



On the degree of polynomial subgroup growth of nilpotent groups

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Abstract

Let N be a finitely generated nilpotent group. The subgroup zeta function $\zeta_N^{\leq}(s)$ and the normal zeta function $\zeta_N^{\triangleleft}(s)$ of N are Dirichlet series enumerating the finite index subgroups or the finite index normal subgroups of N . We present results about their abscissae of convergence α_N^{\leq} and α_N^{\triangleleft} , also known as the degrees of polynomial subgroup growth and polynomial normal subgroup growth of N , respectively. We first prove some upper bounds for the functions $N \mapsto \alpha_N^{\leq}$ and $N \mapsto \alpha_N^{\triangleleft}$ when restricted to the class of torsion-free nilpotent groups of a fixed Hirsch length. We then show that if two finitely generated nilpotent groups have isomorphic \mathbb{C} -Mal'cev completions, then their subgroup (resp. normal) zeta functions have the same abscissa of convergence. This follows, via the Mal'cev correspondence, from a similar result that we establish for zeta functions of rings. This result is obtained by proving that the abscissa of convergence of an Euler product of certain Igusa-type local zeta functions introduced by du Sautoy and Grunewald remains invariant under base change. We also apply this methodology to formulate and prove a version of our result about nilpotent groups for virtually nilpotent groups. As a side application of our result about zeta functions of rings, we present a result concerning the distribution of orders in number fields.

Keywords Subgroup growth · Zeta functions of group and rings

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1 Introduction

Let G be a finitely generated group and let

$$a_n^{\leq}(G) := |\{H \leq G : [G : H] = n\}|, \quad a_n^{\triangleleft}(G) := |\{H \triangleleft G : [G : H] = n\}|$$

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be the number of subgroups or normal subgroups of index n in G . These numbers are finite and the associated Dirichlet series

$$\zeta_G^{\leq}(s) := \sum_{n=1}^{\infty} \frac{a_n^{\leq}(G)}{n^s} = \sum_{H \leq_f G} [G : H]^{-s}, \quad \zeta_G^{\triangleleft}(s) := \sum_{n=1}^{\infty} \frac{a_n^{\triangleleft}(G)}{n^s} = \sum_{H \triangleleft_f G} [G : H]^{-s},$$

are called *the subgroup zeta function* and *the normal zeta function* of G , respectively. The symbol $H \leq_f G$ (resp. $H \triangleleft_f G$) indicates that the summation is over all subgroups (resp. normal subgroups) H of finite index in G . We write $\zeta_G^*(s)$ when we intend to address both types of zeta functions simultaneously. These zeta functions were introduced by Grunewald, Segal and Smith in the landmark paper [12].

Example 1.1 Let $h \in \mathbb{N}$ and let \mathbb{Z}^h be the free abelian group of rank h . Then

$$\zeta_{\mathbb{Z}^h}^*(s) = \zeta(s)\zeta(s - 1) \cdots \zeta(s - h + 1),$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function (cf. [12, Proposition 1.1] or [18, Chapter15]).

Observe that the subgroup zeta function of \mathbb{Z}^h converges on a non-empty region of the complex plane, namely the region $\text{Re}(s) > h$. This is a characteristic property of groups of *polynomial subgroup growth* (PSG), i.e. groups G for which the function $n \mapsto \sum_{i=1}^n a_i^{\leq}(G)$ is bounded by a polynomial function in n . More generally, for $* \in \{\leq, \triangleleft\}$ we set

$$\alpha_G^* := \inf \left\{ \alpha \mid \exists c > 0 \forall n : \sum_{i=1}^n a_i^*(G) \leq cn^\alpha \right\},$$

where conventionally $\inf \emptyset = \infty$. When $\alpha_G^* < \infty$, we call this number *the degree of polynomial (normal) subgroup growth of G* . If $a_n^*(G) \neq 0$ for infinitely many n , then α_G^* coincides with the abscissa of convergence of $\zeta_G^*(s)$, that is, $\zeta_G^*(s)$ defines an analytic function on the region $\text{Re}(s) > \alpha_G^*$ and diverges for any s with $\text{Re}(s) < \alpha_G^*$. A related invariant for groups of polynomial subgroup growth (called *the degree* of the group) was introduced and studied by Shalev; see for instance the influential paper [27]. For other types of growth we refer to [18].

The finitely generated groups of polynomial subgroup growth have been characterized algebraically by Lubotzky, Mann and Segal in [19]. We recall this characterization. Note first that the subgroup growth (or the normal subgroup growth) of a group G is the same as that of the quotient group $G/R(G)$, where $R(G) := \bigcap_{N \triangleleft_f G} N$ is the finite residual of G , so there is no loss of generality in assuming that the groups under consideration are residually finite, i.e. the finite residual is trivial. It is proven in [19] that a finitely generated residually finite group has polynomial subgroup growth if and only if it is virtually soluble of finite rank.

We shall only deal with groups of polynomial subgroup growth that are nilpotent or virtually nilpotent. For these groups we will make some observations about the behavior of the function $G \mapsto \alpha_G^*$. These observations are mainly corollaries of properties of certain Igusa-type zeta functions called *cone integrals* that arise in the analysis of $\zeta_G^*(s)$.

1.1 Zeta functions of \mathfrak{T} -groups

Since their introduction, zeta functions of groups have been investigated mainly for \mathfrak{T} -groups, i.e. finitely generated torsion-free nilpotent groups. If N is a \mathfrak{T} -group, then there is an Euler

product decomposition

$$\zeta_N^*(s) = \prod_{p \text{ prime}} \zeta_{N,p}^*(s) = \prod_{p \text{ prime}} \zeta_{\widehat{N}_p}^*(s), \tag{1.1}$$

where $\zeta_{N,p}^*(s) := \sum_{k=0}^\infty a_{p^k}^*(N)p^{-ks}$ is the local factor of $\zeta_N^*(s)$ at p , and \widehat{N}_p denotes the pro- p completion of N . In addition, $\zeta_{N,p}^*(s)$ is a rational function in p^{-s} [12].

The study of zeta functions of \mathfrak{T} -groups led to the consideration of zeta functions of rings, which we recall below. The book [10] collects comprehensive information about the first stage of the theory of zeta functions of groups and rings. The survey [33] exposes new developments. Let us also mention [4, 16, 23] and Rossmann’s computer-algebra package ZETA [24] (which effectively computes, among other things, many (normal) zeta functions of nilpotent groups of moderate Hirsch length) just to illustrate the current activity on the subject. We summarize some outstanding analytic properties of zeta functions of \mathfrak{T} -groups obtained by du Sautoy and Grunewald.

Theorem 1.2 [8] *Let N be an infinite \mathfrak{T} -group.*

- (1) α_N^* is a rational number and there exists $\delta > 0$ such that $\zeta_N^*(s)$ can be meromorphically continued to the region $\text{Re}(s) > \alpha_N^* - \delta$. The continued function is holomorphic on the line $\text{Re}(s) = \alpha_N^*$ except for a pole at $s = \alpha_N^*$.
- (2) If b_N^* is the order of the pole of the continued function and $g_N^*(s)$ denotes the continuation of $(s - \alpha_N^*)^{b_N^*} \zeta_N^*(s)$, then

$$\sum_{i=1}^n a_i^*(N) \sim \frac{g_N^*(\alpha_N^*)}{\alpha_N^* \cdot (b_N^* - 1)!} \cdot n^{\alpha_N^*} (\log n)^{b_N^* - 1}$$

where we write $f(n) \sim g(n)$ if $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$.

One natural problem is to relate $\alpha_N^*, b_N^* \in \mathbb{R}$ to structural information about N . This was posed as Problem 1.1 in [9] and remains open in general. Example 1.1 shows that if N is a free abelian group of rank $h \geq 1$, then $\alpha_N^* = h$ and $b_N^* = 1$. However, if N is a non-abelian \mathfrak{T} -group, the computation of α_N^* is already a challenge. The values of α_N^* and b_N^* for various \mathfrak{T} -groups are collected in [10]. In [22, Section 6.2], Rossmann computes α_N^{\triangleleft} when N is a \mathfrak{T} -group of maximal nilpotency class. As an example of computation of α_N^{\triangleleft} , we mention the following remarkable calculation by Voll, which was obtained by purely combinatorial methods.

Example 1.3 [32] *Let $F = F_{2,d}$ be the free nilpotent group of class 2 on d generators ($d \geq 2$). Then*

$$\alpha_F^{\triangleleft} = \max \left\{ d, \frac{\binom{d}{2} - j)(d + j) + 1}{\binom{d+1}{2} - j} \mid j = 1, \dots, \binom{d}{2} - 1 \right\}.$$

Remark 1.4 The proof of Theorem 1.2(1) given in [8] expresses α_N^* in terms of some numerical data associated to a principalization of an ideal of polynomials over \mathbb{Q} obtained from N (see Sect. 2 for a review of this). Even though nowadays there are algorithmic resolutions of singularities and principalizations of ideals, they are impractical in this context since the ideals obtained from N are very complicated (several polynomials in several variables), even for quite simple N .

We mentioned that if N is the free abelian group of rank h , then $\alpha_N^{\leq} = \alpha_N^{\triangleleft} = h$. An extension of the notion of rank to the class of \mathfrak{T} -group, or more generally, to the class of polycyclic groups, is the notion of Hirsch length. If N is a polycyclic group, its Hirsch length is the number of infinite factors in a subnormal series with cyclic factors, and it is denoted by $h(N)$. Now, if N is a non-abelian \mathfrak{T} -group, then a simple argument shows that $h(N^{ab}) \leq \alpha_N^{\triangleleft} \leq \alpha_N^{\leq} \leq h(N)$, where N^{ab} denotes the abelianization of N (see [12, Proposition 1]). There are better bounds for α_N^{\triangleleft} when N is a non-abelian \mathfrak{T} -group of nilpotency class 2; cf. [21] and [12, Proposition 6.3]. We also mention the lower bound $\frac{1}{6}h(N) \leq \alpha_N^{\leq}$ ([18, Theorem 5.6.6]), which actually holds for a larger class of groups including the polycyclic groups. All these results are useful for the following problem: given $h > 2$, describe the set S_h^* of possible values for α_N^* as N ranges over the non-abelian \mathfrak{T} -groups of Hirsch length h . According to [9, Proposition 1.1] and Theorem 1.2(a), S_h^* is a finite subset of $[2, h] \cap \mathbb{Q}$. In particular, one can ask what is the maximum of S_h^* . Our first result gives a partial answer to this question.

Theorem A *Let N be a non-abelian \mathfrak{T} -group of Hirsch length h and nilpotency class c . Then $\alpha_N^{\leq} \leq h - \frac{1}{2}$ if $c = 2$ and $\alpha_N^{\leq} \leq h - \frac{1}{c-1}$ if $c > 2$. Also $\alpha_N^{\triangleleft} \leq h - 1$.*

In particular, given $h \in \mathbb{N}$, the only \mathfrak{T} -group N of Hirsch length h with $\alpha_N^{\leq} = h$ is the abelian group \mathbb{Z}^h .

Remark 1.5 The bound $\alpha_N^{\triangleleft} \leq h - 1$ is optimal. Indeed, choose integers $r \geq 0$ and $m > 0$ such that $2m + 1 + r = h$, and let $G(m, r)$ be the product of \mathbb{Z}^r with a central product of m copies of the discrete Heisenberg group $H_3(\mathbb{Z})$. Then $h(G(m, r)^{ab}) = h - 1 \leq \alpha_{G(m,r)}^{\triangleleft} \leq h - 1$, which gives $\alpha_{G(m,r)}^{\triangleleft} = h - 1$. In contrast, by our method in the proof of Theorem A, it seems that our bound for α_N^{\leq} is far from being optimal. The few examples where α_N^{\leq} has been computed show that $\alpha_N^{\leq} \leq h(N) - 1$ if N is a non-abelian \mathfrak{T} -group.

We return with the notation of Theorem 1.2. In trying to understand which structural data of N is reflected in α_N^* and b_N^* it is natural to ask, given two \mathfrak{T} -groups N_1 and N_2 , when $\alpha_{N_1}^* = \alpha_{N_2}^*$, and if this is so, when $b_{N_1}^* = b_{N_2}^*$. According to [12, Proposition 3], if N_1 and N_2 are commensurable \mathfrak{T} -groups (i.e. there exist finite index subgroups $H_1 \leq N_1$ and $H_2 \leq N_2$ such that $H_1 \cong H_2$), then $\alpha_{N_1}^* = \alpha_{N_2}^*$, and moreover $b_{N_1}^* = b_{N_2}^*$ (cf. Proposition 3.7). We rephrase this fact. Recall first that a \mathfrak{T} -group can be embedded as an arithmetic group of a uniquely determined unipotent group scheme over \mathbb{Q} . In addition, two \mathfrak{T} -groups are commensurable if and only if they are isomorphic to arithmetic groups of the same unipotent group scheme over \mathbb{Q} . Thus, the fact that α_N^* and b_N^* are commensurability-invariant can be restated as follows: *If N_1 and N_2 are arithmetic subgroups of the same unipotent group scheme over \mathbb{Q} , then $\alpha_{N_1}^* = \alpha_{N_2}^*$ and $b_{N_1}^* = b_{N_2}^*$.* Our next result is a partial generalization of this property.

Theorem B *Let \mathfrak{N}_1 and \mathfrak{N}_2 be unipotent group schemes over \mathbb{Q} , and let N_i be an arithmetic subgroup of \mathfrak{N}_i for $i = 1, 2$. If \mathfrak{N}_1 and \mathfrak{N}_2 are isomorphic after base change with \mathbb{C} , then $\alpha_{N_1}^* = \alpha_{N_2}^*$ for $* \in \{\leq, \triangleleft\}$.*

Remark 1.6 In contrast, we expect that $b_{N_1}^* = b_{N_2}^*$ does not hold in general. Indeed, Remark 1.9 below shows that this equality fails for zeta functions of rings in general. However, the counter-example that we will show does not belong to the realm of nilpotent Lie rings (or groups).

In other words, Theorem B says that the number α_N^* associated to a \mathfrak{T} -group N is in fact an invariant of the C-Mal'cev completion of N , or simply that it is a geometric invariant. Similar

conclusions in spirit can be deduced from the main result of [7] for the representation zeta function of \mathfrak{T} -groups, and from the main results of [2, 17] for the degree of polynomial representation growth and for the subgroup growth rate of arithmetic groups in simply connected absolutely simple group schemes over \mathbb{Q} . Observe finally that a result similar to Theorem B holds for the degree of polynomial word growth by the formula of Bass-Guivarc’h [3].

