpha}(J,\nu)^2 \equiv \sum_{|\beta|=\kappa} \left| \int_{\mathbb{R}^n} \left(K_y^{\alpha}\right)^{(\kappa)} \left(m_J^{\kappa}\right) d\nu(y) \right|^2 \left\| \triangle_{J;\kappa}^{\omega} x^{\beta} \right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^2$$

$$\Psi^{\alpha}_{\kappa,s}(J,|\nu|)^2 \equiv \left(\frac{\mathrm{P}^{\alpha}_{\kappa+\delta}(J,|\nu|)}{|J|^{\frac{\kappa}{n}}}\right)^2 \left\| \left|x-m^{\kappa}_J\right|^{\kappa} \right\|^2_{W^s_{\mathrm{dyad}}(\mathbf{1}_J\omega)} \frac{\left\|\left|\mathbf{h}^{\omega}_{J;\kappa}\right|\right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}{\ell(J)^{2s}}$$

$$\textit{where } m^{\kappa}_J \in J \textit{ satisfies } \left\| \left|x-m^{\kappa}_J\right|^{\kappa} \right\|^2_{W^s_{\mathrm{dyad}}(\mathbf{1}_J\omega)} = \inf_{m\in J} \left\| |x-m|^{\kappa} \right\|^2_{W^s_{\mathrm{dyad}}(\mathbf{1}_J\omega)}.$$

***Remark 41*** Note that when $s=0$, we have $\frac{\left\|\left|\mathbf{h}^{\omega}_{J;\kappa}\right|\right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}{\ell(J)^{2s}} = \frac{\left\|\mathbf{h}^{\omega}_{J;\kappa}\right\|^2_{L^2(\omega)}}{1} = 1$, and so the above inequality becomes the familiar Monotonicity Lemma. For $s$ close to zero, Lemma 26 shows that $\frac{\left\|\left|\mathbf{h}^{\omega}_{J;\kappa}\right|\right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}{\ell(J)^{2s}} \lesssim 1$, which gives the same familiar form.

***Proof of Lemma 40*** The proof is an easy adaptation of the one-dimensional proof in [14], which was in turn adapted from the proofs in [11, 17], but using a $\kappa^{th}$ order Taylor expansion instead of a first order expansion on the kernel $\left(K^{\alpha}_y\right)(x) = K^{\alpha}(x,y)$. Due to the importance of this lemma, as explained above, we repeat the short argument.

Let $\left\{h^{\omega,a}_{J;\kappa}\right\}_{a\in\Gamma_{J,n,\kappa}}$ be an orthonormal basis of $L^2_{J;\kappa}(\omega)$ consisting of Alpert functions as above. Now we use the Calderón–Zygmund smoothness estimate (1.2), together with Taylor's formula

$$K^{\alpha}_y(x) = \mathrm{Tay}\left(K^{\alpha}_y\right)(x,c) + \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left(K^{\alpha}_y\right)^{(\beta)}(\theta(x,c))(x-c)^{\beta};$$

$$\mathrm{Tay}\left(K^{\alpha}_y\right)(x,c) \equiv K^{\alpha}_y(c) + [(x-c)\cdot\nabla]K^{\alpha}_y(c) + \ldots + \frac{1}{(\kappa-1)!}[(x-c)\cdot\nabla]^{\kappa-1}K^{\alpha}_y(c),$$

and the vanishing means of the Alpert functions $h^{\omega,a}_{J;\kappa}$ for $a\in\Gamma_{J,n,\kappa}$, to obtain

$$\begin{aligned}
\left\langle T^{\alpha}\mu, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)} &= \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} K^{\alpha}(x,y)h^{\omega,a}_{J;\kappa}(x)\,d\omega(x)\right\}d\mu(y) = \int_{\mathbb{R}^n}\left\langle K^{\alpha}_y, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)} d\mu(y)\\
&= \int_{\mathbb{R}^n}\left\langle K^{\alpha}_y(x) - \mathrm{Tay}\left(K^{\alpha}_y\right)\left(x,m^{\kappa}_J\right), h^{\omega,a}_{J;\kappa}(x)\right\rangle_{L^2(\omega)} d\mu(y)\\
&= \int_{\mathbb{R}^n}\left\langle \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left(K^{\alpha}_y\right)^{(\beta)}\left(\theta\left(x,m^{\kappa}_J\right)\right)\left(x-m^{\kappa}_J\right)^{\beta}, h^{\omega,a}_{J;\kappa}(x)\right\rangle_{L^2(\omega)} d\mu(y) \quad \left(\text{some } \theta\left(x,m^{\kappa}_J\right)\in J\right)\\
&= \sum_{|\beta|=\kappa}\left\langle\left[\int_{\mathbb{R}^n}\frac{1}{\kappa!}\left(K^{\alpha}_y\right)^{(\beta)}\left(m^{\kappa}_J\right)d\mu(y)\right]\left(x-m^{\kappa}_J\right)^{\beta}, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)}\\
&\quad + \sum_{|\beta|=\kappa}\left\langle\left[\int_{\mathbb{R}^n}\frac{1}{\kappa!}\left[\left(K^{\alpha}_y\right)^{(\beta)}\left(\theta\left(x,m^{\kappa}_J\right)\right) - \sum_{|\beta|=\kappa}\left(K^{\alpha}_y\right)^{(\beta)}\left(m^{\kappa}_J\right)\right]d\mu(y)\right]\left(x-m^{\kappa}_J\right)^{\beta}, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)}.
\end{aligned}$$

Then using that $\int_{\mathbb{R}^n}\left(K^{\alpha}_y\right)^{(\beta)}\left(m^{\kappa}_J\right)d\mu(y)$ is independent of $x\in J$, and that $\left\langle\left(x-m^{\kappa}_J\right)^{\beta}, \mathbf{h}^{\omega}_{J;\kappa}\right\rangle_{L^2(\omega)} = \left\langle x^{\beta}, \mathbf{h}^{\omega}_{J;\kappa}\right\rangle_{L^2(\omega)}$ by moment vanishing of the Alpert wavelets, we can continue with

$$\begin{aligned}
\left\langle T^{\alpha}\mu, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)} &= \sum_{|\beta|=\kappa}\left[\int_{\mathbb{R}^n}\frac{1}{\kappa!}\left(K^{\alpha}_y\right)^{(\beta)}\left(m^{\kappa}_J\right)d\mu(y)\right]\cdot\left\langle x^{\beta}, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)}\\
&\quad + \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left\langle\left[\int_{\mathbb{R}^n}\left[\left(K^{\alpha}_y\right)^{(\beta)}\left(\theta\left(x,m^{\kappa}_J\right)\right) - \left(K^{\alpha}_y\right)^{(\beta)}\left(m^{\kappa}_J\right)\right]d\mu(y)\right]\left(x-m^{\kappa}_J\right)^{\beta}, h^{\omega,a}_{J;\kappa}\right\rangle_{L^2(\omega)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\left\langle T^{\alpha}\mu, h_{J;\kappa}^{\omega,a}\right\rangle_{L^{2}(\omega)} - \sum_{|\beta|=\kappa}\left[\int_{\mathbb{R}^{n}}\frac{1}{\kappa!}\left(K_{y}^{\alpha}\right)^{(\beta)}\left(m_{J}^{\kappa}\right)d\mu\left(y\right)\right]\cdot\left\langle x^{\beta}, h_{J;\kappa}^{\omega,a}\right\rangle_{L^{2}(\omega)}\right| \\
&\leq \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left|\left\langle\left[\int_{\mathbb{R}^{n}}\sup_{\theta\in J}\left|\left(K_{y}^{\alpha}\right)^{(\beta)}(\theta)-\left(K_{y}^{\alpha}\right)^{(\beta)}\left(m_{J}^{\kappa}\right)\right|d\left|\mu\right|(y)\right]\left|x-m_{J}^{\kappa}\right|^{\kappa}, \left|h_{J;\kappa}^{\omega,a}\right|\right\rangle_{L^{2}(\omega)}\right| \\
&\lesssim \left\|C_{CZ}\frac{\mathrm{P}_{\kappa+\delta}^{\alpha}\left(J,|\mu|\right)}{|J|^{\kappa}}\left|x-m_{J}^{\kappa}\right|^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}\left\|\left|h_{J;\kappa}^{\omega,a}\right|\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)} \\
&\lesssim C_{CZ}\frac{\mathrm{P}_{\kappa+\delta}^{\alpha}\left(J,|\mu|\right)}{|J|^{\kappa}}\left\|\left|x-m_{J}^{\kappa}\right|^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\mathbf{1}_{J}\omega)}\left\|\left|h_{J;\kappa}^{\omega,a}\right|\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}
\end{aligned}$$

where in the last line we have used

$$\begin{aligned}
&\int_{\mathbb{R}^{n}}\sup_{\theta\in J}\left|\left(K_{y}^{\alpha}\right)^{(\beta)}(\theta)-\left(K_{y}^{\alpha}\right)^{(\beta)}\left(m_{J}^{\kappa}\right)\right|d\left|\mu\right|(y) \\
&\quad\lesssim C_{CZ}\int_{\mathbb{R}^{n}}\left(\frac{|J|}{|y-c_{J}|}\right)^{\delta}\frac{d\left|\mu\right|(y)}{|y-c_{J}|^{\kappa+1-\alpha}} = C_{CZ}\frac{\mathrm{P}_{\kappa+\delta}^{\alpha}\left(J,|\mu|\right)}{|J|^{\kappa}}.
\end{aligned}$$

Thus with $v_{J}^{\beta} = \frac{1}{\kappa!}\int_{\mathbb{R}^{n}}\left(K_{y}^{\alpha}\right)^{(\beta)}\left(m_{J}^{\kappa}\right)d\mu\left(y\right)$, and noting that the functions $\left\{v_{J}^{\beta}h_{J;\kappa}^{\omega,a}\right\}_{a\in\Gamma_{J,n,\kappa}}$ are orthonormal in $a\in\Gamma_{J,n,\kappa}$ for each $\beta$ and $J$, we have

$$\begin{aligned}
\left|v_{J}^{\beta}\left\langle x^{\beta}, h_{J;\kappa}^{\omega,a}\right\rangle_{L^{2}(\omega)}\right|^{2} &= \sum_{a\in\Gamma_{J,n,\kappa}}\left|\left\langle x^{\beta}, v_{J}^{\beta}\cdot h_{J;\kappa}^{\omega,a}\right\rangle_{L^{2}(\omega)}\right|^{2} = \left\|\triangle_{J;\kappa}^{\omega}v_{J}^{\beta}x^{\beta}\right\|_{L^{2}(\omega)}^{2} \\
&= \left|v_{J}^{\beta}\right|^{2}\left\|\triangle_{J;\kappa}^{\omega}x^{\beta}\right\|_{L^{2}(\omega)}^{2} = \left|v_{J}^{\beta}\right|^{2}\ell\left(J\right)^{2s}\left\|\triangle_{J;\kappa}^{\omega}x^{\beta}\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2}
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\|\triangle_{J;\kappa}^{\omega}T^{\alpha}\mu\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2} = \left|\widehat{T^{\alpha}\mu}\left(J;\kappa\right)\right|^{2}\ell\left(J\right)^{-2s} = \ell\left(J\right)^{-2s}\sum_{a\in\Gamma_{J,n,\kappa}}\left|\left\langle T^{\alpha}\mu, h_{J;\kappa}^{\omega,a}\right\rangle_{L^{2}(\omega)}\right|^{2} \\
&= \ell\left(J\right)^{-2s}\sum_{|\beta|=\kappa}\left|v_{J}^{\beta}\right|^{2}\ell\left(J\right)^{2s}\left\|\triangle_{J;\kappa}^{\omega}x^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2} \\
&\quad+O\left(\frac{\mathrm{P}_{\kappa+\delta}^{\alpha}\left(J,|\mu|\right)}{|J|^{\frac{\kappa}{n}}}\right)^{2}\ell\left(J\right)^{-2s}\left\|\left|x-m_{J}^{\kappa}\right|^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\mathbf{1}_{J}\omega)}^{2}\sum_{a\in\Gamma_{J,n,\kappa}}\left\|\left|h_{J;\kappa}^{\omega,a}\right|\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^{2}.
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
\left\|\triangle_{J;\kappa}^{\omega}T^{\alpha}\mu\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2} &\leq C_{1}\sum_{|\beta|=\kappa}\left|\frac{1}{\kappa!}\int_{\mathbb{R}^{n}}\left(K_{y}^{\alpha}\right)^{(\beta)}\left(m_{J}\right)d\mu\left(y\right)\right|^{2}\left\|\triangle_{J;\kappa}^{\omega}x^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\omega)}^{2} \\
&\quad+C_{2}\left(\frac{\mathrm{P}_{\kappa+\delta}^{\alpha}\left(J,|\mu|\right)}{|J|^{\frac{\kappa}{n}}}\right)^{2}\left\|\left|x-m_{J}^{\kappa}\right|^{\kappa}\right\|_{W_{\mathrm{dyad}}^{s}(\mathbf{1}_{J}\omega)}^{2}\frac{\left\|\left|\mathbf{h}_{J;\kappa}^{\omega}\right|\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^{2}}{\ell\left(J\right)^{2s}}
\end{aligned}$$

where $\mathbf{h}^{\omega}_{J;\kappa} \equiv \left\{h^{\omega,a}_{J;\kappa}\right\}_{a\in\Gamma_{J,n,\kappa}}$ and

$$\sum_{|\beta|=\kappa} \left| \frac{1}{\kappa!} \int_{\mathbb{R}^n} \left(K^{\alpha}_{y}\right)^{(\beta)} (m_J)\, d\mu(y) \right|^2 \lesssim \left( \frac{\mathrm{P}^{\alpha}_{\kappa}(J,|\mu|)}{|J|^{\frac{\kappa}{n}}} \right)^2 .$$

□

The following Energy Lemma follows from the above Monotonicity Lemma in a standard way, see e.g. [17]. Recall that for a subset $\mathcal{J} \subset \mathcal{D}$, and for a cube $J \in \mathcal{D}$, there are projections $\mathsf{P}^{\omega}_{\mathcal{J}} \equiv \sum_{J'\in\mathcal{J}} \triangle^{\omega}_{J';\kappa}$ and $\mathsf{P}^{\omega}_{J} \equiv \sum_{J'\in\mathcal{D}:\ J'\subset J} \triangle^{\omega}_{J';\kappa}$. Recall also that $\mathbf{h}^{\omega}_{J;\kappa} \equiv \left\{h^{\omega,a}_{J;\kappa}\right\}_{a\in\Gamma_{J,n,\kappa}}$ is the vector of Alpert wavelets associated with the cube $J$.

**Lemma 42** (Energy Lemma) *Fix $\kappa \geq 1$ and a locally finite positive Borel measure $\omega$. Let $J$ be a cube in $\mathcal{D}$. Let $\Psi_J \in W^{-s}_{\mathrm{dyad}}(\omega)$ be supported in $J$ with vanishing $\omega$-means up to order less than $\kappa$. Let $\nu$ be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma > 1$. Let $T^{\alpha}$ be a smooth $\alpha$-fractional singular integral operator with $0 \leq \alpha < n$. Then we have the 'pivotal' bound*

$$\left| \left\langle T^{\alpha}(\varphi\nu), \Psi_J \right\rangle_{L^2(\omega)} \right| \lesssim C_{\gamma} \mathrm{P}^{\alpha}_{\kappa}(J,\nu)\, \ell(J)^{-s} \sqrt{|J|_{\omega}}\, \|\Psi_J\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \frac{\left\| \left| \mathbf{h}^{\omega}_{J;\kappa} \right| \right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}{\ell(J)^{2s}} \tag{3.16}$$

*for any function $\varphi$ with $|\varphi| \leq 1$.*

## 4 The strong -pivotal corona decomposition

To set the stage for control of the stopping form below in the absence of the energy condition, we construct the *strong $\kappa$-pivotal* corona decomposition for $f \in W^{s}_{\mathrm{dyad}}(\mu)$, in analogy with the energy version for $L^2(\sigma)$ and $L^2(\omega)$ used in the two part paper [8, 9] and in [17].

Fix $\gamma > 1$ and define $\mathcal{G}_0 = \left\{F^0_1\right\}$ to consist of the single cube $F^0_1$, and define the first generation $\mathcal{G}_1 = \left\{F^1_k\right\}_k$ of *$\kappa$-pivotal stopping children* of $F^0_1$ to be the *maximal* dyadic subcubes $I$ of $F_0$ satisfying

$$\mathrm{P}^{\alpha}_{\kappa}\left(I, \mathbf{1}_{F^0_1}\sigma\right)^2 |I|_{\omega} \geq \gamma\, |I|_{\sigma} .$$

Then define the second generation $\mathcal{G}_2 = \left\{F^2_k\right\}_k$ of CZ $\kappa$-pivotal $s$-stopping children of $F^0_1$ to be the *maximal* dyadic subcubes $I$ of some $F^1_k \in \mathcal{G}_1$ satisfying

$$\mathrm{P}^{\alpha}_{\kappa}\left(I, \mathbf{1}_{F^1_k}\sigma\right)^2 |I|_{\omega} \geq \gamma\, |I|_{\sigma} .$$

Continue by recursion to define $\mathcal{G}_n$ for all $n \geq 0$, and then set

$$\mathcal{F} \equiv \bigcup_{n=0}^{\infty} \mathcal{G}_n = \left\{F^n_k : n \geq 0, k \geq 1\right\}$$

to be the set of all CZ $\kappa$-pivotal stopping intervals in $F^0_1$ obtained in this way.

### 4.1 Carleson condition for stopping cubes and corona controls

The $\varepsilon$-strong $\sigma$-Carleson condition for $\mathcal{F}$ follows from the usual calculation,

$$\begin{aligned}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F)}\left(\frac{\ell(F)}{\ell(F')}\right)^{\varepsilon}\left|F'\right|_{\sigma} &\leq \frac{1}{\gamma}\left\{\sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F)}\mathrm{P}_{\kappa}^{\alpha}\left(F',\mathbf{1}_{F}\sigma\right)^{2}\left(\frac{\ell(F)}{\ell(F')}\right)^{\varepsilon}\left|F'\right|_{\omega}\right\}\\ &\leq \frac{1}{\gamma}\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}(\sigma,\omega)\left|F\right|_{\sigma}.\end{aligned}$$

Now set $\mathfrak{C}_{\mathcal{F}}^{(\ell)}(F)$ to be the $\ell^{th}$ generation of $\mathcal{F}$-subcubes of $F$, and define $\mathcal{F}(F)=\bigcup_{\ell=0}^{\infty}\mathfrak{C}_{\mathcal{F}}^{(\ell)}(F)$ to be the collection of all $\mathcal{F}$-subcubes of $F$. Then if $\frac{\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}+1}{\gamma}<\frac{1}{2}$, we have the $\varepsilon$-strong $\sigma$-Carleson condition,

$$\begin{aligned}\sum_{F'\in\mathcal{F}(F)}\left(\frac{\ell(F)}{\ell(F')}\right)^{\varepsilon}\left|F'\right|_{\sigma} &= \sum_{\ell=0}^{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(\ell)}(F)}\left(\frac{\ell(F)}{\ell(F')}\right)^{\varepsilon}\left|F'\right|_{\sigma}\\ &\leq \sum_{\ell=0}^{\infty}\left(\frac{\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}+1}{\gamma}\right)^{\ell}\left|F\right|_{\sigma}\leq 2\left|F\right|_{\sigma}. \qquad (4.1)\end{aligned}$$

Using Lemma 28, this Carleson condition delivers a basic method of control by quasiorthogonality (see [9, 17] for the case $s=0$),

$$\sum_{F\in\mathcal{F}}\left|F\right|_{\sigma}\left(\ell(F)^{-s}E_{F}^{\sigma}f\right)^{2}\lesssim\|f\|_{W_{\mathrm{dyad}}^{s}(\sigma)}^{2}, \qquad (4.2)$$

which is used repeatedly in conjunction with orthogonality of Sobolev projections $\triangle_{J;\kappa}^{\omega}g$,

$$\sum_{J\in\mathcal{D}}\left\|\triangle_{J;\kappa}^{\omega}g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^{2}=\|g\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^{2}. \qquad (4.3)$$

Moreover, in each corona

$$\mathcal{C}_{F}\equiv\left\{I\in\mathcal{D}:I\subset F\text{ and }I\not\subset F'\text{ for any }F'\in\mathcal{F}\text{ with }F'\subsetneqq F\right\},$$

we have, from the definition of the stopping times, the $\varepsilon$ -strong $\kappa$-pivotal control,

$$\mathrm{P}_{\kappa}^{\alpha}\left(I,\mathbf{1}_{F}\sigma\right)^{2}\left(\frac{\ell(F)}{\ell(I)}\right)^{\varepsilon}\left|I\right|_{\omega}<\Gamma\left|I\right|_{\sigma},\quad I\in\mathcal{C}_{F}\text{ and }F\in\mathcal{F}. \qquad (4.4)$$

## 5 Reduction of the proof to local forms

To prove Theorem 2, we begin by proving the bilinear form bound,

$$\begin{aligned}&\left|\left\langle T_{\sigma}^{\alpha}f,g\right\rangle_{\omega}\right|\\ &\lesssim\left(\sqrt{A_{2}^{\alpha}(\sigma,\omega)}+\mathfrak{TR}_{T^{\alpha}}^{(\kappa)}(\sigma,\omega)+\mathfrak{TR}_{(T^{\alpha})^{*}}^{(\kappa)}(\omega,\sigma)+\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa}+\mathcal{V}_{2,\varepsilon}^{\alpha,\kappa,*}\right)\|f\|_{W_{\mathrm{dyad}}^{s}(\sigma)}\|g\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.\end{aligned}$$

Following the weighted Haar expansions of Nazarov, Treil and Volberg, we write $f$ and $g$ in weighted Alpert wavelet expansions,

$$\left\langle T_{\sigma}^{\alpha} f, g\right\rangle_{\omega}=\left\langle T_{\sigma}^{\alpha}\left(\sum_{I \in \mathcal{D}} \triangle_{I;\kappa_1}^{\sigma} f\right), \left(\sum_{J \in \mathcal{D}} \triangle_{J;\kappa_2}^{\omega} g\right)\right\rangle_{\omega}. \tag{5.1}$$

Then following [17] and many others, the $L^2$ inner product in ( 5.1) can be expanded as

$$\begin{aligned}\left\langle T_{\sigma}^{\alpha} f, g\right\rangle_{\omega} &=\left\langle T_{\sigma}^{\alpha}\left(\sum_{I \in \mathcal{D}} \triangle_{I;\kappa_1}^{\sigma} f\right), \left(\sum_{J \in \mathcal{D}} \triangle_{J;\kappa_2}^{\omega} g\right)\right\rangle_{\omega} \\ &=\sum_{I \in \mathcal{D} \text{ and } J \in \mathcal{D}}\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa_1}^{\sigma} f\right), \left(\triangle_{J;\kappa_2}^{\omega} g\right)\right\rangle_{\omega}.\end{aligned}$$

Then the sum is further decomposed by first the *Cube Size Splitting*, then using the *Shifted Corona Decomposition*, according to the *Canonical Splitting*. We assume the reader is familiar with the notation and arguments in the first eight sections of [17]. The $n$-dimensional decompositions used in [17] are in spirit the same as the one-dimensional decompositions in [9], as well as the $n$ -dimensional decompositions in [11], but differ in significant details.

