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On the microlocal regularity of the analytic vectors for "sums of squares" of vector fields

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Abstract
We prove via FBI-transform a result concerning the microlocal Gevrey regularity of analytic vectors for operators sums of squares of vector fields with real-valued real analytic coefficients of Hörmander type, thus providing a microlocal version, in the analytic category, of a result due to Derridj (Pac J Math 302(2):511–543, 2019) concerning the problem of the local regularity for the Gevrey vectors for sums of squares of vector fields with real-valued real analytic/Gevrey coefficients.

Keywords Sums of squares · Microlocal regularity · Analytic vectors · Gevrey regularity

Mathematics Subject Classification 35H10 · 35H20 · 35B65

1 Introduction

We deal with the microlocal regularity of the analytic vectors for sum of squares of vector fields. Let $X_1(x, D), \dots, X_m(x, D)$ be vector fields with real-valued real analytic coefficients on U, open neighborhood of the origin in $\mathbb{R}^n$. Let $P(x, D)$ denote the corresponding sum of squares operator

$$P(x, D) = \sum_{j=1}^{m} X_j^2(x, D). \tag{1.1}$$

We assume that the operator P satisfies the Hörmander's condition: the Lie algebra generated by the vector fields and their commutators has the dimension n, equal to the dimension of

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the ambient space. The operator P satisfies the *a priori* estimate

$$\|u\|^2_{1/r} + \sum_{j=1}^{m} \|X_j u\|_0^2 \le C\left(|\langle Pu, u\rangle| + \|u\|_0^2\right), \tag{1.2}$$

which we call, for the sake of brevity, the "subelliptic estimate." Here $u \in C_0^\infty(U)$, $\|\cdot\|_0$ denotes the norm in $L^2(U)$ and $\|\cdot\|_s$ the Sobolev norm of order s in U. Here r is the least integer such that the vector fields, the commutators, the triple commutators etcetera up to the commutators of length r span at any point of the closure of U all the ambient space $\mathbb{R}^n$. The sub-elliptic estimate was proved first by Hörmander in [18] for a Sobolev norm of order $r^{-1} - \varepsilon$ and up to order r^{-1} subsequently by Rothschild and Stein [22] as well as in a pseudodifferential context by Bolley, Camus and Nourrigat in [7].
Let $X_j(x, \xi)$ be the symbol of the vector field X_j. Write $\{X_i, X_k\}$ the Poisson bracket of the symbols of the vector fields X_i, X_k:

$$\{X_i, X_k\}(x, \xi) = \sum_{\ell=1}^{n} \left(\frac{\partial X_i}{\partial \xi_\ell}\frac{\partial X_k}{\partial x_\ell} - \frac{\partial X_k}{\partial \xi_\ell}\frac{\partial X_i}{\partial x_\ell}\right)(x, \xi).$$

Definition 1.1 Let (x_0, ξ_0) be a point in the characteristic set of P:

$$\mathrm{Char}(P) = \{(x, \xi) \in T^*U \setminus \{0\} \,:\, X_j(x, \xi) = 0,\ j = 1, \dots m\}. \tag{1.3}$$

Consider all the iterated Poisson brackets $\{X_i, X_k\}$, $\{X_p, \{X_i, X_k\}\}$ etcetera. We define $\nu(x_0, \xi_0)$ as the length of the shortest iterated Poisson bracket of the symbols of the vector fields which is non zero at (x_0, ξ_0).

We recall

Definition 1.2 Let $P(x, D)$ be as in (1.1). We denote by $G^s(U; P)$ which is the space of the Gevrey vectors of order s with respect to P, the set of all distributions $u \in \mathscr{D}'(U)$ such that for any compact subset K of U there exists a positive constant C_K such that

$$\|P^N u\|_{L^2(K)} \le C_K^{2N+1}((2N)!)^s, \qquad \forall N \in \mathbb{Z}_+. \tag{1.4}$$

When $s = 1$ we set $G^1(U; P) = \mathscr{A}(U; P)$ the set of the analytic vectors with respect to P.

We recall that concerning systems of vector fields with real analytic coefficients satisfying Hörmander's condition the problem of the local regularity of the analytic vectors for such systems was first studied in [11] followed by a more refined version in [16].

In a couple of recent works Derridj, [12] and [13], studied the problem of the local regularity for the Gevrey vectors for operators of Hörmander type of first kind, i.e. sum of squares, and of the second kind or degenerate elliptic parabolic. We prove the minimal microlocal version of the result in [12] in the case of analytic vectors:

Theorem 1.1 *Let P be as in (1.1). Let u be an analytic vector for P, $u \in \mathscr{A}(U; P)$. Let (x_0, ξ_0) be a point in the characteristic set of P and $\nu(x_0, \xi_0)$ its length. Then $(x_0, \xi_0) \notin WF_{\nu(x_0,\xi_0)}(u)$.*

Where $WF_s(u)$, $s \ge 1$, denotes the wave front set of the distribution u; it will be defined in the next section via FBI-transform, Definition 2.1.

Remark 1.2 A few remarks are in order:

(i) the method used to gain the above result can be extended to a class of Hörmander type operators not strictly sums of squares; we consider operators of the form $P(x, D) + \sum_{i=1}^{m} b_j(x)X_j(x, D) + c(x)$ where P is as in (1.1), $b_j(x)$ are real-valued real analytic functions and $c(x)$ is a real analytic complex function;
(ii) the strategy to obtain the above result can be carried over to the case of s-Gevrey vectors with $s \in \mathbb{Z}_+$;
(iii) the result is optimal, see example given in [9].

A few words about the method of proof: it consists in using the FBI transform and the subelliptic inequality on the FBI side obtained in [1]. To do that we use a deformation technique of the Lagrangian associated to the FBI proposed by Grigis and Sjöstrand in [15].

2 Background on FBI and micro-local sub-elliptic estimate for sums of squares

We are going to use a pseudodifferential and FIO (Fourier Integral Operators) calculus introduced by Grigis and Sjöstrand in the paper [15]. We recall below the main definitions and properties to make this paper self-contained and readable. For further details we refer to the paper [15] and notes [23].
FBI Transform. Let $u \in \mathscr{E}'(\Omega)$, where $\mathscr{E}'(\Omega)$ denotes the space of distributions with compact support in Ω, open subset of $\mathbb{R}^n$, which is the dual space of the space of smooth functions in Ω equipped with its natural topology. We define the *FBI transform* of u as

$$Tu(z, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda\psi(z,y)} u(y)dy,$$

where $z \in \mathbb{C}^n$, $\lambda \geq 1$ is a large parameter, $\psi(z, w)$ in $\mathbb{C}^{2n}$ is an holomorphic function such that $\det \partial_z \partial_w \psi \neq 0$, $\Im \partial_w^2 \psi > 0$. To the phase ψ there corresponds a weight function $\phi(z)$, defined as

$$\phi(z) = \sup_{y\in\mathbb{R}^n} -\Im\psi(z, y), \qquad z \in \mathbb{C}^n.$$

Example 1 A typical phase function may be $\psi(z, y) = \frac{i}{2}(z - y)^2$. The corresponding weight function is given by $\phi(z) \doteq \phi_0(z) = \frac{1}{2}(\Im z)^2$.

We recall that T is associated to the following complex canonical transformation:

$$\begin{aligned} \mathscr{H}_T : \mathbb{C}^{2n}_{(w,\theta)} &\longrightarrow \mathbb{C}^{2n}_{(z,\zeta)}, \\ (w, -\partial_w\psi(z, w)) &\mapsto (z, \partial_z\psi(z, w)), \end{aligned} \tag{2.1}$$

with ψ as a generating function.
In particular $\mathscr{H}_T(\mathbb{R}^{2n}) \doteq \Lambda_\phi = \{(z, -2i\partial_z\phi(z)) \,;\, z \in \mathbb{C}^n\}$. In the case of classical phase function, see Example 1, we have

$$\mathscr{H}_0(x, \xi) = (x - i\xi, \xi), \qquad (x, \xi) \in \mathbb{R}^{2n}.$$

We set $\mathscr{H}_0(\mathbb{R}^{2n}) = \Lambda_{\phi_0}$.
We recall the definition of s–Gevrey wave front set of a distribution via classical FBI transform, i.e. using the phase function and the corresponding weight function of the Example 1.

Definition 2.1 Let u be a compactly supported distribution on $\mathbb{R}^n$. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$. We say that $(x_0, \xi_0) \notin WF_s(u)$, $s \geq 1$, if there exist a neighborhood Ω of $x_0 - i\xi_0 \in \mathbb{C}^n$ and positive constants C, ε such that

$$|e^{-\lambda\phi_0(z)} Tu(z, \lambda)| \leq C e^{-\varepsilon\lambda^{1/s}},$$

for every $z \in \Omega$ and $\lambda > 1$.

Pseudodifferential Operators. Let us consider $(z_0, \zeta_0) \in \mathbb{C}^{2n}$ and a real valued real analytic function $\phi(z)$ defined near z_0, such that ϕ is strictly plurisubharmonic and

$$\frac{2}{i}\, \partial_z \phi(z_0) = \zeta_0.$$

Denote by $\vartheta(z, w)$ the holomorphic function defined near $(z_0, \bar{z}_0)$ by

$$\vartheta(z, \bar{z}) = \phi(z). \tag{2.2}$$

Because of the strict plurisubharmonicity of ϕ, we have

$$\det \partial_z \partial_w \vartheta \neq 0 \tag{2.3}$$

and

$$\Re\vartheta(z, \bar{w}) - \frac{1}{2}\left[\phi(z) + \phi(w)\right] \sim -|z - w|^2. \tag{2.4}$$

Let $\lambda \geq 1$ be a large positive parameter. We write

$$\tilde{D} = \frac{1}{\lambda} D, \qquad D = \frac{1}{i}\partial.$$

Denote by $q(z, \zeta, \lambda)$ an analytic classical symbol[1] and by $Q(z, \tilde{D}, \lambda)$ the formal classical pseudodifferential operator associated to q. Using "Kuranishi's trick" [2] one may represent $Q(z, \tilde{D}, \lambda)$ as

$$Qu(z, \lambda) = \left(\frac{\lambda}{2i\pi}\right)^n \int e^{2\lambda(\vartheta(z,\theta) - \vartheta(w,\theta))} \tilde{q}(z, \theta, \lambda) u(w)\, dw\, d\theta. \tag{2.5}$$

Here $\tilde{q}$ denotes the symbol of Q in the actual representation.