Remark 1.7 There is a known classification of nilpotent Lie algebras in dimension ≤ 7 over \mathbb{C} (and not over \mathbb{Q}). Hence Theorem B could be used to completely determine, possibly with computer help, the number α_N^* for all τ -groups N of Hirsch length at most 7.

1.2 Zeta functions of rings

Theorem B is obtained as a consequence of a more general result, namely Theorem C, which employs the concept of zeta functions of rings introduced in [12, Sect. 3]. By a *ring* we shall mean an abelian group L with a bilinear map $L \times L \rightarrow L$ called *multiplication*, e.g. Lie rings and the commutative rings with identity. A *subring* of L is an abelian subgroup A that is closed under multiplication. To allow further applications, in case that L is commutative ring with identity 1, we shall also require (as usual) that $1 \in A$. A two-sided ideal of L will be simply called an *ideal*.

Let L be a ring additively isomorphic to $(\mathbb{Z}^h, +)$ for some $h \geq 1$. For each positive integer n , let $a_n^{\leq}(L)$ and $a_n^{\triangleleft}(L)$ denote the number of subrings or ideals of index n in L . The *subring and the ideal zeta functions* of L are the Dirichlet series $\zeta_L^{\leq}(s) := \sum_{n=1}^{\infty} a_n^{\leq}(L)n^{-s}$ and $\zeta_L^{\triangleleft}(s) := \sum_{n=1}^{\infty} a_n^{\triangleleft}(L)n^{-s}$, respectively. They have a factorization as an Euler product

$$\zeta_L^*(s) = \prod_{p \text{ prime}} \zeta_{L,p}^*(s),$$

where $\zeta_{L,p}^*(s) := \sum_{k=0}^{\infty} a_{p^k}^*(L)p^{-ks}$.

The zeta functions of an arbitrary ring of additive rank 2 were computed in [28, Chapter 7]. The subring zeta function of an arbitrary Lie ring of additive rank 3 was computed in [14]. Zeta functions of nilpotent Lie rings are essentially the same as zeta functions of \mathfrak{T} -groups: via the Mal’cev correspondence one can associate with a \mathfrak{T} -group N of Hirsch length h a nilpotent Lie ring L of additive rank h (and viceversa), and it holds that $\zeta_L^*(s)$ and $\zeta_N^*(s)$ have the same local factors for almost all primes p (see Proposition 3.7). Some information about zeta functions of soluble Lie rings of higher rank can be found in [10, Chapter 3].

Theorem 1.2, formulated there for zeta functions of \mathfrak{T} -groups, was also proved for zeta functions of rings in [8]. In particular, one can consider the pair (α_L^*, b_L^*) , where α_L^* is the abscissa of convergence of $\zeta_L^*(s)$ and b_L^* is the order of the pole of the continued function at $s = \alpha_L^*$. Our next result is:

Theorem C *Let L_1 and L_2 be two rings additively isomorphic to $(\mathbb{Z}^h, +)$. If $L_1 \otimes_{\mathbb{Z}} \mathbb{C} \cong L_2 \otimes_{\mathbb{Z}} \mathbb{C}$ as \mathbb{C} -algebras, then $\alpha_{L_1}^* = \alpha_{L_2}^*$ for $* \in \{\leq, \triangleleft\}$.*

Example 1.8 Let $\mathcal{H} := \langle x, y, z : [x, y] = z, [x, z] = [y, z] = 0 \rangle$ be the discrete Heisenberg Lie ring. For each square free integer k we consider $L_k := \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Q}(\sqrt{k})}$, where $\mathcal{O}_{\mathbb{Q}(\sqrt{k})}$ denotes the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{k})$. This is a 2-step nilpotent Lie ring of additive rank 6. If $k \neq k'$, then $L_k \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_{k'} \otimes_{\mathbb{Z}} \mathbb{Q}$ are not isomorphic as \mathbb{Q} -Lie algebras (cf. [15, Proposition 3.2]). However, $L_k \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{H}^2 \otimes_{\mathbb{Z}} \mathbb{C}$ as \mathbb{C} -Lie algebras (where $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$) and hence $\alpha_{L_k}^* = \alpha_{\mathcal{H}^2}^*$ for all k as above. The zeta functions of \mathcal{H}^2 were

computed in [12, Proposition 8.11] and [31], and it holds that $\alpha_{\mathcal{H}^2}^* = 4$. We can now use Theorem C to conclude that $\alpha_{L^k}^* = 4$ for all square-free integer k .

We remark that a formula for $\zeta_{L^k}^\triangleleft(s)$ was given in [12, Corollary 8.2]. More generally, for any number field K , the local factors of $\zeta_{\mathcal{H}^\triangleleft \otimes_{\mathbb{Z}} \mathcal{O}_K}(s)$ at almost all primes p were computed in [25, 26]. A further analysis on the Euler product of these local factors is required to compute the abscissa of convergence $\alpha_{\mathcal{H}^\triangleleft \otimes_{\mathbb{Z}} \mathcal{O}_K}^\triangleleft$.

As a generalization of the above observation we have the following:

Theorem D *Let L be a ring additively isomorphic to $(\mathbb{Z}^h, +)$, $K \supset \mathbb{Q}$ a number field of degree d and \mathcal{O} its ring of integers. Let $L_{\mathcal{O}} := L \otimes_{\mathbb{Z}} \mathcal{O}$. Then $\alpha_{L_{\mathcal{O}}}^* = \alpha_{L^d}^*$, where L^d denotes the product of d copies of the ring L .*

Proof Note that $L_{\mathcal{O}} \otimes_{\mathbb{Z}} \mathbb{C} = (L \otimes_{\mathbb{Z}} \mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C} \cong L \otimes_{\mathbb{Z}} (\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}) \cong L \otimes_{\mathbb{Z}} \mathbb{C}^d \cong L^d \otimes_{\mathbb{Z}} \mathbb{C}$ as \mathbb{C} -algebras. Hence $\alpha_{L_{\mathcal{O}}}^* = \alpha_{L^d}^*$ by Theorem C. \square

Remark 1.9 Let $L = \mathbb{Z}$, viewed as commutative ring with identity. Then, $L_{\mathbb{Z}[i]} := \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ is the ring of Gaussian integers and $\zeta_{L_{\mathbb{Z}[i]}}^\triangleleft(s)$ is the Dedekind zeta function $\zeta_{\mathbb{Q}(i)}(s)$ of $\mathbb{Q}(i)$. Hence $\alpha_{L_{\mathbb{Z}[i]}}^\triangleleft = 1$ and $b_{L_{\mathbb{Z}[i]}}^\triangleleft = 1$. Note also that the product ring L^2 has ideal zeta function $(\zeta(s))^2$, whence $\alpha_{L^2}^\triangleleft = 1$ and $b_{L^2}^\triangleleft = 2$. Since $L_{\mathbb{Z}[i]} \otimes_{\mathbb{Z}} \mathbb{C} \cong L^2 \otimes_{\mathbb{Z}} \mathbb{C}$, we deduce that Theorem C does not hold for b_L^* in general.

Remark 1.10 The computation of $\alpha_{L^d}^\triangleleft$ is in general a quite difficult task. For example, let $\mathbb{Z}_{\text{ring}}^d$ be the ring that is a product of d copies of the ring of integers \mathbb{Z} . Then $\alpha_{\mathbb{Z}_{\text{ring}}^d}^\triangleleft = 1$ for $d \leq 5$ [13, Theorem 6] while it is unknown for $d > 5$. The computation of $\alpha_{L^d}^\triangleleft$ might be also difficult if L lacks an identity element. For instance, following up Example 1.8 and Corollary D we find that $\alpha_{\mathcal{H}^\triangleleft \otimes_{\mathbb{Z}} \mathcal{O}_K}^\triangleleft = \alpha_{\mathcal{H}^d}^\triangleleft$, where $d = [K : \mathbb{Q}]$. We have $\alpha_{\mathcal{H}^d}^\triangleleft = 2d$ if $d \leq 4$ (see [10, Chapter 1]), however nothing is known for $d > 4$.

Let us give an application of Corollary D to the distribution of orders in number fields. Let K be a number field and let \mathcal{O}_K be its ring of integers. An order is a subring \mathcal{O} of \mathcal{O}_K with identity that is a \mathbb{Z} -module of rank n . Set

$$N_K(n) := |\{\mathcal{O} \subseteq \mathcal{O}_K \mid \text{disc}(\mathcal{O}) \leq n\}|.$$

The asymptotic behavior of $n \mapsto N_K(n)$ was studied in [13]. It was shown, by an application of the results in [8] (the version of Theorem 1.2 for zeta functions of rings), that there exist $C_K \in \mathbb{R}$, $\alpha_K \in \mathbb{Q}$ and $\beta_K \in \mathbb{N}$ such that

$$N_K(n) \sim C_K n^{\alpha_K} (\log n)^{\beta_K - 1}.$$

It was also conjectured that the number α_K only depends on the degree $[K : \mathbb{Q}]$; see [13, Conjecture 1]. We now show that this is a special case of Corollary D.

Theorem E *Let $d := [K : \mathbb{Q}]$, and let $\mathbb{Z}_{\text{ring}}^d$ denote the product of d copies of the ring of integers \mathbb{Z} . Then $\alpha_K = \frac{1}{2} \alpha_{\mathbb{Z}_{\text{ring}}^d}^\triangleleft$. In particular, α_K only depends on the degree $[K : \mathbb{Q}]$.*

Proof Let $\eta_K(s) = \sum_{\mathcal{O} \text{ order}} |\text{disc } \mathcal{O}|^{-s}$. Then $\eta_K(s) = |\text{disc } \mathcal{O}_K|^{-s} \zeta_{\mathcal{O}_K}^\triangleleft(2s)$, where \mathcal{O}_K is seen as a ring with identity. Observe that α_K is the abscissa of convergence of $\eta_K(s)$, hence $\alpha_K = \frac{1}{2} \alpha_{\mathcal{O}_K}^\triangleleft$. Note also that Corollary D with $L = \mathbb{Z}$ (the ring of integers) yields $\alpha_{\mathcal{O}_K}^\triangleleft = \alpha_{\mathbb{Z}_{\text{ring}}^d}^\triangleleft$. Therefore, $\alpha_K = \frac{1}{2} \alpha_{\mathbb{Z}_{\text{ring}}^d}^\triangleleft$. \square

1.3 Methodology

The idea behind the proof of Theorem C is quite simple. If L_1 and L_2 are isomorphic after base change with \mathbb{C} , then $L_1 \otimes_{\mathbb{Z}} K \cong L_2 \otimes_{\mathbb{Z}} K$ for some number field K (Lemma 3.5). Let \mathcal{O} be the ring of integers of K . We associate to the \mathcal{O} -algebra $L_{i\mathcal{O}} := L_i \otimes_{\mathbb{Z}} \mathcal{O}$ the zeta functions $\zeta_{L_{i\mathcal{O}}}^{\leq \mathcal{O}}(s)$ and $\zeta_{L_{i\mathcal{O}}}^{\triangleleft \mathcal{O}}(s)$ enumerating \mathcal{O} -subalgebras or \mathcal{O} -ideals of $L_{i\mathcal{O}}$, and show that $\zeta_{L_i}^*(s)$ and $\zeta_{L_{i\mathcal{O}}}^{\leq \mathcal{O}}(s)$ have the same abscissa of convergence (Corollary 3.3). The proof of this fact makes use of the main tool of the paper: *cone integrals* (see Sect. 2). We show that for each $i = 1, 2$, $\zeta_{L_i}^*(s)$ and $\zeta_{L_{i\mathcal{O}}}^{\leq \mathcal{O}}(s)$ are Euler products of cone integrals with the same cone integral data but over different fields, namely over \mathbb{Q} and over K (Corollary 3.2), and hence they have the same abscissa of convergence. This last assertion follows from Theorem 2.2, which collects several properties about cone integrals. Finally, Theorem 2.2 will also enable us to conclude that $\zeta_{L_{1\mathcal{O}}}^{\leq \mathcal{O}}(s)$ and $\zeta_{L_{2\mathcal{O}}}^{\leq \mathcal{O}}(s)$ have the same abscissa of convergence (Corollary 3.4), from which Theorem C follows.

Theorem 2.2 will be also used to formulate and prove a version of Theorem B for virtually nilpotent groups (Theorem 5.18). This will be possible as the zeta functions of these groups can be expressed as finite sums of series that are Euler products of cone integrals. We omit this version in the introduction and refer the reader to Sect. 5.

1.4 Organization and notation

In Sect. 2, we recall the concept and some important properties of cone integrals over \mathbb{Q} from [8], and we extend them for any number field. We use these results to prove Theorem C in Sect. 3 and to recall how this theorem implies Theorem B. In Sect. 4, we prove Theorem A. This section is, to a large extent, independent from the other ones. Finally, in Sect. 5 we formulate and prove Theorem 5.18, which is the analogous of Theorem B for virtually nilpotent groups.

We write \mathbb{N} for the set $\{1, 2, \dots\}$ and \mathbb{N}_0 for the set $\mathbb{N} \cup \{0\}$. We write $\mathbb{R}_{>0}$ for the set $\{s \in \mathbb{R} : s > 0\}$ and $\mathbb{R}_{\geq 0}$ for the set $\{s \in \mathbb{R} : s \geq 0\}$. The notation $f(n) \sim g(n)$ means that $f(n)/g(n)$ tends to 1 as n tends to infinity.

For a prime p , \mathbb{Z}_p and \mathbb{Q}_p denote the p -adic integers and the p -adic rationals, respectively. For a number field K we denote by \mathcal{O}_K its ring of integers. Given a maximal ideal $\mathfrak{p} \subset \mathcal{O} := \mathcal{O}_K$ we denote by $\mathcal{O}_{\mathfrak{p}}$ and $K_{\mathfrak{p}}$ the \mathfrak{p} -adic completions of \mathcal{O} and K . Given $x \in K_{\mathfrak{p}}$ we denote by $\text{ord}_{\mathfrak{p}}(x) \in \mathbb{Z} \cup \{\infty\}$ its \mathfrak{p} -adic valuation and write $|x|_{\mathfrak{p}} := \mathbf{N}\mathfrak{p}^{-\text{ord}_{\mathfrak{p}}(x)}$ for its \mathfrak{p} -adic norm, where $\mathbf{N}\mathfrak{p} := [\mathcal{O} : \mathfrak{p}]$.

2 A review of cone integrals

Cone integrals are a kind of p -adic integrals which can be seen as a generalization of Igusa local zeta functions. They were introduced and studied by du Sautoy and Grunewald in [8] under the assumption that the base field is \mathbb{Q} . Theorem 2.2 collects the main properties of cone integrals and at the same time it extends them to cone integrals over a general number field. The rest of the section is devoted to explaining how this general formulation follows essentially by the same arguments of [8].