A fundamental result of Nazarov, Treil and Volberg [12] is that all the cubes $I$ and $J$ appearing in the bilinear form above may be assumed to be $(r, \varepsilon)-$ good, where a dyadic interval $K$ is $(r, \varepsilon)-$ good, or simply good, if for *every* dyadic supercube $L$ of $K$, it is the case that **either** $K$ has side length at least $2^{1-r}$ times that of $L$, **or** $K \Subset_{(r,\varepsilon)} L$. We say that a dyadic cube $K$ is $(r, \varepsilon)$-*deeply embedded* in a dyadic cube $L$, or simply $r$-*deeply embedded* in $L$, which we write as $K \Subset_{r,\varepsilon} L$, when $K \subset L$ and both

$$\begin{aligned}\ell(K) &\leq 2^{-r} \ell(L), \\ \operatorname{dist}\left(K, \bigcup_{L' \in \mathfrak{C}_{\mathcal{D}} L} \partial L'\right) &\geq 2 \ell(K)^{\varepsilon} \ell(L)^{1-\varepsilon}.\end{aligned} \tag{5.2}$$

Here is a brief schematic diagram as in [1], summarizing the shifted corona decompositions as used in [1, 17] for Alpert and Haar wavelet expansions of $f$ and $g$. We first introduce parameters as in [1, 17]. We will choose $\varepsilon>0$ sufficiently small later in the argument, and then $r$ must be chosen sufficiently large depending on $\varepsilon$ in order to reduce matters to $(r, \varepsilon)-$ good functions by the Nazarov, Treil and Volberg argument.

**Definition 43** The parameters $\tau$ and $\rho$ are fixed to satisfy

$$\tau>r \text{ and } \rho>r+\tau,$$

where $r$ is the goodness parameter already fixed.

$$\begin{array}{ccccccc}\left\langle T_{\sigma}^{\alpha} f, g\right\rangle_{\omega} &&&&&& \\ \downarrow &&&&&& \\ \mathsf{B}_{\Subset_{\rho}}(f, g) & + & \mathsf{B}_{{}_\rho\Supset}(f, g) & + & \mathsf{B}_{\cap}(f, g) & + & \mathsf{B}_{\diagup}(f, g) \\ \downarrow &&&&&& \\ \mathsf{T}_{\text{diagonal}}(f, g) & + & \mathsf{T}_{\text{farbelow}}(f, g) & + & \mathsf{T}_{\text{farabove}}(f, g) & + & \mathsf{T}_{\text{disjoint}}(f, g) \\ \downarrow && \downarrow &&&& \\ \mathsf{B}_{\Subset_{\rho}}^{F}(f, g) && \mathsf{T}_{\text{farbelow}}^{1}(f, g) & + & \mathsf{T}_{\text{farbelow}}^{2}(f, g) && \\ \downarrow &&&&&& \\ \mathsf{B}_{\text{stop}}^{F}(f, g) & + & \mathsf{B}_{\text{paraproduct}}^{F}(f, g) & + & \mathsf{B}_{\text{neighbour}}^{F}(f, g) & + & \mathsf{B}_{\text{commutator}}^{F}(f, g)\end{array}$$

### 5.1 Cube size splitting

The Nazarov, Treil and Volberg *Cube Size Splitting* of the inner product $\left\langle T_\sigma^\alpha f, g\right\rangle_\omega$ splits the pairs of cubes $(I, J)$ in a simultaneous Alpert decomposition of $f$ and $g$ into four groups determined by relative position, is given by

$$\begin{aligned}\left\langle T_\sigma^\alpha f, g\right\rangle_\omega &= \sum_{I,J\in\mathcal{D}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^\sigma f\right), \left(\triangle_{J;\kappa}^\omega g\right)\right\rangle_\omega \\ &= \sum_{\substack{I,J\in\mathcal{D}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^\sigma f\right), \left(\triangle_{J;\kappa}^\omega g\right)\right\rangle_\omega + \sum_{\substack{I,J\in\mathcal{D}\\ J_{\rho,\varepsilon}\Supset I}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^\sigma f\right), \left(\triangle_{J;\kappa}^\omega g\right)\right\rangle_\omega \\ &\quad + \sum_{\substack{I,J\in\mathcal{D}\\ J\cap I=\emptyset \text{ and } \frac{\ell(J)}{\ell(I)}\notin[2^{-\rho},2^\rho]}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^\sigma f\right), \left(\triangle_{J;\kappa}^\omega g\right)\right\rangle_\omega \\ &\quad + \sum_{\substack{I,J\in\mathcal{D}\\ 2^{-\rho}\le\frac{\ell(J)}{\ell(I)}\le 2^\rho}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^\sigma f\right), \left(\triangle_{J;\kappa}^\omega g\right)\right\rangle_\omega \\ &= \mathsf{B}_{\Subset_{\rho,\varepsilon}}(f,g) + \mathsf{B}_{{}_{\rho,\varepsilon}\Supset}(f,g) + \mathsf{B}_{\cap}(f,g) + \mathsf{B}_{\diagup}(f,g).\end{aligned}$$

Note however that the assumption the cubes $I$ and $J$ are $(r, \varepsilon) -$ good remains in force throughout the proof.

We will now make use of the $\kappa$-cube testing and triple testing constants, defined in (3.6) and (3.7 ), to prove the following bound in the Sobolev setting, which in the case $s = 0$ was proved in [16, see Lemma 31] following the Nazarov, Treil and Volberg arguments for Haar wavelets in [17, see the proof of Lemma 7.1] (see also [9]),

$$\begin{aligned}&\left|\mathsf{B}_{\cap}(f,g) + \mathsf{B}_{\diagup}(f,g)\right| \\ &\le C\left(\mathfrak{T}_{T^\alpha}^{\kappa,s} + \mathfrak{T}_{T^{\alpha,*}}^{\kappa,-s} + \mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2),s}(\sigma,\omega) + \sqrt{A_2^\alpha}\right)\|f\|_{W_{\text{dyad}}^s(\sigma)}\|g\|_{W_{\text{dyad}}^{-s}(\omega)}, \quad (5.3)\end{aligned}$$

where if $\Omega_{\text{dyad}}$ is the set of all dyadic grids,

$$\begin{aligned}&\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2),s}(\sigma,\omega) \\ &\equiv \sup_{\mathcal{D}\in\Omega} \sup_{\substack{Q,Q'\in\mathcal{D}\\ Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q}} \frac{\ell(Q)^s\,\ell(Q')^{-s}}{\sqrt{|Q|_\sigma\,|Q'|_\omega}} \sup_{\substack{f\in\left(\mathcal{P}_Q^{\kappa_1}\right)_{\text{norm}}(\sigma)\\ g\in\left(\mathcal{P}_{Q'}^{\kappa_2}\right)_{\text{norm}}(\omega)}} \left|\int_{Q'} T_\sigma^\alpha\left(\mathbf{1}_Q f\right) g d\omega\right| < \infty\end{aligned}$$

is a weak boundedness constant that in the case $s = 0$ was introduced in [16]. Here we will use the case $\kappa_1 = \kappa_2 = \kappa$. However, we only use that this constant is removed in the final section below using the following bound proved in [16, see (6.25) in Subsection 6.7 and note that only triple testing is needed there by choosing $\ell(Q') \le \ell(Q)$ (using duality and $T^{\alpha,*}$ if needed)], and which holds also in the Sobolev setting using Cauchy–Schwarz and triple testing,

$$\mathcal{WBP}_{T^\alpha}^{(\kappa,\kappa),s}(\sigma,\omega) \le C_\kappa\left(\mathfrak{TR}_{T^\alpha}^{\kappa,s}(\sigma,\omega) + \mathfrak{TR}_{T^{\alpha,*}}^{\kappa,-s}(\omega,\sigma)\right). \quad (5.4)$$

In fact the stronger bound with absolute values inside the sums in (5.3) was proved in the case $s=0$ in the previous references,

$$\sum_{\substack{I,J\in\mathcal{D}\\ J\cap I=\emptyset \text{ and } \frac{\ell(J)}{\ell(I)}\notin[2^{-\rho},2^{\rho}]}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma}f\right),\left(\triangle_{J;\kappa}^{\omega}g\right)\right\rangle_{\omega}\right| + \sum_{\substack{I,J\in\mathcal{D}\\ 2^{-\rho}\leq\frac{\ell(J)}{\ell(I)}\leq 2^{\rho}}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma}f\right),\left(\triangle_{J;\kappa}^{\omega}g\right)\right\rangle_{\omega}\right| \tag{5.5}$$
$$\leq C\left(\mathfrak{T}_{T^{\alpha}}^{\kappa,s}+\mathfrak{T}_{T^{\alpha,*}}^{\kappa,-s}+\mathcal{WBP}_{T^{\alpha}}^{(\kappa,\kappa),s}(\sigma,\omega)+\sqrt{A_2^{\alpha}}\right)\|f\|_{W_{\text{dyad}}^{s}(\sigma)}\|g\|_{W_{\text{dyad}}^{-s}(\omega)}.$$

This bound will be useful later since it yields the same bound for the sum of any subcollection of the index set, and for the convenience of the reader, we prove (5.5) below. Since the *below* and *above* forms $\mathsf{B}_{\Subset_{\rho,\varepsilon}}(f,g)$, $\mathsf{B}_{\rho,\varepsilon\Supset}(f,g)$ are symmetric, matters are then reduced to proving

$$\left|\mathsf{B}_{\Subset_{\rho,\varepsilon}}(f,g)\right|\lesssim\left(\mathfrak{T}_{T^{\alpha}}^{s}+\mathfrak{T}_{T^{\alpha,*}}^{-s}+\sqrt{A_2^{\alpha}}\right)\|f\|_{W_{\text{dyad}}^{s}(\sigma)}\|g\|_{W_{\text{dyad}}^{-s}(\omega)}. \tag{5.6}$$

We introduce some notation in order to prove (5.5). For weighted Alpert wavelet projections $\triangle_{I;\kappa}^{\sigma}$, we write the projection $\mathbb{E}_{I';\kappa}^{\sigma}\triangle_{I;\kappa}^{\sigma}f$ onto the child $I'\in\mathfrak{C}_{\mathcal{D}}(I)$ as $M_{I';\kappa}^{\sigma}\mathbf{1}_{I_{\pm}}$, where $M_{I';\kappa}^{\sigma}$ is a polynomial of degree less than $\kappa$ restricted to $I'$. Then we let $P_{I';\kappa}^{\sigma}\equiv\frac{M_{I';\kappa}^{\sigma}}{\left\|M_{I';\kappa}^{\sigma}\right\|_{\infty}}$ be its normalization on $I'$. From (2.8) we have the estimate,

$$\left\|\mathbb{E}_{I';\kappa}^{\sigma}\triangle_{I;\kappa}^{\sigma}f\right\|_{\infty}\leq C\frac{\left\|\triangle_{I;\kappa}^{\sigma}f\right\|_{L^{2}(\sigma)}}{\sqrt{|I'|_{\sigma}}}\approx C\frac{\left|\widehat{f}_{\kappa}(I)\right|}{\sqrt{|I'|_{\sigma}}}. \tag{5.7}$$

*Proof of* (5.5). To handle the second term in (5.5) we first decompose it into

$$\left\{\sum_{\substack{I,J\in\mathcal{D}:\ J\subset 3I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}+\sum_{\substack{I,J\in\mathcal{D}:\ I\subset 3J\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}+\sum_{\substack{I,J\in\mathcal{D}\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)\\ J\not\subset 3I \text{ and } I\not\subset 3J}}\right\}\left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma}f\right),\triangle_{J;\kappa}^{\omega}g\right\rangle_{\omega}\right|$$
$$\equiv A_1+A_2+A_3.$$

The proof of the bound for term $A_3$ is similar to that of the bound for the first term in (5.5), and so we will defer its proof until after the second term has been proved.

We now consider term $A_1$ as term $A_2$ is symmetric. To handle this term we will write the Alpert functions $h_{I;\kappa}^{\sigma}$ and $h_{J;\kappa}^{\omega}$ as linear combinations of polynomials times indicators of the children of their supporting cubes, denoted $I_\theta$ and $J_{\theta'}$ respectively. Then we use the testing condition on $I_\theta$ and $J_{\theta'}$ when they *overlap*, i.e. their interiors intersect; we use the weak boundedness property on $I_\theta$ and $J_{\theta'}$ when they *touch*, i.e. their interiors are disjoint but their closures intersect (even in just a point); and finally we use the $A_2^{\alpha}$ condition when $I_\theta$ and $J_{\theta'}$ are *separated*, i.e. their closures are disjoint. We will suppose initially that the side length of $J$ is at most the side length $I$, i.e. $\ell(J)\leq\ell(I)$, the proof for $J=\pi I$ being similar but for one point mentioned below.

So suppose that $I_\theta$ is a child of $I$ and that $J_{\theta'}$ is a child of $J$. If $J_{\theta'}\subset I_\theta$ we have using (5.7),

$$\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_\theta}\triangle_{I;\kappa}^{\sigma}f\right),\mathbf{1}_{J_{\theta'}}\triangle_{J;\kappa}^{\omega}g\right\rangle_{\omega}\right|\lesssim\frac{\left|\widehat{f}_{\kappa}(I)\right|}{\sqrt{|I|_{\sigma}}}\left|\left\langle T_{\sigma}^{\alpha}\left(P_{I_\theta;\kappa}^{\sigma}\mathbf{1}_{I_\theta}\right),P_{J_{\theta'};\kappa}^{\omega}\mathbf{1}_{J_{\theta'}}\right\rangle_{\omega}\right|\frac{\left|\widehat{g}_{\kappa}(J)\right|}{\sqrt{|J|_{\omega}}}$$

$$\lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\left(\int_{J_{\theta'}}\left|T_\sigma^\alpha\left(P_{I_\theta;\kappa}^\sigma \mathbf{1}_{I_\theta}\right)\right|^2 d\omega\right)^{\frac{1}{2}}\left|\widehat{g_\kappa}(J)\right|$$

$$\lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\mathfrak{T}_{T_\alpha}^{\kappa,s}|I|_\sigma^{\frac{1}{2}}\left|\widehat{g_\kappa}(J)\right| \lesssim \mathfrak{T}_{T_\alpha}^{\kappa,s}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|,$$

where $\widehat{f_\kappa}(I)$ denotes the vector Alpert coefficient of $f$ at the dyadic cube $I$. The point referred to above is that when $J = \pi I$ we write

$$\left\langle T_\sigma^\alpha\left(P_{I_\theta;\kappa}^\sigma \mathbf{1}_{I_\theta}\right), P_{J_{\theta'};\kappa}^\omega \mathbf{1}_{J_{\theta'}}\right\rangle_\omega = \left\langle P_{I_\theta;\kappa}^\sigma \mathbf{1}_{I_\theta}, T_\omega^{\alpha,*}\left(P_{J_{\theta'};\kappa}^\omega \mathbf{1}_{J_{\theta'}}\right)\right\rangle_\sigma$$

and get the dual testing constant $\mathfrak{T}_{T_\alpha^*}^{\kappa,s}$. If $J_{\theta'}$ and $I_\theta$ touch, then $\ell(J_{\theta'}) \le \ell(I_\theta)$ and we have $J_{\theta'} \subset 3I_\theta \backslash I_\theta$, and so

$$\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \bigtriangleup_{I;\kappa}^\sigma f\right), \mathbf{1}_{J_{\theta'}} \bigtriangleup_{J;\kappa}^\omega g\right\rangle_\omega\right| \lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\left|\left\langle T_\sigma^\alpha\left(P_{I_\theta;\kappa}^\sigma \mathbf{1}_{I_\theta}\right), P_{J_{\theta'};\kappa}^\omega \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right|\frac{\left|\widehat{g_\kappa}(J)\right|}{\sqrt{|J|_\omega}}$$

$$\lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\mathcal{WBP}_{T^\alpha}^{(\kappa,\kappa),s}\sqrt{|I|_\sigma |J|_\omega}\frac{\left|\widehat{g_\kappa}(J)\right|}{\sqrt{|J|_\omega}} = \mathcal{WBP}_{T^\alpha}^{(\kappa,\kappa),s}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|.$$

Finally, if $J_{\theta'}$ and $I_\theta$ are separated, and if $K$ is the smallest (not necessarily dyadic) cube containing both $J_{\theta'}$ and $I_\theta$, then $\operatorname{dist}(I_\theta, J_{\theta'}) \approx \ell(K)$ and we have

$$\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \bigtriangleup_{I;\kappa}^\sigma f\right), \mathbf{1}_{J_{\theta'}} \bigtriangleup_{J;\kappa}^\omega g\right\rangle_\omega\right| \lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right|\frac{\left|\widehat{g_\kappa}(J)\right|}{\sqrt{|J|_\omega}}$$

$$\lesssim \frac{\left|\widehat{f_\kappa}(I)\right|}{\sqrt{|I|_\sigma}}\frac{1}{\operatorname{dist}(I_\theta, J_{\theta'})^{n-\alpha}}|I_\theta|_\sigma |J_{\theta'}|_\omega \frac{\left|\widehat{g_\kappa}(J)\right|}{\sqrt{|J|_\omega}} = \frac{\sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega}}{\operatorname{dist}(I_\theta, J_{\theta'})^{n-\alpha}}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|$$

$$\lesssim \sqrt{A_2^\alpha}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|.$$

Now we sum over all the children of $J$ and $I$ satisfying $2^{-\rho}\ell(I) \le \ell(J) \le 2^\rho \ell(I)$ for which $J \subset 3I$ to obtain that

$$A_1 \lesssim \left(\mathfrak{T}_{T^\alpha}^{\kappa,s} + \mathfrak{T}_{T^{\alpha,*}}^{\kappa,-s} + \mathcal{WBP}_{T^\alpha}^{(\kappa,\kappa),s}(\sigma,\omega) + \sqrt{A_2^\alpha}\right)\sum_{\substack{I,J\in\mathcal{D}:\ J\subset 3I \\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|.$$

It is at this point that the Sobolev norms make their appearance, through an application of the Cauchy–Schwarz inequality to obtain

$$\sum_{\substack{I,J\in\mathcal{D}:\ J\subset 3I \\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}}\left|\widehat{f_\kappa}(I)\right|\left|\widehat{g_\kappa}(J)\right|$$

$$\le C_\rho\left(\sum_{\substack{I,J\in\mathcal{D}:\ J\subset 3I \\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}}\left|\widehat{f_\kappa}(I)\right|^2 \ell(I)^{-2s}\right)^{\frac{1}{2}}\left(\sum_{\substack{I,J\in\mathcal{D}:\ J\subset 3I \\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}}\left|\widehat{g_\kappa}(J)\right|^2 \ell(J)^{2s}\right)^{\frac{1}{2}}$$

$$\lesssim \|f\|_{W_{\mathrm{dyad}}^s(\sigma)}\|g\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.$$

This completes our proof of the bound for the second term in (5.5), save for the deferral of term $A_3$, which we bound below.