To realize the above operator we need a prescription for the integration path[3]. This is accomplished by transforming the classical integration path via the Kuranishi change of variables and eventually applying Stokes theorem:

$$Q^\Omega u(z, \lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{2\lambda\vartheta(z,\bar{w})} \tilde{q}(z, \bar{w}, \lambda) u(w) e^{-2\lambda\phi(w)} L(dw), \tag{2.6}$$

where $L(dw) = (2i)^{-n} dw \wedge d\bar{w}$ is the Lebesgue measure in $\mathbb{R}^{2n}$, the integration path is $\theta = \bar{w}$ and $\Omega \times \overline{\Omega}$ is a small neighborhood of $(z_0, \bar{z}_0)$. We remark that $Q^\Omega u(z)$ is an holomorphic function of z.

1 For more details on the subject see [23], Section 1; see also [17].

2 For more details on the "Kuranishi's trick" see [19] Proposition 2.1.3 and [23] Remarque 4.3.

3 For a detailed discussion about the integration paths see [23].

Definition 2.2 Let Ω be an open subset of $\mathbb{C}^n$. We denote by $H_\phi(\Omega)$ the space of all functions $u(z, \lambda)$ holomorphic with respect to z, such that for every $\epsilon > 0$ and for every compact $K \subset\subset \Omega$ there exists a constant $C > 0$ such that

$$|u(z, \lambda)| \leq Ce^{\lambda(\phi(z)+\varepsilon)},$$

for $z \in K$ and $\lambda \geq 1$.

A few remarks are in order.

(i) If $\tilde{q}$ is a classical symbol of order zero, $Q^\Omega(z, \tilde{D}, \lambda)$ is uniformly bounded as $\lambda \to +\infty$, from $H_\phi(\Omega)$ into itself.

(ii) If the principal symbol is real, $Q^\Omega(z, \tilde{D}, \lambda)$ is formally self adjoint operator in $L^2(\Omega, e^{-2\lambda\phi(z)}L(dz))$.

(iii) The definition (2.5) of the realization of a pseudodifferential operator on an open subset Ω of $\mathbb{C}^n$ is not the classical one. Via the Kuranishi trick it can be reduced to the classical definition. On the other hand using the function ϑ allows us to use a weight function not explicitly related to an FBI phase. This is useful since in the proof we deform the I-Lagrangian, R-Symplectic variety Λ_{ϕ_0}, corresponding e.g. to the classical FBI phase, and obtain a *deformed* weight function which is useful in the a priori estimate.

We also recall that the identity operator can be realized as

$$I^\Omega u(z, \lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{2\lambda\vartheta(z,\bar{w})} i(z, \bar{w}, \lambda) u(w, \lambda) e^{-2\lambda\phi(w)} L(dw), \tag{2.7}$$

for a suitable analytic classical symbol $i(z, \zeta, \lambda)$. Moreover we have the following estimate (see [15] and [23], Sect. 12)

$$\|I^\Omega u - u\|_{\phi - d^2/C} \leq C'\|u\|_{\phi + d^2/C}, \tag{2.8}$$

for suitable positive constants C and C', for $u \in L^2(\Omega)$ and holomorphic in Ω . Here we denoted by

$$d(z) = \operatorname{dist}(z, \complement\Omega), \tag{2.9}$$

the distance of z to the boundary of Ω, and by

$$\|u\|_f^2 = \int_\Omega |u(z)|^2 e^{-2\lambda f(z)} L(dz). \tag{2.10}$$

We also recall the following important result on the composition of two pseudodifferential operators.

Proposition 2.1 ([15], Proposition 1.3). *Let Q_1 and Q_2 be of order zero. Then they can be composed and*

$$Q_1^\Omega \circ Q_2^\Omega = (Q_1 \circ Q_2)^\Omega + R^\Omega,$$

where R^Ω is an error term, i.e. an operator whose norm is $\mathscr{O}(1)$ as an operator from $H_{\phi+(1/C)d^2}$ to $H_{\phi-(1/C)d^2}$

The *a priori* Estimate. Let $X_j(z, \zeta)$, $j = 1, \ldots, m$, be classical analytic symbols of order one defined in Ω open neighborhood of $(z_0, \zeta_0) \in \Lambda_\phi$ in $\mathbb{C}^{2n}$. We assume also that the $X_{j|_{\Lambda_\phi}}$ are real valued. Let

$$P(z, \tilde{D}) = \sum_{j=1}^m X_j^2(z, \tilde{D}). \tag{2.11}$$

According to [15] the Ω-realization of P can be written as

$$P^{\Omega} = \sum_{j=1}^{m} (X_j^{\Omega})^2 + \mathscr{O}(\lambda^2), \tag{2.12}$$

where $\mathscr{O}(\lambda^2)$ is continuous from $H_{\tilde{\phi}}$ to $H_{\phi-(1/C)d^2}$ with norm bounded by $C'\lambda^2$, $\tilde{\phi}$ given by

$$\tilde{\phi}(z) = \phi(z) + \frac{1}{C} d^2(z),$$

and d has been defined in (2.9).
Following [1] we state the FBI version of the estimate (1.2).

Theorem 2.1 *Let (x_0, ξ_0) be in Char(P) and $\nu \doteq \nu(x_0, \xi_0)$, Definition 1.1. Let $\mathscr{H}_T(x_0, \xi_0) = (z_0, \zeta_0) \in \Lambda_\phi$ and P^Ω be as in (2.12). Let Ω_1 open neighborhood of (z_0, ζ_0) such that $\Omega_1 \subset\subset \Omega$. Then*

$$\lambda^{\frac{2}{\nu}} \|u\|_\phi^2 + \sum_{j=1}^{m} \|X_j^\Omega u\|_\phi^2 \leq C \left(\langle P^\Omega u, u\rangle_\phi + \lambda^\alpha \|u\|^2_{\phi,\Omega\setminus\Omega_1} \right), \tag{2.13}$$

where α is a positive integer and $u \in L^2(\Omega, e^{-2\phi(z)} L(dz))$.

3 Proof of the Theorem 1.1

In order to prove the result we want take advantage of Theorem 2.1. We consider the sum of squares operator

$$Q(x, D_t, D) = \sum_{j=0}^{m} X_j^2 = D_t^2 + P(x, D), \tag{3.1}$$

in $\tilde{\mathcal{O}} =]-\delta_0, \delta_0[\times \mathcal{O}$, $\delta_0 > 0$.
We study the microlocal properties of the solutions of the problem $Qv = f$, $f \in C^\omega(\tilde{\mathcal{O}})$.
We denote by $\tilde{\Sigma}$ the characteristic set of Q given by

$$\begin{aligned} \tilde{\Sigma} &= \{(t, x, \tau, \xi) \in T^*\tilde{\mathcal{O}}\setminus\{0\} \,:\, Q(t, x, \tau, \xi) = 0\} \\ &= \{(t, x, \tau, \xi) \in T^*\tilde{\mathcal{O}}\setminus\{0\} \,:\, \tau = 0,\ X_j(x, \xi) = 0,\ j = 1, \ldots, m\}. \end{aligned} \tag{3.2}$$

We remark that $\nu_{(t_0,x_0,0,\xi_0)}$,$(t_0, x_0, 0, \xi_0) \in \tilde{\Sigma}$, is equal to $\nu_{(x_0,\xi_0)}$, $(x_0, \xi_0) \in \Sigma$, where Σ denotes the characteristic set of $P(x, D)$.
We construct a deformation of Λ_{ϕ_0} following the ideas in [15], see also [1]. Let $(0, x_0, 0, \xi_0) \in \tilde{\Sigma}$ and ν its length.
We perform an FBI-transform of the form

$$Tu(z, \lambda) = \int_{\mathbb{R}^{n+1}} e^{i\lambda\psi(z,t,x)} u(t, x) dt dx, \qquad z = (z_0, z_1) \in \mathbb{C}^{1+n},$$

where $u(t, x)$ is a compactly supported distribution and $\psi(z, t, x)$ is a phase function. Even though it does not really matter which phase function we use, the classical phase function will be employed:

$$\psi_0(z, t, x) = \frac{i}{2}\left[(z_0 - t)^2 + (z_1 - x)^2\right]. \tag{3.3}$$

Let Ω be an open neighborhood of the point $\pi_z \circ \mathscr{H}_T(0, x_0, 0, \xi_0)$ in $\mathbb{C}^{1+n}$. Here π_z denotes the space projection $\pi_z : \mathbb{C}_z^{1+n} \times \mathbb{C}_\zeta^{1+n} \to \mathbb{C}_z^{1+n}$, $\zeta = (\zeta_0, \zeta_1)$, and $\mathscr{H}_T$ is the complex canonical transformation associated to T, (2.1). We recall that in the case of FBI with classical phase function, Example 1, we have $\mathscr{H}_0(t, x, \tau, \xi) = (t - i\tau, x - i\xi, \tau, \xi)$.