Definition 2.1 Let K be a number field and let m be a positive integer. A finite collection $\mathcal{D} = (f_0, g_0, f_1, g_1, \dots, f_l, g_l)$ of non-zero polynomials in $K[x_1, \dots, x_m]$ is called a *cone*

integral data over K . Let $\mathcal{O} = \mathcal{O}_K$, and let $\mathfrak{p} \subset \mathcal{O}$ be a maximal ideal. Then the integral

$$Z_{\mathcal{D}}(s, \mathfrak{p}) = \int_{\mathcal{M}(\mathcal{D}, \mathfrak{p})} |f_0(\mathbf{x})|_{\mathfrak{p}}^s |g_0(\mathbf{x})|_{\mathfrak{p}} |d\mathbf{x}|_{\mathfrak{p}},$$

where

$$\mathcal{M}(\mathcal{D}, \mathfrak{p}) = \{\mathbf{x} \in \mathcal{O}_{\mathfrak{p}}^m \mid \text{ord}_{\mathfrak{p}}(f_i(\mathbf{x})) \leq \text{ord}_{\mathfrak{p}}(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\},$$

and $|d\mathbf{x}|_{\mathfrak{p}} = |dx_1 \wedge \dots \wedge dx_m|_{\mathfrak{p}}$ is the normalized additive Haar measure on $\mathcal{O}_{\mathfrak{p}}^m$, is called a cone integral over K , with cone integral data \mathcal{D} .

It is easy to see that for each maximal ideal $\mathfrak{p} \subset \mathcal{O}$, the integral $Z_{\mathcal{D}}(s, \mathfrak{p})$ can be expressed as a power series, say $Z_{\mathcal{D}}(s, \mathfrak{p}) = \sum_{i=0}^{\infty} a_{\mathfrak{p},i} \mathbf{N}\mathfrak{p}^{-is}$, where each $a_{\mathfrak{p},i}$ is a non-negative rational number. We associate to \mathcal{D} the Dirichlet series

$$Z_{\mathcal{D}}(s) := \prod_{\substack{\mathfrak{p} \subset \mathcal{O} \text{ maximal} \\ a_{\mathfrak{p},0} \neq 0}} a_{\mathfrak{p},0}^{-1} Z_{\mathcal{D}}(s, \mathfrak{p}),$$

and denote its abscissa of convergence by $\alpha_{\mathcal{D}}$. A function $Z(s)$ such that $Z(s) = Z_{\mathcal{D}}(s)$ is said to be an Euler product of cone integrals over K with cone integral data \mathcal{D} .

Theorem 2.2 Assume that $Z_{\mathcal{D}}(s)$ is not the constant function. Then the following holds.

- (1) Each $Z_{\mathcal{D}}(s, \mathfrak{p})$ is a rational function in $\mathbf{N}\mathfrak{p}^{-s}$ with rational coefficients.
- (2) $\alpha_{\mathcal{D}}$ is a rational number, and the abscissa of convergence of each $Z_{\mathcal{D}}(s, \mathfrak{p})$ is strictly to the left of $\alpha_{\mathcal{D}}$.
- (3) There exists $\delta > 0$ such that $Z_{\mathcal{D}}(s)$ has meromorphic continuation to the region $\{s \in \mathbb{C} : \text{Re}(s) > \alpha_{\mathcal{D}} - \delta\}$, and the continued function is holomorphic on the line $\text{Re}(s) = \alpha_{\mathcal{D}}$ except at $s = \alpha_{\mathcal{D}}$, where it has a pole, say of order $b_{\mathcal{D}}$.
- (4) Let $Z(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series such that $Z(s) = Z_{\mathcal{D}}(s - h)$ for some h , and assume that its abscissa of convergence $\alpha = \alpha_{\mathcal{D}} + h$ is positive. Then there exist $c, c' \in \mathbb{R}$ such that

$$\sum_{n=1}^N a_n \sim c N^{\alpha} (\log N)^{b_{\mathcal{D}}-1} \quad \text{and} \quad \sum_{n=1}^N \frac{a_n}{n^{\alpha}} \sim c' (\log N)^{b_{\mathcal{D}}}.$$

- (5) Let K' be a number field including K , and let \mathcal{D}' be the same collection \mathcal{D} viewed as cone integral data over K' . Then $\alpha_{\mathcal{D}} = \alpha_{\mathcal{D}'}$.

Remark 2.3 It is not true in general that $b_{\mathcal{D}} = b_{\mathcal{D}'}$ in (5). In fact, consider the cone integral data $\mathcal{D} = (f_0, g_0, f_1, g_1, f_2, g_2, f_3, g_3)$ over \mathbb{Q} , with polynomials in $\mathbb{Q}[x_{11}, x_{22}, x_{12}]$, where

$$\begin{aligned} f_0 &= x_{11}x_{22}, & g_0 &= x_{11}, & f_1 &= x_{11}, & g_1 &= x_{12}, & f_2 &= x_{11}x_{22}, \\ g_2 &= x_{12}^2 + x_{11}^2, & f_3 &= x_{11}, & g_3 &= x_{22}. \end{aligned}$$

Let \mathcal{D}' be the same collection \mathcal{D} viewed as cone integral data over $\mathbb{Q}[i]$. An easy computation shows that

$$Z_{\mathcal{D}}(s) = \zeta(s+2)L(\chi, s+2) \quad \text{and} \quad Z_{\mathcal{D}'}(s) = (\zeta(s+2)L(\chi, s+2))^2,$$

where $\chi : \mathbb{N} \rightarrow \mathbb{C}$ is the Dirichlet character given by $\chi(n) = 1$ if $n \equiv 1 \pmod{4}$, $\chi(n) = -1$ if $n \equiv -1 \pmod{4}$ and $\chi(n) = 0$ otherwise, and $L(s, \chi)$ is the associated L -function. In this example we have $\alpha_{\mathcal{D}} = \alpha_{\mathcal{D}'} = -1$, whereas $b_{\mathcal{D}} = 1$ and $b_{\mathcal{D}'} = 2$. One can show, in the notation of Sect. 3, that $Z_{\mathcal{D}}(s-2)$ is the ideal zeta function of $\mathbb{Z}[i]$, whereas $Z_{\mathcal{D}'}(s-2)$ is the $\mathbb{Z}[i]$ -ideal zeta function of $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i]$.

Properties (1)-(4) in Theorem 2.2 were proved in [8] in the case $K = \mathbb{Q}$. Notice that (1) also follows from a result of Denef [5]. The general case and (5) are somehow implicit in the arguments of [8]. The rest of this section is devoted to making this more precise. Let us first state an important corollary that will be used in our study of zeta functions of groups and rings.

Corollary 2.4 *Let \mathcal{D}_1 and \mathcal{D}_2 be two cone integral data over K .*

- (1) *If $Z_{\mathcal{D}_1}(s, \mathfrak{p}) = Z_{\mathcal{D}_2}(s, \mathfrak{p})$ for almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}_K$, then $\alpha_{\mathcal{D}_1} = \alpha_{\mathcal{D}_2}$ and $b_{\mathcal{D}_1} = b_{\mathcal{D}_2}$.*
- (2) *If there exists a number field K' including K such that $Z_{\mathcal{D}'_1}(s - h, \mathfrak{p}') = Z_{\mathcal{D}'_2}(s - h, \mathfrak{p}')$ for almost all maximal ideals $\mathfrak{p}' \subset \mathcal{O}_{K'}$, where \mathcal{D}'_i denotes the same collection \mathcal{D}_i viewed as cone integral data over K' , then $\alpha_{\mathcal{D}_1} = \alpha_{\mathcal{D}_2}$.*

Proof Note that (1) is an immediate consequence of Theorem 2.2(2). As for (2), Theorem 2.2(5) implies first that $\alpha_{\mathcal{D}_i} = \alpha_{\mathcal{D}'_i}$, and (1) shows that $\alpha_{\mathcal{D}'_1} = \alpha_{\mathcal{D}'_2}$. It follows that $\alpha_{\mathcal{D}_1} = \alpha_{\mathcal{D}_2}$. \square

From now on, we follow [8, Sects. 2, 3, 4] with slight modifications in the notation. The interested reader may also consult [6] for further details. Let K^o be a number field, let $\mathcal{O}^o = \mathcal{O}_{K^o}$, and let $\mathcal{D}^o = (f_0, g_0, \dots, f_l, g_l)$ be a cone integral data over K^o . Let (Y^o, h^o) be a resolution for the polynomial $F = \prod_{i=0}^l f_i g_i \in K^o[x_1, \dots, x_m]$ over K^o . Thus Y^o is a closed subscheme of some projective space over $\mathbb{A}_{K^o}^m$, say $Y^o \subset \mathbb{A}_{K^o}^m \times_{K^o} \mathbb{P}_{K^o}^k$, h^o is the restriction to Y^o of the projection $\mathbb{A}_{K^o}^m \times_{K^o} \mathbb{P}_{K^o}^k \rightarrow \mathbb{A}_{K^o}^m$, and the following holds:

- (i) Y^o is smooth over $\text{Spec}(K^o)$;
- (ii) h^o is an isomorphism over $\mathbb{A}_{K^o}^m \setminus V(F)$, where $V(F) \subset \mathbb{A}_{K^o}^m$ is the vanishing set of F ;
- (iii) the reduced scheme $((h^o)^{-1}(V(F)))_{\text{red}}$ associated to $(h^o)^{-1}(V(F))$ has only normal crossings as subscheme of Y^o .

Let $\{E_\iota^o : \iota \in T\}$ be the set of irreducible components of $((h^o)^{-1}(V(F)))_{\text{red}}$. These, with the structure of reduced subscheme, are smooth hypersurfaces of Y^o by (iii). For each $\iota \in T$, let $N_\iota(f_j)$ and $N_\iota(g_j)$ be, respectively, the multiplicities of E_ι^o in the divisor of $f_j \circ h$ and $g_j \circ h$ ($j = 0, 1, \dots, l$), and let $\nu_\iota - 1$ be the multiplicity of E_ι^o in the divisor of $(h^o)^*(dx_1 \wedge \dots \wedge dx_m)$.

We next define

$$\overline{D_T} := \left\{ u \in \mathbb{R}_{\geq 0}^T : \sum_{\iota \in T} N_\iota(f_j)u(\iota) \leq \sum_{\iota \in T} N_\iota(g_j)u(\iota), \text{ for } j = 1, \dots, l \right\}, \tag{2.1}$$

and for a subset $I \subset T$ we define

$$D_I := \{u \in \overline{D_T} : u(\iota) = 0 \text{ if and only if } \iota \in T \setminus I\},$$

so we have $\overline{D_T} = \bigcup_{I \subset T} D_I$, a disjoint union.

Note that $\overline{D_T}$ is a rational convex polyhedral cone, so there are integral generators $\mathbf{e}_1, \dots, \mathbf{e}_q \in \mathbb{N}_0^T \cap \overline{D_T}$ for its extremal edges such that $\mathbb{N}_0^T \cap \mathbb{R}_{\geq 0}\mathbf{e}_i = \mathbb{N}_0\mathbf{e}_i$ for $i = 1, \dots, q$. The following constants will be important:

$$A_k := \sum_{\iota \in T} \mathbf{e}_k(\iota)N_\iota(f_0) \in \mathbb{N}_0, \quad B_k := \sum_{\iota \in T} \mathbf{e}_k(\iota)(N_\iota(g_0) + \nu_\iota) \in \mathbb{N}, \quad k = 1, \dots, q. \tag{2.2}$$

The cone $\overline{D_T}$ has a simplicial decomposition, say $\overline{D_T} = R_0 \cup R_1 \cup \dots \cup R_q \cup \dots \cup R_w$, such that $R_0 = \{0\}$, $R_i = \mathbb{R}_{>0} \mathbf{e}_i$ for $i = 1, \dots, q$, and $\dim R_i > 1$ for $i > q$. For each $k = 0, \dots, w$ there is a subset $M_k \subset \{1, \dots, q\}$ such that

$$R_k = \left\{ \sum_{j \in M_k} \alpha_j \mathbf{e}_j : \alpha_j > 0 \text{ for all } j \in M_k \right\}.$$

Now for each $I \subset T$ there is a subset $W_I \subset \{0, \dots, w\}$ such that $D_I = \bigcup_{k \in W_I} R_k$. Since $\overline{D_T}$ is the disjoint union of the D_I 's, $\{0, 1, \dots, w\}$ is also the disjoint union of the W_I 's; thus, for each $k = 0, \dots, w$, there is a unique subset $I \subset T$ such that $k \in W_I$, and this subset will be denoted by I_k .

Proposition 2.5 *Let K be a number field including K^o , let $\mathcal{O} = \mathcal{O}_K$, and let \mathcal{D} be the same collection \mathcal{D}^o viewed as cone integral data over K . If $\mathfrak{p} \subset \mathcal{O}$ is a maximal ideal such that (Y^o, h^o) has good reduction modulo $\mathfrak{p}^o := \mathfrak{p} \cap \mathcal{O}^o \subset \mathcal{O}^o$, then*

$$Z_{\mathcal{D}}(s, \mathfrak{p}) = \sum_{k=0}^w (\mathbf{N}\mathfrak{p} - 1)^{|I_k|} \mathbf{N}\mathfrak{p}^{-m} c_{\mathfrak{p}, I_k} \prod_{j \in M_k} \frac{\mathbf{N}\mathfrak{p}^{-(A_j s + B_j)}}{1 - \mathbf{N}\mathfrak{p}^{-(A_j s + B_j)}}, \tag{2.3}$$

where for a subset $I \subset T$, $c_{\mathfrak{p}, I} := |\{a \in \overline{Y^o}(\mathcal{O}/\mathfrak{p}) : a \in \overline{E_i^o}(\mathcal{O}/\mathfrak{p}) \text{ if and only if } i \in I\}|$, and for a closed subscheme $Z^o \subset Y^o$, $\overline{Z^o}$ denotes its reduction modulo \mathfrak{p}^o .

Though this proposition is more general than [8, Corollary 3.2], its proof is the same, so we do not repeat it. However, we mention some comments. Let (Y, h) and $(E_i)_{i \in T}$ be the base changes of (Y^o, h^o) and $(E_i^o)_{i \in T}$ by $\text{Spec}(K_{\mathfrak{p}}) \rightarrow \text{Spec}(K^o)$. Note that each E_i is a smooth hypersurface in Y , but $(E_i)_{i \in T}$ is not necessarily the collection of irreducible components of $h^{-1}(V(F))$ as E_i might not be irreducible. Nevertheless, it is easy to show that

$$\begin{aligned} \text{div}(f_j \circ h) &= \sum_{i \in T} N_i(f_j) E_i \quad \text{and} \quad \text{div}(g_j \circ h) = \sum_{i \in T} N_i(g_j) E_i \quad \text{for } j = 0, \dots, l, \\ \text{div}(h^*(dx_1 \wedge \dots \wedge dx_m)) &= \sum_{i \in T} (v_i - 1) E_i, \end{aligned}$$

so one can argue as in [8, Sects. 2 and 3] by using the collection $(E_i)_{i \in T}$ and not the irreducible components of $h^{-1}(V(F))$. In fact, this seems to be what is really done in [8].