Now we turn to the sum of separated cubes in (5.5). We split the pairs $(I, J) \in \mathcal{D}^{\sigma} \times \mathcal{D}^{\omega}$ occurring in the first term in (5.5) into two groups, those with side length of $J$ smaller than side length of $I$, and those with side length of $I$ smaller than side length of $J$, treating only the former case, the latter being symmetric. Thus we prove the following bound:

$$\mathcal{A}(f,g) \equiv \sum_{\substack{I,J\in\mathcal{D}\\ I\cap J=\emptyset \text{ and } \ell(J)\leq 2^{-\rho}\ell(I)}} \left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma} f\right), \triangle_{J;\kappa}^{\omega} g\right\rangle_{\omega}\right| \lesssim \sqrt{A_2^{\alpha}}\, \|f\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \|g\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.$$

We apply the 'pivotal' bound (3.13) from the Energy Lemma to estimate the inner product $\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma} f\right), \triangle_{J;\kappa}^{\omega} g\right\rangle_{\omega}$ and obtain,

$$\left|\left\langle T_{\sigma}^{\alpha}\left(\triangle_{I;\kappa}^{\sigma} f\right), \triangle_{J;\kappa}^{\omega} g\right\rangle_{\omega}\right| \lesssim \mathrm{P}_{\kappa}^{\alpha}\left(J, \left|\triangle_{I;\kappa}^{\sigma} f\right|\sigma\right) \ell(J)^{-s} \sqrt{|J|_{\omega}} \left\|\triangle_{J;\kappa}^{\omega} g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.$$

Denote by dist the $\ell^{\infty}$ distance in $\mathbb{R}^n$: $\mathrm{dist}(x,y) = \max_{1\leq j\leq n}\left|x_j - y_j\right|$. We now estimate separately the long-range and mid-range cases where $\mathrm{dist}(J,I) \geq \ell(I)$ holds or not, and we decompose $\mathcal{A}$ accordingly:

$$\mathcal{A}(f,g) \equiv \mathcal{A}^{\mathrm{long}}(f,g) + \mathcal{A}^{\mathrm{mid}}(f,g).$$

**The long-range case**: We begin with the case where $\mathrm{dist}(J,I)$ is at least $\ell(I)$, i.e. $J\cap 3I = \emptyset$. Since $J$ and $I$ are separated by at least $\max\{\ell(J), \ell(I)\}$, we have the inequality

$$\begin{aligned}
\mathrm{P}_{\kappa}^{\alpha}\left(J, \left|\triangle_{I;\kappa}^{\sigma} f\right|\sigma\right) &\approx \int_I \frac{\ell(J)}{|y-c_J|^{n+1-\alpha}} \left|\triangle_{I;\kappa}^{\sigma} f(y)\right| d\sigma(y)\\
&\lesssim \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \ell(I)^{s} \frac{\ell(J)\sqrt{|I|_{\sigma}}}{\mathrm{dist}(I,J)^{n+1-\alpha}},
\end{aligned}$$

since $\int_I \left|\triangle_{I;\kappa}^{\sigma} f(y)\right| d\sigma(y) \leq \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{L^2(\sigma)} \sqrt{|I|_{\sigma}}$ and $\left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} = \ell(I)^{-s} \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{L^2(\sigma)}$. Thus with $A(f,g) = \mathcal{A}^{\mathrm{long}}(f,g)$ we have

$$\begin{aligned}
A(f,g) &\lesssim \sum_{I\in\mathcal{D}} \sum_{J:\, \ell(J)\leq\ell(I):\, \mathrm{dist}(I,J)\geq\ell(I)} \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \left\|\triangle_{J;\kappa}^{\omega} g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}\\
&\quad\times \left(\frac{\ell(I)}{\ell(J)}\right)^{s} \frac{\ell(J)}{\mathrm{dist}(I,J)^{n+1-\alpha}} \sqrt{|I|_{\sigma}}\sqrt{|J|_{\omega}}\\
&\equiv \sum_{(I,J)\in\mathcal{P}} \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \left\|\triangle_{J;\kappa}^{\omega} g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)} A(I,J);
\end{aligned}$$

$$\text{with } A(I,J) \equiv \left(\frac{\ell(I)}{\ell(J)}\right)^{s} \frac{\ell(J)}{\mathrm{dist}(I,J)^{n+1-\alpha}} \sqrt{|I|_{\sigma}}\sqrt{|J|_{\omega}};$$
$$\text{and } \mathcal{P} \equiv \{(I,J)\in\mathcal{D}\times\mathcal{D} : \ell(J)\leq\ell(I) \text{ and } \mathrm{dist}(I,J)\geq\ell(I)\}.$$

Now let $\mathcal{D}_N \equiv \left\{K\in\mathcal{D} : \ell(K) = 2^N\right\}$ for each $N\in\mathbb{Z}$. For $N\in\mathbb{Z}$ and $t\in\mathbb{Z}_+$, we further decompose $A(f,g)$ by pigeonholing the sidelengths of $I$ and $J$ by $2^N$ and $2^{N-t}$ respectively:

$$\begin{aligned}
A(f,g) &= \sum_{t=0}^{\infty}\sum_{N\in\mathbb{Z}} A_N^t(f,g);\\
A_N^t(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_N^t} \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \left\|\triangle_{J;\kappa}^{\omega} g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)} A(I,J)
\end{aligned}$$

$$\text{where } \mathcal{P}_N^t \equiv \{(I,J) \in \mathcal{D}_N \times \mathcal{D}_{N-t} : \text{dist}(I,J) \geq \ell(I)\}.$$

Now $A_N^t(f,g) = A_N^t\left(\mathsf{P}_{N;\kappa}^{\sigma} f, \mathsf{P}_{N-t;\kappa}^{\omega} g\right)$ where $\mathsf{P}_{M;\kappa}^{\mu} = \sum_{K\in\mathcal{D}_M} \triangle_{K;\kappa}^{\mu}$ denotes Alpert projection onto the linear span Span $\left\{h_{K;\kappa}^{\mu,a}\right\}_{K\in\mathcal{D}_M, a\in\Gamma_{K,n,\kappa}}$, and so by orthogonality of the projections $\left\{\mathsf{P}_{M;\kappa}^{\mu}\right\}_{M\in\mathbb{Z}}$ we have

$$\begin{aligned}
\left|\sum_{N\in\mathbb{Z}} A_N^t(f,g)\right| &= \sum_{N\in\mathbb{Z}} \left|A_N^t\left(\mathsf{P}_{N;\kappa}^{\sigma} f, \mathsf{P}_{N-t;\kappa}^{\omega} g\right)\right| \\
&\leq \sum_{N\in\mathbb{Z}} \left\|A_N^t\right\| \left\|\mathsf{P}_{N;\kappa}^{\sigma} f\right\|_{W_{\text{dyad}}^s(\sigma)} \left\|\mathsf{P}_{N-t;\kappa}^{\omega} g\right\|_{W_{\text{dyad}}^{-s}(\omega)} \\
&\leq \left\{\sup_{N\in\mathbb{Z}} \left\|A_N^t\right\|\right\} \left(\sum_{N\in\mathbb{Z}} \left\|\mathsf{P}_{N;\kappa}^{\sigma} f\right\|_{W_{\text{dyad}}^s(\sigma)}^2\right)^{\frac{1}{2}} \left(\sum_{N\in\mathbb{Z}} \left\|\mathsf{P}_{N-t;\kappa}^{\omega} g\right\|_{W_{\text{dyad}}^{-s}(\omega)}^2\right)^{\frac{1}{2}} \\
&\leq \left\{\sup_{N\in\mathbb{Z}} \left\|A_N^t\right\|\right\} \|f\|_{W_{\text{dyad}}^s(\sigma)} \|g\|_{W_{\text{dyad}}^{-s}(\omega)}.
\end{aligned}$$

Thus it suffices to show an estimate uniform in $N$ with geometric decay in $t$, and we will show

$$\left|A_N^t(f,g)\right| \leq C2^{-t}\sqrt{A_2^{\alpha}} \|f\|_{W_{\text{dyad}}^s(\sigma)} \|g\|_{W_{\text{dyad}}^{-s}(\omega)}, \quad \text{for } t \geq 0 \text{ and } N \in \mathbb{Z}. \tag{5.8}$$

We now pigeonhole the distance between $I$ and $J$:

$$\begin{aligned}
A_N^t(f,g) &= \sum_{\ell=0}^{\infty} A_{N,\ell}^t(f,g); \\
A_{N,\ell}^t(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_{N,\ell}^t} \left\|\triangle_{I;\kappa}^{\sigma} f\right\|_{W_{\text{dyad}}^s(\sigma)} \left\|\triangle_{J;\kappa}^{\omega} g\right\|_{W_{\text{dyad}}^{-s}(\omega)} A(I,J) \\
\text{where } \mathcal{P}_{N,\ell}^t &\equiv \left\{(I,J) \in \mathcal{D}_N \times \mathcal{D}_{N-s} : \text{dist}(I,J) \approx 2^{N+\ell}\right\}.
\end{aligned}$$

If we define $\mathcal{H}\left(A_{N,\ell}^t\right)$ to be the bilinear form on $\ell^2 \times \ell^2$ with matrix $[A(I,J)]_{(I,J)\in\mathcal{P}_{N,\ell}^t}$, then it remains to show that the norm $\left\|\mathcal{H}\left(A_{N,\ell}^t\right)\right\|_{\ell^2\to\ell^2}$ is bounded by $C2^{-t(1-s)-\ell}\sqrt{A_2^{\alpha}}$. In turn, this is equivalent to showing that the norm $\left\|\mathcal{H}\left(B_{N,\ell}^t\right)\right\|_{\ell^2\to\ell^2}$ of the bilinear form $\mathcal{H}\left(B_{N,\ell}^t\right) \equiv \mathcal{H}\left(A_{N,\ell}^t\right)^{\text{tr}} \mathcal{H}\left(A_{N,\ell}^t\right)$ on the sequence space $\ell^2$ is bounded by $C^2 2^{-2t(1-s)-2\ell} A_2^{\alpha}$. Here $\mathcal{H}\left(B_{N,\ell}^t\right)$ is the quadratic form with matrix kernel $\left[B_{N,\ell}^t(J,J')\right]_{J,J'\in\mathcal{D}_{N-s}}$ having entries:

$$B_{N,\ell}^t(J,J') \equiv \sum_{I\in\mathcal{D}_N:\ \text{dist}(I,J)\approx\text{dist}(I,J')\approx 2^{N+\ell}} A(I,J)\, A\left(I,J'\right), \quad \text{for } J,J' \in \mathcal{D}_{N-t}.$$

We are reduced to showing,

$$\left\|\mathcal{H}\left(B_{N,\ell}^t\right)\right\|_{\ell^2\to\ell^2} \leq C2^{-2t(1-s)-2\ell} A_2^{\alpha} \quad \text{for } t \geq 0,\ \ell \geq 0 \text{ and } N \in \mathbb{Z},$$

which is an estimate in which Alpert projections no longer play a role, and this estimate is proved as in [12], and more precisely as in [17]. Note that the only arithmetic difference in the argument here is that in the estimates, the parameter $t > 0$ is replaced by $t(1-s) > 0$, which has no effect on the conclusion. This completes our proof of the long-range estimate

$$\mathcal{A}^{\mathrm{long}}(f,g) \lesssim \sqrt{A_2^\alpha}\,\|f\|_{W^s_{\mathrm{dyad}}(\sigma)}\,\|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\,.$$

At this point we pause to complete the bound for $A_3$ in the second term in (5.5). Indeed, the deferred term $A_3$ can be handled using the above argument since $3J \cap I = \emptyset = J \cap 3I$ implies that we can use the Energy Lemma as we did above.

**The mid range case**: Let

$$\mathcal{P} \equiv \left\{(I,J) \in \mathcal{D}\times\mathcal{D} : J \text{ is good, } \ell(J) \le 2^{-\rho}\ell(I),\ J \subset 3I\setminus I\right\}.$$

For $(I,J) \in \mathcal{P}$, the 'pivotal' bound (3.13 ) from the Energy Lemma gives

$$\left|\left\langle T^\alpha_\sigma\left(\triangle^\sigma_{I;\kappa} f\right), \triangle^\omega_J g\right\rangle_\omega\right| \lesssim \mathrm{P}^\alpha_\kappa\left(J, \left|\triangle^\sigma_{I;\kappa} f\right|\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\,\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\,.$$

Now we pigeonhole the lengths of $I$ and $J$ and the distance between them by defining

$$\mathcal{P}^t_{N,d} \equiv \left\{(I,J) \in \mathcal{D}\times\mathcal{D} : J \text{ is good, } \ell(I) = 2^N,\ \ell(J) = 2^{N-t},\ J \subset 3I\setminus I,\ 2^{d-1} \le \operatorname{dist}(I,J) \le 2^d\right\}.$$

Note that the closest a good cube $J$ can come to $I$ is determined by the goodness inequality, which gives this bound for $2^d \ge \operatorname{dist}(I,J)$:

$$2^d \ge \frac{1}{2}\ell(I)^{1-\varepsilon}\,\ell(J)^\varepsilon = \frac{1}{2}2^{N(1-\varepsilon)}2^{(N-t)\varepsilon} = \frac{1}{2}2^{N-\varepsilon t};$$
$$\text{which implies } N - \varepsilon t - 1 \le d \le N,$$

where the last inequality holds because we are in the case of the mid-range term. Thus we have

$$\begin{aligned}
&\sum_{(I,J)\in\mathcal{P}}\left|\left\langle T^\alpha_\sigma\left(\triangle^\sigma_{I;\kappa} f\right), \triangle^\omega_{J;\kappa} g\right\rangle_\omega\right| \\
&\lesssim \sum_{(I,J)\in\mathcal{P}}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\mathrm{P}^\alpha_\kappa\left(J, \left|\triangle^\sigma_{I;\kappa} f\right|\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega} \\
&= \sum_{t=\rho}^\infty\sum_{N\in\mathbb{Z}}\sum_{d=N-\varepsilon t-1}^{N}\sum_{(I,J)\in\mathcal{P}^t_{N,d}}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\mathrm{P}^\alpha_\kappa\left(J, \left|\triangle^\sigma_{I;\kappa} f\right|\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}.
\end{aligned}$$

Now we use

$$\begin{aligned}
\mathrm{P}^\alpha\left(J, \left|\triangle^\sigma_{I;\kappa} f\right|\sigma\right) &= \int_I \frac{\ell(J)}{\left(\ell(J) + |y - c_J|\right)^{n+1-\alpha}}\left|\triangle^\sigma_{I;\kappa} f(y)\right| d\sigma(y) \\
&\lesssim \frac{2^{N-t}}{2^{d(n+1-\alpha)}}\left\|\triangle^\sigma_I f\right\|_{W^s_{\mathrm{dyad}}(\sigma)}\ell(I)^s\sqrt{|I|_\sigma}
\end{aligned}$$

and apply Cauchy–Schwarz in $J$ and use $J \subset 3I\setminus I$ to get

$$\sum_{(I,J)\in\mathcal{P}}\left|\left\langle T^\alpha_\sigma\left(\triangle^\sigma_{I;\kappa} f\right), \triangle^\omega_{J;\kappa} g\right\rangle_\omega\right|$$

$$\lesssim \sum_{t=\rho}^{\infty} \sum_{N\in\mathbb{Z}} \sum_{d=N-\varepsilon t-1}^{N} \sum_{I\in\mathcal{D}_N} \frac{2^{N-t(1-s)}2^{N(n-\alpha)}}{2^{d(n+1-\alpha)}} \left\| \triangle^{\sigma}_{I;\kappa} f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} \frac{\sqrt{|I|_{\sigma}}\sqrt{|3I\setminus I|_{\omega}}}{2^{N(n-\alpha)}}$$
$$\times \sqrt{\sum_{\substack{J\in\mathcal{D}_{N-t}\\ J\subset 3I\setminus I \text{ and } \operatorname{dist}(I,J)\approx 2^d}} \left\| \triangle^{\omega}_{J;\kappa} g \right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}$$
$$\lesssim (1+\varepsilon t) \sum_{t=\rho}^{\infty} \sum_{N\in\mathbb{Z}} \frac{2^{N-t(1-s)}2^{N(n-\alpha)}}{2^{(N-\varepsilon t)(n+1-\alpha)}} \sqrt{A_2^{\alpha}} \sum_{I\in\mathcal{D}_N} \left\| \triangle^{\sigma}_{I;\kappa} f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)}$$
$$\sqrt{\sum_{\substack{J\in\mathcal{D}_{N-t}\\ J\subset 3I\setminus I}} \left\| \triangle^{\omega}_{J;\kappa} g \right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}}$$
$$\lesssim (1+\varepsilon t) \sum_{t=\rho}^{\infty} 2^{-t[1-s-\varepsilon(n+1-\alpha)]} \sqrt{A_2^{\alpha}} \left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} \left\| g \right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$
$$\lesssim \sqrt{A_2^{\alpha}} \left\| f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} \left\| g \right\|_{W^{-s}_{\mathrm{dyad}}(\omega)},$$

where in the third line above we have used $\sum_{d=N-\varepsilon t-1}^{N} 1 \lesssim 1+\varepsilon t$, and in the last line

$$\frac{2^{N-t(1-s)}2^{N(n-\alpha)}}{2^{(N-\varepsilon t)(n+1-\alpha)}} = 2^{-t[1-s-\varepsilon(n+1-\alpha)]},$$

followed by Cauchy–Schwarz in $I$ and $N$, using that we have bounded overlap in the triples of $I$ for $I \in \mathcal{D}_N$. We have also assumed here that $0 < \varepsilon < \frac{1-s}{n+1-\alpha}$, and this completes the proof of (5.5). □

### 5.2 Shifted corona decomposition

To prove (5.6), we recall the *Shifted Corona Decomposition*, as opposed to the *parallel* corona decomposition used in [16], associated with the Calderón–Zygmund $\kappa$-pivotal stopping cubes $\mathcal{F}$ introduced above. But first we must invoke standard arguments, using the full $\kappa$-cube testing conditions (3.7), to permit us to assume that $f$ and $g$ are supported in a finite union of dyadic cubes $F_0$ on which they have vanishing moments of order less than $\kappa$.

#### 5.2.1 The initial reduction using full testing

For this construction, we will follow the treatment as given in [19]. We first restrict $f$ and $g$ to be supported in a large common cube $Q_\infty$. Then we cover $Q_\infty$ with $2^n$ pairwise disjoint cubes $I_\infty \in \mathcal{D}$ with $\ell(I_\infty) = \ell(Q_\infty)$. We now claim we can reduce matters to consideration of the $2^{2n}$ forms

$$\sum_{I\in\mathcal{D}:\ I\subset I_\infty} \sum_{J\in\mathcal{D}:\ J\subset J_\infty} \int_{\mathbb{R}^n} \left(T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa} f\right) \triangle^{\omega}_{J;\kappa} g d\omega,$$

as both $I_\infty$ and $J_\infty$ range over the dyadic cubes as above. First we note that when $I_\infty$ and $J_\infty$ are distinct, the corresponding form is included in the sum $\mathsf{B}_{\cap}(f,g) + \mathsf{B}_{\diagup}(f,g)$, and hence controlled. Thus it remains to consider the forms with $I_\infty = J_\infty$ and use the cubes $I_\infty$

as the starting cubes in our corona construction below. Indeed, we have from (2.5) that

$$f=\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\triangle^\sigma_{I;\kappa}f+\mathbb{E}^\sigma_{I_\infty;\kappa}f,$$
$$g=\sum_{J\in\mathcal{D}:\ J\subset I_\infty}\triangle^\omega_{J;\kappa}g+\mathbb{E}^\omega_{I_\infty;\kappa}g,$$

which can then be used to write the bilinear form $\int\left(T^\alpha_\sigma f\right)gd\omega$ as a sum of the forms

$$\int_{\mathbb{R}^n}\left(T^\alpha_\sigma f\right)gd\omega=\sum_{I_\infty}\left\{\sum_{I,J\in\mathcal{D}:\ I,J\subset I_\infty}\int_{\mathbb{R}^n}\left(T^\alpha_\sigma\triangle^\sigma_{I;\kappa}f\right)\triangle^\omega_{J;\kappa}gd\omega\right. \tag{5.9}$$
$$+\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\int_{\mathbb{R}^n}\left(T^\alpha_\sigma\triangle^\sigma_{I;\kappa}f\right)\mathbb{E}^\omega_{I_\infty;\kappa}gd\omega+\sum_{J\in\mathcal{D}:\ J\subset I_\infty}\int_{\mathbb{R}^n}\left(T^\alpha_\sigma\mathbb{E}^\sigma_{I_\infty;\kappa}f\right)\triangle^\omega_{J;\kappa}gd\omega$$
$$\left.+\int_{\mathbb{R}^n}\left(T^\alpha_\sigma\mathbb{E}^\sigma_{I_\infty;\kappa}f\right)\mathbb{E}^\omega_{I_\infty;\kappa}gd\omega\right\},$$

taken over the $2^n$ cubes $I_\infty$ above.