Denoting by $\tilde{Q}$ our operator after the FBI we have that $\tilde{Q}_{|\Lambda_{\phi_0}} = Q$, $\Lambda_{\phi_0} = \mathscr{H}_0(\mathbb{R}^{2(1+n)})$. We have that $\pi_z \circ \mathscr{H}_0(0, x_0, 0, \xi_0) = (0, x_0 - i\xi_0) = (0, w_0) \in \mathbb{C}^{1+n}$. We perturb canonically ϕ_0. For $\lambda \geq 1$ let us consider a real analytic function defined near the point $\mathscr{H}_0(0, x_0, 0, \xi_0) \in \Lambda_{\phi_0}$, say $h(z, \zeta, \lambda)$. Solve, for small positive s, the Hamilton-Jacobi problem

$$\begin{cases} 2\dfrac{\partial \phi}{\partial s}(s, z, \lambda) = h\left(z, \dfrac{2}{i}\dfrac{\partial \phi}{\partial z}(s, z, \lambda), \lambda\right) \\ \phi(0, z, \lambda) = \phi_0(z) \end{cases}. \tag{3.4}$$

Set

$$\phi_s(z, \lambda) = \phi(s, z, \lambda),$$

we have the canonical map $\Lambda_{\phi_0} \to \Lambda_{\phi_s}$ where

$$\Lambda_{\phi_s} = \exp(isH_h)\,\Lambda_{\phi_0}.$$

We choose the function h as

$$h(z, \zeta, \lambda) = h\left(z, \frac{2}{i}\frac{\partial \phi_0}{\partial z}(z), \lambda\right) + \lambda^{-1} h_1(z, \zeta)\left(\zeta - \frac{2}{i}\frac{\partial \phi_0}{\partial z}(z)\right),$$

where $h_1(z, \zeta)$ is an holomorphic function and

$$h\left(z, \frac{2}{i}\frac{\partial \phi_0}{\partial z}(z), \lambda\right) = h(z, \zeta, \lambda)_{|\Lambda_{\phi_0}} = (z_0'')^2 + \lambda^{-\frac{\nu-1}{\nu}}\left((z_0')^2 + |z_1 - w_0|^2\right), \tag{3.5}$$

$z_0 = z_0' + iz_0'' \in \mathbb{C}$.
Since $\mathbb{R}^{2(1+n)}$ and Λ_{ϕ_0} are isometric, keep in mind the definition of Λ_{ϕ_0}, it is easier to construct the function h in $\mathbb{R}^{2(n+1)}$ near the characteristic point:

$$h(t, x, \tau, \xi, \lambda) = \tau^2 + \lambda^{-1+\frac{1}{\nu}}\left[t^2 + |x - x_0|^2 + |\xi - \xi_0|^2\right]. \tag{3.6}$$

The function ϕ_s can be expanded as a power series in the variable s using both equation (3.4) and the Faà di Bruno formula to obtain

$$\phi_s(z, \lambda) = \phi_0(z) + \frac{s}{2}\, h(\cdot, \cdot, \lambda)\big|_{\Lambda_{\phi_0}} + \mathscr{O}(\lambda^{-1}s^2), \tag{3.7}$$

where h on Λ_{ϕ_0} is given by (3.5). Our purpose is to use the estimate (2.13) where the weight function ϕ has been replaced by the weight ϕ_s. This is possible using the phase ϑ_s in (2.5) and realizing the operator as in (2.6). Here ϑ_s is defined as the holomorphic extension of $\vartheta_s(z, \bar{z}) = \phi_s(z)$.

We need to restrict the symbol of Q to Λ_{ϕ_s}; we denote by Q^s the symbols of Q restricted to Λ_{ϕ_s}. Noting that

$$\begin{aligned} X_j^2\left(x, \frac{2}{i}\partial_x \phi_s(x, \lambda)\right), \lambda) &= X_j^2\left(x, \frac{2}{i}\partial_x \phi_0(x), \lambda\right) \\ &+ 2sX_j\left(x, \frac{2}{i}\partial_x \phi_0(x), \lambda\right)\left\langle \partial_\xi X_j(x, \frac{2}{i}\partial_x \phi_0(x), \lambda), \frac{2}{i}\partial_x \partial_s \phi_s(x, \lambda)\big|_{s=0}\right\rangle \\ &+ \mathscr{O}(s^2 \lambda^{\frac{2}{\nu}}). \end{aligned}$$

We deduce that

$$\begin{aligned}Q^s &= Q + s\sum_{j=0}^{m} X_j\{h, X_j\} + s^2\sum_{j=0}^{m}\{h, X_j\}^2 + \mathcal{O}(s^2\lambda^{\frac{2}{\nu}})\\ &= Q(x,\xi) + sR(x,\xi,\lambda) + \mathscr{O}(s^2\lambda^{\frac{2}{\nu}}).\end{aligned} \tag{3.8}$$

The analytic extension of Q^s is the symbol appearing in the Ω-realization of Q^s, $Q^{s\,\Omega}$. We point out that the principal symbol of Q^s satisfies the assumptions of Theorem 2.1 and, using the a priori inequality (2.13), we can deduce an estimate of the form (2.13) for Q^s in the H_{ϕ_s} spaces. We have

$$\begin{aligned}&\lambda^{\frac{2}{\nu}}\|u\|^2_{\phi_s} + \sum_{j=0}^{m}\|X_j^{\Omega}u\|^2_{\phi_s}\\ &\le C\left(|\langle(Q^{s\,\Omega} - sR^{\Omega} - \mathscr{O}(s^2\lambda^{\frac{2}{\nu}}))u, u\rangle_{\phi_s}| + \lambda^{\alpha}\|u\|^2_{\phi_s,\Omega\backslash\Omega_1}\right).\end{aligned}$$

The third term in the right hand side of the scalar product above is easily absorbed on the left provided s is small enough. Let us consider the second term in the scalar product above. By Proposition 2.1, we have

$$R^{\Omega} = \sum_{j=0}^{m} a_j^{\Omega}(x,\tilde{D},\lambda)X_j^{\Omega}(x,\tilde{D},\lambda) + \mathscr{O}(\lambda),$$

where $a_j^{\Omega}(x,\tilde{D},\lambda)$, $j = 0,\dots,m$, are zero order operators and $\mathscr{O}(\lambda)$ denotes an operator from $H_{\phi_s+\frac{1}{C}d^2}$ to $H_{\phi_s-\frac{1}{C}d^2}$ whose norm is bounded by $C\lambda$. Hence

$$s|\langle R^{\Omega}u, u\rangle_{\phi_s}| \le Cs\Big(\lambda^{\frac{2}{\nu}}\|u\|^2_{\phi_s} + \sum_{j=0}^{m}\|X_j^{\Omega}u\|^2_{\phi_s} + \lambda^2\|u\|^2_{\tilde{\phi}_s}\Big),$$

where $\tilde{\phi}_s = \phi_s + \frac{1}{C}d^2$. Hence we deduce that there exist a neighborhood Ω_0 of $(0, w_0)$, a positive number δ and a positive integer α such that, for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega \subset \Omega_0$, there exists a constant $C > 0$ such that, for $0 < s < \delta$, we have

$$\lambda^{\frac{2}{\nu}}\|u\|_{\phi_s,\Omega_1} \le C\left(\|Q^{s\,\Omega}u\|_{\phi_s,\Omega_2} + \lambda^{\alpha}\|u\|_{\phi_s,\Omega\backslash\Omega_1}\right). \tag{3.9}$$

We now prove that if Qu is analytic at $(0, x_0, 0, \xi_0)$ then the point $(0, x_0, 0, \xi_0)$ does not belong to $WF_\nu(u)$.

Since Qu is real analytic the first term in the right hand side of (3.9) can be estimated by $Ce^{-\lambda/C}$ for a positive constant C. We have to estimate the second term on the right hand side of the above inequality. We have

$$\phi_s(z,\lambda) = \phi_0(z) + \frac{s}{2}h(z, \frac{2}{i}\frac{\partial\phi}{\partial z}(0,z),\lambda) + \mathcal{O}(\lambda^{-1}s^2).$$

Hence

$$\phi_s(z,\lambda) - \phi_0(z) \sim \frac{s}{2}\left[(z_0'')^2 + \lambda^{-\frac{\nu-1}{\nu}}\left((z_0')^2 + |z_1 - w_0|^2\right)\right].$$

Since $z = (z_0, z_1) \in \Omega\backslash\Omega_1$, i.e. far from $(0, w_0)$, there exists a positive constant β such that

$$h_{|\Lambda_{\phi_0}\cap\Omega\backslash\Omega_1} \ge 2\lambda^{-1+1/\nu}\beta > 0.$$

We have

$$\phi_s(z,\lambda)_{|\Omega\setminus\Omega_1} \geq \phi_0(z) + s\lambda^{-1+\frac{1}{\nu}}\beta + \mathcal{O}(\lambda^{-1}s^2).$$

The second term on the right hand side of (3.9) can be estimated by

$$\|u\|^2_{\phi_s,\Omega\setminus\Omega_1} \leq C_1(s)e^{-\lambda^{\frac{1}{\nu}}s\frac{\beta}{2}},$$

where $C_1(s) > 0$. From (3.9) and the above argument there is a positive constant C_2 such that

$$\|u\|^2_{\phi_s,\Omega_1} \leq C_2e^{-\lambda^{\frac{1}{\nu}}s\frac{\beta}{2}}.$$

Let Ω_3 a sufficient small neighborhood of the point $(0, w_0)$, $\Omega_3 \Subset \Omega_1$, such that for a fixed small positive s

$$\phi_s(z,\lambda) - \phi_0(z) \leq \frac{s\beta}{4}\lambda^{-1+\frac{1}{\nu}} + \lambda^{-1}C_3(s),$$

$z \in \Omega_3$, $\lambda \geq 1$.
Then there are two positive constants, $\tilde{C}$ and ϵ, such that

$$\|u\|^2_{\phi_0,\Omega_3} \leq \tilde{C}e^{-\epsilon\lambda^{\frac{1}{\nu}}}.$$