The following consequence was also obtained at the end of [8, Sect. 2] from a different formula for $Z_{\mathcal{D}}(s, \mathfrak{p})$.

Corollary 2.6 *Write each $Z_{\mathcal{D}}(s, \mathfrak{p})$ as a power series $\sum_{i=0}^{\infty} a_{\mathfrak{p}, i} \mathbf{N}\mathfrak{p}^{-is}$. Then, $a_{\mathfrak{p}, 0} \neq 0$ for almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}$.*

Proof It is enough to show that $a_{\mathfrak{p}, 0} \neq 0$ for almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}$ satisfying the hypothesis of Proposition 2.5. We compute the term of the formula (2.3) for $k = 0$. Note that $M_0 = \emptyset$ and that $I_0 = \emptyset$, so the term for $k = 0$ is $\mathbf{N}\mathfrak{p}^{-m} c_{\mathfrak{p}, \emptyset} = \mathbf{N}\mathfrak{p}^{-m} |\{a \in \overline{Y^o}(\mathcal{O}/\mathfrak{p}) \setminus \bigcup_{i \in T} \overline{E_i^o}(\mathcal{O}/\mathfrak{p})\}| = \mathbf{N}\mathfrak{p}^{-m} |\mathbb{A}^m(\mathcal{O}/\mathfrak{p}) \setminus V(\overline{F})|$, where \overline{F} denotes the reduction of F modulo \mathfrak{p}^o and $V(\overline{F}) \subset \mathbb{A}^m(\mathcal{O}/\mathfrak{p})$ denotes the zero set of \overline{F} . Note that by dimension arguments, $|\mathbb{A}^m(\mathcal{O}/\mathfrak{p}) \setminus V(\overline{F})| > 0$ for almost all \mathfrak{p} . The proof follows as the term for $k = 0$ contributes to $a_{\mathfrak{p}, 0}$. \square

Proposition 2.5 provides a formula for $Z_{\mathcal{D}}(s, \mathfrak{p})$ for almost all maximal ideals. A formula for the exceptional primes (when $K = K^o = \mathbb{Q}$) is given in [8, Proposition 3.3], but this

seems to be incorrect as it was pointed out in [1, Remark 4.6]. Nevertheless, Theorem 2.2 does not use this proposition but rather its consequence [8, Corollary 3.4], which is correct as claimed also in [1, Remark 4.6]. We will provide a proof of this in Proposition 2.8 below. We shall need the following result of Stanley (cf. [29, Chapter 1]), which is formulated here in our special setting.

Theorem 2.7 *Let $I \subset T$ be a non-empty subset, and let $\Phi_I \in \mathbb{Z}^{l \times l}$ be the matrix defined by*

$$(\Phi_I)_{j,\iota} = N_\iota(g_j) - N_\iota(f_j), \quad j = 1, \dots, l, \iota \in I.$$

Given a vector $v \in \mathbb{Z}^l$, we consider the generating series

$$F_{\Phi,v}((X_i)_{i \in I}) := \sum_{u \in \mathbb{N}_0^l : \Phi_I \cdot u \geq v} X^u,$$

where $X^u = \prod_{i \in I} X_i^{u(i)}$ and $(X_i)_{i \in I}$ is a collection of variables. If $\{u \in \mathbb{N}_0^l : \Phi \cdot u \geq v\}$ is non-empty, then $F_{\Phi,v}((X_i)_{i \in I})$ is a rational function whose denominator divides

$$\prod_{j \in \cup_{k \in W_I} M_k} \left(1 - \prod_{\iota \in I} X^{\mathbf{e}_j(\iota)} \right).$$

Here the inequality $\Phi_I \cdot u \geq v$ means component-wise. Observe that $\{\mathbf{e}_j : j \in \cup_{k \in W_I} M_k\}$ is just the set of integral generators of the extremal edges of the cone $\{u \in \mathbb{N}_0^l : \Phi_I \cdot u \geq 0\}$. These vectors can be thought of as vectors in \mathbb{N}_0^l since they are zero outside I .

Proposition 2.8 *Let K be a number field including K^o , let $\mathcal{O} = \mathcal{O}_K$, and let \mathcal{D} be the collection \mathcal{D}^o viewed as cone integral data over K . If $\mathfrak{p} \subset \mathcal{O}$ is any maximal ideal, then $Z_{\mathcal{D}}(s, \mathfrak{p})$ is a rational function in $\mathbb{N}\mathfrak{p}^{-s}$ with rational coefficients, and its denominator divides*

$$\prod_{j=1}^q (1 - \mathbb{N}\mathfrak{p}^{-(A_j s + B_j)}).$$

In particular, the abscissa of convergence of $Z_{\mathcal{D}}(s, \mathfrak{p})$ is either $-\infty$ or one of the rational numbers $-\frac{B_j}{A_j}$ where $j = 1, \dots, q$ and $A_j \neq 0$.

Proof The same notation and comments after Proposition 2.5 apply here. Now h induces a morphism of analytic manifolds $Y(K_{\mathfrak{p}}) \rightarrow K_{\mathfrak{p}}^m$ that we still denote by h . This is an isomorphism above $\{\mathbf{x} \in K_{\mathfrak{p}}^m : F(\mathbf{x}) \neq 0\}$. Write $Y(\mathcal{O}_{\mathfrak{p}})$ for $h^{-1}(\mathcal{O}_{\mathfrak{p}}^m)$. Then $Y(\mathcal{O}_{\mathfrak{p}})$ can be expressed as a disjoint union of a finite number of coordinate charts, say $\{(U_b, (y_1, \dots, y_m)) : b \in \mathcal{B}\}$, such that for each $b \in \mathcal{B}$ the following holds (see [20, Sect. 2]):

- (1) The image of $(y_1, \dots, y_m) : U_b \rightarrow K_{\mathfrak{p}}^m$ is exactly $(\mathfrak{p}^{e_b} \mathcal{O}_{\mathfrak{p}})^m$ for some $e_b \in \mathbb{N}_0$.
- (2) Let $I_b = \{\iota \in T : E_\iota(K_{\mathfrak{p}}) \cap U_b \neq \emptyset\}$. Then there exists an injective function $\mu_b : I_b \rightarrow \{1, \dots, m\}$ such that $E_\iota(K_{\mathfrak{p}}) \cap U_b = \{y_{\mu_b(\iota)} = 0\}$ for all $\iota \in I_b$.

(3) There are non-negative integers $c_b(f_j), c_b(g_j)$ for $j = 0, \dots, l$ and d_b , such that the following holds on U_b :

$$\begin{aligned}
 |f_j \circ h|_{\mathfrak{p}} &= \mathbf{N}\mathfrak{p}^{-c_b(f_j)} \prod_{\iota \in I_b} |y_{\mu(\iota)}|_{\mathfrak{p}}^{N_{\iota}(f_j)}, \\
 |g_j \circ h|_{\mathfrak{p}} &= \mathbf{N}\mathfrak{p}^{-c_b(g_j)} \prod_{\iota \in I_b} |y_{\mu(\iota)}|_{\mathfrak{p}}^{N_{\iota}(g_j)}, \\
 |h^*(dx_1 \wedge \dots \wedge dx_m)|_{\mathfrak{p}} &= \mathbf{N}\mathfrak{p}^{-d_b} \prod_{\iota \in I_b} |y_{\mu(\iota)}|_{\mathfrak{p}}^{v_{\iota}-1} |dy_1 \wedge \dots \wedge dy_m|_{\mathfrak{p}}.
 \end{aligned}$$

It follows that $Z_{\mathcal{D}}(s, \mathfrak{p}) = \sum_{b \in \mathcal{B}} J_b(s)$, where

$$J_b(s) = \int_{V_b} \mathbf{N}\mathfrak{p}^{-c_b(f_0)s - c_b(g_0) - d_b} \prod_{\iota \in I_b} |y_{\mu_b(\iota)}|_{\mathfrak{p}}^{N_{\iota}(f_0)s + N_{\iota}(g_0) + v_{\iota} - 1} |dy_1 \wedge \dots \wedge dy_m|_{\mathfrak{p}}$$

and V_b is the subset of U_b defined by the conditions

$$c_b(g_j) - c_b(f_j) + \sum_{\iota \in I_b} (N_{\iota}(g_j) - N_{\iota}(f_j)) \text{ord}_{\mathfrak{p}}(y_{\mu_b(\iota)}) \geq 0, \quad j = 1, \dots, l.$$

If $I_b = \emptyset$, then clearly $J_b(s) = r\mathbf{N}\mathfrak{p}^{-c_b(f_0)s}$ for some rational number r . We now analyse $J_b(s)$ for $I_b \neq \emptyset$. By (1) we can assume that $U_b = (\mathfrak{p}^{e_b} \mathcal{O}_{\mathfrak{p}})^m$ with coordinates y_1, \dots, y_m . Let $\Phi_I = \Phi_{I_b}$ be the matrix of Theorem 2.7, let $c_b(f) - c_b(g) \in \mathbb{Z}^l$ denote the vector whose j -entry is $c_b(f_j) - c_b(g_j)$, and let $v := c_b(f) - c_b(g) - \Phi \cdot (e_b)_{\iota \in T} \in \mathbb{Z}^l$. It follows that

$$\begin{aligned}
 J_b(s) &= (1 - \mathbf{N}\mathfrak{p}^{-1})^{|I_b|} \\
 &\cdot \mathbf{N}\mathfrak{p}^{-c_b(f_0)s - c_b(g_0) - d_b - e_b(m - |I_b|)} \cdot \sum_{\substack{u \in (e_b \mathbb{N}_0)^{I_b} \\ \Phi \cdot u \geq c_b(f) - c_b(g)}} \prod_{\iota \in I_b} \mathbf{N}\mathfrak{p}^{-(sN_{\iota}(f_0) + N_{\iota}(g_0) + v_{\iota})u(\iota)} \\
 &= (\mathbf{N}\mathfrak{p}^{e_b} - \mathbf{N}\mathfrak{p}^{e_b - 1})^{|I_b|} \cdot \mathbf{N}\mathfrak{p}^{-c_b(f_0)s - c_b(g_0) - d_b - e_b m - \sum_{\iota \in I_b} (sN_{\iota}(f_0) + N_{\iota}(g_0) + v_{\iota})e_b} \\
 &\cdot \sum_{\substack{u \in \mathbb{N}_0^l \\ \Phi \cdot u \geq v}} \prod_{\iota \in I_b} \mathbf{N}\mathfrak{p}^{-(sN_{\iota}(f_0) + N_{\iota}(g_0) + v_{\iota})u(\iota)}.
 \end{aligned}$$

By Theorem 2.7 this is a rational function in $\mathbf{N}\mathfrak{p}^{-s}$ with rational coefficients, and whose denominator divides

$$\begin{aligned}
 &\prod_{j \in \bigcup_{k \in W_{I_b}} M_k} \left(1 - \prod_{\iota \in I_b} \mathbf{N}\mathfrak{p}^{-(sN_{\iota}(f_0) + N_{\iota}(g_0) + v_{\iota})\mathbf{e}_j(\iota)} \right) \\
 &= \prod_{j \in \bigcup_{k \in W_{I_b}} M_k} \left(1 - \prod_{\iota \in T} \mathbf{N}\mathfrak{p}^{-(sN_{\iota}(f_0) + N_{\iota}(g_0) + v_{\iota})\mathbf{e}_j(\iota)} \right) \\
 &= \prod_{j \in \bigcup_{k \in W_{I_b}} M_k} \left(1 - \mathbf{N}\mathfrak{p}^{-(sA_j + B_j)} \right).
 \end{aligned}$$

This completes the proof of the proposition. □

With Proposition 2.5, Corollary 2.6 and Proposition 2.8, the proof of properties (1)–(4) in Theorem 2.2 goes now exactly as in [8, Sect. 4]. In particular, one obtains the following formula for $\alpha_{\mathcal{D}}$ which is independent of the field extension $K \supset K_0$, and which proves (5) in Theorem 2.2.

Proposition 2.9 $\alpha_{\mathcal{D}} = \max \left\{ \frac{1-B_j}{-A_j} : j = 1, \dots, l, A_j \neq 0 \right\}$.

3 Zeta functions of rings as cone integrals

Let R be a commutative ring with identity. By an R -algebra we shall mean an R -module L endowed with an R -bilinear map $L \times L \rightarrow L$ called multiplication, e.g. an R -Lie algebra. An R -subalgebra of L is an R -submodule A that is closed under multiplication. Left, right and two-sided R -ideals are defined similarly. To simplify the presentation, by an R -ideal we shall refer to a left R -ideal. Nevertheless, everything we say about left R -ideals is also valid for right R -ideals and for two-sided R -ideals. To allow applications such as Theorem E we shall also consider R -algebras with an identity, and in that case we will require that the R -subalgebras contain the identity. A \mathbb{Z} -algebra will be also called a ring.

Let L be an R -algebra (or an R -algebra with identity). The R -subalgebra zeta function and the R -ideal zeta function of L are by definition the formal series

$$\zeta_L^{\leq R}(s) = \sum_{L' \leq_R L} [L : L']^{-s} \quad \text{and} \quad \zeta_L^{\triangleleft R}(s) = \sum_{L' \triangleleft_R L} [L : L']^{-s}$$

where L' runs only over those R -subalgebras or R -ideals of finite additive index respectively. We write $\zeta_L^{*R}(s)$ to address both type of zeta functions simultaneously. We will suppress the subindex R when $R = \mathbb{Z}$.

Let K be a number field and let $\mathcal{O} = \mathcal{O}_K$. Let L be an \mathcal{O} -algebra (or an \mathcal{O} -algebra with identity) whose underlying \mathcal{O} -module is free of rank $h \geq 1$. For a maximal ideal $\mathfrak{p} \subset \mathcal{O}$ we write $L_{\mathfrak{p}} := L \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$. This is an $\mathcal{O}_{\mathfrak{p}}$ -algebra and it is easy to show that

$$\zeta_L^{*\mathcal{O}}(s) = \prod_{\substack{\mathfrak{p} \subset \mathcal{O} \\ \text{maximal}}} \zeta_{L_{\mathfrak{p}}}^{*\mathcal{O}_{\mathfrak{p}}}(s).$$

We now fix a basis for L as \mathcal{O} -module and hence identify L with \mathcal{O}^h . Let (c_{ij}^k) be the structure coefficients of L with respect to the canonical basis $\{e_1, \dots, e_h\}$, that is, $e_i \cdot e_j = \sum_{k=1}^h c_{ij}^k e_k$. We denote by $\text{Tr}_h(\mathcal{O}_{\mathfrak{p}})$ the set of upper-triangular matrices with entries in $\mathcal{O}_{\mathfrak{p}}$ and write $|dm|_{\mathfrak{p}}$ for the normalized additive Haar measure of $\text{Tr}_h(\mathcal{O}_{\mathfrak{p}})$. Given a matrix $m = (m_{ij})$ we denote by $m' = (m'_{ij})$ its adjoint.