The second, third and fourth sums in (5.9) can be controlled by the full testing conditions (3.7), e.g.

$$\left|\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\int_{\mathbb{R}^n}\left(T^\alpha_\sigma\triangle^\sigma_{I;\kappa}f\right)\mathbb{E}^\omega_{I_\infty;\kappa}gd\omega\right|=\left|\int_{I_\infty}\left(\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\triangle^\sigma_{I;\kappa}f\right)T^{\alpha,*}_\omega\left(\mathbb{E}^\omega_{I_\infty;\kappa}g\right)d\sigma\right| \tag{5.10}$$
$$\leq\left\|\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\triangle^\sigma_{I;\kappa}f\right\|_{W^s_{\mathrm{dyad}}(\sigma)}\left\|\mathbf{1}_{I_\infty}T^{\alpha,*}_\omega\left(\mathbb{E}^\omega_{I_\infty;\kappa}g\right)\right\|_{W^{-s}_{\mathrm{dyad}}(\sigma)}$$
$$=\left\|\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\triangle^\sigma_{I;\kappa}f\right\|_{W^s_{\mathrm{dyad}}(\sigma)}\left\|\mathbb{E}^\omega_{I_\infty;\kappa}g\right\|_{L^\infty}\left\|\mathbf{1}_{I_\infty}T^{\alpha,*}_\omega\left(\frac{\mathbb{E}^\omega_{I_\infty;\kappa}g}{\left\|\mathbb{E}^\omega_{I_\infty;\kappa}g\right\|_{L^\infty}}\right)\right\|_{W^{-s}_{\mathrm{dyad}}(\sigma)}$$
$$\lesssim\mathfrak{T}^\kappa_{T^{\alpha,*}_\omega}\|f\|_{W^s_{\mathrm{dyad}}(\sigma)}\|g\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

and similarly for the third and fourth sum.

### 5.2.2 The shifted corona

Define the two Alpert corona projections,

$$\mathsf{P}^\sigma_{\mathcal{C}_F}\equiv\sum_{I\in\mathcal{C}_F}\triangle^\sigma_{I;\kappa_1}\text{ and }\mathsf{P}^\omega_{\mathcal{C}_F^{\tau-\mathrm{shift}}}\equiv\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\triangle^\omega_{J;\kappa_2}$$

where

$$\mathcal{C}_F^{\tau-\mathrm{shift}}\equiv\left[\mathcal{C}_F\setminus\mathcal{N}^\tau_{\mathcal{D}}(F)\right]\cup\bigcup_{F'\in\mathfrak{C}_{\mathcal{F}}(F)}\left[\mathcal{N}^\tau_{\mathcal{D}}\left(F'\right)\setminus\mathcal{N}^\tau_{\mathcal{D}}(F)\right]; \tag{5.11}$$
$$\text{where }\mathcal{N}^\tau_{\mathcal{D}}(F)\equiv\left\{J\in\mathcal{D}:J\subset F\text{ and }\ell(J)>2^{-\tau}\ell(F)\right\}.$$

Thus the *shifted* corona $\mathcal{C}_F^{\tau-\text{shift}}$ has the top $\tau$ levels from $\mathcal{C}_F$ removed, and includes the first $\tau$ levels from each of its $\mathcal{F}$-children, except if they have already been removed. We must restrict the Alpert supports of $f$ and $g$ to good cubes, as defined e.g. in [16], so that with the superscript good denoting this restriction,

$$\mathsf{P}_{\mathcal{C}_F}^{\sigma} f = \sum_{I \in \mathcal{C}_F^{\text{good}}} \triangle_{I;\kappa_1}^{\sigma} \text{ and } \mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^{\omega} g = \sum_{J \in \mathcal{C}_F^{\text{good},\tau-\text{shift}}} \triangle_{J;\kappa_2}^{\omega}$$

where $\mathcal{C}_F^{\text{good}} \equiv \mathcal{C}_F \cap \mathcal{D}^{\text{good}}$ and $\mathcal{C}_F^{\text{good},\tau-\text{shift}} \equiv \mathcal{C}_F^{\tau-\text{shift}} \cap \mathcal{D}^{\text{good}}$, and $\mathcal{D}^{\text{good}}$ consists of the $(r, \varepsilon)-$ good cubes in $\mathcal{D}$.

A simple but important property is the fact that the $\tau$-shifted coronas $\mathcal{C}_F^{\tau-\text{shift}}$ have overlap bounded by $\tau$:

$$\sum_{F \in \mathcal{F}} \mathbf{1}_{\mathcal{C}_F^{\tau-\text{shift}}}(J) \leq \tau, \quad J \in \mathcal{D}. \tag{5.12}$$

It is convenient, for use in the canonical splitting below, to introduce the following shorthand notation for $F, G \in \mathcal{F}$:

$$\left\langle T_\sigma^\alpha \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_G^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} \equiv \sum_{\substack{I \in \mathcal{C}_F \text{ and } J \in \mathcal{C}_G^{\tau-\text{shift}} \\ J \Subset_\rho I}} \left\langle T_\sigma^\alpha \left(\triangle_{I;\kappa}^{\sigma} f\right), \left(\triangle_{J;\kappa}^{\omega} g\right) \right\rangle_\omega .$$

## 5.3 Canonical splitting

We then proceed with the *Canonical Splitting* as in [17], but with Alpert wavelets in place of Haar wavelets,

$$\begin{aligned}
\mathsf{B}_{\Subset_\rho}(f, g) &= \sum_{F, G \in \mathcal{F}} \left\langle T_\sigma \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_G^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} \\
&= \sum_{F \in \mathcal{F}} \left\langle T_\sigma \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} + \sum_{\substack{F, G \in \mathcal{F} \\ G \subsetneqq F}} \left\langle T_\sigma \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_G^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} \\
&\quad + \sum_{\substack{F, G \in \mathcal{F} \\ G \supsetneqq F}} \left\langle T_\sigma \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_G^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} + \sum_{\substack{F, G \in \mathcal{F} \\ F \cap G = \emptyset}} \left\langle T_\sigma \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right), \mathsf{P}_{\mathcal{C}_G^{\tau-\text{shift}}}^{\omega} g \right\rangle_\omega^{\Subset_\rho} \\
&\equiv \mathsf{T}_{\text{diagonal}}(f, g) + \mathsf{T}_{\text{far below}}(f, g) + \mathsf{T}_{\text{far above}}(f, g) + \mathsf{T}_{\text{disjoint}}(f, g) .
\end{aligned}$$

The two forms $\mathsf{T}_{\text{far above}}(f, g)$ and $\mathsf{T}_{\text{disjoint}}(f, g)$ each vanish just as in [17], since there are no pairs $(I, J) \in \mathcal{C}_F \times \mathcal{C}_G^{\emptyset-\text{shift}}$ with both (**i**) $J \Subset_\rho I$ and (**ii**) either $F \subsetneqq G$ or $G \cap F = \emptyset$.

### 5.3.1 The far below form

Here is a generalization to weighted Sobolev spaces of the Intertwining Proposition from [16, Proposition 36 on page 35], that uses strong $\kappa$-pivotal conditions with Alpert wavelets. Recall that $0 < \varepsilon < 1$ and $r$ is chosen sufficiently large depending on $\varepsilon$. The argument given here is considerably simpler than that in [16].

**Proposition 44** (The Intertwining Proposition) *Suppose $\sigma, \omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, that $\sigma$ is doubling, and that $\mathcal{F}$ satisfies an $\theta_\sigma^{\text{rev}}$-strong $\sigma$-Carleson condition. Then for a smooth $\alpha$-fractional singular integral $T^\alpha$, and for* good *functions $f \in W^s_{\text{dyad}}(\sigma)$ and $g \in W^{-s}_{\text{dyad}}(\omega)$, and with $\kappa_1, \kappa_2 \geq 1$ sufficiently large, we have the following bound for $\mathsf{T}_{\textit{far below}}(f,g) = \sum_{F\in\mathcal{F}} \sum_{I:\, I\supsetneqq F} \left\langle T^\alpha_\sigma \bigtriangleup^\sigma_{I;\kappa_1} f, P^\omega_{\mathcal{C}_F^{\mathbf{\phi}-\text{shift}}} g \right\rangle_\omega$:*

$$\left|\mathsf{T}_{\textit{far below}}(f,g)\right| \lesssim \left(\mathcal{V}^{\alpha,\kappa_1}_{2,\theta^{\text{rev}}_\sigma} + \sqrt{A^\alpha_2}\right) \|f\|_{W^s_{\text{dyad}}(\sigma)} \|g\|_{W^{-s}_{\text{dyad}}(\omega)} . \tag{5.13}$$

***Proof*** We write

$$\begin{aligned}
f_F &\equiv \sum_{I:\, I\supsetneqq F} \bigtriangleup^\sigma_{I;\kappa_1} f = \sum_{m=1}^\infty \sum_{I:\, \pi^m_{\mathcal{F}}F \subsetneqq I \subset \pi^{m+1}_{\mathcal{F}}F} \bigtriangleup^\sigma_{I;\kappa_1} f \\
&= \sum_{m=1}^\infty \sum_{I:\, \pi^m_{\mathcal{F}}F \subsetneqq I \subset \pi^{m+1}_{\mathcal{F}}F} \mathbf{1}_{\theta(I)} \left(\mathbb{E}^\sigma_{I;\kappa_1} f - \mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1} f\right) \\
&= \sum_{m=1}^\infty \sum_{I:\, \pi^m_{\mathcal{F}}F \subsetneqq I \subset \pi^{m+1}_{\mathcal{F}}F} \mathbf{1}_{\theta(I)} \left(\mathbb{E}^\sigma_{I;\kappa_1} f\right) - \sum_{m=1}^\infty \mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F \setminus \pi^m_{\mathcal{F}}F} \left(\mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1} f\right) \\
&\equiv \beta_F - \gamma_F
\end{aligned}$$

and then

$$\sum_{F\in\mathcal{F}} \left\langle T^\alpha_\sigma f_F, g_F\right\rangle_\omega = \sum_{F\in\mathcal{F}} \left\langle T^\alpha_\sigma \beta_F, g_F\right\rangle_\omega + \sum_{F\in\mathcal{F}} \left\langle T^\alpha_\sigma \gamma_F, g_F\right\rangle_\omega .$$

Now we use the pivotal bound (3.13),

$$\left|\left\langle T^\alpha(\varphi\nu), \Psi_J\right\rangle_{L^2(\omega)}\right| \lesssim \mathrm{P}^\alpha_\kappa(J,\nu)\,\ell(J)^{-s}\sqrt{|J|_\omega}\,\|\Psi_J\|_{W^{-s}_{\text{dyad}}(\omega)} ,$$

the pivotal stopping control (4.4),

$$\mathrm{P}^\alpha_\kappa(I, \mathbf{1}_F\sigma)^2 \left(\frac{\ell(F)}{\ell(I)}\right)^\varepsilon |I|_\omega < \Gamma\, |I|_\sigma , \quad I \in \mathcal{C}_F \text{ and } F \in \mathcal{F},$$

and (3.14), namely

$$\mathrm{P}^\alpha_k\left(J, \sigma\mathbf{1}_{K\setminus I}\right) \lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{k-\varepsilon(n+k-\alpha)} \mathrm{P}^\alpha_k\left(I, \sigma\mathbf{1}_{K\setminus I}\right),$$

to obtain that

$$\begin{aligned}
\left|\sum_{F\in\mathcal{F}} \left\langle T^\alpha_\sigma \gamma_F, g_F\right\rangle_\omega\right| &\lesssim \sum_{F\in\mathcal{F}} \mathrm{P}^\alpha_\kappa\left(F, \sum_{m=1}^\infty \mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^m_{\mathcal{F}}F} \left|\mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1} f\right| \sigma\right) \ell(F)^{-s}\sqrt{|F|_\omega}\,\|g_F\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&= \sum_{m=1}^\infty \sum_{F\in\mathcal{F}} \left\|\mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1} f\right\|_\infty \mathrm{P}^\alpha_\kappa\left(F, \mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^m_{\mathcal{F}}F}\sigma\right) \ell(F)^{-s}\sqrt{|F|_\omega}\,\|g_F\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&\leq \sum_{m=1}^\infty \sum_{F\in\mathcal{F}} \left\|\mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1} f\right\|_\infty \left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{\kappa-\varepsilon(n+\kappa-\alpha)} \\
&\quad\times \mathrm{P}^\alpha_\kappa\left(\pi^m_{\mathcal{F}}F, \mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^m_{\mathcal{F}}F}\sigma\right) \ell(F)^{-s}\sqrt{|F|_\omega}\,\|g_F\|_{W^{-s}_{\text{dyad}}(\omega)}
\end{aligned}$$

equals

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1}f\right\|_{\infty}\left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{\kappa-\varepsilon(n+\kappa-\alpha)-s}\ell\left(\pi^m_{\mathcal{F}}F\right)^{-s}\\
&\quad\times\left\{\mathrm{P}^{\alpha}_{\kappa}\left(\pi^m_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^m_{\mathcal{F}}F}\sigma\right)\sqrt{\left|\pi^m_{\mathcal{F}}F\right|_{\omega}}\right\}\sqrt{\frac{|F|_{\omega}}{\left|\pi^m_{\mathcal{F}}F\right|_{\omega}}}\,\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\\
&\leq\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1}f\right\|_{\infty}\left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{\eta+\eta'}\\
&\quad\times\ell\left(\pi^m_{\mathcal{F}}F\right)^{-s}\left\{\mathcal{V}^{\alpha,\kappa}_{2,\theta^{\mathrm{rev}}_{\sigma}}\sqrt{\left|\pi^m_{\mathcal{F}}F\right|_{\sigma}}\right\}\sqrt{\frac{|F|_{\omega}}{\left|\pi^m_{\mathcal{F}}F\right|_{\omega}}}\,\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)},
\end{aligned}$$

where we have used the pivotal stopping inequality, and written

$$\kappa-\varepsilon(n+\kappa-\alpha)-s+\theta^{\mathrm{rev}}_{\sigma}=\eta+\eta',$$

with $\eta,\eta'>0$ to be chosen later. Note that this requires the Alpert parameter $\kappa$ to satisfy

$$\kappa>\frac{\varepsilon(n-\alpha)+s-\theta^{\mathrm{rev}}_{\sigma}}{1-\varepsilon}. \tag{5.14}$$

Then by Cauchy–Schwarz we have

$$\begin{aligned}
&\left|\sum_{F\in\mathcal{F}}\left\langle T^{\alpha}_{\sigma}\gamma_F,g_F\right\rangle_{\omega}\right|\\
&\lesssim\mathcal{V}^{\alpha,\kappa}_{2,\theta^{\mathrm{rev}}_{\sigma}}\left(\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1}f\right\|^2_{\infty}\ell\left(\pi^m_{\mathcal{F}}F\right)^{-2s}\left|\pi^m_{\mathcal{F}}F\right|_{\sigma}\left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{2\eta}\frac{|F|_{\omega}}{\left|\pi^m_{\mathcal{F}}F\right|_{\omega}}\right)^{\frac{1}{2}}\\
&\quad\times\left(\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{2\eta'}\|g_F\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}\right)^{\frac{1}{2}}.
\end{aligned}$$

The square of the first factor satisfies

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi^{m+1}_{\mathcal{F}}F;\kappa_1}f\right\|^2_{\infty}\left|\pi^m_{\mathcal{F}}F\right|_{\sigma}\ell\left(\pi^m_{\mathcal{F}}F\right)^{-2s}\left(\frac{\ell(F)}{\ell(\pi^m_{\mathcal{F}}F)}\right)^{2\eta}\frac{|F|_{\omega}}{\left|\pi^m_{\mathcal{F}}F\right|_{\omega}}\\
&=\sum_{F'\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F';\kappa_1}f\right\|^2_{\infty}\ell\left(F'\right)^{-2s}\left|F'\right|_{\sigma}\sum_{\substack{F\in\mathcal{F}\\F\subset F'}}\left(\frac{\ell(F)}{\ell(F')}\right)^{2\eta}\frac{|F|_{\omega}}{|F'|_{\omega}}\\
&\lesssim\sum_{F'\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F';\kappa_1}f\right\|^2_{\infty}\ell\left(F'\right)^{-2s}\left|F'\right|_{\sigma}=\sum_{F''\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{F'';\kappa_1}f\right\|^2_{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F'')}\ell\left(F'\right)^{-2s}\left|F'\right|_{\sigma}\\
&\lesssim\sum_{F''\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{F'';\kappa_1}f\right\|^2_{\infty}\ell\left(F''\right)^{-2s}\left|F''\right|_{\sigma}\lesssim\|f\|^2_{W^{s}_{\mathrm{dyad}}(\sigma)}
\end{aligned}$$

where the first inequality in the last line follows from the strong $\sigma$ -Carleson condition, and the second inequality follows from the Quasiorthogonality Lemma 28. The square of the second factor satisfies

$$\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left(\frac{\ell(F)}{\ell\left(\pi_{\mathcal{F}}^{m}F\right)}\right)^{2\eta'}\|g_F\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}\lesssim\sum_{F\in\mathcal{F}}\|g_F\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}\leq\|g\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

where we have used $\eta'>0$.

It remains to bound $\sum_{F\in\mathcal{F}}\left\langle T^{\alpha}_{\sigma}\beta_F,g_F\right\rangle_{\omega}$ where $\beta_F=\sum_{m=1}^{\infty}\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\mathbf{1}_{\theta(I)}\left(\mathbb{E}^{\sigma}_{I;\kappa_1}f\right)$. The difference between the previous estimate and this one is that the averages $\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\left|\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}^{m+1}F;\kappa_1}f\right|$ inside the Poisson kernel have been replaced with the sum of averages $\sum_{m=1}^{\infty}\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\mathbf{1}_{\theta(I)}\left|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right|$, but where the sum is taken over pairwise disjoint sets $\{\theta(I)\}_{\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}$. We start with

$$\begin{aligned}&\left|\sum_{F\in\mathcal{F}}\left\langle T^{\alpha}_{\sigma}\beta_F,g_F\right\rangle_{\omega}\right|\\&\quad\lesssim\sum_{F\in\mathcal{F}}\mathrm{P}^{\alpha}_{\kappa}\left(F,\sum_{m=1}^{\infty}\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\mathbf{1}_{\theta(I)}\left|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right|\sigma\right)\ell(F)^{-s}\sqrt{|F|_{\omega}}\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\\&\quad=\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}\mathrm{P}^{\alpha}_{\kappa}\left(F,\mathbf{1}_{\theta(I)}\sigma\right)\ell(F)^{-s}\sqrt{|F|_{\omega}}\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\equiv S.\end{aligned}$$

Then we use

$$\begin{aligned}&\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}\mathrm{P}^{\alpha}_{\kappa}\left(F,\mathbf{1}_{\theta(I)}\sigma\right)\\&\qquad\leq\left(\sup_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}\right)\mathrm{P}^{\alpha}_{\kappa}\left(F,\sum_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\mathbf{1}_{\theta(I)}\sigma\right)\\&\qquad=\left(\sup_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}\right)\mathrm{P}^{\alpha}_{\kappa}\left(F,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma\right),\end{aligned}$$

and obtain that

$$\begin{aligned}S\leq\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}&\left(\sup_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}\right)\\&\mathrm{P}^{\alpha}_{\kappa}\left(F,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma\right)\ell(F)^{-s}\sqrt{|F|_{\omega}}\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}.\end{aligned}$$

Now we define $G_m[F]\in\left(\pi_{\mathcal{F}}^{m}F,\pi_{\mathcal{F}}^{m+1}F\right]$ so that $\sup_{I:\,\pi_{\mathcal{F}}^{m}F\subsetneqq I\subset\pi_{\mathcal{F}}^{m+1}F}\left\|\mathbb{E}^{\sigma}_{I;\kappa_1}f\right\|_{\infty}=\left\|\mathbb{E}^{\sigma}_{G_m[F];\kappa_1}f\right\|_{\infty}$, and dominate $S$ by

$$S\leq\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\left\|\mathbb{E}^{\sigma}_{G_m[F];\kappa_1}f\right\|_{\infty}\mathrm{P}^{\alpha}_{\kappa}\left(F,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma\right)\ell(F)^{-s}\sqrt{|F|_{\omega}}\|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

$$\begin{aligned}
&\lesssim \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^{\sigma}_{G_m[F];\kappa_1} f \right\|_{\infty} \left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta} \mathrm{P}^{\alpha}_{\kappa} \left( G_m[F], \mathbf{1}_{\pi^{m+1}_{\mathcal{F}} F \setminus \pi^{m}_{\mathcal{F}} F} \sigma \right) \\
&\quad \times \ell(F)^{-s} \sqrt{|F|_{\omega}} \, \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \\
&= \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^{\sigma}_{G_m[F];\kappa_1} f \right\|_{\infty} \left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta} \\
&\quad \times \left\{ \mathrm{P}^{\alpha}_{\kappa} \left( G_m[F], \mathbf{1}_{\pi^{m+1}_{\mathcal{F}} F \setminus \pi^{m}_{\mathcal{F}} F} \sigma \right) \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{\varepsilon} \sqrt{|G_m[F]|_{\omega}} \right\} \\
&\quad \times \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{-\varepsilon} \ell(F)^{-s} \sqrt{\frac{|F|_{\omega}}{|G_m[F]|_{\omega}}} \, \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)},
\end{aligned}$$

and then continue with

$$\begin{aligned}
&\lesssim \mathcal{V}^{\alpha}_{2,\theta^{\mathrm{rev}}_{\sigma}} \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^{\sigma}_{G_m[F];\kappa_1} f \right\|_{\infty} \left( \frac{\ell(F)}{\ell(G[F])} \right)^{\eta} \ell(F)^{-s} \left\{ \sqrt{|G_m[F]|_{\sigma}} \right\} \\
&\quad \times \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{-\varepsilon} \sqrt{\frac{|F|_{\omega}}{|G_m[F]|_{\omega}}} \, \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \\
&= \mathcal{V}^{\alpha}_{2,\theta^{\mathrm{rev}}_{\sigma}} \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^{\sigma}_{G[F];\kappa_1} f \right\|_{\infty} \ell(G_m[F])^{-s} \sqrt{|G_m[F]|_{\sigma}} \\
&\quad \times \left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta - s} \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{-\varepsilon} \sqrt{\frac{|F|_{\omega}}{|G_m[F]|_{\omega}}} \, \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)} \\
&\approx \mathcal{V}^{\alpha}_{2,\theta^{\mathrm{rev}}_{\sigma}} \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}} \left\| \triangle^{\sigma}_{G_m[F];\kappa_1} f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} \\
&\quad \times \left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta - s} \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{-\varepsilon} \sqrt{\frac{|F|_{\omega}}{|G_m[F]|_{\omega}}} \, \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}.
\end{aligned}$$