Now we consider the problem

$$\begin{cases} \left(D_t^2 + P(x,D)\right)U(t,x) = 0, \\ U(0,x) = u(x), \end{cases} \tag{3.10}$$

in $\tilde{\mathscr{O}} =]-\delta_0, \delta_0[\times \mathscr{O}$, $\delta_0 > 0$, where $u(x)$ is an analytic vector for $P(x,D)$:

$$\|P^k u\|_0 \leq C^{2k+1}(2k)!. \tag{3.11}$$

The function

$$U(t,x) = \sum_{k\geq 0} \frac{t^{2k}}{2k!} P^k u(x)$$

is a solution of the above problem. We choose $\delta_0 < \sqrt{2}C$.
In order to complete the proof of the Theorem 1.1 we have to show that $(0, x_0, 0, \xi_0) \notin WF_{s_0}(U)$ if and only if $(x_0, \xi_0) \notin WF_{s_0}(u)$ for every $s_0 \geq 1$.
This result was showed in the case $s_0 = 1$ via Fourier transform in [8], Proposition 3.3. We give, for any $s_0 \in [1, +\infty)$, a proof via the classical FBI transform.
Step one: if $(x_0, \xi_0) \notin WF_{s_0}(u)$ then $(0, x_0, 0, \xi_0) \notin WF_{s_0}(U)$.
By hypothesis we have that $(x_0, \xi_0) \notin WF_{s_0}(u)$ if and only of there exist Ω open neighborhood of the point $x_0 - i\xi_0$ in $\mathbb{C}^n$ and positive constants C_1 and ε_1 such that

$$|e^{-\lambda\phi_0(z)}T(\chi u)(z,\lambda)| \leq C_1e^{-\varepsilon_1\lambda^{1/s_0}}, \qquad \forall z \in \Omega, \tag{3.12}$$

where χ is a $C_0^\infty(\mathscr{O})$ identically one in a neighborhood of x_0.
We have to show that there is Ξ open neighborhood of the point $(0, x_0 - i\xi_0)$ in $\mathbb{C}^{n+1}$ and positive constants C_2 and ε_2 such that

$$|e^{-\lambda\phi_0(w,z)}T(\chi U)(w,z,\lambda)| \leq C_2e^{-\varepsilon_2\lambda^{1/s_0}}, \qquad \forall (w,z) \in \Xi, \tag{3.13}$$

where $\chi(t,x)=\chi_0(t)\theta_0(x)$, here $\chi_0(t)$ is $C_0^\infty(]-\delta_1,\delta_1[)$, $0<\delta_1<\delta_0$, such that $\chi_0(t)\equiv 1$ in $]-\delta_2,\delta_2[$, $0<\delta_2<\delta_1/2$, and $\theta_0(x)$ is $C_0^\infty(\mathscr{B}_{r_0}(x_0))$, $\mathscr{B}_{r_0}(x_0)=\{x\in\mathbb{R}^n : |x-x_0|<r_0\}$, $r_0\le \operatorname{dist}\left(x_0, \complement\pi_{z'}(\Omega)\right)$ such that $\theta_0(x)\equiv 1$ in $\mathscr{B}_{r_1}(x_0)$, $0<r_1<r_0$.
We have

$$\begin{aligned} T(\chi U)(w,z,\lambda) &= \iint e^{-\frac{\lambda}{2}(w-s)^2} e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)U(s,y)\,ds dy \\ &= \sum_{N=0}^{\infty} \underbrace{\frac{1}{(2N)!}\iint e^{-\frac{\lambda}{2}(w-s)^2} e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)s^{2N}P^N u(y)\,ds dy}_{\doteq \mathscr{P}_N(w,z)} . \end{aligned}$$

Let $\tilde{r}_0>0$ such that $\tilde{r}_0 \ll r_1$. We take z in the FBI transform such that $z'=\Re(z)\in \mathscr{B}_{\tilde{r}_0-\varepsilon}(x_0)$, where $0<\varepsilon<\tilde{r}_0$.
Case $N=0$, since there are two positive constants A and $\tilde{\varepsilon}_0$ such that

$$\left|\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)ds\right| \le e^{\frac{\lambda}{2}(w'')^2} A e^{-\lambda\tilde{\varepsilon}_0},$$

taking advantage from (3.12) there is a positive constant C_3 such that

$$\begin{aligned} &\left| e^{-\lambda\phi_0(w,z)} \iint e^{-\frac{\lambda}{2}(w-s)^2} e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)u(y)\,ds dy\right| \\ &\quad \le C_3 e^{-\varepsilon_1\lambda^{1/s_0}} e^{-\tilde{\varepsilon}_0\lambda}. \end{aligned} \tag{3.14}$$

In order to make the proof more readable, before looking at the general case, we analyze the cases $N=1$ and $N=2$.
Case $N=1$; we have

$$\mathscr{P}_1(w,z) \doteq \frac{1}{2}\iint e^{-\frac{\lambda}{2}(w-s)^2} e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)s^2 Pu(y)\,ds dy. \tag{3.15}$$

We introduce $\theta_1(y)\in C_0^\infty\left(\mathscr{B}_{r_1}(x_0)\right)$ such that $\operatorname{supp}\left(\theta_1\right)\subseteq \mathscr{B}_{r_1}(x_0)$, where $\theta_0(y)\equiv 1$, and $\theta_1(y)\equiv 1$ in $\mathscr{B}_{r_2}(x_0)$, where $\tilde{r}_0\le r_2<r_1<r_0$. We have

$$\begin{aligned} \mathscr{P}_1(w,z) &= \frac{1}{2}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^2 e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y) \\ &\quad\times P\left[\left(\theta_1(y)+\left(1-\theta_1(y)\right)\right)u(y)\right] ds dy \\ &= \frac{1}{2}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^2\left(P^* e^{-\frac{\lambda}{2}(z-y)^2}\right)\theta_1(y)\,u(y)\,ds dy \\ &\quad+\frac{1}{2}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^2\left[P^*\left(e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y)\right)\right] \\ &\quad\times\left(1-\theta_1(y)\right)u(y)\,ds dy; \end{aligned} \tag{3.16}$$

P^* denotes the adjoint of P.
Case $N=2$; we have

$$\mathscr{P}_2(w,z) \doteq \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2} e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)s^4 P^2 u(y)\,ds dy. \tag{3.17}$$

We introduce $\theta_1(y)$ in $C_0^\infty\left(\mathscr{B}_{r_1}(x_0)\right)$ and $\theta_2(y)$ in $C_0^\infty\left(\mathscr{B}_{r_2}(x_0)\right)$ such that $\operatorname{supp}\left(\theta_1\right)\subseteq \mathscr{B}_{r_1}(x_0)$, where $\theta_0(y)\equiv 1$, $\theta_1(y)\equiv 1$ in $\mathscr{B}_{r_2}(x_0)$, $\operatorname{supp}\left(\theta_2\right)\subseteq\mathscr{B}_{r_2}(x_0)$, where $\theta_1(y)\equiv 1$, and $\theta_2(y) \equiv 1$ in $\mathscr{B}_{r_3}(x_0)$, where $\tilde{r}_0 \le r_3 < r_2 < r_1 < r_0$. We have

$$\begin{aligned}
\mathscr{P}_2(w,z) &= \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4 e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y) \\
&\quad \times P\left[\left(\theta_1(y) + \left(1-\theta_1(y)\right)\right)Pu(y)\right]dsdy \\
&= \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left(P^* e^{-\frac{\lambda}{2}(z-y)^2}\right)\theta_1(y)\,Pu(y)\,dsdy \\
&\quad + \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left[P^*\left(e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y)\right)\right] \\
&\quad \times \left(1-\theta_1(y)\right)Pu(y)\,dsdy \\
&= \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left(P^* e^{-\frac{\lambda}{2}(z-y)^2}\right)\theta_1(y) \\
&\quad \times P\left[\left(\theta_2(y) + \left(1-\theta_2(y)\right)\right)u(y)\right]dsdy \\
&\quad + \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left[P^*\left(e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y)\right)\right] \\
&\quad \times \left(1-\theta_1(y)\right)Pu(y)\,dsdy \\
&= \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left[\left(P^*\right)^2 e^{-\frac{\lambda}{2}(z-y)^2}\right]\theta_2(y)\,u(y)\,dsdy \\
&\quad + \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left[P^*\left(\left(P^* e^{-\frac{\lambda}{2}(z-y)^2}\right)\right.\right. \\
&\quad \left.\left.\times\theta_1(y)\right)\right]\left(1-\theta_2(y)\right)u(y)\,dsdy \\
&\quad + \frac{1}{4!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^4\left[P^*\left(e^{-\frac{\lambda}{2}(z-y)^2}\theta_0(y)\right)\right] \\
&\quad \times \left(1-\theta_1(y)\right)Pu(y)\,dsdy.
\end{aligned} \tag{3.18}$$

The idea is to introduce a sequence of cut-off functions, the support of the subsequent nested where the previous is identically equal to one, in order to move all the powers of P on the exponential function in a neighborhood of x_0 with the purpose of taking advantage of (3.12). However this will give rise to other terms which still involve powers of P acting on u, but in a region far from x_0. We handle the general case as above. We introduce a family of smooth functions $\{\theta_j(y)\}_{1\le j\le N}$ such that such that supp $\left(\theta_j\right) \subseteq \mathscr{B}_{r_j}(x_0)$ and $\theta_j(y) \equiv 1$ in $\mathscr{B}_{r_{j+1}}(x_0)$, $\tilde{r}_0 \le r_{N+1} < r_N < \cdots < r_2 < r_1 < r_0$. So, we see that for every j less or equal than $N+1$, one has that: i less than j implies $\theta_i \equiv 1$ on a neighborhood of θ_j. In order to construct the functions θ_j we follow the same strategy used to construct the Ehrenpreis-Hörmander cut-off functions. More precisely we choose $r_j = r_0 - (r_0 - \tilde{r}_0)\frac{j}{N+1}$, we have $r_j - r_{j+1} = \frac{r_0-\tilde{r}_0}{N+1}$. Let ψ be a function in $\mathscr{D}(\mathbb{R}^n)$ with support in $\mathscr{B}_{1/4}(0) \doteq \{y \in \mathbb{R}^n : |y| \le 1/4\}$ such that $\psi \ge 0$ and $\int \psi\, dy = 1$. For every $\gamma > 0$ we write $\psi_\gamma(y) = \gamma^{-n}\psi\left(\frac{x}{\gamma}\right)$. Let χ_j be the characteristic function of the set $\{y \in \mathbb{R}^n : \text{dist}\left(y; \mathscr{B}_{r_{j+1}}(x_0)\right) < \frac{r_0-\tilde{r}_0}{2(N+1)}\}$. We set

$$\theta_j = \psi_{\frac{r_0-\tilde{r}_0}{(N+1)}} * \psi_{\frac{r_0-\tilde{r}_0}{(N+1)}} * \chi_j.$$

These functions have the desired properties. Moreover we have

$$\|D_{y_i}\theta_j\|_\infty \le \|D_{y_i}\psi_{\frac{r_0-\tilde r_0}{(N+1)}}\|_{L^1}\|\psi_{\frac{r_0-\tilde r_0}{(N+1)}}\|_{L^1}\|\chi_j\|_\infty \le C_0\frac{N+1}{r_0-\tilde r_0},$$

$$\|D_{y_i}D_{y_k}\theta_j\|_\infty \le \|D_{y_i}\psi_{\frac{r_0-\tilde r_0}{(N+1)}}\|_{L^1}\|D_{y_k}\psi_{\frac{r_0-\tilde r_0}{(N+1)}}\|_{L^1}\|\chi_j\|_\infty \le \left(C_0\frac{N+1}{r_0-\tilde r_0}\right)^2,$$

where $C_0 = \sup_{1\le i\le n}\|D_{y_i}\psi\|_{L^1(\mathscr{B}_{1/4}(0))}$.