Proposition 3.1 *For each maximal ideal $\mathfrak{p} \subset \mathcal{O}$ it holds that*

$$\zeta_{L_{\mathfrak{p}}}^{*\mathcal{O}_{\mathfrak{p}}}(s) = (1 - \mathbf{N}_{\mathfrak{p}}^{-h})^{-1} \int_{\mathcal{M}^{\leq}(\mathcal{O}_{\mathfrak{p}})} |m_{11}|_{\mathfrak{p}}^{s-1} \cdots |m_{hh}|_{\mathfrak{p}}^{s-h} |dm|_{\mathfrak{p}},$$

where $\mathcal{M}^{\leq}(\mathcal{O}_{\mathfrak{p}}) \subset \text{Tr}_h(\mathcal{O}_{\mathfrak{p}})$ denotes the set of those upper-triangular matrices (m_{ij}) such that

$$\text{ord}_{\mathfrak{p}}(m_{11} \cdots m_{hh}) \leq \text{ord}_{\mathfrak{p}} \left(\sum_{t=1}^h \sum_{r=i}^h \sum_{s=j}^h m_{ir} c_{rs}^t m_{js} m'_{tk} \right) \quad \text{for all } i, j, k = 1, \dots, h,$$

and $\mathcal{M}^\triangleleft(\mathcal{O}_p) \subset \text{Tr}_h(\mathcal{O}_p)$ denotes the set of those upper-triangular matrices (m_{ij}) such that

$$\text{ord}_p(m_{11} \cdots m_{hh}) \leq \text{ord}_p \left(\sum_{s=1}^h \sum_{r=i}^h m_{ir} c_{rj}^s m'_{sk} \right) \quad \text{for all } i, j, k = 1, \dots, h.$$

If in addition L is an \mathcal{O} -algebra with identity, say $1 = (u_1, \dots, u_h)$, then we also have to add the following extra conditions in the definition of $\mathcal{M}^\leq(\mathcal{O}_p)$:

$$\text{ord}_p(m_{11} \cdots m_{hh}) \leq \text{ord}_p \left(\sum_{i=1}^h u_i m'_{ij} \right) \quad \forall j = 1, \dots, h.$$

Proof The first part of the proposition when $K = \mathbb{Q}$ is proved in [8, Theorem 5.5] (see also [12, Sec. 3]). That proof can be easily extended to this general case. By following that proof in the case that L is an \mathcal{O} -algebra with identity, we see that the conditions that we add to the definition of $\mathcal{M}^\leq(\mathcal{O}_p)$ are simply a translation of the condition that the \mathcal{O}_p -submodule of L_p generated by the rows of the matrix (m_{ij}) contains $1 = (u_1, \dots, u_h)$. This is necessary since we are requiring that the R -subalgebras of an R -algebra with identity must contain 1. \square

As an immediate consequence we obtain

Corollary 3.2 *Let L be an \mathcal{O} -algebra (or an \mathcal{O} -algebra with identity) that is isomorphic to \mathcal{O}^h as \mathcal{O} -module. Then there exists a cone integral data \mathcal{D}^* over K such that*

$$\zeta_L^{*\mathcal{O}}(s) = Z_{\mathcal{D}^*}(s - h),$$

and such that for any finite extension $K' \supset K$, say with ring of integers \mathcal{O}' , we have

$$\zeta_{L'}^{*\mathcal{O}'}(s) = Z_{\mathcal{D}'^*}(s - h),$$

where L' is the \mathcal{O}' -algebra $L \otimes_{\mathcal{O}} \mathcal{O}'$ and \mathcal{D}'^* is the same collection \mathcal{D}^* viewed as cone integral data over K' .

As a combination of this corollary and Theorem 2.2 we now obtain

Corollary 3.3 *Let L be an \mathcal{O} -algebra (or an \mathcal{O} -algebra with identity) that is isomorphic to \mathcal{O}^h as \mathcal{O} -module. Assume that $\zeta_L^{*\mathcal{O}}(s)$ is not a constant function, and let $\alpha_L^{*\mathcal{O}}$ be its abscissa of convergence. Then the following holds.*

- (1) $\alpha_L^{*\mathcal{O}}$ is a rational number and there exists $\delta > 0$ such that $\zeta_L^{*\mathcal{O}}(s)$ can be meromorphically continued to the region $\text{Re}(s) > \alpha_L^{*\mathcal{O}} - \delta$. Moreover, the continued function is holomorphic on the line $\text{Re}(s) = \alpha_L^{*\mathcal{O}}$ except at $s = \alpha_L^{*\mathcal{O}}$ where it has a pole.
- (2) Let K' be a number field including K , \mathcal{O}' its ring of integers, and $L' = L \otimes_{\mathcal{O}} \mathcal{O}'$. Then $\zeta_L^{*\mathcal{O}}(s)$ and $\zeta_{L'}^{*\mathcal{O}'}(s)$ have the same abscissa of convergence.

Another consequence is

Corollary 3.4 *Let L_1 and L_2 be two \mathcal{O} -algebras (or \mathcal{O} -algebras with identity) that are isomorphic to \mathcal{O}^h as \mathcal{O} -modules. Let $b_{L_i}^{*\mathcal{O}}$ be the order of the pole of $\zeta_{L_i}^{*\mathcal{O}}(s)$ at $s = \alpha_{L_i}^{*\mathcal{O}}$. If $L_1 \otimes_{\mathcal{O}} K$ and $L_2 \otimes_{\mathcal{O}} K$ are isomorphic K -algebras, then $\alpha_{L_1}^{*\mathcal{O}} = \alpha_{L_2}^{*\mathcal{O}}$ and $b_{L_1}^{*\mathcal{O}} = b_{L_2}^{*\mathcal{O}}$.*

Proof Let \mathcal{D}_i^* be the cone integral data of Corollary 3.2 for L_i . By Corollary 2.2(1) it is enough to prove that $Z_{\mathcal{D}_1^*}(s - h, \mathfrak{p}) = Z_{\mathcal{D}_2^*}(s - h, \mathfrak{p})$ for almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}$. This follows from the fact that $L_{1p} := L_1 \otimes_{\mathcal{O}} \mathcal{O}_p$ and $L_{2p} := L_2 \otimes_{\mathcal{O}} \mathcal{O}_p$ are isomorphic

$\mathcal{O}_{\mathfrak{p}}$ -algebras for almost all \mathfrak{p} . To prove this fact, we may assume that $L_1 = L_2 = \mathcal{O}^h$ as \mathcal{O} -modules and so $L_1 \otimes_{\mathcal{O}} K = L_2 \otimes_{\mathcal{O}} K = K^h$. Let $\alpha \in \text{GL}_h(K)$ be a K -algebra isomorphism $\alpha : L_1 \otimes_{\mathcal{O}} K \rightarrow L_2 \otimes_{\mathcal{O}} K$. Then for almost all maximal ideals \mathfrak{p} it holds that $\alpha, \alpha^{-1} \in \text{GL}_h(\mathcal{O}_{\mathfrak{p}})$, and hence α induces an isomorphism $L_{1\mathfrak{p}} \cong L_{2\mathfrak{p}}$. This completes the proof. \square

In order to prove Theorem C we shall need the following result.

Lemma 3.5 *Let K be a field and let A_1 and A_2 be finite dimensional K -algebras. If for some field extension $K' \supset K$ the K' -algebras $A_1 \otimes_K K'$ and $A_2 \otimes_K K'$ are isomorphic over K' , then this also holds for some finite extension $K' \supset K$.*

Proof Note that A_1 and A_2 have necessarily the same dimension over K , hence we can assume that $A_1 = A_2 = K^h$ as vector spaces over K for some h . Let $I \subset \text{GL}_h \times_{\mathbb{Z}} \text{Spec}(K)$ be the subfunctor such that if K' is any commutative algebra over K with identity, then $I(K')$ is the set of K' -algebra isomorphisms $A_1 \otimes_K K' \rightarrow A_2 \otimes_K K'$. It is easy to see that I is represented by a closed subscheme of $\text{GL}_h \times_{\mathbb{Z}} \text{Spec}(K)$. The hypothesis implies that I is not the empty scheme. Therefore, if K' is the residue field at a closed point of I , which is a finite extension of K , we have $I(K') \neq \emptyset$. \square

Theorem C follows from the following

Theorem 3.6 *Let K be a number field and \mathcal{O} its ring of integers. Let L_1 and L_2 be two \mathcal{O} -algebras (or \mathcal{O} -algebras with identity) that are isomorphic to \mathcal{O}^h as \mathcal{O} -modules for some $h > 0$. If $L_1 \otimes_{\mathcal{O}} K'$ and $L_2 \otimes_{\mathcal{O}} K'$ are isomorphic K' -algebras for some field extension $K' \supset K$, then $\zeta_{L_1}^{*\mathcal{O}}(s)$ and $\zeta_{L_2}^{*\mathcal{O}}(s)$ have the same abscissa of convergence.*

Proof By Lemma 3.5 we can assume that the field K' of the hypothesis is a finite extension of K . Let \mathcal{O}' be its ring of integers and let $L'_i = L_i \otimes_{\mathcal{O}} \mathcal{O}'$. By Corollary 3.3, $\zeta_{L'_i}^{*\mathcal{O}'}(s)$ and $\zeta_{L'_i}^{*\mathcal{O}'}(s)$ have the same abscissa of convergence for $i = 1, 2$. By Corollary 3.4, $\zeta_{L'_1}^{*\mathcal{O}'}(s)$ and $\zeta_{L'_2}^{*\mathcal{O}'}(s)$ have the same abscissa of convergence since $L'_1 \otimes_{\mathcal{O}'} K' = L'_2 \otimes_{\mathcal{O}'} K'$ as K' -algebras. It follows that $\zeta_{L_1}^{*\mathcal{O}}(s)$ and $\zeta_{L_2}^{*\mathcal{O}}(s)$ have the same abscissa of convergence. \square

We now explain how Theorem B follows from Theorem C. Let \mathfrak{N} be a unipotent group scheme over \mathbb{Q} and let N be an arithmetic subgroup of \mathfrak{N} . We define \mathfrak{n} to be the Lie algebra of \mathfrak{N} , which is a nilpotent Lie algebra of dimension $h(N)$ over \mathbb{Q} .

Proposition 3.7 *Let L be any Lie subring of \mathfrak{n} additively isomorphic to \mathbb{Z}^h , with $h = h(N)$, such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{n}$. Then $\zeta_N^*(s)$ and $\zeta_L^*(s)$ have the same abscissa of convergence, that is $\alpha_N^* = \alpha_L^*$. In addition, $b_N^* = b_L^*$.*

Proof By Corollary 3.4 it is enough to prove this by just one L . By [12, Theorem 4.1], there is L satisfying the hypothesis such that for almost all primes p , $\zeta_N^*(s)$ and $\zeta_L^*(s)$ have the same local factor at p . Now, by Corollary 3.2, there exists a cone integral data \mathcal{D}_L^* such that $\zeta_L^*(s) = Z_{\mathcal{D}_L^*}(s - h)$. By [30, Corollary 1], there exists a cone integral data \mathcal{D}_N^* such that $\zeta_N^*(s) = Z_{\mathcal{D}_N^*}(s - h)$. It follows that $Z_{\mathcal{D}_N^*}(s - h, p) = Z_{\mathcal{D}_L^*}(s - h, p)$ for almost all p . The proposition follows now from Corollary 2.4(1). \square

Since the category of unipotent algebraic groups over a field K of characteristic zero and the category of finite dimensional nilpotent Lie algebras over K are equivalent, it is now clear that Theorem B follows from Theorem C.

4 An upper bound for α_N^* for non-abelian \mathfrak{T} -groups

In this section we prove Theorem A. By Proposition 3.7, it is enough to prove the analogous result for nilpotent Lie rings.

Lemma 4.1 *Let \mathbb{Z}^e be the free abelian group of rank e , $e > 0$, and let δ be a positive number such that $\delta < e$. Then there is a constant $k = k(e, \delta)$ such that*

$$\sum_{n=1}^N \frac{a_n^{\leq}(\mathbb{Z}^e)}{n^{e-\delta}} \leq kN^\delta, \quad \forall N \in \mathbb{N}.$$

Proof Consider the Dirichlet series $Z(s) = \sum_{n=1}^\infty \frac{b_n}{n^s}$, where $b_n = \frac{a_n^{\leq}(\mathbb{Z}^e)}{n^{e-\delta}}$. Then $Z(s) = \zeta_{\mathbb{Z}^e}^{\leq}(s + e - \delta) = Z_{\mathcal{D}}(s - \delta)$, where \mathcal{D} is the cone integral data of Corollary 3.2 for the zeta function $\zeta_{\mathbb{Z}^e}^{\leq}(s)$ (we view \mathbb{Z}^e as an abelian Lie ring). The abscissa of convergence of $Z(s)$ is $\delta > 0$, and the order of the pole of $Z(s)$ at $s = \delta$, which is the order of the pole of $\zeta_{\mathbb{Z}^e}^{\leq}(s)$ at e , is 1. This follows from the formula for $\zeta_{\mathbb{Z}^e}^{\leq}(s)$ given in Example 1.1. Thus, by Theorem 2.2(4), there exists a constant k such that $\sum_{n=1}^N b_n \leq kN^\delta$ for all N . This proves the lemma. \square

We now set some notation. If L is a ring additively isomorphic to \mathbb{Z}^h , then $\alpha_L^\triangleleft \leq \alpha_L^{\leq} \leq \alpha_{\mathbb{Z}^h}^{\leq} = h$. We denote $\delta_L^* = h - \alpha_L^*$ for $*$ in $\{\leq, \triangleleft\}$. If L is a nilpotent Lie ring, then $\gamma_i(L)$ denotes the i -th term of the lower central series. The nilpotency class of L is the first positive integer c such that $\gamma_{c+1}(L) = 0$. If A is an ideal of L we also define a lower series as follows: $\gamma_1(L, A) = A$ and $\gamma_i(L, A) = [\gamma_{i-1}(L, A), L]$ for $i > 1$.