Since there is geometric gain in the product

$$\left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta - s} \left( \frac{\ell\left(\pi^{m+1}_{\mathcal{F}} F\right)}{\ell(G_m[F])} \right)^{-\varepsilon} \sqrt{\frac{|F|_{\omega}}{|G_m[F]|_{\omega}}} \leq \left( \frac{\ell(F)}{\ell(G_m[F])} \right)^{\eta - s} \lesssim 2^{-m(\eta - s)},$$

provided $\eta > s$, an application of Cauchy–Schwarz finishes the proof since $G_m[F]$ is uniquely determined by $F$ and $m$ in the tower $\left(\pi^{m}_{\mathcal{F}} F, \pi^{m+1}_{\mathcal{F}} F\right]$:

$$\mathcal{V}^{\alpha}_{2,\theta^{\mathrm{rev}}_{\sigma}} \sum_{m=1}^{\infty} 2^{-m(\eta - s)} \sum_{F \in \mathcal{F}} \left\| \triangle^{\sigma}_{G_m[F];\kappa_1} f \right\|_{W^{s}_{\mathrm{dyad}}(\sigma)} \|g_F\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

$$\leq \mathcal{V}^{\alpha}_{2,\theta^{\text{rev}}_{\sigma}} \sum_{m=1}^{\infty} 2^{-m(\eta-s)} \left( \sum_{F\in\mathcal{F}} \left\| \triangle^{\sigma}_{G_m[F];\kappa_1} f \right\|^2_{W^s_{\text{dyad}}(\sigma)} \right)^{\frac{1}{2}} \left( \sum_{F\in\mathcal{F}} \|g_F\|^2_{W^{-s}_{\text{dyad}}(\omega)} \right)^{\frac{1}{2}}$$

$$\leq \mathcal{V}^{\alpha}_{2,\theta^{\text{rev}}_{\sigma}} \sum_{m=1}^{\infty} 2^{-m(\eta-s)} \|f\|_{W^s_{\text{dyad}}(\sigma)} \|g\|_{W^{-s}_{\text{dyad}}(\omega)} = C\mathcal{V}^{\alpha}_{2,\theta^{\text{rev}}_{\sigma}} \|f\|_{W^s_{\text{dyad}}(\sigma)} \|g\|_{W^{-s}_{\text{dyad}}(\omega)} .$$

Note that we have used $\eta > s$, which requires a bit more on $\kappa$ than was used in (5.14), namely that

$$\kappa - \varepsilon (n + \kappa - \alpha) - s + \theta^{\text{rev}}_{\sigma} = \eta + \eta' > s,$$

which requires

$$\kappa > \frac{\varepsilon (n - \alpha) + 2s - \theta^{\text{rev}}_{\sigma}}{1 - \varepsilon}. \tag{5.15}$$

□

### 5.3.2 The diagonal form

To handle the diagonal term $\mathsf{T}_{\text{diagonal}}(f, g)$, we decompose according to the stopping times $\mathcal{F}$,

$$\mathsf{T}_{\text{diagonal}}(f, g) = \sum_{F\in\mathcal{F}} \mathsf{B}^F_{\Subset_\rho}(f, g) \equiv \left\langle T^{\alpha}_{\sigma} \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right), \mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\text{shift}}_F} g \right\rangle^{\Subset_\rho}_{\omega},$$

and it is enough, using Cauchy–Schwarz and quasiorthogonality (4.2) in $f$, together with orthogonality (4.3) in both $f$ and $g$, to prove the 'below form' bound involving the usual cube testing constant,

$$\left|\mathsf{B}^F_{\Subset_\rho}(f, g)\right| \lesssim \left(\mathfrak{T}^s_{T^\alpha} + \sqrt{A^{\alpha}_2}\right) \left(\ell(F)^{-s} \left\|\mathbb{E}^{\sigma}_{F;\kappa} f\right\|_{\infty} \sqrt{|F|_{\sigma}} + \left\|\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right\|_{W^s_{\text{dyad}}(\sigma)}\right) \left\|\mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\text{shift}}_F} g\right\|_{W^{-s}_{\text{dyad}}(\omega)} . \tag{5.16}$$

Indeed, using quasiorthogonality, Lemma 28, and orthogonality of projections $\mathsf{P}^{\sigma}_{\mathcal{C}_F} f$ and $\mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\text{shift}}_F} g$ this then gives the estimate,

$$\left|\mathsf{T}_{\text{diagonal}}(f, g)\right| \lesssim \left(\mathfrak{T}^s_{T^\alpha} + \sqrt{A^{\alpha}_2}\right) \|f\|_{W^s_{\text{dyad}}(\sigma)} \|g\|_{W^{-s}_{\text{dyad}}(\omega)} . \tag{5.17}$$

Thus at this point we have essentially reduced the proof of Theorem 2 to

(1) proving (5.16),
(2) and controlling the triple polynomial testing condition (3.7) by the usual cube testing condition and the classical Muckenhoupt condition.

In the next section we address the first issue by proving the inequality (5.16) for the below forms $\mathsf{B}^F_{\Subset_\rho}(f, g)$. In the final section, we address the second issue and complete the proofs of our theorems by drawing together all of the estimates.

## 6 The Nazarov, Treil and Volberg reach for Alpert wavelets

It will be convenient to denote our fractional singular integral operators by $T^{\lambda}$, $0 \leq \lambda < n$, instead of $T^{\alpha}$, thus freeing up $\alpha$ for the familiar role of denoting multi-indices in $\mathbb{Z}_+^n$. Before getting started, we note that for a doubling measure $\mu$, a cube $I$ and a polynomial $P$, we have

$$\|P\mathbf{1}_I\|_{L^\infty(\mu)} = \sup_{x\in I} |P(x)|$$

because $\mu$ charges all open sets, and so in particular, $\|P\mathbf{1}_I\|_{L^\infty(\sigma)} = \|P\mathbf{1}_I\|_{L^\infty(\omega)}$.

We will often follow the analogous arguments in [1], and point out the places where significant new approaches are needed. See [1] for a review of the classical reach of Nazarov, Treil and Volberg using Haar wavelet projections $\triangle_I^\sigma$, namely the beautiful and ingenious 'thinking outside the box' idea of the paraproduct / stopping / neighbour decomposition of Nazarov, Treil and Volberg [12] using Haar wavelets.

When using weighted Alpert wavelet projections $\triangle_{I;\kappa}^\sigma$ instead, the projection $\mathbb{E}_{I';\kappa}^\sigma \triangle_{I;\kappa}^\sigma f$ onto the child $I' \in \mathfrak{C}_{\mathcal{D}}(I)$ equals $M_{I';\kappa}\mathbf{1}_{I_\pm}$ where $M = M_{I';\kappa}$ is a polynomial of degree less than $\kappa$ restricted to $I'$, and hence no longer commutes with the operator $T_\sigma^\lambda$—unless it is the constant polynomial. We now recall the modifications used in [1], where they obtained,

$$\begin{aligned}
\mathsf{B}_{\Subset_{\rho,\varepsilon};\kappa}^F(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\lambda \triangle_{I;\kappa}^\sigma f, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\boldsymbol{\emptyset}-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\lambda \left(\mathbf{1}_{I_J} \triangle_{I;\kappa}^\sigma f\right), \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&\quad + \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\boldsymbol{\emptyset}-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \left\langle T_\sigma^\lambda \left(\mathbf{1}_{\theta(I_J)} \triangle_{I;\kappa}^\sigma f\right), \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&\equiv \mathsf{B}_{\text{home};\kappa}^F(f,g) + \mathsf{B}_{\text{neighbour};\kappa}^F(f,g).
\end{aligned}$$

They further decomposed the $\mathsf{B}_{\text{home};\kappa}^F$ form using

$$M_{I'} = M_{I';\kappa} \equiv \mathbf{1}_{I'} \triangle_{I;\kappa}^\sigma f = \mathbb{E}_{I';\kappa}^\sigma \triangle_{I;\kappa}^\sigma f, \tag{6.1}$$

to obtain

$$\begin{aligned}
\mathsf{B}_{\text{home};\kappa}^F(f,g) &= \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\lambda \left(M_{I_J}\mathbf{1}_{I_J}\right), \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle M_{I_J} T_\sigma^\lambda \mathbf{1}_{I_J}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&\quad + \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle \left[T_\sigma^\lambda, M_{I_J}\right]\mathbf{1}_{I_J}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle M_{I_J} T_\sigma^\lambda \mathbf{1}_F, \triangle_{J;\kappa}^\omega g\right\rangle_\omega
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle M_{I_J} T_\sigma^\lambda \mathbf{1}_{F\setminus I_J}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&+ \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle \left[T_\sigma^\lambda, M_{I_J}\right]\mathbf{1}_{I_J}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&\equiv \mathsf{B}_{\text{paraproduct};\kappa}^F(f,g) - \mathsf{B}_{\text{stop};\kappa}^F(f,g) + \mathsf{B}_{\text{commutator};\kappa}^F(f,g).
\end{aligned}
$$

Altogether then we have the weighted Alpert version of the Nazarov, Treil and Volberg paraproduct decomposition that was obtained by Alexis, Sawyer and Uriarte-Tuero in [1],

$$
\begin{aligned}
\mathsf{B}_{\Subset_{\rho,\varepsilon};\kappa}^F(f,g) &= \mathsf{B}_{\text{paraproduct};\kappa}^F(f,g) + \mathsf{B}_{\text{stop};\kappa}^F(f,g) + \mathsf{B}_{\text{commutator};\kappa}^F(f,g) \\
&\quad + \mathsf{B}_{\text{neighbour};\kappa}^F(f,g).
\end{aligned}
$$

### 6.1 The paraproduct form

Following [1], we first pigeonhole the sum over pairs $I$ and $J$ arising in the paraproduct form according to which child $K\in\mathfrak{C}_\mathcal{D}(I)$ contains $J$,

$$
\mathsf{B}_{\text{paraproduct};\kappa}^F(f,g) = \sum_{I\in\mathcal{C}_F}\sum_{K\in\mathfrak{C}_\mathcal{D}(I)}\sum_{\substack{J\in\mathcal{C}_F^{\tau-\text{shift}}:\ J\Subset_{\rho,\varepsilon} I\\ J\subset K}} \left\langle M_{K;\kappa} T_\sigma^\lambda \mathbf{1}_F, \triangle_{J;\kappa}^\omega g\right\rangle_\omega .
$$

The form $\mathsf{B}_{\text{paraproduct};\kappa}^F(f,g)$ will be handled using the telescoping property in part (2) of Theorem 9, to sum the restrictions to a cube $J\in\mathcal{C}_F^{\tau-\text{shift}}$ of the polynomials $M_{K;\kappa}$ on a child $K\in\mathfrak{C}_\mathcal{D}(I)$ of $I$, over the relevant cubes $I$, to obtain a restricted polynomial $\mathbf{1}_J P_{K;\kappa}$ that is controlled by $E_F^\sigma|f|$, and then passing the polynomial $M_{J;\kappa}$ over to $\triangle_{J;\kappa}^\omega g$. More precisely, for each $J\in\mathcal{C}_F^{\boldsymbol{\phi}-\text{shift}}$, let $I_J^\natural$ denote the smallest $L\in\mathcal{C}_F$ such that $J\Subset_{\rho,\varepsilon} L$ provided it exists. Note that $J$ is at most $\tau$ levels below the bottom of the corona $\mathcal{C}_F$, and since $\rho>2\tau$, we have that *either* $\pi_\mathcal{D}^{(\rho)}J\in\mathcal{C}_F$ *or* that $\pi_\mathcal{D}^{(\rho)}J\supsetneqq F$. Let $I_J^\flat$ denote the $\mathcal{D}$-child of $I_J^\natural$ that contains $J$, provided $I_J^\natural$ exists. We have

$$
\begin{aligned}
\sum_{I\in\mathcal{C}_F:\ I_J^\natural\subset I} \mathbf{1}_J M_{K;\kappa} &= \mathbf{1}_J \sum_{I\in\mathcal{C}_F:\ I_J^\natural\subset I} M_{K;\kappa} = \mathbf{1}_J\left(\mathbb{E}_{I_J^\flat;\kappa}^\sigma f - \mathbb{E}_{F;\kappa}^\sigma f\right) \\
&\equiv \begin{cases} \mathbf{1}_J P_{J;\kappa} & \text{if } \quad I_J^\natural \text{ exists} \\ 0 & \text{if } I_J^\natural \text{ doesn't exist}\end{cases}.
\end{aligned}
\tag{6.2}
$$

Then we write

$$
\mathsf{B}_{\text{paraproduct};\kappa}^F(f,g) = \sum_{\substack{I\in\mathcal{C}_F,\ I_J^\flat\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon} I}} \left\langle M_{I_J;\kappa} T_\sigma^\lambda \mathbf{1}_F, \triangle_{J;\kappa}^\omega g\right\rangle_\omega .
$$

From (6.2) we obtain

$$
\left\|\mathbf{1}_J P_{J;\kappa}\right\|_{L^\infty(\sigma)} \le \left\|\mathbb{E}_{I_J^\flat;\kappa}^\sigma f\right\|_{L^\infty(\sigma)} + \left\|\mathbb{E}_{F;\kappa}^\sigma f\right\|_{L^\infty(\sigma)}, \tag{6.3}
$$

and so

$$\mathsf{B}^F_{\mathrm{paraproduct};\kappa}(f,g) = \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} \left\langle \mathbf{1}_J \left( \sum_{\substack{I\in\mathcal{C}_F:\ I_J^\flat\in\mathcal{C}_F \\ J\Subset_{\rho,\varepsilon} I}} \sum_{\substack{I'\in\mathfrak{C}_{\mathcal{D}}(I) \\ J\subset I'}} M_{I';\kappa} \right) T_\sigma^\lambda \mathbf{1}_F, \triangle^\omega_{J;\kappa} g \right\rangle_\omega$$
$$= \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} \left\langle \mathbf{1}_J P_{J;\kappa} T_\sigma^\lambda \mathbf{1}_F, \triangle^\omega_{J;\kappa} g\right\rangle_\omega$$

to obtain

$$\left|\mathsf{B}^F_{\mathrm{paraproduct};\kappa}(f,g)\right| = \left| \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} \left\langle T_\sigma^\lambda \mathbf{1}_F, P_{J;\kappa}\triangle^\omega_{J;\kappa} g\right\rangle_\omega \right| = \left| \left\langle T_\sigma^\lambda \mathbf{1}_F, \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} P_{J;\kappa}\triangle^\omega_{J;\kappa} g \right\rangle_\omega \right|$$
$$\le \left\| T_\sigma^\lambda \mathbf{1}_F \right\|_{W^s_{\mathrm{dyad}}(\mu)} \left\| \mathbb{E}^\sigma_{F;\kappa} f\right\|_{L^\infty(\sigma)} \left\| \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} \frac{P_{J;\kappa}}{E_F^\sigma |f|} \triangle^\omega_{J;\kappa} g \right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$
$$\le \mathfrak{T}^s_{T^\lambda} \ell(F)^{-s}\sqrt{|F|_\sigma} \left\| \mathbb{E}^\sigma_{F;\kappa} f\right\|_{L^\infty(\sigma)} \left\| \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} \frac{P_{J;\kappa}}{E_F^\sigma |f|} \triangle^\omega_{J;\kappa} g \right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}.$$

Now we will use an almost orthogonality argument that reflects the fact that for $J'$ small compared to $J$, the function $M_{J';\kappa}\triangle^\omega_{J';\kappa} g$ has vanishing $\omega$ -mean, and the polynomial $\mathbf{1}_J P^{\mathrm{corona}}_{J;\kappa}\triangle^\omega_{J;\kappa} g$ is relatively smooth at the scale of $J'$, together with the fact that the polynomials

$$R_{J;\kappa} \equiv \frac{P_{J;\kappa}}{E_F^\sigma |f|}$$

of degree at most $\kappa-1$, have $L^\infty$ norm uniformly bounded by the constant $C$ appearing in (6.3). We begin by writing

$$\left\| \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} R_{J;\kappa}\triangle^\omega_{J;\kappa} g \right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)} = \left\langle \sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}} R_{J;\kappa}\triangle^\omega_{J;\kappa} g, \sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}} R_{J';\kappa}\triangle^\omega_{J';\kappa} g \right\rangle_{W^{-s}_{\mathrm{dyad}}(\omega)}$$
$$= \left\{ \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\mathrm{shift}} \\ J'\subset J}} + \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\mathrm{shift}} \\ J\subsetneqq J'}} \right\} \left\langle R_{J;\kappa}\triangle^\omega_{J;\kappa} g, R_{J';\kappa}\triangle^\omega_{J';\kappa} g\right\rangle_{W^{-s}_{\mathrm{dyad}}(\omega)} \equiv A+B.$$

By symmetry, it suffices to estimate term $A$. We will use the definitions

$$\|f\|^2_{W^s_{\mathrm{dyad}}(\mu)} \equiv \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\| \triangle^\mu_{Q;\kappa} f\right\|^2_{L^2(\mu)},$$
$$\langle f,h\rangle_{W^s_{\mathrm{dyad}}(\mu)} \equiv \sum_{Q\in\mathcal{D}} \ell(Q)^{-2s} \left\langle \triangle^\mu_{Q;\kappa} f, \triangle^\mu_{Q;\kappa} h\right\rangle_{L^2(\mu)},$$

together with the fact that when $\kappa = 1$, $\triangle^{\omega}_{Q;1} R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g$ has one vanishing moment, to obtain

$$A = \sum_{\substack{J,J' \in \mathcal{C}_F^{\tau-\text{shift}} \\ J' \subset J}} \sum_{\substack{Q \in \mathcal{D} \\ Q \subset J}} \ell(Q)^{2s} \int_{\mathbb{R}^n} \left( \triangle^{\omega}_{Q;1} R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right) \left( \triangle^{\omega}_{Q;1} R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right) d\omega$$

$$\lesssim \sum_{J \in \mathcal{C}_F^{\tau-\text{shift}}} \left\| \triangle^{\omega}_{J;\kappa} g \right\|^2_{W^{-s}_{\text{dyad}}(\omega)} = \sum_{J \in \mathcal{C}_F^{\tau-\text{shift}}} \ell(J)^{2s} \left\| \triangle^{\omega}_{J;\kappa} g \right\|^2_{L^2(\omega)} = \left\| \mathsf{P}^{\omega}_{\mathcal{C}_F^{\tau-\text{shift}}} g \right\|^2_{W^{-s}_{\text{dyad}}(\omega)} .$$

Note that here it is important to know that $W^{-s}_{\text{dyad}}(\omega)$ equals both $W^{-s}_{\mathcal{D};\kappa}(\omega)$ and $W^{-s}_{\mathcal{D};1}(\omega)$.