Remark 3.2 One may also choose a sequence of θ_j independent of N, by repeating the above construction and taking the convolution with $\psi_{(r_0-\tilde r_0)/2^j}$.
Moreover the θ_j can be constructed by just one convolution, i.e. $\theta_j = \psi_{\frac{r_0-\tilde r_0}{2^j}} * \chi$. This will be more evident in the next few steps.

We set $X_j(x,D) = \sum_{\ell=1}^{n} a_{\ell,j}(x)D_i$. We have

$$\begin{aligned}
\mathscr{P}_N(w,z) &= \frac{1}{(2N)!}\iint e^{-\frac{\lambda}{2}(w-s)^2}e^{-\frac{\lambda}{2}(z-y)^2}\chi_0(s)\theta_0(y)s^{2N}P^N u(y)\,dsdy\\
&= \frac{1}{(2N)!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\left[\left(P^*\right)^N e^{-\frac{\lambda}{2}(z-y)^2}\right]\theta_N(y)u(y)\,dsdy\\
&\quad+\frac{1}{(2N)!}\sum_{j=1}^{N}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\Big\{P^*\Big[\Big(\left(P^*\right)^{j-1}e^{-\frac{\lambda}{2}(z-y)^2}\Big)\theta_{j-1}(y)\Big]\Big\}\\
&\quad\times\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dsdy\\
&= \frac{1}{(2N)!}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\left[\left(P^*\right)^N e^{-\frac{\lambda}{2}(z-y)^2}\right]\theta_N(y)u(y)\,dsdy\\
&\quad+\frac{1}{(2N)!}\sum_{j=1}^{N}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\Big(\left(P^*\right)^{j}e^{-\frac{\lambda}{2}(z-y)^2}\Big)\theta_{j-1}(y)\\
&\quad\times\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dsdy\\
&\quad+\frac{1}{(2N)!}\sum_{j=1}^{N}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\Big(\left(P^*\right)^{j-1}e^{-\frac{\lambda}{2}(z-y)^2}\Big)\left(P\theta_{j-1}(y)\right)\\
&\quad\times\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dsdy\\
&\quad+\frac{1}{(2N)!}\sum_{j=1}^{N}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\left[\sum_{k=1}^{m}\Big(X_k\left(P^*\right)^{j-1}e^{-\frac{\lambda}{2}(z-y)^2}\Big)\right.\\
&\quad\left.\times\left(X_k\theta_{j-1}(y)\right)\right]\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dsdy\\
&\quad+\frac{2}{(2N)!}\sum_{j=1}^{N}\iint e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\left[\Big(\left(P^*\right)^{j-1}e^{-\frac{\lambda}{2}(z-y)^2}\Big)\right.\\
&\quad\left.\times\sum_{k=1}^{m}f_k\left(X_k\theta_{j-1}(y)\right)\right]\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dsdy\\
&= I_1+I_2+I_3+I_4+I_5,
\end{aligned} \tag{3.19}$$

where $f_k = \frac{1}{i}\sum_{\ell=1}^{n} a_{\ell,k}^{(e_\ell)}(y)$, e_ℓ, $\ell, = 1, \dots, n$, in the upper index denotes the derivatives in the direction ℓ.

Before estimating the above terms a few remarks are in order:

(i) each step no more than two derivatives act on $\theta_j(y)$;

(ii) let $y \in \operatorname{supp}(\theta_{j-1}^{(\alpha)}) \cap \operatorname{supp}\left(1-\theta_j\right)$, $0 \le |\alpha| \le 2$, since Ω is a complex neighborhood of $x_0 - i\xi_0$ such that $\pi_{z'}(\Omega) \subset \mathscr{B}_{\tilde{r}_0-\varepsilon}$, we have that $(z'-y)^2 \ge \varepsilon^2$;

(iii) without loss of generality we may write

$$\left(P^*(y,D)\right)^N = \sum_{|\beta|\le 2N} a_{2N,\beta}(y) D^\beta$$

where $a_{2N,\beta}(y)$ are analytic functions such that for any compact set K in U we have

$$\left|a_{2N,\beta}^{(\gamma)}(y)\right| \le C_K^{3N-|\beta|+|\gamma|}\,(2N-|\beta|+|\gamma|)! \qquad \forall\, y \in K \text{ and } \gamma \in \mathbb{Z}_+^n. \tag{3.20}$$

iv) the following identity holds

$$\left(\frac{d}{dy_k}\right)^{\beta_k} e^{-\frac{\lambda}{2}(z_k-y_k)^2} = e^{-\frac{\lambda}{2}(z_k-y_k)^2} \sum_{\ell_k=0}^{\lfloor\frac{\beta_k}{2}\rfloor} \frac{\beta_k!(i)^{2(\beta_k-\ell_k)}}{\ell_k!\,(\beta_k-2\ell_k)!2^{\ell_k}} \lambda^{\beta_k-\ell_k}\,(z_k-y_k)^{\beta_k-2\ell_k}$$

$$= e^{-\frac{\lambda}{2}(z_k-y_k)^2}(i)^{\beta_k}\left(\frac{\lambda}{2}\right)^{\beta_k/2} \sum_{\ell_k=0}^{\lfloor\frac{\beta_k}{2}\rfloor} \frac{\beta_k!}{\ell_k!\,(\beta_k-2\ell_k)!}\left[i\sqrt{2\lambda}\,(z_k-y_k)\right]^{\beta_k-2\ell_k}.$$

Estimate of the term I_2. Since we are far from x_0 we expect exponential decay. We have

$$I_2 = \frac{1}{(2N)!}\sum_{j=1}^{N}\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}ds \int \sum_{|\beta|\le 2j}\frac{1}{(i)^{|\beta|}} a_{2j,\beta}(y) e^{\frac{\lambda}{2}(z'')^2+i\lambda(y-z')z''}$$
$$\times\left[\prod_{\nu=1}^{n}\left(\sum_{\gamma_\nu=0}^{\lfloor\frac{\beta_\nu}{2}\rfloor}\frac{\beta_\nu!i^{2(\beta_\nu-\gamma_\nu)}}{\gamma_\nu!\,(\beta_\nu-2\gamma_\nu)!2^{|\gamma_\nu|}}\left(\lambda^{\beta_\nu-\gamma_\nu}\,(z_\nu-y_\nu)^{\beta_\nu-2\gamma_\nu}\right)e^{-\frac{\lambda}{2}(z_\nu-y_\nu)^2}\right)\right]$$
$$\times\,\theta_{j-1}(y)\left(1-\theta_j(y)\right)P^{N-j}u(y)\,dy.$$

We remark that the integral with respect the variable s is the FBI transform of $\chi_0(s)s^{2N}$. We take $\Re(w) \in]-\delta_2-\sqrt{\tilde{\varepsilon}_0}, \delta_2+\sqrt{\tilde{\varepsilon}_0}[$, $\tilde{\varepsilon}_0$ sufficiently small positive constant. Splitting the domain of integration in the regions where $\chi_0(s) \ne 1$ and $\chi_0(s) = 1$ and changing, in the last one region, the integration path as in the Remark 3.3, so that it is in the strip $\sigma = s + i\sigma''$, $|\sigma''| < \delta_2/2$, where we consider the holomorphic extension of s^{2N}, we can conclude that there is a positive constants A such that

$$\left|\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}ds\right| \le e^{\frac{\lambda}{2}(w'')^2}A\delta_1^{2N}e^{-\lambda\tilde{\varepsilon}_0}.$$

Since $y \in \mathscr{B}_{r_{j-1}}(x_0)\backslash\mathscr{B}_{r_{j+1}}(x_0)$ we have $(z'-y)^2 \ge \varepsilon_0$. We obtain

$$\left|\lambda^{\beta_\nu-\gamma_\nu}\,(z_\nu-y_\nu)^{\beta_\nu-2\gamma_\nu}\,e^{-\frac{\lambda}{2}(z'_\nu-y_\nu)^2}\right|$$
$$\le 2\cdot 2^{\frac{3}{2}(\beta_\nu-2\gamma_\nu)}\left(\frac{8}{\varepsilon_0}\right)^{\beta_\nu-\gamma_\nu}(\beta_\nu!)^{\frac{1}{2}}\left[(\beta_\nu-2\gamma_\nu)!\right]^{\frac{1}{2}}e^{-\frac{\varepsilon_0}{16}\lambda},$$

where we can assume that $|z''_\nu| \le 1$. Since $\beta_\nu! \le 2^{\beta_\nu+2\gamma_\nu}\left[(\beta_\nu - 2\gamma_\nu)!\right](\gamma_\nu!)^2$, we have

$$\sum_{\gamma_\nu=0}^{\lfloor \frac{\beta_\nu}{2} \rfloor} \frac{\beta_\nu!}{\gamma_\nu!\,(\beta_\nu - 2\gamma_\nu)!2^{|\gamma_\nu|}} \left|\left(\lambda^{\beta_\nu-\gamma_\nu}\,(z_\nu - y_\nu)^{\beta_\nu-2\gamma_\nu}\right) e^{-\frac{\lambda}{2}(z_\nu-y_\nu)^2}\right| \le 4\cdot(\beta_\nu!)\left(\frac{32}{\varepsilon_0}\right)^{\beta_\nu} e^{-\frac{n\varepsilon_0}{16}\lambda},$$

then

$$\prod_{\nu=1}^{n}\left(\sum_{\gamma_\nu=0}^{\lfloor \frac{\beta_\nu}{2} \rfloor} \frac{\beta_\nu!}{\gamma_\nu!\,(\beta_\nu - 2\gamma_\nu)!2^{|\gamma_\nu|}} \left|\left(\lambda^{\beta_\nu-\gamma_\nu}\,(z_\nu - y_\nu)^{\beta_\nu-2\gamma_\nu}\right) e^{-\frac{\lambda}{2}(z_\nu-y_\nu)^2}\right|\right) \le 4^n\,(\beta!)\left(\frac{32}{\varepsilon_0}\right)^{|\beta|} e^{-\frac{n\varepsilon_0}{16}\lambda}.$$