Proposition 4.2 *Let L be a non-abelian nilpotent Lie ring additively isomorphic to \mathbb{Z}^h , let c be its nilpotency class, let $Z = \{x \in L : nx \in \gamma_c(L) \text{ for some } n \in \mathbb{N}\}$, and let e be the additive rank of $\gamma_c(L)$. Then*

$$\delta_L^{\leq} \geq \frac{\delta_{L/Z}^{\leq} + e}{1 + (c - 1)e} \quad \text{and} \quad \delta_L^{\triangleleft} \geq \frac{\delta_{L/Z}^{\triangleleft} + e}{1 + e}.$$

Proof Notice that Z is an ideal of L included in the centre of L , the quotient ring L/Z is additively isomorphic to \mathbb{Z}^{h-e} , and the index $k_1 := [Z : \gamma_c(L)]$ is finite. Since Z is central, any subgroup of Z is automatically an ideal of L .

If A is a finite index subring of L , then $A + Z$ is a subring including Z , and $A \cap Z$ is a finite index subgroup of Z . If A is an ideal, then $A + Z$ is also an ideal. In any case, we have $[L : A] = [L : A + Z][Z : A \cap Z]$. It is easy to show that if A is a subring, then $\gamma_c(A + Z) \subset A \cap Z$, and if A is an ideal, then $\gamma_c(L, A + Z) \subset A \cap Z$. Thus, for any positive δ with $\delta < h$ we have

$$\begin{aligned} \zeta_L^{\leq}(h - \delta) &= \sum_{A \leq_f L} [L : A]^{-h+\delta} = \sum_{\substack{B \leq_f L: \\ Z \subset B}} [L : B]^{-h+\delta} \sum_{\substack{C \leq_f Z: \\ \gamma_c(B,C) \subset C}} [Z : C]^{-h+\delta} \mu_{B,C}^{\leq}, \\ \zeta_L^{\triangleleft}(h - \delta) &= \sum_{A \triangleleft_f L} [L : A]^{-h+\delta} = \sum_{\substack{B \triangleleft_f L: \\ Z \subset B}} [L : B]^{-h+\delta} \sum_{\substack{C \leq_f Z: \\ \gamma_c(L,B) \subset C}} [Z : C]^{-h+\delta} \mu_{B,C}^{\triangleleft}, \end{aligned}$$

where $\mu_{B,C}^{\leq} = |\{A \leq_f L : A + Z = B, A \cap Z = C\}|$ and $\mu_{B,C}^{\triangleleft} = |\{A \triangleleft_f L : A + Z = B, A \cap Z = C\}|$. By [18, Lemma 1.3.1] we have $\mu_{B,C}^{\triangleleft} \leq \mu_{B,C}^{\leq} \leq |\text{Hom}(B/Z, Z/C)| \leq$

$[Z : C]^{h-e}$. Thus,

$$\begin{aligned} \zeta_L^{\leq}(h - \delta) &\leq \sum_{\substack{B \leq_f L: \\ Z \subset B}} [L : B]^{-h+\delta} \sum_{\substack{C \leq_f Z: \\ \gamma_c(B) \subset C}} [Z : C]^{-e+\delta}, \\ \zeta_L^{\triangleleft}(h - \delta) &\leq \sum_{\substack{B \triangleleft_f L: \\ Z \subset B}} [L : B]^{-h+\delta} \sum_{\substack{C \leq_f Z: \\ \gamma_c(L, B) \subset C}} [Z : C]^{-e+\delta}. \end{aligned} \tag{4.1}$$

Given $B \leq_f L$ with $Z \subset B$, we claim that $\gamma_c(B) \supset [L : B]^{c-1} \gamma_c(L)$. In fact, there are $x_1, \dots, x_{h-e} \in Z$ whose classes modulo Z form a basis of the \mathbb{Z} -module L/Z , and there are positive integers $d_1 | d_2 | \dots | d_{h-e}$ such that B/Z is generated by the classes of $d_1 x_1, \dots, d_{h-e} x_{h-e}$. So $[L : B] = d_1 d_2 \dots d_{h-e}$. Notice that $\gamma_c(L)$ is the subgroup of Z generated by all the elements of the form $[x_{i_1}, \dots, x_{i_c}] := [[\dots [x_{i_1}, x_{i_2}], x_{i_3}], \dots], x_{i_c}$ with not all i_1, \dots, i_c equal, and $\gamma_c(B)$ is generated by all the elements of the form $d_{i_1} \dots d_{i_c} [x_{i_1}, \dots, x_{i_c}]$ with not all i_1, \dots, i_c equal. Now, if i_1, \dots, i_c are not all equal to each other, then the product of two factors in $d_{i_1} \dots d_{i_c}$ divides $[L : B]$ and the other factors are also divisors of $[L : B]$. Thus, $d_{i_1} \dots d_{i_c}$ divides $[L : B]^{c-1}$ and hence $[L : B]^{c-1} [x_{i_1}, \dots, x_{i_c}] \in \gamma_c(B)$. It follows that $\gamma_c(B) \supset [L : B]^{c-1} \gamma_c(L)$, as claimed. Note that if B is in addition an ideal, then $\gamma_c(L, B) \supset \gamma_c(L, [L : B]L) = [L : B] \gamma_c(L)$. Since $[L : [L : B]^t \gamma_c(L)] = k_1 [\gamma_c(L) : [L : B]^t \gamma_c(L)] = k_1 [L : B]^{te}$ for any $t \in \mathbb{N}$, we conclude that

$$\begin{aligned} \sum_{\substack{C \leq_f Z: \\ \gamma_c(B) \subset C}} [Z : C]^{-e+\delta} &\leq \sum_{\substack{C \leq_f Z \\ [Z:C] \leq k_1 [L:B]^{(c-1)e}}} [Z : C]^{-e+\delta} \leq k_2 (k_1 [L : B]^{(c-1)e})^\delta, \\ \sum_{\substack{C \leq_f Z: \\ \gamma_c(L, B) \subset C}} [Z : C]^{-e+\delta} &\leq \sum_{\substack{C \leq_f Z \\ [Z:C] \leq k_1 [L:B]^e}} [Z : C]^{-e+\delta} \leq k_2 (k_1 [L : B]^e)^\delta, \end{aligned} \tag{4.2}$$

where k_2 is the constant provided by Lemma 4.1.

A combination of (4.1) and (4.2) yields

$$\begin{aligned} \zeta_L^{\leq}(h - \delta) &\leq k(\delta) \sum_{Z \subset B \leq_f L} [L : B]^{-h+\delta+\delta(c-1)e} = k(\delta) \zeta_{L/Z}^{\leq}(h - \delta(1 + (c - 1)e)), \\ \zeta_L^{\triangleleft}(h - \delta) &\leq k(\delta) \sum_{Z \subset B \triangleleft_f L} [L : B]^{-h+\delta+\delta e} = k(\delta) \zeta_{L/Z}^{\triangleleft}(h - \delta(1 + e)), \end{aligned}$$

for some constant $k(\delta)$. It follows from the first inequality that $\zeta_L^{\leq}(h - \delta)$ converges if $h - \delta(1 + ce) > \alpha_{L/Z}^{\leq}$, i.e., if $\delta < \frac{h - \alpha_{L/Z}^{\leq}}{1 + ce} = \frac{\delta_{L/Z}^{\leq} + e}{1 + (c-1)e}$. This proves that $\delta_{L/Z}^{\leq}$ is at least $\frac{\delta_{L/Z}^{\leq} + e}{1 + (c-1)e}$. Similarly, from the second inequality we deduce that δ_L^{\triangleleft} is at least $\frac{\delta_{L/Z}^{\triangleleft} + e}{1 + e}$. \square

Theorem A follows from the next one.

Theorem 4.3 *Let L be a non-abelian nilpotent Lie ring additively isomorphic to \mathbb{Z}^h , and let c be its nilpotency class.*

- (1) *If $c = 2$, then $\alpha_L^{\leq} \leq h - \frac{1}{2}$. If $c > 2$ then $\alpha_L^{\leq} \leq h - \frac{1}{c-1}$.*
- (2) *$\alpha_L^{\triangleleft} \leq h - 1$.*

Proof We use the notation of Proposition 4.2. Observe that the nilpotency class of L/Z is $c - 1$.

We prove (1). Assume first that $c = 2$. By Proposition 4.2, $\delta_L^{\leq} \geq \frac{e}{1+e} \geq \frac{1}{2}$, hence $\alpha_L^{\leq} \leq h - \frac{1}{2}$. We now assume that $c > 2$ and prove that $\alpha_L^{\leq} \leq h - \frac{1}{c-1}$, or equivalently that $\delta_L^{\leq} \geq \frac{1}{c-1}$, by induction on c . If $c = 3$, then $\delta_L^{\leq} \geq \frac{\delta_{L/Z}^{\leq} + e}{1+2e} \geq \frac{1/2+e}{1+2e} = \frac{1}{2} = \frac{1}{c-1}$. Assume next that $c > 3$ and that the result has been proved for $c - 1$ (in particular for L/Z). Then $\delta_L^{\leq} \geq \frac{\delta_{L/Z}^{\leq} + e}{1+(c-1)e} \geq \frac{1/(c-2)+e}{1+(c-1)e} > \frac{1/(c-1)+e}{1+(c-1)e} = \frac{1}{c-1}$. This completes the induction and the proof of (1).

We now prove (2), which is equivalent to $\delta_L^{\triangleleft} \geq 1$, by induction on c . If $c = 2$, then the result follows from [12, Proposition 6.3]. Assume now that $c > 2$ and that the result has been proved for $c - 1$. By Proposition 4.2 we have $\delta_L^{\triangleleft} \geq \frac{\delta_{L/Z}^{\triangleleft} + e}{1+e} \geq \frac{1+e}{1+e} = 1$. This completes the induction and the proof of (2). \square

5 A version of Theorem B for virtually nilpotent groups

Let G be a finitely generated virtually nilpotent group and let N be its Fitting subgroup, that is, the maximal nilpotent normal subgroup. It is known that $\alpha_G^* \leq \alpha_N^* + 1$ (cf. [18, Proposition 5.6.4], [30, Theorem 3]), and the next example shows that the equality might hold.

Example 5.1 Let $N = \mathbb{Z}$ and $G = \mathbb{Z} \rtimes \text{Aut}(\mathbb{Z})$. Then $\zeta_N^{\leq}(s) = \zeta(s)$ and $\zeta_G^{\leq}(s) = 2^{-s}\zeta(s) + \zeta(s - 1)$. In particular, $\alpha_N^{\leq} = 1$ and $\alpha_G^{\leq} = 2$.

It follows that α_G^* is not longer commensurability-invariant within the class of finitely generated virtually nilpotent groups. This notwithstanding, it is possible to formulate a version of Theorem B for virtually nilpotent groups. First of all, given G and N as above, we may assume that N is torsion-free, that is, a \mathfrak{T} -group. In fact, the torsion subgroup $t(N)$ of N is a finite normal subgroup of G and the next lemma shows that $\alpha_G^* = \alpha_{G/t(N)}^*$.

Lemma 5.2 *Let G be a group of finite rank and T a finite normal subgroup of G . Write $Q = G/T$. Then $\alpha_G^* = \alpha_Q^*$ and $\zeta_{G,p}^*(s) = \zeta_{Q,p}^*(s)$ for every prime p not dividing $|T|$.*

Proof The equality $\alpha_G^{\leq} = \alpha_Q^{\leq}$ is proved in [18, Proposition 5.6.2]. We adapt that proof to show that $\alpha_G^{\triangleleft} = \alpha_Q^{\triangleleft}$. Clearly $\alpha_G^{\triangleleft} \geq \alpha_Q^{\triangleleft}$, so we only need to focus on the reverse inequality.

Fix a positive integer n . A normal subgroup $H \triangleleft G$ of index n determines normal subgroups $H \cap T \triangleleft T$ and $HT \triangleleft G$. The index $[T : H \cap T]$ divides both n and $|T|$, and we have $[G : HT] = n/[T : H \cap T]$. Now fix a common divisor, say t , of $|T|$ and n . Fix also $D \triangleleft T$ and $B \triangleleft G$ such that $T \subset B$, $[T : D] = t$ and $[G : B] = n/t$. If there is $H \triangleleft G$ such that $H \cap T = D$ and $HT = B$, then necessarily $[G : H] = n$, D is normal in G , and H/D is a complement of T/D in HT/D . Therefore, there are at most $|\text{Der}(B/T, T/D)|$ possibilities for H , and this number is turn bounded by $[T : D]^{\text{rk}(Q)} \leq |T|^{\text{rk}(Q)}$; cf. [18, Lemma 1.3.1]. It follows that

$$a_n^{\triangleleft}(G) \leq \sum_{t|n, t||T|} a_t^{\triangleleft}(T) a_{n/t}^{\triangleleft}(Q) |T|^{\text{rk}(Q)} \leq |T|^{\text{rk}(Q)} \zeta_T^{\triangleleft}(0) \sum_{t|n, t||T|} a_{n/t}^{\triangleleft}(Q).$$

By using the above inequality we now conclude that for any positive integer n ,

$$\sum_{j=1}^n a_j^{\triangleleft}(G) \leq |T|^{\text{rk}(Q)} \zeta_T^{\triangleleft}(0) \sum_{t||T|} \sum_{t|j \leq n} a_{j/t}^{\triangleleft}(Q) \leq |T|^{\text{rk}(Q)+1} \zeta_T^{\triangleleft}(0) \sum_{j=1}^n a_j^{\triangleleft}(Q)$$

and this clearly implies that $\alpha_G^{\triangleleft} \leq \alpha_Q^{\triangleleft}$.

The equality $\zeta_{G,p}^*(s) = \zeta_{Q,p}^*(s)$ for a prime $p \nmid |T|$ holds because the index of any subgroup $H \leq G$ is divisible by $[TH : H] = [T : T \cap H]$, hence if $[G : H]$ is a power of p we have $[T : T \cap H] = 1$, that is $H \supseteq T$. \square

We return to the set-up introduced at the beginning of the section. We will also assume from now on that N is a \mathfrak{T} -group. Consider now the induced group extension $S : 1 \rightarrow N \rightarrow G \xrightarrow{\pi} F \rightarrow 1$. It is easy to check that

$$\zeta_G^{\leq}(s) = \sum_{E \leq F} [F : E]^{-s} \zeta_{S,E}^{\leq}(s), \quad \zeta_G^{\triangleleft}(s) = \sum_{E \triangleleft F} [F : E]^{-s} \zeta_{S,E}^{\triangleleft}(s),$$

where

$$\zeta_{S,E}^{\leq}(s) := \sum_{A \leq_f G : \pi(AN) = E} [\pi^{-1}(E) : A]^{-s}, \quad \zeta_{S,E}^{\triangleleft}(s) := \sum_{A \triangleleft_f G : \pi(AN) = E} [\pi^{-1}(E) : A]^{-s}.$$

We denote by $\alpha_{S,E}^{\leq}$ or $\alpha_{S,E}^{\triangleleft}$ the abscissae of convergence of $\zeta_{S,E}^{\leq}(s)$ and $\zeta_{S,E}^{\triangleleft}(s)$. We recall the following result.