Indeed, if $J'$ is small compared to $J$, and $J' \subset J_{J'} \subset J$, then there are just three possibilities for $Q$, namely $Q \cap J' = \emptyset$, $Q \supsetneqq J'$, and $Q \subset J'$. If $Q \cap J' = \emptyset$ then the integral vanishes by support considerations. If $Q \supsetneqq J'$ then the integral vanishes since $\triangle^{\omega}_{Q;1} \left( R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right)$ is constant on $J'$, $R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g$ has $\omega$-mean 0 on $J'$, and $\triangle^{\omega}_{Q;1}$ is a projection. Thus we are left with the case where $Q \subset J' \subset J$. We have

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \left( \triangle^{\omega}_{Q;1} \left( R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right) \right) \left( \triangle^{\omega}_{Q;1} \left( R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right) \right) d\omega \right| \\ &= \left\| R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right\|_{L^\infty(\omega)} \left\| R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right\|_{L^\infty(\omega)} \\ &\quad \times \left| \int_{\mathbb{R}^n} \left( \frac{\triangle^{\omega}_{Q;1} \left( R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right)}{\left\| R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right\|_{L^\infty(\omega)}} \right) \left( \frac{\triangle^{\omega}_{Q;1} \left( R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right)}{\left\| R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right\|_{L^\infty(\omega)}} \right) d\omega \right| , \end{aligned}$$

and now if $Q \subset J'$, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \left( \frac{\triangle^{\omega}_{Q;1} \left( R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right)}{\left\| R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right\|_{L^\infty(\omega)}} \right) \left( \frac{\triangle^{\omega}_{Q;1} \left( R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right)}{\left\| R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right\|_{L^\infty(\omega)}} \right) d\omega \\ &= \int_{\mathbb{R}^n} \left( \frac{\triangle^{\omega}_{Q;1} R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g}{\left\| R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right\|_{L^\infty(\omega)}} \right) \left( \frac{R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g}{\left\| R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right\|_{L^\infty(\omega)}} \right) d\omega \end{aligned}$$

where $\triangle^{\omega}_{Q;1}$ has one vanishing mean, and hence

$$\begin{aligned} &\left| \int \left( \triangle^{\omega}_{Q;1} \left( R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right) \right) \left( \triangle^{\omega}_{Q;1} \left( R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right) \right) d\omega \right| \\ &\leq \frac{\ell(Q)}{\ell(J)} \left\| R_{J;\kappa} \triangle^{\omega}_{J;\kappa} g \right\|_{L^\infty(\omega)} \frac{\ell(Q)}{\ell(J')} \left\| R_{J';\kappa} \triangle^{\omega}_{J';\kappa} g \right\|_{L^\infty(\omega)} |Q|_\omega \\ &\lesssim \frac{\left\| \triangle^{\omega}_{J;\kappa} g \right\|_{L^2(\omega)}}{\sqrt{|J|_\omega}} \frac{\left\| \triangle^{\omega}_{J';\kappa} g \right\|_{L^2(\omega)}}{\sqrt{|J'|_\omega}} \frac{\ell(Q)}{\ell(J)} \frac{\ell(Q)}{\ell(J')} |Q|_\omega \\ &= \frac{\ell(Q)}{\ell(J)} \frac{\ell(Q)}{\ell(J')} \sqrt{\frac{|Q|_\omega}{|J|_\omega}} \sqrt{\frac{|Q|_\omega}{|J'|_\omega}} \left\| \triangle^{\omega}_{J;\kappa} g \right\|_{L^2(\omega)} \left\| \triangle^{\omega}_{J';\kappa} g \right\|_{L^2(\omega)} \\ &\leq \sqrt{\frac{|Q|_\omega}{|J|_\omega}} \frac{\ell(Q)}{\ell(J)} \left\| \triangle^{\omega}_{J';\kappa} g \right\|_{L^2(\omega)} \left\| \triangle^{\omega}_{J;\kappa} g \right\|_{L^2(\omega)} , \end{aligned}$$

by (2.8), i.e.

$$\left\| R_{J';\kappa} \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{\infty}(\omega)} \sqrt{|J'|_{\omega}} \lesssim \left\| \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{\infty}(\omega)} \sqrt{|J'|_{\omega}} \lesssim \left\| \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{2}(\omega)},$$
$$\left\| R_{J;\kappa} \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{\infty}(\omega)} \sqrt{|J|_{\omega}} \lesssim \left\| \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{\infty}(\omega)} \sqrt{|J|_{\omega}} \lesssim \left\| \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{2}(\omega)}.$$

Note that it was necessary to invoke the Haar wavelets with $\kappa = 1$ in order to obtain this inequality for $s \neq 0$.

We now note for use in the next estimate that

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{D}\\ Q\subset J'}} \ell(Q)^{2s} \sqrt{\frac{|Q|_{\omega}}{|J|_{\omega}}} \frac{\ell(Q)}{\ell(J)} &= \sum_{m=0}^{\infty} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset J' \text{ and } \ell(Q)=2^{-m}\ell(J')}} \ell(Q)^{2s} \sqrt{\frac{|Q|_{\omega}}{|J|_{\omega}}} \frac{\ell(Q)}{\ell(J)} \\
&= \sum_{m=0}^{\infty} \ell(J')^{2s} 2^{-2sm} \frac{2^{-m}\ell(J')}{\ell(J)} \sqrt{\frac{1}{|J|_{\omega}}} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset J' \text{ and } \ell(Q)=2^{-m}\ell(J')}} 1\sqrt{|Q|_{\omega}} \\
&\leq \sum_{m=0}^{\infty} \ell(J')^{2s} 2^{-2sm} \frac{2^{-m}\ell(J')}{\ell(J)} \sqrt{\frac{1}{|J|_{\omega}}} 2^{\frac{m}{2}} \sqrt{\sum_{\substack{Q\in\mathcal{D}\\ Q\subset J' \text{ and } \ell(Q)=2^{-m}\ell(J')}} |Q|_{\omega}} \\
&= \ell(J')^{2s} \frac{\ell(J')}{\ell(J)} \sqrt{\frac{|J'|_{\omega}}{|J|_{\omega}}} \sum_{m=0}^{\infty} 2^{\frac{m}{2}} 2^{-2sm} 2^{-m} \leq C_s \ell(J')^{2s} \frac{\ell(J')}{\ell(J)} \sqrt{\frac{|J'|_{\omega}}{|J|_{\omega}}}.
\end{aligned}$$

Thus recalling that we restricted attention to the case $J' \subset J$ by symmetry, we have

$$\begin{aligned}
&\sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\text{shift}}\\ J'\subset J}} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset J}} \ell(Q)^{2s} \left| \int_{\mathbb{R}^n} \left( \bigtriangleup^{\omega}_{Q;1} R_{J;\kappa} \bigtriangleup^{\omega}_{J;\kappa} g \right) \left( \bigtriangleup^{\omega}_{Q;1} R_{J';\kappa} \bigtriangleup^{\omega}_{J';\kappa} g \right) d\omega \right| \\
&= \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\text{shift}}\\ J'\subset J}} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset J'}} \ell(Q)^{2s} \left| \int_{\mathbb{R}^n} \left( \bigtriangleup^{\omega}_{Q;1} R_{J;\kappa} \bigtriangleup^{\omega}_{J;\kappa} g \right) \left( \bigtriangleup^{\omega}_{Q;1} R_{J';\kappa} \bigtriangleup^{\omega}_{J';\kappa} g \right) d\omega \right| \\
&\lesssim \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\text{shift}}\\ J'\subset J}} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset J'}} \ell(Q)^{2s} \sqrt{\frac{|Q|_{\omega}}{|J|_{\omega}}} \frac{\ell(Q)}{\ell(J)} \left\| \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{2}(\omega)} \left\| \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{2}(\omega)} \\
&\lesssim \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\text{shift}}\\ J'\subset J}} \ell(J')^{2s} \frac{\ell(J')}{\ell(J)} \sqrt{\frac{|J'|_{\omega}}{|J|_{\omega}}} \left\| \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{2}(\omega)} \left\| \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{2}(\omega)}
\end{aligned}$$

which is

$$\begin{aligned}
&= \sum_{m=0}^{\infty} 2^{-m} 2^{-2sm} \sum_{\substack{J,J'\in\mathcal{C}_F^{\tau-\text{shift}}\\ \ell(J')=2^{-m}\ell(J)}} \ell(J)^{2s} \sqrt{\frac{|J'|_{\omega}}{|J|_{\omega}}} \left\| \bigtriangleup^{\omega}_{J';\kappa} g \right\|_{L^{2}(\omega)} \left\| \bigtriangleup^{\omega}_{J;\kappa} g \right\|_{L^{2}(\omega)} \\
&\leq \sum_{m=0}^{\infty} 2^{-m} 2^{-2sm} \sum_{J\in\mathcal{C}_F^{\tau-\text{shift}}} \ell(J)^{2s} \sum_{J'\in\mathcal{C}_F^{\mathbf{\tau}-\text{shift}}:\ \ell(J')=2^{-m}\ell(J)}
\end{aligned}$$

$$\times\sqrt{\frac{|J'|_\omega}{|J|_\omega}}\left\|\triangle^\omega_{J';\kappa}g\right\|_{L^2(\omega)}\left\|\triangle^\omega_{J;\kappa}g\right\|_{L^2(\omega)}$$
$$\lesssim\sum_{m=0}^{\infty}2^{-m}2^{-2sm}\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J)^{2s}\sqrt{\sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}:\ \ell(J')=2^{-m}\ell(J)}\frac{|J'|_\omega}{|J|_\omega}\left\|\triangle^\omega_{J;\kappa}g\right\|^2_{L^2(\omega)}}$$
$$\times\sqrt{\sum_{J':\ \ell(J')=2^{-m}\ell(J)}\left\|\triangle^\omega_{J';\kappa}g\right\|^2_{L^2(\omega)}}$$

which is at most

$$\lesssim\sum_{m=0}^{\infty}2^{-m}2^{-2sm}\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J)^{2s}\left\|\triangle^\omega_{J;\kappa}g\right\|_{L^2(\omega)}$$
$$\times\sqrt{\sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}:\ \ell(J')=2^{-m}\ell(J)}\left\|\triangle^\omega_{J';\kappa}g\right\|^2_{L^2(\omega)}}$$
$$\lesssim\sum_{m=0}^{\infty}2^{-m}2^{-2sm}\sqrt{\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J)^{2s}\left\|\triangle^\omega_{J;\kappa}g\right\|^2_{L^2(\omega)}}$$
$$\times\sqrt{\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J)^{2s}\sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}:\ \ell(J')=2^{-m}\ell(J)}\left\|\triangle^\omega_{J';\kappa}g\right\|^2_{L^2(\omega)}}$$
$$\lesssim\sqrt{\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J)^{2s}\left\|\triangle^\omega_{J;\kappa}g\right\|^2_{L^2(\omega)}}\sum_{m=0}^{\infty}2^{-m}2^{-2sm}2^{ms}$$
$$\times\sqrt{\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}:\ \ell(J')=2^{-m}\ell(J)}\ell(J')^{2s}\left\|\triangle^\omega_{J';\kappa}g\right\|^2_{L^2(\omega)}}$$
$$\leq\left\|\mathsf{P}^\omega_{\mathcal{C}_F^{\tau-\mathrm{shift}}}g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}\sum_{m=0}^{\infty}2^{-m}2^{-2sm}2^{ms}$$
$$\sqrt{\sum_{J'\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\ell(J')^{2s}\left\|\triangle^\omega_{J';\kappa}g\right\|^2_{L^2(\omega)}}=C_s\left\|\mathsf{P}^\omega_{\mathcal{C}_F^{\tau-\mathrm{shift}}}g\right\|^2_{W^{-s}_{\mathrm{dyad}}(\omega)}$$

provided that $s>-1$. Altogether we have shown

$$\left|\mathsf{B}^F_{\mathrm{paraproduct};\kappa}(f,g)\right|\lesssim\mathfrak{T}^s_{T^\lambda}\ell(F)^{-s}\sqrt{|F|_\sigma}\left\|\mathbb{E}^\sigma_{F;\kappa}f\right\|_{L^\infty(\sigma)}\left\|\sum_{J\in\mathcal{C}_F^{\tau-\mathrm{shift}}}\frac{P_{J;\kappa}}{E^\sigma_F|f|}\triangle^\omega_{J;\kappa}g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}$$
$$\lesssim\mathfrak{T}^s_{T^\lambda}\ell(F)^{-s}\sqrt{|F|_\sigma}\left\|\mathbb{E}^\sigma_{F;\kappa}f\right\|_{L^\infty(\sigma)}C_s\left\|\mathsf{P}^\omega_{\mathcal{C}_F^{\tau-\mathrm{shift}}}g\right\|_{W^{-s}_{\mathrm{dyad}}(\omega)}.$$

For future reference we record the fact that the main inequality proved above,

$$\left\| \sum_{J\in\mathcal{C}_F^{\tau-\text{shift}}} R_{J;\kappa} \bigtriangleup_{J;\kappa}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 \lesssim \sum_{J\in\mathcal{C}_F^{\tau-\text{shift}}} \left\| \bigtriangleup_{J;\kappa}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 ,$$

continues to hold if $\mathcal{C}_F^{\tau-\text{shift}}$ is replaced by an arbitrary subset $\mathcal{J}$ of the dyadic grid $\mathcal{D}$, and if the polynomials $R_{J;\kappa}$ are replaced with any family of polynomials $\left\{W_{J;\kappa}\right\}_{J\in\mathcal{J}}$ such that for all $J \in \mathcal{J}$,

$$\begin{aligned} \deg W_{J;\kappa} &< \kappa, \\ \int_J W_{J;\kappa} d\omega &= 0, \\ \sup_J \left|W_{J;\kappa}\right| &\leq C. \end{aligned}$$

More precisely the above arguments prove,

$$\left\| \sum_{J\in\mathcal{J}} W_{J;\kappa} \bigtriangleup_{J;\kappa}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 \lesssim \sum_{J\in\mathcal{J}} \left\| \bigtriangleup_{J;\kappa}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 . \tag{6.4}$$

Since the weighted Sobolev wavelets $\left\{\bigtriangleup_{J;\kappa}^{\omega}\right\}_{J\in\mathcal{D}}$ are pairwise orthogonal, we have

$$\sum_{J\in\mathcal{C}_F^{\tau-\text{shift}}} \left\| \bigtriangleup_{J;\kappa}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 = \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)}^2 ,$$

and so we obtain

$$\left|\mathsf{B}_{\text{paraproduct};\kappa}^{F}(f,g)\right| \lesssim \mathfrak{T}_{T^\lambda}^{s} \ell(F)^{-s} \left\|\mathbb{E}_{F;\kappa}^{\sigma} f\right\|_{L^\infty(\sigma)} \sqrt{|F|_\sigma} \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^{\omega} g \right\|_{W_{\text{dyad}}^{-s}(\omega)} \tag{6.5}$$

as required by (5.16).

We next turn to the commutator inner products $\left\langle \left[T_\sigma^\lambda, M_{I';\kappa}\right] \mathbf{1}_{I'}, \bigtriangleup_{J;\kappa}^{\omega} g\right\rangle_\omega$ arising in $\mathsf{B}_{\text{commutator};\kappa}^{F}(f,g)$, followed by the neighbour and stopping inner products.

***Remark 45*** The arguments for the commutator, neighbour and stopping forms follow closely the analogous arguments in [1] where the case $s = 0$ is handled. Nevertheless, there are differences arising when $s \neq 0$, and so we give complete details for the convenience of the reader.

## 6.2 The commutator form

Fix $\kappa \geq 1$. In this subsection we use $\alpha$ to denote a multiindex in $\mathbb{R}^n$, and so we will instead use $\lambda$ to denote the fractional order of the Calderon–Zygmund operator. Assume now that $K^\lambda$ is a general standard $\lambda$-fractional kernel in $\mathbb{R}^n$, and $T^\lambda$ is the associated Calderón–Zygmund operator, and that $P_{\alpha,a,I'}(x) = \left(\frac{x-a}{\ell(I')}\right)^\alpha = \left(\frac{x_1-a_1}{\ell(I')}\right)^{\alpha_1} \ldots \left(\frac{x_n-a_n}{\ell(I')}\right)^{\alpha_n}$, where $|\alpha| \leq \kappa - 1$ and

$I' \in \mathfrak{C}_{\mathcal{D}}(I)$, $I \in \mathcal{C}_F$. We recall from [1] the formula

$$x^\alpha - y^\alpha = \sum_{k=1}^{n} (x_k - y_k) \sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha,\beta,\gamma} x^\beta y^\gamma .$$

Continuing to follow [1], we then have

$$\begin{aligned}
&\mathbf{1}_{I'}(x)\left[P_{\alpha,a,I'} T_\sigma^\lambda\right]\mathbf{1}_{I'}(x) = \mathbf{1}_{I'}(x)\int_{\mathbb{R}^n} K^\lambda(x-y)\left\{P_{\alpha,a,I'}(x) - P_{\alpha,a,I'}(y)\right\}\mathbf{1}_{I'}(y)\,d\sigma(y)\\
&= \mathbf{1}_{I'}(x)\int_{\mathbb{R}^n} K^\lambda(x-y)\left\{\sum_{k=1}^{n}\left(\frac{x_k-y_k}{\ell(I')}\right)\sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha,\beta,\gamma}\left(\frac{x-a}{\ell(I')}\right)^\beta\left(\frac{y-a}{\ell(I')}\right)^\gamma\right\}\mathbf{1}_{I'}(y)\,d\sigma(y)\\
&= \sum_{k=1}^{n}\sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha,\beta,\gamma}\mathbf{1}_{I'}(x)\left[\int \Phi_k^\lambda(x-y)\left\{\left(\frac{y-a}{\ell(I')}\right)^\gamma\right\}\mathbf{1}_{I'}(y)\,d\sigma(y)\right]\left(\frac{x-a}{\ell(I')}\right)^\beta,
\end{aligned}$$

where $\Phi_k^\lambda(x-y) = K^\lambda(x-y)\left(\frac{x_k-y_k}{\ell(I')}\right)$. So $\left[P_{\alpha,a,I'}, T_\sigma^\lambda\right]\mathbf{1}_{I'}(x)$ is a 'polynomial' of degree $|\alpha|-1$ with *variable* coefficients. Now we take the inner product of the commutator with $\triangle_{J;\kappa}^\omega g$ for some $J \subset I'$, and split the inner product into two pieces,

$$\begin{aligned}
&\left\langle\left[P_{\alpha,a,I'} T_\sigma^\lambda\right]\mathbf{1}_{I'}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega = \int_{\mathbb{R}^n}\left[P_{\alpha,a,I'} T_\sigma^\lambda\right]\mathbf{1}_{I'}(x)\,\triangle_{J;\kappa}^\omega g(x)\,d\omega(x) \qquad (6.6)\\
&= \int_{\mathbb{R}^n}\left[P_{\alpha,a,I'} T_\sigma^\lambda\right]\mathbf{1}_{I'\setminus 2J}(x)\,\triangle_{J;\kappa}^\omega g(x)\,d\omega(x)\\
&\quad + \int_{\mathbb{R}^n}\left[P_{\alpha,a,I'} T_\sigma^\lambda\right]\mathbf{1}_{I'\cap 2J}(x)\,\triangle_{J;\kappa}^\omega g(x)\,d\omega(x)\\
&\equiv \mathrm{Int}^{\lambda,\natural}(J) + \mathrm{Int}^{\lambda,\flat}(J),
\end{aligned}$$

where we are suppressing the dependence on $\alpha$ and $I'$. For the first term $\mathrm{Int}^{\lambda,\natural}(J)$ we write

$$\mathrm{Int}^{\lambda,\natural}(J) = \sum_{k=1}^{n}\sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha,\beta,\gamma}\,\mathrm{Int}_{k,\beta,\gamma}^{\lambda,\natural}(J),$$

where with the choice $a = c_J$ the center of $J$, we define

$$\begin{aligned}
\mathrm{Int}_{k,\beta,\gamma}^{\lambda,\natural}(J) &\equiv \int_J\left[\int_{I'\setminus 2J}\Phi_k^\lambda(x-y)\left(\frac{y-c_J}{\ell(I')}\right)^\gamma d\sigma(y)\right]\left(\frac{x-c_J}{\ell(I')}\right)^\beta \triangle_{J;\kappa}^\omega g(x)\,d\omega(x)\\
&= \int_{I'\setminus 2J}\left\{\int_J \Phi_k^\lambda(x-y)\left(\frac{x-c_J}{\ell(I')}\right)^\beta \triangle_{J;\kappa}^\omega g(x)\,d\omega(x)\right\}\left(\frac{y-c_J}{\ell(I')}\right)^\gamma d\sigma(y).
\end{aligned} \tag{6.7}$$

While these integrals need no longer vanish, we will show they are suitably small, using that the function $\left(\frac{x-c_J}{\ell(I')}\right)^\beta \triangle_{J;\kappa}^\omega g(x)$ is supported in $J$ and has vanishing $\omega$-means up to order $\kappa-|\beta|-1$, and that the function $\Phi_k^\lambda(z)$ is appropriately smooth away from $z=0$,

$$\left|\nabla^m \Phi_k^\lambda(z)\right| \le C_{m,n}\frac{1}{|z|^{m+n-\lambda-1}\,\ell(I')}.$$

Indeed, we have the following estimate for the integral in braces in (6.7), keeping in mind that $y \in I'\setminus 2J$ and $x \in J$, where the term in the second line below vanishes because

$h(x) = \left(\frac{x-c_J}{\ell(I')}\right)^\beta \triangle^\omega_{J;\kappa} g(x)$ has vanishing $\omega$-means up to order $\kappa - 1 - |\beta|$, and the fourth line uses that $h(x)$ is supported in $J$:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \Phi^\lambda_k (x-y)\, h(x)\, d\omega(x)\right| \\
&= \left|\int_{\mathbb{R}^n} \left\{\sum_{m=0}^{\kappa-|\beta|-1} \frac{1}{m!} ((x-c_J)\cdot\nabla)^m \Phi^\lambda_k (c_J - y)\right\} h(x)\, d\omega(x)\right. \\
&\quad \left. + \int_{\mathbb{R}^n} \frac{1}{(\kappa-|\beta|)!} ((x-c_J)\cdot\nabla)^{\kappa-|\beta|} \Phi^\lambda_k \left(\eta^\omega_J\right) h(x)\, d\omega(x)\right| \\
&\lesssim \|h\|_{L^1(\omega)} \frac{\ell(J)^{\kappa-|\beta|}}{[\ell(J) + \operatorname{dist}(y,J)]^{\kappa-|\beta|+n-\lambda-1}\, \ell(I')} \\
&\lesssim \left(\frac{\ell(J)}{\ell(I')}\right)^{|\beta|} \frac{\ell(J)^{\kappa-|\beta|}}{[\ell(J) + \operatorname{dist}(y,J)]^{\kappa-|\beta|+n-\lambda-1}\, \ell(I')} \sqrt{|J|_\omega}\, |\widehat{g}(J)|,
\end{aligned}$$

since

$$\begin{aligned}
\|h\|_{L^1(\omega)} &= \int_J \left|\left(\frac{x-c_J}{\ell(I')}\right)^\beta \triangle^\omega_{J;\kappa} g(x)\right| d\omega(x) \\
&\le \left(\frac{\ell(J)}{\ell(I')}\right)^{|\beta|} \left\|\triangle^\omega_{J;\kappa} g\right\|_{L^1(\omega)} \lesssim \left(\frac{\ell(J)}{\ell(I')}\right)^{|\beta|} \sqrt{|J|_\omega}\, |\widehat{g}(J)|.
\end{aligned}$$