We obtain

$$|e^{-\lambda\phi_0(w,z)} I_2| \le 4^n A\,\frac{(2\pi)^n r_0^{n-1}}{\Gamma\left(\frac{n}{2}\right)}\,e^{-\tilde{\varepsilon}_0\lambda}\,e^{-\frac{n\varepsilon_0}{16}\lambda}\delta_1^N \times \sum_{j=1}^{N}\frac{1}{(2N)!}\left(\sum_{|\beta|\le 2j} C_1^{3j-|\beta|+1}\,(2j-|\beta|)!\beta!\left(\frac{32}{\varepsilon_0}\right)^{|\beta|}\right)\tilde{C}_2^{2(N-j)+1}\,[2(N-j)]!,$$

where C_1 and $\tilde{C}_2$ are the constants in (3.20) and in (3.11), respectively, with $K = \overline{\mathscr{B}_{r_0}(x_0)}$. Without loss of generality we may assume that C_1 and $\tilde{C}_2$ are greater then 2. Since $(2j - |\beta|)! \le (2j)!\,(|\beta|!)^{-1}$ and $[2(N-j)]! \le (2N)!\,((2j)!)^{-1}$, we have

$$|e^{-\lambda\phi_0(w,z)} I_2| \le 2\cdot 8^n\,\frac{(2\pi)^n r_0^{n-1}}{\Gamma\left(\frac{n}{2}\right)}\,AC_1\tilde{C}_2\,e^{-\tilde{\varepsilon}_0\lambda}\,e^{-\frac{n\varepsilon_0}{16}\lambda}\delta_1^N\left(\frac{32\,C_1^{\frac{3}{2}}\tilde{C}_2}{\varepsilon_0}\right)^{2N}.$$

Taking δ_1 small enough we conclude that there are two positive constants C_2 and ε_2, independent by N, such that

$$|e^{-\lambda\phi_0(w,z)} I_2| \le C_2\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda}. \tag{3.21}$$

Estimate of the terms I_3, I_4 **and** I_5. The only difference from I_2 is that either two derivatives or one derivative act on the functions $\theta_j(y)$. These terms are treated analogously to the term I_2. Then there are positive constants, C_3, C_4 and C_5, independent of N, such that

$$|e^{-\lambda\phi_0(w,z)} I_3| \le C_3(N+1)^2\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda}, \tag{3.22}$$

and

$$|e^{-\lambda\phi_0(w,z)} I_4| \le C_4(N+1)\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda} \tag{3.23}$$

and

$$|e^{-\lambda\phi_0(w,z)} I_4| \le C_5(N+1)\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda}. \tag{3.24}$$

Estimate of the term I_1. Roughly speaking we are studying the micro-local regularity of the product of an analytic function with u at the point (x_0, ξ_0). In order to estimate this term we take advantage from the following theorem which characterizes micro-local smoothness in terms of $(s_0 - 1)$-almost analytic extendability in certain wedges.

Theorem 3.1 (see Theorem 2.3 in [3]). *Let* $u \in \mathscr{D}'(U)$. *Then* $(x_0, \xi_0) \notin WF_{s_0}(u)$ *if and only if there exist a neighborhood* U_0 *of* x_0, *open acute cones* $\Gamma^1, \dots, \Gamma^k$ *in* $\mathbb{R}^n \setminus \{0\}$ *and* $(s_0 - 1)$*-almost analytic functions* f_j *on* $U_0 + i\Gamma^j_{\varepsilon_1}$, $\Gamma^j_{\varepsilon_1} = \Gamma^j \cap \{\xi : |\xi| < \varepsilon_1\}$, *of temperate growth such that* $u = \sum_{j=1}^k bf_j$ *near* x_0 *and* $\xi_0 \cdot \Gamma^j < 0$ *for all* j.

Analogous results in the smooth and analytic category can be found in [4, 5]. We point out that in the analytic case the f_j are holomorphic functions. We recall

Definition 3.1 Let $f \in G^{s_0}(U)$, U open subset of $\mathbb{R}^n$, and suppose $\tilde{U}$ is a open neighborhood of U in $\mathbb{C}^n$. A function $\tilde{f}(y, \eta) \in C^\infty(\tilde{U})$ is called an $(s_0 - 1)$-almost analytic extension of f if $\tilde{f}(y, 0) = f(y)\,\forall\, y \in U$ and for every compact K in U there exists positive constants C_K and small ε_K such that

$$\left|\partial_{\bar{z}_j} \tilde{f}\right| \le C_K e^{-\varepsilon_K |\eta|^{-\frac{1}{s_0-1}}}, \qquad j = 1, \dots, n,$$

holds for $y \in K$ and η in ball of radius ε_K.

The $(s_0 - 1)$-almost analytic extension of a Gevrey function f can be obtained in the following way

$$\tilde{f}(y + i\eta) = \sum_\gamma f^{(\gamma)}(y) \frac{i^{|\gamma|} y^\gamma}{\gamma!} \Theta\left(\tilde{c}|\gamma|^{s_0-1}|\eta|\right),$$

where Θ is in $C_0^\infty(\mathbb{R})$ such that supp$\Theta \subset [-1, 1]$ and $\Theta(y) \equiv 1$ on $[-1/2, 1/2]$. For other details see [10] or [3]. We point out that, by hypothesis, we can construct in a suitable region an $(s_0 - 1)$-almost analytic extension of u. We have to estimate

$$I_1 = \frac{1}{(2N)!} \int e^{-\frac{\lambda}{2}(w-s)^2} \chi_0(s) s^{2N} ds \sum_{|\beta| \le 2N} \sum_{\substack{\gamma \le \lfloor \frac{\beta}{2} \rfloor \\ \gamma \in \mathbb{Z}^n_+}} \frac{\beta!\, i^{|\beta| - 2|\gamma|}}{\gamma!\,(\beta - 2\gamma!)\, 2^\gamma} \lambda^{|\beta| - |\gamma|}$$
$$\times \int a_{2N,\beta}(y) e^{-\frac{\lambda}{2}(z-y)^2} \prod_{\nu=1}^n (z_\nu - y_\nu)^{\beta_\nu - 2\gamma_\nu} \theta_N(y) u(y)\, dy. \tag{3.25}$$

In order to handle the integral with respect the variable y we follow the classical strategy developed by Bros and Iagolnitzer, [21]. We split the integration domain in two parts: $\mathscr{B}_{r_N}(x_0) \setminus \mathscr{B}_{\tilde{r}_0}(x_0)$ and $\mathscr{B}_{\tilde{r}_0}(x_0)$. We have

$$\int a_{2N,\beta}(y) e^{-\frac{\lambda}{2}(z-y)^2} \prod_{\nu=1}^n (z_\nu - y_\nu)^{\beta_\nu - 2\gamma_\nu} \theta_N(y) u(y)\, dy$$
$$= \int_{\mathscr{B}_{r_N}(x_0) \setminus \mathscr{B}_{\tilde{r}_0}(x_0)} a_{2N,\beta}(y) e^{-\frac{\lambda}{2}(z-y)^2} \prod_{\nu=1}^n (z_\nu - y_\nu)^{\beta_\nu - 2\gamma_\nu} \theta_N(y) u(y)\, dy$$
$$+ \int_{\mathscr{B}_{\tilde{r}_0}(x_0)} a_{2N,\beta}(y) e^{-\frac{\lambda}{2}(z-y)^2} \prod_{\nu=1}^n (z_\nu - y_\nu)^{\beta_\nu - 2\gamma_\nu} u(y)\, dy. \tag{3.26}$$

In the first region $\left(z'-y\right)^2 \geq \varepsilon_0$, this will give the analytic exponential decay in this region. Our purpose is to verify that the second integral gives the desired Gevrey exponential decay. Since $(x_0,\xi_0) \notin WF_{s_0}(u)$ without loss of generality we may assume that u is a boundary value of $\tilde{u}(\zeta)$, (s_0-1)-almost analytic function on $\mathscr{B}_{\tilde{r}_0}(x_0)+i\Gamma_{\varepsilon_2}$, $\Gamma_{\varepsilon_2}=\{\eta\in\Gamma : |\eta|<\varepsilon_2\}$, where Γ is an open cone such that $\eta\cdot\xi_0<0$ for all $\eta\in\Gamma$. We point out that, for a fixed a neighborhood of ξ_0, we can choose Γ such that $\eta\cdot\xi<0$ for all $\eta\in\Gamma$ and ξ in the neighborhood of ξ_0. On the other hand $a_{2N,\beta}(y)$ are analytic functions, we can construct their holomorphic extension $\tilde{a}_{2N,\beta}(\zeta)$ in $\mathbb{C}^n_\zeta$, where $\zeta=y+i\eta$ and $|\eta|\leq\varepsilon_3$. We take ε_2 such that $\varepsilon_2\leq\varepsilon_3$.
Let $\vartheta(y)\in C_0^\infty(\mathbb{R}^n)$ with support equal to $\mathscr{B}_{\tilde{r}_0}(x_0)$ such that $0\leq\vartheta(y)\leq 1$ and $\vartheta(x_0)=1$. Let $\eta^0\in\Gamma_{\varepsilon_2}$, we define the n-dimensional manifold, S_{η^0,ε_4}, in $\mathbb{C}^n_\zeta$, $\zeta=y+i\eta$, given by