Theorem 5.3 [30] *Let $*$ \in $\{\leq, \triangleleft\}$, and let $E \leq F$, where E is normal if $*$ $=$ \triangleleft . Then there exists a cone integral data \mathcal{D}_E^* over \mathbb{Q} such that $\zeta_{S,E}^*(s) = Z_{\mathcal{D}_E^*}(s - h(N) - |E| + 1)$. Therefore, $\alpha_{S,E}^*$ is a rational number and $\zeta_{S,E}^*(s)$ has meromorphic continuation to a region of the form $\text{Re}(s) > \alpha_{S,E}^* - \delta$ for some $\delta > 0$.*

It follows that α_G^* is a rational number and that $\zeta_G^*(s)$ has meromorphic continuation to a region of the form $\text{Re}(s) > \alpha_G^* - \delta$ for some $\delta > 0$.

5.1 The case of virtually abelian groups

To motivate the formulation of Theorem B for virtually nilpotent groups, we will make a digression and discuss the case where N is abelian. We will change the notation and write T instead of N . In this case, a formula for $\zeta_{S,E}^*(s)$ (up to a finite number of local factors) was given in [11], and this suffices to read off the abscissa of convergence (Theorem 2.2(2)). We will recall this result (see Proposition 5.7 below) after introducing some notation.

Let F be a finite group and let V be a $\mathbb{Q}[F]$ -module of finite dimension over \mathbb{Q} .

- (1) Let $\mathbb{Q}[F] = A_0 \oplus A_1 \oplus \dots \oplus A_r$ be a decomposition of $\mathbb{Q}[F]$ into simple components. Then A_i is isomorphic to a matrix algebra $M_{m_i}(D_i)$ for some central division algebra D_i over a number field K_i . Assume that $A_0 = \mathbb{Q} \cdot \sum_{\gamma \in F} \gamma$, so that $D_0 = K_0 = \mathbb{Q}$ and $m_0 = 1$.
- (2) Let $n_i^2 = \dim_{K_i} A_i = m_i^2 e_i^2$, where $e_i^2 = \dim_{K_i} D_i$.
- (3) Let $V = V_0 \oplus V_1 \oplus \dots \oplus V_r$, where $V_i = A_i V$. Then $V_i \cong (D_i^{m_i})^{k_i}$ (as A_i -modules) for some integer $k_i \geq 0$. Note that $V_0 \cong \mathbb{Q}^{k_0}$ is the set of fixed points of F .

Define

$$\zeta_{F \curvearrowright V}^{\leq}(s) := \prod_{j=0}^{k_0-1} \zeta(s-j) \cdot \prod_{i=1}^r \prod_{j=0}^{k_i e_i - 1} \zeta_{K_i}(n_i(s-1) - j),$$

$$\zeta_{F \curvearrowright V}^{\triangleleft}(s) := \prod_{j=0}^{k_0-1} \zeta(s-j) \cdot \prod_{i=1}^r \prod_{j=0}^{k_i e_i - 1} \zeta_{K_i}(n_i s - j),$$

where $\zeta_{K_i}(s)$ is the Dedekind zeta function of K_i . Let $\alpha_{F \curvearrowright V}^*$ denote the abscissa of convergence of $\zeta_{F \curvearrowright V}^*(s)$. Since $\zeta_{K_i}(s)$ has abscissa of convergence 1, we conclude that

$$\alpha_{F \curvearrowright V}^{\leq} = \max \left\{ k_0, \frac{k_1}{m_1} + 1, \dots, \frac{k_r}{m_r} + 1 \right\}, \quad \alpha_{F \curvearrowright V}^{\triangleleft} = \max \left\{ k_0, \frac{k_1}{m_1}, \dots, \frac{k_r}{m_r} \right\}.$$

Remark 5.4 A quick analysis shows that either $\alpha_{F \curvearrowright V}^{\leq} \leq \dim V$ or else $\alpha_{F \curvearrowright V}^{\leq} = \dim V + 1$, in which case the action of F on V is non-trivial, every $f \in F$ acts on V either as the identity or as minus the identity, and $\zeta_{F \curvearrowright V}^{\leq}(s) = \zeta_V(s - 1)$, where $\zeta_V(s) := \prod_{j=0}^{\dim V - 1} \zeta(s - j)$.

Remark 5.5 Note that after base change with \mathbb{C} , A_i decomposes as a product of $[K_i : \mathbb{Q}]$ copies of $M_{m_i e_i}(\mathbb{C})$, and V_i decomposes accordingly as a product of $[K_i : \mathbb{Q}]$ copies of $(\mathbb{C}^{m_i e_i})^{e_i k_i}$. Since $\frac{e_i k_i}{e_i m_i} = \frac{k_i}{m_i}$, the numbers $\alpha_{F \curvearrowright V}^{\leq}$ and $\alpha_{F \curvearrowright V}^{\triangleleft}$ can still be read off from the $\mathbb{C}[F]$ -module $V \otimes_{\mathbb{Q}} \mathbb{C}$. Hence:

Corollary 5.6 *Let V_1 and V_2 be $\mathbb{Q}[F]$ -modules of finite dimension over \mathbb{Q} . If $V_1 \otimes_{\mathbb{Q}} \mathbb{C} \cong V_2 \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic as $\mathbb{C}[F]$ -modules, then $\alpha_{F \curvearrowright V_1}^* = \alpha_{F \curvearrowright V_2}^*$.*

Now, let G be a finitely generated virtually abelian group with torsion-free Fitting subgroup $T \triangleleft G$ (hence $T \cong \mathbb{Z}^h$ for some h), and let $S : 1 \rightarrow T \rightarrow G \xrightarrow{\pi} F \rightarrow 1$ be the associated group extension. Set $V := T \otimes_{\mathbb{Z}} \mathbb{Q}$, which is naturally a $\mathbb{Q}[F]$ -module. Note that if $E \triangleleft F$ is a normal subgroup, then the 0-homology $H_0(E, V)$ of E with coefficients in V is also a $\mathbb{Q}[F]$ -module.

Proposition 5.7 [11, Sect. 2] *For each subgroup $E \leq F$, the series $\zeta_{S,E}^{\leq}(s)$ and $\zeta_{E \curvearrowright V}^{\leq}(s)$ have the same local factor at p for almost all primes p . For each normal subgroup $E \triangleleft F$, the series $\zeta_{S,E}^{\triangleleft}(s)$ and $\zeta_{F \curvearrowright H_0(E,V)}^{\triangleleft}(s)$ have the same local factor at p for almost all primes p .*

Corollary 5.8 *For each $E \leq F$ we have $\alpha_{S,E}^{\leq} = \alpha_{E \curvearrowright V}^{\leq}$, and for each $E \triangleleft F$ we have $\alpha_{S,E}^{\triangleleft} = \alpha_{F \curvearrowright H_0(E,V)}^{\triangleleft}$.*

Proof By Theorem 5.3 and Theorem 2.2(2), we can disregard a finite number of local factors in the computation of $\alpha_{S,E}^*$. By the definition of $\zeta_{E \curvearrowright V}^{\leq}(s)$ and $\zeta_{F \curvearrowright H_0(E,V)}^{\triangleleft}(s)$, we can also disregard a finite number of local factors in the computation of $\alpha_{E \curvearrowright V}^{\leq}$ or $\alpha_{F \curvearrowright H_0(E,V)}^{\triangleleft}$. Thus, the corollary follows from Proposition 5.7. \square

Corollary 5.9 *Either $\alpha_G^{\leq} = \alpha_T^{\leq}$ or else $\alpha_G^{\leq} = \alpha_T^{\leq} + 1$. Moreover, the latter occurs if and only if some element of F acts as minus the identity on T .*

Proof We have $\alpha_G^{\leq} = \max\{\alpha_{S,E}^{\leq} : E \leq F\}$ and we know that $\alpha_G^{\leq} \geq \alpha_T^{\leq} = h$, where h is the rank of T . By Corollary 5.8 and Remark 5.4 we find that $\alpha_{S,E}^{\leq}$ is either $h + 1$ or $\alpha_{S,E}^{\leq} \leq h$. This proves that α_G^{\leq} is either h or $h + 1$. If $\alpha_G^{\leq} = h + 1$, then $\alpha_{S,E}^{\leq} = h + 1$ for some non-trivial $E \leq F$, and by Remark 5.4 there is a non-trivial element of E that acts on T as minus the identity. Conversely, if there is $f \in F$ that acts on T as minus the identity, then we can replace f by some power and assume that $f^2 = 1$. If $E = \langle f \rangle$, then $\alpha_{S,E}^{\leq} = h + 1$ by Remark 5.4. \square

Proposition 5.10 *For each $i \in \{1, 2\}$, let G_i be a finitely generated virtually abelian group with torsion-free Fitting subgroup T_i , and let $S_i : 1 \rightarrow T_i \rightarrow G_i \rightarrow F_i \rightarrow 1$ be the induced group extension. Assume that there is a \mathbb{C} -linear isomorphism $\alpha : T_1 \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow T_2 \otimes_{\mathbb{Z}} \mathbb{C}$ and a group isomorphism $\gamma : P_1 \rightarrow P_2$ such that $\alpha(f \cdot v) = \gamma(f) \cdot \alpha(v)$ for all $v \in T_1 \otimes_{\mathbb{Z}} \mathbb{C}$ and $f \in F_1$. Then for each $E_1 \leq F_1$ it holds that $\alpha_{S_1,E_1}^{\leq} = \alpha_{S_2,\gamma(E_1)}^{\leq}$, and for each $E_1 \triangleleft F_1$ it holds that $\alpha_{S_1,E_1}^{\triangleleft} = \alpha_{S_2,\gamma(E_1)}^{\triangleleft}$. In particular, $\alpha_{G_1}^{\leq} = \alpha_{G_2}^{\leq}$ and $\alpha_{G_1}^{\triangleleft} = \alpha_{G_2}^{\triangleleft}$.*

Proof This follows from Corollary 5.8 and Corollary 5.6. □

5.2 The R -Mal'cev completion for virtually nilpotent groups

Proposition 5.10 is our version of Theorem B for virtually abelian groups, and we want to formulate a similar result for virtually nilpotent groups. To do this we recall the notion of Mal'cev completion for virtually nilpotent groups [30, Section 1]. We begin by reviewing the definition of nilpotent R -powered groups.

Definition 5.11 Let $c \in \mathbb{N}$. A commutative ring R is said to be c -binomial if $R \rightarrow R \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective and if $\binom{r}{k} := \frac{r(r-1)\dots(r-k+1)}{k!}$ belongs to R for all $r \in R$ and $k = 1, \dots, c$. For such a ring, a nilpotent group N of nilpotency class $\leq c$ is said to be R -powered if for all $r \in R$ and $n \in N$, an element $n^r \in N$ has been defined such that the following holds:

- (i) $n^1 = n, n^{r_1+r_2} = n^{r_1}n^{r_2}, (n^{r_1})^{r_2} = n^{r_1r_2}$ for all $n \in N, r_1, r_2 \in R$.
- (ii) $m^{-1}n^r m = (m^{-1}nm)^r$ for all $m, n \in N, r \in R$.
- (iii) The Hall-Petresco formula holds for all k -tuples (n_1, \dots, n_k) of elements of N and all $r \in R$ [34, Chap., 6].

Note that (iii) makes sense by [34, Theorem 6.1] since N has nilpotency class $\leq c$ and therefore only the first c binomials $\binom{r}{1}, \dots, \binom{r}{c}$ appear in the formula.

A morphism $\varphi : N \rightarrow M$ of nilpotent R -powered groups of nilpotency class $\leq c$ is a group homomorphism such that $\varphi(n^r) = \varphi(n)^r$ for all $n \in N$ and $r \in R$. They will be called R -morphisms.

Definition 5.12 Let N be a \mathfrak{T} -group, say of nilpotency class c . Let R be a c -binomial ring. The R -Mal'cev completion of N is a nilpotent R -powered group N^R (necessarily of the same nilpotency class as N) together with a homomorphism $\iota : N \rightarrow N^R$ satisfying the following universal property: if M is another nilpotent R -powered group of nilpotency class $\leq c$ and $\varphi : N \rightarrow M$ is a group homomorphism, then there exists a unique R -morphism $\tilde{\varphi} : N^R \rightarrow M$ such that $\tilde{\varphi} \circ \iota = \varphi$.

The theory of nilpotent R -powered groups, in particular the proof of the existence of the R -Mal'cev completion for \mathfrak{T} -groups, is expounded in [34, Chapters 10 and 11] under the assumption that R is a binomial domain (i.e. $\binom{r}{k} \in R$ for all $r \in R$ and all $k \in \mathbb{N}$). However, everything can be extended without further modifications to c -binomial rings in the case of nilpotency class $\leq c$.

Remark 5.13 The unipotent group scheme \mathfrak{N} over \mathbb{Q} defined by a \mathfrak{T} -group N is precisely the group scheme that represents the functor $K \mapsto N^K$ from commutative \mathbb{Q} -algebras to groups. If \mathfrak{N}_1 and \mathfrak{N}_2 are the unipotent group schemes over \mathbb{Q} defined respectively by two \mathfrak{T} -groups N_1 and N_2 , then \mathfrak{N}_1 and \mathfrak{N}_2 are isomorphic after base change with a field $K \supset \mathbb{Q}$ if and only if N_1^K and N_2^K are isomorphic as nilpotent K -powered groups.

Definition 5.14 Given $c \in \mathbb{N}$ and a c -binomial ring R , we define a category $\mathcal{V}_{c,R}$ as follows. The objects are group extensions $S : 1 \rightarrow N \rightarrow G \rightarrow F \rightarrow 1$, where N is a nilpotent R -powered group of nilpotency class $\leq c$, F is a finite group, and it is required that for any $g \in G$, the automorphism of N induced by conjugation by g is an R -automorphism. The morphisms in $\mathcal{V}_{c,R}$ are morphisms of short exact sequences of groups $(u, v, w) : S \rightarrow S'$ such that u is an R -morphism. We also call them R -morphisms.