Now recall the orthonormal basis $\left\{h^{\sigma,a}_{I;\kappa}\right\}_{a\in\Gamma}$ of $L^2_{I;\kappa}(\sigma)$ for any $I \in \mathcal{D}$. For a $\mathcal{D}$-child $I'$ of a cube $I$, we consider the polynomial

$$Q^\sigma_{I';\kappa} \equiv \mathbf{1}_{I'} \sum_{a\in\Gamma} c_a h^{\sigma,a}_{I;\kappa}$$

where $c_a = \frac{\left\langle f, h^{\sigma,a}_{I;\kappa}\right\rangle}{\left|\widehat{f}(I)\right|}$, so that $Q^\sigma_{I';\kappa}$ is a renormalization of the polynomial $M_{I';\kappa}$ introduced earlier. We have

$$\left|\widehat{f}(I)\right| \mathbf{1}_{I'} \sum_{a\in\Gamma} c_a h^{\sigma,a}_{I;\kappa} = \left|\widehat{f}(I)\right| \mathbf{1}_{I'} Q^\sigma_{I';\kappa} = \mathbb{E}^\sigma_{I';\kappa} f - \mathbf{1}_{I'} \mathbb{E}^\sigma_{I;\kappa} f,$$

where

$$\sum_{a\in\Gamma} |c_a|^2 = \sum_{a\in\Gamma} \left|\frac{\left\langle f, h^{\sigma,a}_{I;\kappa}\right\rangle}{\left|\widehat{f}(I)\right|}\right|^2 = 1.$$

Recall also that from (2.8) we have

$$\left|\widehat{f}(I)\right| \left\|Q^\sigma_{I';\kappa}\right\|_\infty = \left\|\mathbf{1}_{I'} \sum_{a\in\Gamma} \left\langle f, h^{\sigma,a}_{I;\kappa}\right\rangle h^{\sigma,a}_{I;\kappa}\right\|_\infty = \left\|\mathbf{1}_{I'} \triangle^\sigma_{I;\kappa} f\right\|_\infty \approx \frac{\left|\widehat{f}(I)\right|}{\sqrt{|I|_\sigma}}.$$

Hence for $c_J \in J \subset I'$, if we write

$$Q^\sigma_{I';\kappa}(y) = \sum_{|\alpha|<\kappa} b_\alpha \left(\frac{y-c_J}{\ell(I')}\right)^\alpha = \sum_{|\alpha|<\kappa} b_\alpha P_{\alpha,c_J}(y),$$

and then rescale to the unit cube and invoke the fact that any two norms on a finite dimensional vector space are equivalent, we obtain

$$\sum_{|\alpha|<\kappa} |b_\alpha| \approx \left\| Q^{\sigma}_{I';\kappa} \right\|_\infty \approx \frac{1}{\sqrt{|I|_\sigma}}, \qquad I' \in \mathfrak{C}_{\mathcal{D}}(I). \tag{6.8}$$

We then write

$$\left\langle \left[ Q_{I';\kappa}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega = \sum_{|\alpha|<\kappa} b_\alpha \left\langle \left[ P_{\alpha,c_J}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega$$

and note that

$$\begin{aligned} \left| \left\langle \left[ Q_{I';\kappa}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega \right| &\le \sum_{|\alpha|<\kappa} \left| b_\alpha \left\langle \left[ P_{\alpha,c_J}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega \right| \\ &\lesssim \frac{1}{\sqrt{|I|_\sigma}} \max_{|\alpha|<\kappa} \left| \left\langle \left[ P_{\alpha,c_J}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega \right|, \end{aligned}$$

so that it remains to estimate each inner product $\mathrm{Int}^{\lambda,\natural}(J) = \left\langle \left[ P_{\alpha,c_J}, T^{\lambda}_{\sigma} \right] \mathbf{1}_{I'\setminus 2J}, \triangle^{\omega}_{J;\kappa} g \right\rangle_\omega$ as follows:

$$\left| \mathrm{Int}^{\lambda,\natural}(J) \right| = \left| \sum_{k=1}^{n} \sum_{|\beta|+|\gamma|=|\alpha|-1} c_{\alpha,\beta,\gamma} \mathrm{Int}^{\lambda,\natural}_{k,\beta,\gamma}(J) \right| \lesssim \max_{|\beta|+|\gamma|=|\alpha|-1} \left| \mathrm{Int}^{\lambda,\natural}_{k,\beta,\gamma}(J) \right|,$$

where $|\beta| + |\gamma| = |\alpha| - 1$, and the estimates above imply,

$$\begin{aligned} \left| \mathrm{Int}^{\lambda,\natural}_{k,\beta,\gamma}(J) \right| &\le \int_{I'\setminus 2J} \left| \int \Phi^{\lambda}_{k}(x-y) \left( \frac{x-c_J}{\ell(I')} \right)^{\beta} \triangle^{\omega}_{J;\kappa} g(x)\, d\omega(x) \right| \left| \left( \frac{y-c_J}{\ell(I')} \right)^{\gamma} \right| d\sigma(y) \\ &\lesssim \int_{I'\setminus 2J} \left( \frac{\ell(J)}{\ell(I')} \right)^{|\beta|} \frac{\ell(J)^{\kappa-|\beta|}}{[\ell(J)+\mathrm{dist}(y,J)]^{\kappa-|\beta|+n-\lambda-1}\, \ell(I)} \\ &\quad \times \sqrt{|J|_\omega}\, |\widehat{g}(J)| \left( \frac{\ell(J)+\mathrm{dist}(y,J)}{\ell(I')} \right)^{|\gamma|} d\sigma(y) \\ &= \int_{I'\setminus 2J} \left( \frac{\ell(J)}{\ell(I')} \right)^{|\alpha|-1} \left( \frac{\ell(J)}{\ell(J)+\mathrm{dist}(y,J)} \right)^{\kappa-|\alpha|+1} \\ &\quad \times \frac{1}{[\ell(J)+\mathrm{dist}(y,J)]^{n-\lambda-1}\, \ell(I')} \sqrt{|J|_\omega}\, |\widehat{g}(J)|\, d\sigma(y) \\ &= \left( \frac{\ell(J)}{\ell(I')} \right)^{|\alpha|-1} \sqrt{|J|_\omega}\, |\widehat{g}(J)| \\ &\quad \times \left\{ \int_{I'\setminus 2J} \left( \frac{\ell(J)}{\ell(J)+\mathrm{dist}(y,J)} \right)^{\kappa-|\alpha|+1} \frac{1}{[\ell(J)+\mathrm{dist}(y,J)]^{n-\lambda-1}\, \ell(I')} d\sigma(y) \right\}. \end{aligned}$$

Now we fix $t \in \mathbb{N}$, and estimate the sum of $\left| \mathrm{Int}^{\lambda,\natural}(J) \right|$ over $J \subset I'$ with $\ell(J) = 2^{-t}\ell(I')$ by splitting the integration in $y$ according to the size of $\ell(J) + \mathrm{dist}(y,J)$, to obtain the following bound:

$$\sum_{J\subset I':\, \ell(J)=2^{-t}\ell(I')} \left| \mathrm{Int}^{\lambda,\natural}(J) \right|$$

$$\lesssim 2^{-t(|\alpha|-1)} \sum_{J\subset I':\,\ell(J)=2^{-t}\ell(I')} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \left\{ \int_{I'\setminus 2J} \left( \frac{\ell(J)}{\ell(J)+\operatorname{dist}(y,J)} \right)^{\kappa-|\alpha|+1} \times \frac{d\sigma(y)}{[\ell(J)+\operatorname{dist}(y,J)]^{n-\lambda-1}\,\ell(I')} \right\}$$

$$\lesssim 2^{-t(|\alpha|-1)} \sum_{J\subset I':\,\ell(J)=2^{-t}\ell(I')} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \left\{ \sum_{s=1}^{t} \int_{2^{s+1}J\setminus 2^s J} \left(2^{-s}\right)^{\kappa-|\alpha|+1} \frac{d\sigma(y)}{(2^s\ell(J))^{n-\lambda-1}\,\ell(I')} \right\}$$

$$\lesssim 2^{-t|\alpha|} \sum_{J\subset I':\,\ell(J)=2^{-t}\ell(I')} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|+1} 2^{-s(n-\lambda-1)} \frac{\left|2^s J\right|_\sigma}{\ell(J)^{n-\lambda}},$$

which gives upon pigeonholing the sum in $J$ according to membership in the grandchildren of $I'$ at depth $t-s$:

$$\begin{aligned}
&\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} \left|\mathrm{Int}^{\lambda,\natural}(J)\right| \\
&\lesssim 2^{-t|\alpha|} \sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|+n-\lambda} \frac{|2^s J|_\sigma}{\ell(J)^{n-\lambda}} \\
&= 2^{-t|\alpha|} \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|} \sum_{K\in\mathfrak{C}_{\mathcal{D}}^{(t-s)}(I')} \sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(s)}(K)} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \frac{|2^s J|_\sigma}{\ell(K)^{n-\lambda}} \\
&\lesssim 2^{-t|\alpha|} \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|} \sum_{K\in\mathfrak{C}_{\mathcal{D}}^{(t-s)}(I')} \frac{|3K|_\sigma}{\ell(K)^{n-\lambda}} \sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(s)}(K)} \sqrt{|J|_\omega}\,|\widehat{g}(J)| \\
&\lesssim 2^{-t|\alpha|} \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|} \sum_{K\in\mathfrak{C}_{\mathcal{D}}^{(t-s)}(I')} \frac{|3K|_\sigma}{\ell(K)^{n-\lambda}} \sqrt{|K|_\omega} \sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(s)}(K)} |\widehat{g}(J)|^2}.
\end{aligned}$$

Now we use the $A_2^\lambda$ condition and doubling for $\sigma$ to obtain the bound

$$\frac{|3K|_\sigma}{\ell(K)^{n-\lambda}}\sqrt{|K|_\omega} \lesssim \frac{\sqrt{|3K|_\sigma\,|3K|_\omega}}{\ell(3K)^{n-\lambda}}\sqrt{|3K|_\sigma} \le \sqrt{A_2^\lambda}\sqrt{|3K|_\sigma} \lesssim \sqrt{A_2^\lambda}\sqrt{|K|_\sigma}.$$

Thus we have

$$\begin{aligned}
\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} \left|\mathrm{Int}^{\lambda,\natural}(J)\right| &\lesssim 2^{-t|\alpha|}\sqrt{A_2^\lambda} \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|} \sum_{K\in\mathfrak{C}_{\mathcal{D}}^{(t-s)}(I')} \sqrt{|K|_\sigma} \sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(s)}(K)} |\widehat{g}(J)|^2} \\
&\lesssim 2^{-t|\alpha|}\sqrt{A_2^\lambda} \sum_{s=1}^{t} \left(2^{-s}\right)^{\kappa-|\alpha|} \sqrt{|I'|_\sigma} \sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} |\widehat{g}(J)|^2} \\
&\lesssim 2^{-t}\sqrt{A_2^\lambda}\sqrt{|I'|_\sigma} \sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} |\widehat{g}(J)|^2},
\end{aligned}$$

since $1 \leq |\alpha| \leq \kappa - 1$ (the commutator vanishes if $|\alpha| = 0$) shows that both $2^{-t|\alpha|} \leq 2^{-t}$ and

$$\kappa - |\alpha| \geq 1 > 0.$$

We now claim the same estimate holds for the sum of $\left|\mathrm{Int}^{\lambda,\flat}(J)\right|$ over $J \subset I'$ with $\ell(J) = 2^{-t}\ell\left(I'\right)$, namely

$$\begin{aligned}
\left|\mathrm{Int}^{\lambda,\flat}_{k,\beta,\gamma}(J)\right| &\lesssim \left|\int_J \left(\int_{2J} \Phi^{\lambda}_k(x-y)\left(\frac{y-c_J}{\ell(I')}\right)^{\gamma} d\sigma(y)\right)\left(\frac{x-c_J}{\ell(I')}\right)^{\beta} \triangle^{\omega}_{J;\kappa}\, g(x)\, d\omega(x)\right| \\
&\leq \int_J \left(\int_{2J} \frac{1}{\ell(I')\,|x-y|^{n-\lambda-1}}\left|\frac{y-c_J}{\ell(I')}\right|^{|\gamma|} d\sigma(y)\right)\left|\frac{x-c_J}{\ell(I')}\right|^{|\beta|}\left|\triangle^{\omega}_{J;\kappa} g(x)\right| d\omega(x) \\
&\lesssim \left(\frac{\ell(J)}{\ell(I')}\right)^{|\gamma|+|\beta|} \frac{1}{\ell(I')\sqrt{|J|_{\omega}}}\,|\widehat{g}(J)| \int_J\int_{2J} \frac{d\sigma(y)\, d\omega(x)}{|x-y|^{n-\lambda-1}}.
\end{aligned}$$

In order to estimate the double integral using the $A^{\lambda}_2$ condition we cover the band $|x-y| \leq C2^{-m}\ell(J)$ by a collection of cubes $Q\left(z_m, C2^{-m}\ell(J)\right) \times Q\left(z_m, C2^{-m}\ell(J)\right)$ in $CJ \times CJ$ with centers $(z_m, z_m)$ and bounded overlap. Then we have

$$\begin{aligned}
\int_J\int_{2J} \frac{d\sigma(y)\, d\omega(x)}{|x-y|^{n-\lambda-1}} &\leq \sum_{m=0}^{\infty} \iint_{\substack{x,y\in 2J \\ |x-y|\approx 2^{-m}\ell(J)}} \frac{d\sigma(y)\, d\omega(x)}{\left(2^{-m}\ell(J)\right)^{n-\lambda-1}} \\
&\approx \sum_{m=0}^{\infty} \sum_{Q(z_m,C2^{-m}\ell(J))\times Q(z_m,C2^{-m}\ell(J))} \int_{Q(z_m,C2^{-m}\ell(J))\times Q(z_m,C2^{-m}\ell(J))} \frac{d\sigma(y)\, d\omega(x)}{\left(2^{-m}\ell(J)\right)^{n-\lambda-1}} \\
&\leq \frac{1}{\ell(J)^{n-\lambda-1}} \sum_{m=0}^{\infty} 2^{m(n-\lambda-1)} \sum_{z_m} \left|Q\left(z_m, C2^{-m}\ell(J)\right)\right|_{\sigma} \left|Q\left(z_m, C2^{-m}\ell(J)\right)\right|_{\omega} \\
&\leq \frac{1}{\ell(J)^{n-\lambda-1}} \sum_{m=0}^{\infty} 2^{m(n-\lambda-1)} \sum_{z_m} \sqrt{\left|Q\left(z_m, C2^{-m}\ell(J)\right)\right|_{\sigma} \left|Q\left(z_m, C2^{-m}\ell(J)\right)\right|_{\omega}} \sqrt{A^{\lambda}_2}\left(2^{-m}\ell(J)\right)^{n-\lambda} \\
&\lesssim \sqrt{A^{\lambda}_2}\,\ell(J) \sum_{m=0}^{\infty} 2^{m(n-\lambda-1)} 2^{-m(n-\lambda)} \sqrt{|CJ|_{\sigma}\,|CJ|_{\omega}} \lesssim \sqrt{A^{\lambda}_2}\,\ell(J)\sqrt{|J|_{\sigma}\,|J|_{\omega}}.
\end{aligned}$$

Thus altogether, we have

$$\begin{aligned}
\left|\mathrm{Int}^{\lambda,\flat}_{k,\beta,\gamma}(J)\right| &\lesssim \left(\frac{\ell(J)}{\ell(I')}\right)^{|\gamma|+|\beta|} \frac{1}{\ell(I')\sqrt{|J|_{\omega}}}\,|\widehat{g}(J)| \sqrt{A^{\lambda}_2}\,\ell(J)\sqrt{|J|_{\sigma}\,|J|_{\omega}} \\
&\leq \sqrt{A^{\lambda}_2}\left(\frac{\ell(J)}{\ell(I')}\right)^{|\alpha|} |\widehat{g}(J)|\sqrt{|J|_{\sigma}} \leq \sqrt{A^{\lambda}_2}\,\frac{\ell(J)}{\ell(I')}\,|\widehat{g}(J)|\sqrt{|J|_{\sigma}},
\end{aligned}$$

since $|\alpha| \geq 1$ (otherwise the commutator vanishes). Now

$$\begin{aligned}
\sum_{J\in\mathfrak{C}^{(t)}_{\mathcal{D}}(I')} \left|\mathrm{Int}^{\lambda,\flat}(J)\right| &\lesssim \sum_{J\in\mathfrak{C}^{(t)}_{\mathcal{D}}(I')} \sqrt{A^{\lambda}_2}\,\frac{\ell(J)}{\ell(I')}\left[\ell(J)^{-s}\,\ell(J)^{s}\right]|\widehat{g}(J)|\sqrt{|J|_{\sigma}} \\
&\leq \ell\left(I'\right)^{-s} 2^{-t(1-s)}\sqrt{A^{\lambda}_2}\sqrt{|I'|_{\sigma}}\sqrt{\sum_{J\in\mathfrak{C}^{(t)}_{\mathcal{D}}(I')} \ell(J)^{2s}\,|\widehat{g}(J)|^2},
\end{aligned}$$

and so altogether we have

$$\begin{aligned}
&\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')} \left|\left\langle\left[P_{\alpha,c_J},T_\sigma^\lambda\right]\mathbf{1}_{I'},\triangle_{J;\kappa}^\omega g\right\rangle_\omega\right| \\
&\quad=\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\left|\mathrm{Int}^\lambda(J)\right| \\
&\quad\leq\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\left|\mathrm{Int}^{\lambda,\natural}(J)\right|+\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\left|\mathrm{Int}^{\lambda,\flat}(J)\right| \\
&\quad\lesssim 2^{-t(1-s)}\ell\left(I'\right)^{-s}\sqrt{A_2^\lambda}\sqrt{|I'|_\sigma}\sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\ell(J)^{2s}\left|\widehat{g}(J)\right|^2}.
\end{aligned}$$

Finally then we obtain from this and (6.8),

$$\begin{aligned}
\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\left|\left\langle\left[Q_{I';\kappa},T_\sigma^\lambda\right]\mathbf{1}_{I'},\triangle_{J;\kappa}^\omega g\right\rangle_\omega\right| &= \sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\left|\sum_{|\alpha|\leq\kappa-1}b_\alpha\left\langle\left[P_{\alpha,c_J},T_\sigma^\lambda\right]\mathbf{1}_{I'},\triangle_{J;\kappa}^\omega g\right\rangle_\omega\right| \\
&\lesssim 2^{-t(1-s)}\ell\left(I'\right)^{-s}\sqrt{A_2^\lambda}\sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I')}\ell(J)^{2s}\left|\widehat{g}(J)\right|^2}.
\end{aligned}$$

Now using $M_{I_J;\kappa}=\left|\widehat{f}(I)\right|Q_{I_J;\kappa}$, and applying the above estimates with $I'=I_J$, we can sum over $t$ and $I\in\mathcal{C}_F$ to obtain

$$\left|\mathsf{B}_{\text{commutator};\kappa}^F(f,g)\right|\leq\sum_{\substack{I\in\mathcal{C}_F\text{ and }J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon}I}}\left|\widehat{f}(I)\right|\left|\left\langle\left[T_\sigma^\lambda,Q_{I_J;\kappa}\right]\mathbf{1}_{I_J},\triangle_{J;\kappa}^\omega g\right\rangle_\omega\right| \tag{6.9}$$

$$\begin{aligned}
&\lesssim\sum_{t=r}^\infty\sum_{I\in\mathcal{C}_F}2^{-t(1-s)}\ell\left(I'\right)^{-s}\sqrt{A_2^\lambda}\left|\widehat{f}(I)\right|\sqrt{\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I_J)\text{ and }J\in\mathcal{C}_F^{\tau-\text{shift}}}\ell(J)^{2s}\left|\widehat{g}(J)\right|^2} \\
&\lesssim\sqrt{A_2^\lambda}\sum_{t=r}^\infty 2^{-t(1-s)}\sqrt{\sum_{I\in\mathcal{C}_F}\ell(I)^{-2s}\left|\widehat{f}(I)\right|^2}\sqrt{\sum_{I\in\mathcal{C}_F}\sum_{J\in\mathfrak{C}_{\mathcal{D}}^{(t)}(I_J)\text{ and }J\in\mathcal{C}_F^{\tau-\text{shift}}}\ell(J)^{2s}\left|\widehat{g}(J)\right|^2} \\
&\lesssim\sqrt{A_2^\lambda}\sum_{t=r}^\infty 2^{-t}\left\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\right\|_{W_{\text{dyad}}^s(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^\omega g\right\|_{W_{\text{dyad}}^{-s}(\omega)} \\
&\lesssim\sqrt{A_2^\lambda}\left\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\right\|_{W_{\text{dyad}}^s(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_F^{\emptyset-\text{shift}}}^\omega g\right\|_{W_{\text{dyad}}^{-s}(\omega)}.
\end{aligned}$$

Thus the commutator form $\mathsf{B}_{\text{commutator};\kappa}^F(f,g)$ is controlled by $A_2^\lambda$ alone.