$$y\mapsto\zeta=y+i\varepsilon_4\vartheta(y)\eta^0,$$

where $\varepsilon_4\in\mathbb{R}_+$ and is sufficiently small so that S_{η^0,ε_4} is contained in $\mathscr{B}_{\tilde{r}_0}(x_0)+i\Gamma_{\varepsilon_2}$. We remark that the boundary of S_{η^0,ε_4} is equal to $\partial\mathscr{B}_{\tilde{r}_0}(x_0)$. By the Stokes theorem we have

$$\begin{aligned}
&\int_{\mathscr{B}_{\tilde{r}_0}(x_0)} a_{2N,\beta}(y)e^{-\frac{\lambda}{2}(z-y)^2}\prod_{\nu=1}^n(z_\nu-y_\nu)^{\beta_\nu-2\gamma_\nu}\,u(y)\,dy\\
&\quad=-\int_{S_{\eta^0,\varepsilon_4}}\tilde{a}_{2N,\beta}(\zeta)e^{-\frac{\lambda}{2}(z-\zeta)^2}\prod_{\nu=1}^n(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\,\tilde{u}(\zeta)\,d\zeta\\
&\qquad+\int_{D_{\eta^0}} d\left(\tilde{a}_{2N,\beta}(\zeta)e^{-\frac{\lambda}{2}(z-\zeta)^2}\prod_{\nu=1}^n(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\,\tilde{u}(\zeta)\right)\wedge d\zeta,
\end{aligned}$$

where $D_{\eta^0}=\bigcup_{0<t<\varepsilon_4} S_{\eta^0,t}\subseteq\mathscr{B}_{\tilde{r}_0}(x_0)+i\Gamma_{\varepsilon_2}$ and $\partial D_{\eta^0}=\mathscr{B}_{\tilde{r}_0}(x_0)\cup S_{\eta^0,\varepsilon_4}$.
Since

$$\begin{aligned}
d\zeta_j&=\sum_{\substack{k=1\\k\neq j}}^n(it\,\vartheta^{(e_k)}(y)\,\eta^0_j)dy_k+(1+it\,\vartheta^{(e_j)}(y)\,\eta^0_j)dy_j+(i\,\vartheta(y)\,\eta^0_j)dt,\\
d\bar{\zeta_i}&=\sum_{\substack{k=1\\k\neq j}}^n(-it\,\vartheta^{(e_k)}(y)\,\eta^0_j)dy_k+(1-it\,\vartheta^{(e_j)}(y)\,\eta^0_j)dy_j-(i\,\vartheta(y)\,\eta^0_j)dt,
\end{aligned}$$

where $\vartheta^{(e_j)}(y)=\left(\partial_{y_j}\vartheta\right)(y)$, and $\tilde{a}_{2N,\beta}(\zeta)$ are holomorphic functions, analytic extensions of $a_{2N,\beta}(y)$, in D_{η^0}, we have

$$\begin{aligned}
&\left(d\left(\tilde{a}_{2N,\beta}(\zeta)e^{-\frac{\lambda}{2}(z-\zeta)^2}\prod_{\nu=1}^n(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\,\tilde{u}(\zeta)\right)\wedge d\zeta\right)_{|S_{\eta^0,t}}\\
&=\sum_{j=1}^n\left(\tilde{a}_{2N,\beta}(\zeta)e^{-\frac{\lambda}{2}(z-\zeta)^2}\prod_{\nu=1}^n(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\,\frac{\partial\tilde{u}}{\partial\bar{\zeta}_j}(\zeta)d\bar{\zeta_j}\wedge d\zeta\right)_{|S_{\eta^0,t}}\\
&=\sum_{j=1}^n\left(\tilde{a}_{2N,\beta}(\zeta)e^{-\frac{\lambda}{2}(z-\zeta)^2}\prod_{\nu=1}^n(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\,\frac{\partial\tilde{u}}{\partial\bar{\zeta}_i}(\zeta)\right)_{|S_{\eta^0,t}}\det\left(A_j(y,t,\eta^0)\right)\,dt\,dy,
\end{aligned}$$

where $A_j(y,t,\eta^0)$ is the $(n+1)\times(n+1)-$matrix

$$\begin{pmatrix} -it\vartheta^{(e_1)}(y)\eta_j^0 & \cdots & -it\vartheta^{(e_{j-1})}(y)\eta_j^0 & 1-it\vartheta^{(e_j)}(y)\eta_j^0 & -it\vartheta^{(e_{j+1})}(y)\eta_j^0 & \cdots & -i\vartheta(y)\eta_j^0 \\ 1+it\vartheta^{(e_1)}(y)\eta_1^0 & it\vartheta^{(e_2)}(y)\eta_1^0 & \cdots & \cdots & \cdots & \cdots & i\vartheta(y)\eta_1^0 \\ it\vartheta^{(e_1)}(y)\eta_1^0 & 1+it\vartheta^{(e_2)}(y)\eta_2^0 & it\vartheta^{(e_3)}(y)\eta_2^0 & \cdots & \cdots & \cdots & i\vartheta(y)\eta_2^0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ it\vartheta^{(e_1)}(y)\eta_n^0 & \cdots & \cdots & \cdots & it\vartheta^{(e_{n-1})}(y)\eta_n^0 & 1+it\vartheta^{(e_n)}(y)\eta_n^0 & i\vartheta(y)\eta_n^0 \end{pmatrix}$$

We obtain

$$e^{-\lambda\phi_0(w,z)}I_1 = e^{-\lambda\phi_0(w,z)}I_{1,1} + e^{-\lambda\phi_0(w,z)}I_{1,2} + e^{-\lambda\phi_0(w,z)}I_{1,3},$$

where

$$\begin{aligned} e^{-\lambda\phi_0(w,z)}I_{1,1} &= \frac{1}{(2N)!}e^{-\frac{\lambda}{2}\left[(w'')^2+(z'')^2\right]}\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\,ds \\ &\quad\times \sum_{|\beta|\le 2N}\sum_{\substack{\gamma\le\lfloor\frac{\beta}{2}\rfloor\\ \gamma\in\mathbb{Z}_+^n}} \frac{\beta!\,i^{|\beta|-2|\gamma|}}{\gamma!\,(\beta-2\gamma!)\,2^{\gamma}}\lambda^{|\beta|-|\gamma|}\int_{\mathscr{B}_{r_N}(x_0)\setminus\mathscr{B}_{\tilde r_0}(x_0)} e^{-\frac{\lambda}{2}(z-y)^2}a_{2N,\beta}(y) \\ &\quad\times \prod_{\nu=1}^{n}(z_\nu-y_\nu)^{\beta_\nu-2\gamma_\nu}\,\theta_N(y)u(y)\,dy, \\ e^{-\lambda\phi_0(w,z)}I_{1,2} &= \frac{1}{(2N)!}e^{-\frac{\lambda}{2}\left[(w'')^2+(z'')^2\right]} \\ &\quad\times\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\,ds\sum_{|\beta|\le 2N}\sum_{\substack{\gamma\le\lfloor\frac{\beta}{2}\rfloor\\ \gamma\in\mathbb{Z}_+^n}} \frac{\beta!\,i^{|\beta|-2|\gamma|}}{\gamma!\,(\beta-2\gamma!)\,2^{\gamma}}\lambda^{|\beta|-|\gamma|} \\ &\quad\times\int_{\mathscr{B}_{\tilde r_0}(x_0)}\left[e^{-\frac{\lambda}{2}(z-\zeta)^2}\tilde a_{2N,\beta}(\zeta)\prod_{\nu=1}^{n}(z_\nu-\zeta_\nu)^{\beta_\nu-2\gamma_\nu}\tilde u(\zeta)\right]_{\zeta=y+i\varepsilon_4\vartheta(y)\eta^0}\det\left(B(y,\varepsilon_4,\eta^0)\right)\,dy, \end{aligned}$$

where $B(y,\varepsilon_4,\eta^0)$ is the $n\times n$-matrix

$$\begin{pmatrix} 1+i\varepsilon_4\vartheta^{(e_1)}(y)\eta_1^0 & i\varepsilon_4\vartheta^{(e_2)}(y)\eta_1^0 & \cdots & \cdots & \cdots & i\varepsilon_4\vartheta^{(e_n)}(y)\eta_1^0 \\ i\varepsilon_4\vartheta^{(e_1)}(y)\eta_1^0 & 1+i\varepsilon_4\vartheta^{(e_2)}(y)\eta_2^0 & i\varepsilon_4\vartheta^{(e_3)}(y)\eta_2^0 & \cdots & \cdots & i\varepsilon_4\vartheta^{(e_n)}(y)\eta_2^0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ i\varepsilon_4\vartheta^{(e_1)}(y)\eta_n^0 & \cdots & \cdots & \cdots & i\varepsilon_4\vartheta^{(e_{n-1})}(y)\eta_n^0 & 1+i\varepsilon_4\vartheta^{(e_n)}(y)\eta_n^0 \end{pmatrix}$$

and

$$\begin{aligned} e^{-\lambda\phi_0(w,z)}I_{1,3} =& \frac{1}{(2N)!}e^{-\frac{\lambda}{2}\left[(w'')^2+(z'')^2\right]}\int e^{-\frac{\lambda}{2}(w-s)^2}\chi_0(s)s^{2N}\,ds \\ &\times\sum_{|\beta|\le 2N}\sum_{\substack{\gamma\le\lfloor\frac{\beta}{2}\rfloor\\ \gamma\in\mathbb{Z}_+^n}} \frac{\beta!\,i^{|\beta|-2|\gamma|}}{\gamma!\,(\beta-2\gamma!)\,2^{\gamma}}\lambda^{|\beta|-|\gamma|}\sum_{\ell=1}^{n} I_{1,3,\beta,\gamma,\ell}, \end{aligned}$$

where

$$
\begin{aligned}
&I_{1,3,\beta,\gamma,\ell} = \\
&\int_0^{\varepsilon_4} \int_{S_{\eta^0,t}} e^{-\frac{\lambda}{2}(z-\zeta)^2} \tilde{a}_{2N,\beta}(\zeta) \prod_{\nu=1}^{n} (z_\nu - \zeta_\nu)^{\beta_\nu - 2\gamma_\nu} \left(\bar{\partial}\tilde{u}\right)(\zeta)\, (i\vartheta\left(\frac{\zeta+\bar{\zeta}}{2}\right)\eta^0_\ell)\, dt \wedge d\zeta \\
&= \int_0^{\varepsilon_4} \int_{\mathscr{B}_{\tilde{r}_0}(x_0)} e^{-\frac{\lambda}{2}(z-y-it\vartheta(y)\eta^0)^2} \tilde{a}_{2N,\beta}\big(y+it\vartheta(y)\eta^0\big) \left(\bar{\partial}\tilde{u}\right)(y+it\vartheta(y)\eta^0) \\
&\quad \times \prod_{\nu=1}^{n} \left(z_\nu - y_\nu - it\vartheta(y)\eta^0_\nu\right)^{\beta_\nu - 2\gamma_\nu} \det\left(A_\ell(y,t,\eta^0)\right)\, dt\, dy.
\end{aligned}
$$