Definition 5.15 Let $S : 1 \rightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} F \rightarrow 1$ be an object of $\mathcal{V}_{c, \mathbb{Z}}$, where N is a \mathfrak{T} -group, and let R be a c -binomial ring. The R -Mal'cev completion of S is an object S^R of $\mathcal{V}_{c, R}$ together with a morphism of short exact sequences $(i, j, k) : S \rightarrow S^R$ satisfying the following universal property: if $(u, v, w) : S \rightarrow T$ is a morphism of short exact sequences, where T is an object of $\mathcal{V}_{c, R}$, then there exists a unique R -morphism $(\tilde{u}, \tilde{v}, \tilde{w}) : S^R \rightarrow T$ such that $(\tilde{u}, \tilde{v}, \tilde{w}) \circ (i, j, k) = (u, v, w)$.

The following construction of the R -Mal'cev completion of S was given in [30, Section 1] under the assumption that R is binomial. However, everything remains valid in our situation. We may assume that ι is an inclusion and that π is a quotient map. Let $s : F \rightarrow G$ be a section of π (i.e. $\pi \circ s = \text{id}_F$) such that $s(1) = 1$ and $s(f^{-1}) = s(f)^{-1}$ for all $f \in F$. Then there are maps $\sigma : F \rightarrow \text{Aut}(N)$ and $\psi : F \times F \rightarrow N$ such that for all $f, f' \in F$ and $n \in N$ we have $\sigma(f)(n) = s(f)ns(f)^{-1}$ and $s(f)s(f') = \psi(f, f')s(ff')$. The pair (σ, ψ) is called *the cocycle associated to S and the section s* . It satisfies the following cocycle conditions:

$$\begin{aligned} \sigma(f)\sigma(f') &= \mu(\psi(f, f'))\sigma(ff') \quad \forall f, f' \in F \\ \psi(f, f')\psi(ff', f'') &= \sigma(f)(\psi(f, f'))\psi(f, f'f'') \quad \forall f, f', f'' \in F. \end{aligned} \tag{5.1}$$

The group G can be identified with the group $N \times_{(\sigma, \psi)} F$ whose underlying set is $N \times F$ and where the operations are given by

$$(n, f) \cdot (n', f') = (n\sigma(f)(n')\psi(f, f'), ff'). \tag{5.2}$$

Under this identification, N becomes $N \times \{1\}$.

We now consider the R -Mal'cev completion $N \hookrightarrow N^R$ of N . Note that σ can be extended to a map $F \rightarrow \text{Aut}_R(N)$, and ψ can be seen as a map $F \times F \rightarrow N^R$. We still denote these extensions by σ and ψ . We obtain a group $N^R \times_{(\sigma, \psi)} F$ whose underlying set is $N^R \times F$ and where the operations are given by (5.2). Indeed, we get a group extension $1 \rightarrow N^R \rightarrow N^R \times_{(\sigma, \psi)} F \rightarrow F \rightarrow 1$, which is an object in $\mathcal{V}_{c, R}$. This extension together with the inclusions $N \hookrightarrow N^R, N \times_{(\sigma, \psi)} F \hookrightarrow N^R \times_{(\sigma, \psi)} F, F = F$, is in fact the R -Mal'cev completion S^R of S .

Remark 5.16 For $i \in \{1, 2\}$ let G_i be a finitely generated virtually nilpotent group with torsion-free Fitting subgroup N_i , and let $S_i : 1 \rightarrow N_i \rightarrow G_i \rightarrow F_i \rightarrow 1$ be the associated group extension. We may assume that $G_i = N_i \times_{(\sigma_i, \psi_i)} F_i$, with (σ_i, ψ_i) satisfying the cocycle (5.1). Let $c \in \mathbb{N}$ be an upper bound for the nilpotency classes of N_1 and N_2 , and let R be a c -binomial domain. We describe what an R -isomorphism between S_1^R and S_2^R is. By definition this is a morphism of exact sequences $(u, v, w) : S_1^R \rightarrow S_2^R$, where u is an R -isomorphism and $w : F_1 \rightarrow F_2$ is an isomorphism. Note that v has the form

$$v(n, f) = (u(n)\tau(f), w(f)) \tag{5.3}$$

for some map $\tau : F_1 \rightarrow N_2^R$. One can easily check that given an R -isomorphism $u : N_1^R \rightarrow N_2^R$, a group isomorphism $w : F_1 \rightarrow F_2$, and a map $\tau : F_1 \rightarrow N_2^R$, if we define v as in (5.3), then (u, v, w) is an R -isomorphism between S_1^R and S_2^R if and only if for all $f, f' \in F$ and $n \in N$,

$$u(\sigma_1(f)(n))u(\psi(f, f'))\tau(ff') = \tau(f)\sigma_2(w(f))(u(n))\sigma_2(w(f))(\tau(f'))\psi_2(w(f), w(f')). \tag{5.4}$$

Remark 5.17 Let G be a finitely generated virtually nilpotent group with torsion-free Fitting subgroup N , and let $S : 1 \rightarrow N \rightarrow G \rightarrow F \rightarrow 1$ be the associated group extension.

We may assume that $G = N \times_{(\sigma, \psi)} F$, with (σ, ψ) satisfying the cocycle condition (5.1). For each \mathbb{Q} -commutative algebra K we define $\mathfrak{G}(K) := N^K \times_{(\sigma, \psi)} F$. Then \mathfrak{G} is an affine group scheme over \mathbb{Q} isomorphic as scheme to $\dot{\cup}_{f \in F} \mathbb{A}_{\mathbb{Q}}^h$, where $h := h(N)$. It has \mathfrak{N} , the unipotent group scheme defined by N , as its connected component. It is easy to check that \mathfrak{G} is well-defined up to \mathbb{Q} -isomorphism. We call \mathfrak{G} the group scheme defined by G . Let G' be another finitely generated virtually nilpotent group with torsion-free Fitting subgroup N' , let S' be the associated group extension, and let \mathfrak{G}' be the group scheme over \mathbb{Q} defined by G' . It is not difficult to show that \mathfrak{G} and \mathfrak{G}' are isomorphic after base change with a field $K \supset \mathbb{Q}$ if and only if S_1^K and S_2^K are K -isomorphic.

5.3 Theorem B for virtually nilpotent groups

We can now state the analogous of Theorem B for virtually nilpotent groups, which also extends Proposition 5.10.

Theorem 5.18 *Let G_i be a finitely generated virtually nilpotent group with torsion-free Fitting subgroup N_i and let $S_i : 1 \rightarrow N_i \rightarrow G_i \rightarrow F_i \rightarrow 1$ be the induced group extension for $i = 1, 2$. Assume that there is a \mathbb{C} -isomorphism $(u, v, w) : S_1^{\mathbb{C}} \cong S_2^{\mathbb{C}}$. Then for each $E_1 \leq F_1$ we have $\alpha_{S_1, E_1}^{\leq} = \alpha_{S_2, w(E_1)}^{\leq}$, and for each $E_1 \triangleleft F_1$ we have $\alpha_{S_1, E_1}^{\triangleleft} = \alpha_{S_2, w(E_1)}^{\triangleleft}$.*

The rest of the section is devoted to the proof of this theorem. The idea is similar to that of the proof of Theorem C given in Section 3.

Lemma 5.19 *For $i \in \{1, 2\}$, let G_i be a finitely generated virtually nilpotent group with torsion-free Fitting subgroup N_i , and let $S_i : 1 \rightarrow N_i \rightarrow G_i \rightarrow F_i \rightarrow 1$ be the associated group extension. Assume that $F_1 \cong F_2$ and let $w_0 : F_1 \rightarrow F_2$ be an isomorphism. If for some field extension $K \supset \mathbb{Q}$ there is K -isomorphism $(u, v, w) : S_1^K \cong S_2^K$ with $w = w_0$, then this also holds for some number field.*

Proof We can assume that $G_i = N_i \times_{(\sigma_i, \psi_i)} F_i$, with (σ_i, ψ_i) satisfying the cocycle condition (5.1). Let I_{w_0} be the functor from the category of commutative algebras over \mathbb{Q} to the category of sets that is defined by:

$$I_{w_0}(K) = \{(u, \tau) \mid u : N_1^K \rightarrow N_2^K \text{ is a } K\text{-isomorphism and } \tau : F_1 \rightarrow N_2^K \text{ satisfies 5.4}\}. \tag{5.5}$$

By using the log isomorphism between the unipotent group scheme over \mathbb{Q} defined by N_i and its Lie algebra (viewed as functor on commutative algebras over \mathbb{Q}) and the fact that, owing to nilpotency, the Baker-Campbell-Hausdorff formula is finite, one can easily show that I_{w_0} is represented by a closed subscheme of $\text{GL}_h \times \prod_{f \in F_1} \mathbb{A}^h \times \text{Spec}(\mathbb{Q})$, where $h := h(N_1) = h(N_2)$. The hypothesis implies that I_{w_0} is not the empty scheme by Remark 5.16. If K is the residue field at a closed point of I_{w_0} , then K is a number field and $I_{w_0}(K) \neq \emptyset$. Again by Remark 5.16, a pair $(u, \tau) \in I_{w_0}(K)$ yields a K -isomorphism $(u, v, w) : S_1^K \rightarrow S_2^K$, with v defined by (5.3) and $w = w_0$. \square

Definition 5.20 Let R be a c -binomial ring, and let $S : 1 \rightarrow N \rightarrow G \xrightarrow{\pi} F \rightarrow 1$ be an object of $\mathcal{V}_{c,R}$. For each subgroup $E \leq F$ we define the following formal series:

$$\zeta_{S,E}^{\leq}(s) := \sum_{\substack{A \leq_f G: \\ \pi(A) = E \wedge A \cap N \leq_R N}} [\pi^{-1}(E) : A]^{-s}, \quad \zeta_{S,E}^{\triangleleft}(s) := \sum_{\substack{A \triangleleft_f G: \\ \pi(A) = E \wedge A \cap N \leq_R N}} [\pi^{-1}(E) : A]^{-s}, \tag{5.6}$$

where the notation $A \cap N \leq_R N$ means that $A \cap N$ is an R -subgroup of N , that is, a subgroup such that $n^r \in A \cap N$ for all $n \in A \cap N$ and $r \in R$.

Proposition 5.21 *Let G be a finitely generated virtually nilpotent group with torsion-free Fitting subgroup N , say of nilpotency class c and Hirsch length h , and let $S : 1 \rightarrow N \rightarrow G \xrightarrow{\pi} F \rightarrow 1$ be the associated extension. Let $*$ $\in \{\leq, \triangleleft\}$. Then for each $E \leq F$, with E normal if $*$ $= \triangleleft$, there exists a cone integral data \mathcal{D}_E^* over \mathbb{Q} such that the following holds.*

- (1) *For each prime p we have $\zeta_{S_{\mathbb{Z}_p, E}}^*(s) = (1 - p^{-1})^h Z_{\mathcal{D}_E^*}(s - h - |E| + 1, p)$.*
- (2) *Let K be a number field and \mathcal{O} its ring of integers. For each maximal ideal $\mathfrak{p} \subset \mathcal{O}$ for which $\mathcal{O}_{\mathfrak{p}}$ is c -binomial (e.g. $c! \notin \mathfrak{p}$) we have $\zeta_{S_{\mathcal{O}_{\mathfrak{p}}, E}}^*(s) = (1 - \mathbf{N}\mathfrak{p}^{-1})^{-h} Z_{\mathcal{D}_E^* \otimes_{\mathbb{Q}} K}(s - h - |E| + 1, \mathfrak{p})$, where $\mathcal{D}_E^* \otimes_{\mathbb{Q}} K$ denotes the collection \mathcal{D}_E^* viewed as cone integral data over K .*

Proof A construction of a collection \mathcal{D}_E^* that satisfies (1) was obtained in [30, Section 2.2]. With the same proof one can show that this collection also satisfies (2). □

Proof of Theorem 5.18: Lemma 5.19 enables us to replace \mathbb{C} by a number field K in the hypothesis of the theorem without modifying w . We can assume that $G_i = N_i \times_{(\sigma_i, \psi_i)} F_i$, with (σ_i, ψ_i) satisfying the cocycle condition (5.1). Let $\tau : F_1 \rightarrow N_2^K$ be the map of Remark 5.16 defined from (u, v, w) . Let $\{x_1, \dots, x_h\}$ be a Mal'cev basis for N_1 and $\{y_1, \dots, y_h\}$ a Mal'cev basis for N_2 .

We denote by T the set of maximal ideals $\mathfrak{p} \subset \mathcal{O} := \mathcal{O}_K$ satisfying the following conditions:

- (1) $\mathcal{O}_{\mathfrak{p}}$ is c -binomial (e.g. $c! \notin \mathfrak{p}$) so that $N_1^{\mathcal{O}_{\mathfrak{p}}}$ and $N_2^{\mathcal{O}_{\mathfrak{p}}}$ are $\mathcal{O}_{\mathfrak{p}}$ -powered groups;
- (2) $u(x_i), \tau(f) \in N_2^{\mathcal{O}_{\mathfrak{p}}}$ for all $i = 1, \dots, h$ and $f \in F_1$;
- (3) $u^{-1}(y_i) \in N_1^{\mathcal{O}_{\mathfrak{p}}}$ for all $i = 1, \dots, h$;

Note that almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}$ are in T . For such a \mathfrak{p} , conditions (2) and (3) imply that u induces an $\mathcal{O}_{\mathfrak{p}}$ -isomorphism $N_1^{\mathcal{O}_{\mathfrak{p}}} \rightarrow N_2^{\mathcal{O}_{\mathfrak{p}}}$. According to Remark 5.16, (u, v, w) induces an isomorphism $S_1^{\mathcal{O}_{\mathfrak{p}}} \rightarrow S_2^{\mathcal{O}_{\mathfrak{p}}}$.

Fix $E_1 \leq F_1$, with E_1 normal if $*$ $= \triangleleft$, and set $E_2 := w(E_1)$. We consider the cone integral data $\mathcal{D}_{E_i}^*$ of Proposition 5.21 applied to S_i . From the above paragraph we deduce that $Z_{\mathcal{D}_{E_1}^* \otimes K}(s - h - |E_1| + 1, \mathfrak{p}) = Z_{\mathcal{D}_{E_2}^* \otimes K}(s - h - |E_2| + 1, \mathfrak{p})$ for almost all maximal ideals $\mathfrak{p} \subset \mathcal{O}$, where $\mathcal{D}_{E_i}^* \otimes K$ denotes the same collection $\mathcal{D}_{E_i}^*$ viewed as cone integral data over K . Therefore, by Corollary 2.4(2), $Z_{\mathcal{D}_{E_1}^*}(s - h - |E_1| + 1)$ and $Z_{\mathcal{D}_{E_2}^*}(s - h - |E_2| + 1)$ have the same abscissa of convergence. Finally, by Proposition 5.21, $\zeta_{S_1, E_1}^*(s)$ and $\zeta_{S_2, E_2}^*(s)$ have the same abscissa of convergence. □

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