### 6.3 The neighbour form

In this form we can obtain the required bound, which uses only the $A_2^\lambda$ constant, by taking absolute values inside the sum, and following [1] again, we argue as in the case of Haar wavelets in [17, end of Subsection 8.4]. We begin with $M_{I';\kappa}=\mathbf{1}_{I'}\triangle_{I;\kappa}^\sigma f$ as in (6.1) to obtain

$$\left|\mathsf{B}^{F}_{\text{neighbour};\kappa}(f,g)\right| \leq \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \left|\left\langle T_\sigma^\lambda\left(\mathbf{1}_{\theta(I_J)}\,\triangle^\sigma_{I;\kappa} f\right),\triangle^\omega_{J;\kappa} g\right\rangle_\omega\right|$$

$$\leq \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \sum_{I'\equiv\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \left|\left\langle T_\sigma^\lambda\left(M_{I';\kappa}\mathbf{1}_{I'}\right),\triangle^\omega_{J;\kappa} g\right\rangle_\omega\right|.$$

We now control this by the pivotal bound (3.13) on the inner product with $\nu = \left\|M_{I';\kappa}\right\|_{L^\infty(\sigma)}\mathbf{1}_{I'}d\sigma$, and then estimating by the usual Poisson kernel,

$$\left|\left\langle T_\sigma^\lambda\left(M_{I';\kappa}\mathbf{1}_{I'}\right),\triangle^\omega_{J;\kappa} g\right\rangle_\omega\right| \lesssim \mathrm{P}^\lambda_\kappa\left(J,\left\|M_{I';\kappa}\right\|_{L^\infty(\sigma)}\mathbf{1}_{I'}\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)}$$

$$\leq \left\|M_{I';\kappa}\right\|_{L^\infty(\sigma)}\mathrm{P}^\lambda\left(J,\mathbf{1}_{I'}\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)},$$

and the estimate $\left\|M_{I';\kappa}\right\|_{L^\infty(\sigma)} \approx \frac{1}{\sqrt{|I'|_\sigma}}\left|\widehat{f}(I)\right|$ from (2.8), along with (3.14), namely

$$\mathrm{P}^\lambda_m\left(J,\sigma\mathbf{1}_{K\setminus I}\right) \lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{m-\varepsilon(n+m-\lambda)}\mathrm{P}^\alpha_m\left(I,\sigma\mathbf{1}_{K\setminus I}\right),$$

to obtain

$$\begin{aligned}
&\left|\mathsf{B}^{F}_{\text{neighbour};\kappa}(f,g)\right| \\
&\lesssim \sum_{\substack{I\in\mathcal{C}_F \text{ and } J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \sum_{I'\equiv\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \frac{\left|\widehat{f}(I)\right|}{\sqrt{|I'|_\sigma}}\mathrm{P}^\lambda\left(J,\mathbf{1}_{I'}\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&= \sum_{I\in\mathcal{C}_F}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{\substack{J\in\mathcal{C}_F^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I \text{ and } J\subset I_0}} \frac{\left|\widehat{f}(I)\right|}{\sqrt{|I_\theta|_\sigma}}\mathrm{P}^\lambda\left(J,\mathbf{1}_{I_\theta}\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&= \sum_{t=r}^\infty\sum_{I\in\mathcal{C}_F}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{\substack{J\in\mathcal{C}_F^{\tau-\text{shift}} \text{ and } \ell(J)=2^{-t}\ell(I)\\ J\Subset_{\rho,\varepsilon} I \text{ and } J\subset I_0}} \frac{\left|\widehat{f}(I)\right|}{\sqrt{|I_\theta|_\sigma}}\mathrm{P}^\lambda\left(J,\mathbf{1}_{I_\theta}\sigma\right)\ell(J)^{-s}\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&\lesssim \sum_{t=r}^\infty\sum_{I\in\mathcal{C}_F}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{\substack{J\in\mathcal{C}_F^{\tau-\text{shift}} \text{ and } \ell(J)=2^{-t}\ell(I)\\ J\Subset_{\rho,\varepsilon} I \text{ and } J\subset I_0}} \frac{\left|\widehat{f}(I)\right|}{\sqrt{|I_\theta|_\sigma}}\left\{\left(2^{-t}\right)^{1-\varepsilon(n+1-\lambda)-s}\ell(I)^s\,\mathrm{P}^\lambda\left(I_0,\mathbf{1}_{I_\theta}\sigma\right)\right\} \\
&\quad\times\sqrt{|J|_\omega}\left\|\triangle^\omega_{J;\kappa} g\right\|_{W^{-s}_{\text{dyad}}(\omega)} \\
&= \sum_{I\in\mathcal{C}_F}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{t=r}^\infty A\left(I,I_0,I_\theta,t\right),
\end{aligned}$$

where

$$\begin{aligned}
&A\left(I,I_0,I_\theta,t\right) \\
&= \left(2^{-t}\right)^{1-\varepsilon(n+1-\lambda)-s}\frac{\ell(I)^s\left|\widehat{f}(I)\right|}{\sqrt{|I'|_\sigma}}\mathrm{P}^\lambda\left(I_0,\mathbf{1}_{I_\theta}\sigma\right)
\end{aligned}$$

$$\times \sum_{\substack{J \in \mathcal{C}_F^{\boldsymbol{\sigma}-\text{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \Subset_{\rho,\varepsilon} I \text{ and } J \subset I_0}} \sqrt{|J|_\omega}\,\ell(J)^s \left\|\triangle_{J;\kappa}^\omega g\right\|_{W_{\text{dyad}}^{-s}(\omega)}.$$

Now recall that the case $s = 0$ of the following estimate was proved in [17, see from the bottom of page 120 to the top of page 122],

$$\left|\sum_{I \in \mathcal{C}_F} \sum_{\substack{I_0, I_\theta \in \mathfrak{C}_{\mathcal{D}}(I) \\ I_0 \neq I_\theta}} \sum_{s=r}^\infty A(I, I_0, I_\theta, s)\right| \lesssim \sqrt{A_2^\lambda}\,\left\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\right\|_{W_{\text{dyad}}^s(\sigma)} \left\|\mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^\omega g\right\|_{W_{\text{dyad}}^{-s}(\omega)},$$

where the quantity $A(I, I_0, I_\theta, t)$ was defined there with $s = 0$ (in our notation) by

$$\left(2^{-t}\right)^{1-\varepsilon(n+1-\lambda)} \left|E_{I_\theta}^\sigma \triangle_{I;1}^\sigma f\right| \mathrm{P}^\lambda\left(I_0, \mathbf{1}_{I_\theta}\sigma\right) \sum_{\substack{J \in \mathcal{C}_F^{\tau-\text{shift}} \text{ and } \ell(J)=2^{-s}\ell(I) \\ J \Subset_{\rho,\varepsilon} I \text{ and } J \subset I_0}} \sqrt{|J|_\omega}\,\left\|\triangle_{J;\kappa}^\omega g\right\|_{L^2(\omega)}.$$

When $\sigma$ is doubling, the reader can check that $\left|E_{I_\theta}^\sigma \triangle_{I;1}^\sigma f\right| \approx \frac{|\widehat{f}(I)|}{\sqrt{|I_\theta|_\sigma}}$, and then that the proof in [17, see from the bottom of page 120 to the top of page 122] applies almost verbatim to our situation when $|\varepsilon(n+1-\lambda)+s| < 1$. This proves the required bound for the neighbour form,

$$\left|\mathsf{B}_{\text{neighbour};\kappa}^F(f, g)\right| \sqrt{A_2^\lambda}\,\left\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\right\|_{W_{\text{dyad}}^s(\sigma)} \left\|\mathsf{P}_{\mathcal{C}_F^{\tau-\text{shift}}}^\omega g\right\|_{W_{\text{dyad}}^{-s}(\omega)}, \quad |s| < 1, \tag{6.10}$$

since we can always take $\varepsilon(n+1-\lambda)$ as small as we wish.

### 6.4 The stopping form

To bound the stopping form following the argument in [1], we only need the $\kappa$-pivotal constant $\mathcal{V}_2^{\lambda,\kappa}$, not the strong $\kappa$-pivotal constant $\mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}$, together with the argument for the Haar stopping form due to Nazarov, Treil and Volberg. Nevertheless, we will use the strong constant $\mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}$ below for convenience. Recall that

$$\left|\widehat{f}(I)\right| Q_{I';\kappa} = \mathbb{E}_{I';\kappa}^\sigma f - \mathbf{1}_{I'} \mathbb{E}_{I;\kappa}^\sigma f.$$

We begin the proof by pigeonholing the ratio of side lengths of $I$ and $J$ in the stopping form:

$$\begin{aligned}
\mathsf{B}_{\text{stop};\kappa}^F(f, g) &\equiv \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\text{shift}} \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left|\widehat{f}(I)\right| \left\langle Q_{I';\kappa} T_\sigma^\lambda \mathbf{1}_{F \setminus I'}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&= \sum_{t=0}^\infty \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\text{shift}} \text{and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left|\widehat{f}(I)\right| \left\langle Q_{I';\kappa} T_\sigma^\lambda \mathbf{1}_{F \setminus I'}, \triangle_{J;\kappa}^\omega g\right\rangle_\omega \\
&\equiv \sum_{s=0}^\infty \mathsf{B}_{\text{stop};\kappa,t}^F(f, g)
\end{aligned}$$

where using Lemma 26, we have $\left\| \left| \triangle_{J;\kappa}^{\omega} g \right| \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)} \lesssim \left\| \triangle_{J;\kappa}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$, and so

$$\left| \mathsf{B}_{\mathrm{stop};\kappa,t}^{F}(f,g) \right| \leq \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left| \widehat{f}(I) \right| \left\| Q_{I';\kappa} \right\|_{L^\infty(\omega)} \left| \left\langle T_\sigma^\lambda \mathbf{1}_{F \setminus I'}, \triangle_{J;\kappa}^{\omega} g \right\rangle_\omega \right|$$

$$\lesssim \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left| \widehat{f}(I) \right| \frac{1}{\sqrt{|I'|_\sigma}} \mathrm{P}_\kappa^\lambda \left(J, \mathbf{1}_{F \setminus I'} \sigma\right) \ell(J)^{-s} \sqrt{|J|_\omega} \left\| \triangle_{J;\kappa}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$$

$$\leq \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \ell(I)^{-s} \left| \widehat{f}(I) \right| \frac{1}{\sqrt{|I'|_\sigma}} \left(2^{-t}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)-s}$$

$$\times \mathrm{P}_\kappa^\lambda \left(I, \mathbf{1}_{F \setminus I'} \sigma\right) \sqrt{|J|_\omega} \left\| \triangle_{J;\kappa}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$$

$$\lesssim \left(2^{-t}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)-s} \sqrt{\sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \ell(I)^{-2s} \left| \widehat{f}(I) \right|^2 \frac{1}{|I'|_\sigma} \mathrm{P}_\kappa^\lambda \left(I, \mathbf{1}_{F \setminus I'} \sigma\right)^2 |J|_\omega}$$

$$\times \sqrt{\sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left\| \triangle_{J}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^2}.$$

Now we note that

$$\sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-s}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \left\| \triangle_{J}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^2$$

$$\leq \sum_{J \in \mathcal{C}_F^{\tau-\mathrm{shift}}} \left\| \triangle_{J}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^2 = \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}^2$$

and use the stopping control bound $\mathrm{P}_\kappa^\lambda (I, \mathbf{1}_F \sigma)^2 |I|_\omega \leq \Gamma |I|_\sigma \lesssim 2\mathcal{V}_{2,\varepsilon}^{\lambda}(\sigma,\omega) \left|I'\right|_\sigma$ in the corona $\mathcal{C}_F$, to obtain

$$\left| \mathsf{B}_{\mathrm{stop};\kappa,t}^{F}(f,g) \right| \lesssim \left(2^{-t(1-s)}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)} \mathcal{V}_{2,\varepsilon}^{\lambda}(\sigma,\omega)$$

$$\times \sqrt{\sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I)} \sum_{\substack{J \in \mathcal{C}_F^{\tau-\mathrm{shift}} \text{ and } \ell(J)=2^{-t}\ell(I) \\ J \subset I' \text{ and } J \Subset_{\rho,\varepsilon} I}} \ell(I)^{-2s} \left| \widehat{f}(I) \right|^2 \frac{|J|_\sigma}{|I'|_\sigma}} \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$$

$$\lesssim \left(2^{-t(1-s)}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)} \mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}(\sigma,\omega) \sqrt{\sum_{I \in \mathcal{C}_F} \ell(I)^{-2s} \left| \widehat{f}(I) \right|^2} \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$$

$$\lesssim \left(2^{-t}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)-s} \mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}(\sigma,\omega) \left\| \mathsf{P}_{\mathcal{C}_F}^{\sigma} f \right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.$$

Finally then we sum in $t$ to obtain

$$\left| \mathsf{B}_{\mathrm{stop};\kappa}^{F}(f,g) \right| \leq \mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}(\sigma,\omega) \sum_{t=0}^{\infty} \left| \mathsf{B}_{\mathrm{stop};\kappa,t}^{F}(f,g) \right| \tag{6.11}$$

$$\lesssim \mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}(\sigma,\omega) \sum_{t=0}^{\infty} \left(2^{-t}\right)^{\kappa-\varepsilon(n+\kappa-\lambda)-s} \left\| \mathsf{P}_{\mathcal{C}_F}^{\sigma} f \right\|_{W_{\mathrm{dyad}}^{s}(\sigma)} \left\| \mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega} g \right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}$$

$$\lesssim \mathcal{V}_{2,\varepsilon}^{\lambda,\kappa}(\sigma,\omega)\left\|\mathsf{P}_{\mathcal{C}_F}^{\sigma}f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_F^{\tau-\mathrm{shift}}}^{\omega}g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)},$$

if we take $0<\varepsilon<\frac{\kappa+s}{n+\kappa-\lambda}$, which can always be done for $\kappa$ large if $|s|<1$.

## 7 Conclusion of the proofs

Collecting all the estimates proved above, namely (5.3), (5.10), (5.4), (5.13), Lemma 36, (5.17), (6.5), (6.9), (6.10) and ( 6.11), we obtain just as in [1] that for any dyadic grid $\mathcal{D}$, and any admissible truncation of $T^{\alpha}$,

$$\begin{aligned}\left|\left\langle T_{\sigma}^{\alpha}\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}f,\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}g\right\rangle_{\omega}\right| &\leq C\left(\mathfrak{TR}_{T^{\alpha}}^{\kappa,s}(\sigma,\omega)+\mathfrak{TR}_{T^{\alpha},*}^{\kappa,-s}(\omega,\sigma)+A_2^{\alpha}(\sigma,\omega)+\varepsilon_3\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)\right)\\ &\quad\times\left\|\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)}\left\|\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}.\end{aligned}$$

Thus for any admissible truncation of $T^{\alpha}$, using the above two theorems, we obtain

$$\begin{aligned}\mathfrak{N}_{T^{\alpha}}(\sigma,\omega) &\leq C\sup_{\mathcal{D}}\frac{\left|\left\langle T_{\sigma}^{\alpha}\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}f,\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}g\right\rangle_{\omega}\right|}{\left\|\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}f\right\|_{W_{\mathrm{dyad}}^{s}(\sigma)}\left\|\mathsf{P}_{\mathrm{good}}^{\mathcal{D}}g\right\|_{W_{\mathrm{dyad}}^{-s}(\omega)}} \qquad (7.1)\\ &\leq C\left(\mathfrak{TR}_{T^{\alpha}}^{\kappa,s}(\sigma,\omega)+\mathfrak{TR}_{T^{\alpha},*}^{\kappa,-s}(\omega,\sigma)+A_2^{\alpha}(\sigma,\omega)\right)+C\varepsilon_3\mathfrak{N}_{T^{\alpha}}(\sigma,\omega).\end{aligned}$$

Our next task is to use the doubling hypothesis to replace the triple $\kappa$-testing constants by the usual cube testing constants, and we follow almost verbatim the argument in [1] for the case $s=0$. Recall that the $\kappa$-cube testing conditions use the $Q$-normalized monomials $m_Q^{\beta}(x)\equiv\mathbf{1}_Q(x)\left(\frac{x-c_Q}{\ell(Q)}\right)^{\beta}$, for which we have $\left\|m_Q^{\beta}\right\|_{L^{\infty}}\approx 1$.

**Theorem 46** *Suppose that $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}^n$, with $\sigma$ doubling, and let $\kappa\in\mathbb{N}$. If $T^{\alpha}$ is a bounded operator from $W^s(\sigma)$ to $W^s(\omega)$, then for every $0<\varepsilon_2<1$, there is a positive constant $C(\kappa,\varepsilon_2)$ such that*

$$\mathfrak{TR}_{T^{\alpha}}^{\kappa,s}(\sigma,\omega)\leq C(\kappa,\varepsilon_2)\left[\mathfrak{T}_{T^{\alpha}}^{s}(\sigma,\omega)+\sqrt{A_2^{\alpha}(\sigma,\omega)}\right]+\varepsilon_2\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)\quad\kappa\geq 1,$$

*and where the constants $C(\kappa,\varepsilon_2)$ depend only on $\kappa$ and $\varepsilon$, and not on the operator norm $\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)$.*

***Proof*** Fix a dyadic cube $I$. If $P$ is an $I$-normalized polynomial of degree less than $\kappa$ on the cube $I$, i.e. $\|P\|_{L^{\infty}}\approx 1$, then we can approximate $P$ by a step function

$$S\equiv\sum_{I'\in\mathfrak{C}_{\mathcal{D}}^{(m)}(I)}a_{I'}\mathbf{1}_{I'},$$

satisfying

$$\|S-\mathbf{1}_I P\|_{L^{\infty}(\sigma)}<\frac{\varepsilon_2}{2}$$

provided we take $m\geq 1$ sufficiently large depending on $n$ and $\kappa$, but independent of the cube $I$. Then using the above lemma with $C2^{\frac{m}{2}}\varepsilon_1\leq\frac{\varepsilon_2}{2}$, and the estimate $|a_{I'}|\lesssim\|P\|_{L^{\infty}}\lesssim 1$, we have

$$\begin{aligned}
\left\| T_\sigma^\alpha \mathbf{1}_I P \right\|_{W_{\mathrm{dyad}}^s(\sigma)} &\leq \left\| \sum_{I' \in \mathfrak{C}_{\mathcal{D}}^{(m)}(I)} a_{I'} T_\sigma^\alpha \mathbf{1}_{I'} \right\|_{W_{\mathrm{dyad}}^s(\sigma)} \\
&\quad + \left\| T_\sigma^\alpha \left[ (S - P) \mathbf{1}_I \right] \right\|_{W_{\mathrm{dyad}}^s(\sigma)} \sqrt{\int_{3I} \left| T_\sigma^\alpha \left[ (S - P) \mathbf{1}_I \right] \right|^2 d\omega} \\
&\leq C \sum_{I' \in \mathfrak{C}_{\mathcal{D}}^{(m)}(I)} |a_{I'}| \left\| T_\sigma^\alpha \mathbf{1}_{I'} \right\|_{W_{\mathrm{dyad}}^s(\sigma)} + \frac{\varepsilon_2}{2} \mathfrak{N}_{T^\alpha}(\sigma, \omega) \sqrt{|I|_\sigma} \\
&\leq C \sum_{I' \in \mathfrak{C}_{\mathcal{D}}^{(m)}(Q)} \mathfrak{T}_{T^\alpha}^s(\sigma, \omega) \sqrt{|I'|_\sigma} + \frac{\varepsilon_2}{2} \mathfrak{N}_{T^\alpha}(\sigma, \omega) \sqrt{|I|_\sigma} \\
&\leq C \left\{ \mathfrak{T}_{T^\alpha}^s(\sigma, \omega) + \frac{\varepsilon_2}{2} \mathfrak{N}_{T^\alpha}(\sigma, \omega) \right\} \sqrt{|I|_\sigma}.
\end{aligned}$$

□

Combining this with (7.1) we obtain

$$\mathfrak{N}_{T^\alpha}(\sigma, \omega) \leq C \left( \mathfrak{T}_{T^\alpha}^s(\sigma, \omega) + \mathfrak{T}_{T^{\alpha,*}}^{-s}(\omega, \sigma) + \sqrt{A_2^\alpha(\sigma, \omega)} \right) + C(\varepsilon_2 + \varepsilon_3) \mathfrak{N}_{T^\alpha}(\sigma, \omega).$$

Since $\mathfrak{N}_{T^\alpha}(\sigma, \omega) < \infty$ for each truncation, we may absorb the final summand on the right into the left hand side provided $C(\varepsilon_2 + \varepsilon_3) < \frac{1}{2}$, to obtain

$$\mathfrak{N}_{T^\alpha}(\sigma, \omega) \lesssim \mathfrak{T}_{T^\alpha}^s(\sigma, \omega) + \mathfrak{T}_{T^{\alpha,*}}^{-s}(\omega, \sigma) + \sqrt{A_2^\alpha(\sigma, \omega)}.$$

By the definition of boundedness of $T^\alpha$ in (34), this completes the proof of Theorem 2.

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