Since $\left(z'-y\right)^2 \geq \varepsilon_0$ in $\mathscr{B}_{r_N}(x_0)\backslash\mathscr{B}_{\tilde{r}_0}(x_0)$, using the same strategy as for the term I_2, we obtain

$$
\left| e^{-\lambda\phi_0(w,z)} I_{1,1} \right| \leq C_5 \left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda},
$$

where C_5 is a positive constant independent of N.
A quick inspection of the terms $I_{1,2}$ and $I_{1,3}$ highlights that the main differences with respect to the already treated terms, are the behavior of the phase function on the integration path as well as the presence of $\bar{\partial}u$. We point out that setting $a^\ell_{p,q}(y,t,\eta^0)$ and $b_{k,m}(y,\varepsilon_4,\eta^0)$, $p,q \in \{1,\ldots,n+1\}$ and $k,m \in \{1,\ldots,n\}$, the entries of the matrixes $A_\ell(y,t,\eta^0)$ and $B(y,\varepsilon_4,\eta^0)$ respectively, since we can estimate the entries $|a^\ell_{p,q}(y,t,\eta^0)|$ and $|b_{k,m}(y,\varepsilon_4,\eta^0)|$ by $(1+\sup_i \|\vartheta^{(e_i)}\|_\infty)$ we have

$$
\begin{aligned}
\left|\det\left(A_\ell(y,t,\eta^0)\right)\right| &\leq \sum_{\sigma\in S_{n+1}} \prod_{p=1}^{n+1} \left| a^\ell_{p,\sigma(q)}(y,t,\eta^0) \right| \\
&\leq \frac{(n+1)!\,[(n+1)!+1]}{2} (1+\sup_i \|\vartheta^{(e_i)}\|_\infty)^{n+1}, \\
\left|\det\left(B(y,\varepsilon_4,\eta^0)\right)\right| &\leq \sum_{\sigma\in S_n} \prod_{k=1}^{n} \left| b_{k,\sigma(k)}(y,\varepsilon_4,\eta^0) \right| \\
&\leq \frac{n!\,(n!+1)}{2} (1+\sup_i \|\vartheta^{(e_i)}\|_\infty)^{n}.
\end{aligned}
$$

We focus on the exponential function:

$$
e^{-\frac{\lambda}{2}\Re(z-y-it\vartheta(y)\eta^0)^2} = e^{\frac{\lambda}{2}(z'')^2} e^{-\frac{\lambda}{2}(z'-y)^2} e^{-\lambda t\vartheta(y) z''\eta^0 + \frac{\lambda}{2}(t\vartheta(y))^2|\eta^0|^2}.
$$

Since z'' is in a neighborhood of $-\xi_0$ then $z''\eta^0 > 0$. Hence there is a positive constant c such that $z''\eta^0 > c|z''||\eta^0|$; moreover since we can assume that there is a strictly positive constant a such that $|z''| \geq a$ then $z''\eta^0 > c_1|\eta^0|$, $c_1 > 0$. We can estimate the above quantity with

$$
e^{\frac{\lambda}{2}(z'')^2} e^{-\frac{\lambda}{2}\left[(z'-y)^2 + t\vartheta(y)|\eta^0|\left(2c_1 - t\vartheta(y)|\eta^0|\right)\right]}.
$$

Choosing t sufficiently small we have that $2c_1 - t\vartheta(y)|\eta^0| > 0$. In the case $t = \varepsilon_4$, we obtain the analytic exponential decay for $I_{1,2}$; more precisely the same strategy used to handle the term I_2 gives that there are two positive constants C_6 and $\tilde{\varepsilon}_2$, independent of N, such that

$$\left|e^{-\lambda\phi_0(w,z)}I_{1,2}\right| \leq C_6\left(\frac{1}{2}\right)^N e^{-\tilde{\varepsilon}_2\lambda}.$$

In order to estimate the last term, $\left|e^{-\lambda\phi_0(w,z)}I_{1,3}\right|$, we can apply once again the strategy used to estimate I_2. The only difference is that we have to take care of the term $\left|\left(\bar{\partial}\tilde{u}\right)(y+it\vartheta(y)\eta^0)\right|$. Keeping in mind that $\tilde{u}$ is an (s_0-1)-almost analytic extension of u, we have

$$\begin{aligned}
&\left|e^{-\frac{\lambda}{2}(z-\zeta)^2}\right|\left|\left(\bar{\partial}\tilde{u}\right)(y+it\vartheta(y)\eta^0)\right| \\
&\quad\leq Ce^{\frac{\lambda}{2}(z'')^2}e^{-\frac{\lambda}{2}(z'-y)^2}e^{-\lambda c_2 t\vartheta(y)|\eta^0|}e^{-\varepsilon_K\left(t\vartheta(y)|\eta^0|\right)^{-\frac{1}{s_0-1}}} \\
&\quad\leq Ce^{\frac{\lambda}{2}(z'')^2-\frac{\lambda}{2}(z'-y)^2}e^{-\tilde{\varepsilon}_K\lambda^{1/s_0}},
\end{aligned}$$

where $\tilde{\varepsilon}_K = c_2\gamma_1^{(s_0-1)/s_0} + \gamma_1^{-1/s_0}$, $\gamma_1 = \varepsilon_K/(c_2(s_0-1))$, and ε_K is as in the Definition 3.1 with $K = \overline{\mathscr{B}_{\tilde{r}_0}(x_0)}$. The estimate in the exponential is obtained taking $\inf_b\left(\lambda c_1 b + \varepsilon_K b^{-\frac{1}{s_0-1}}\right)$, where $b = t\vartheta(y)|\eta^0|$. Using this estimate we conclude that there are two positive constants C_7 and ε_4 such that

$$\left|e^{-\lambda\phi_0(w,z)}I_{1,3}\right| \leq C_7\left(\frac{1}{2}\right)^N e^{-\varepsilon_4\lambda^{1/s_0}}.$$

We deduce that there is a positive constant C_8 such that

$$|e^{-\lambda\phi_0(w,z)}I_1| \leq C_8\left(\frac{1}{2}\right)^N e^{-\varepsilon_4\lambda^{1/s_0}}. \tag{3.27}$$

Remark 3.3 The estimate of the second term on the right hand side of (3.26), i.e. in the region $\mathscr{B}_{\tilde{r}_0}(x_0)$, can be obtained in a similar way introducing the family of homeomorphisms

$$\mathscr{H}_t : \mathscr{B}_{\tilde{r}_0}(x_0) \ni y \to \left(y_1 + it\eta_1^0, \ldots, y_n + it\eta_n^0\right) \in \mathbb{C}^n_\zeta,$$

where $\eta^0 \in \Gamma_{\varepsilon_2}$. Also in this case $\mathscr{H}_t\left(\mathscr{B}_{\tilde{r}_0}(x_0)\right)$ is a n-dimensional manifold of $\mathbb{C}^n_\zeta$ for every $t \in [0,1]$. Setting

$$\begin{aligned}
V(y,t\eta^0) = \tilde{a}_{2N,\beta}(y+it\eta^0)e^{-\frac{\lambda}{2}(z-y-it\eta^0)^2} \times \\
\prod_{\nu=1}^{n}\left(z_\nu - y_\nu - it\eta_\nu^0\right)^{\beta_\nu-2\gamma_\nu}\tilde{u}(y+it\eta^0),
\end{aligned}$$

and applying, also in this case, the Stokes' theorem

$$\int_{\mathscr{H}_1\left(\mathscr{B}_{\tilde{r}_0}(x_0)\right)} V(\zeta)\,d\zeta - \int_{\mathscr{H}_0\left(\mathscr{B}_{\tilde{r}_0}(x_0)\right)} V(\zeta)\,d\zeta = \int_{\mathscr{V}} d\left(V(\zeta)\right)\wedge d\zeta,$$

where $\mathscr{V} = [0,1]\times\mathscr{H}_0\left(\mathscr{B}_{\tilde{r}_0}(x_0)\right)$, the estimate (3.27) can be obtained following step by step the strategy employed above.

By (3.21), (3.22), (3.23), (3.24) and (3.27) we have

$$|e^{-\lambda\phi_0(w,z)}\mathscr{P}_N(w,z)| \leq C_8\left(\frac{1}{2}\right)^N e^{-\varepsilon_4\lambda^{1/s_0}} + C_2\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda} + [C_3(N+1)+C_4+C_5](N+1)\left(\frac{1}{2}\right)^N e^{-\varepsilon_2\lambda}.$$

Summing up we obtain that there are two positive constants C and ε such that

$$|e^{-\lambda\phi_0(w,z)}T(\chi U)(w,z,\lambda)| \leq Ce^{-\varepsilon\lambda^{1/s_0}},$$

for all (w,z) in a neighborhood of $(0, x_0 - i\xi_0) \in \mathbb{C}^{1+n}$.
Step two: if $(0, x_0, 0, \xi_0) \notin WF_{s_0}(U)$ then $(x_0, \xi_0) \notin WF_{s_0}(u)$. In the analytic category the result was obtained in [8] via Fourier transform and taking advantage from the Theorem 8.2.4 in [20]. Via FBI transform it is a consequence of a result in [21] on the restriction of a distribution to a sub-manifold. More precisely we remark that for every $\tau_0 \neq 0$ the points of the form $(t_0, x_0, \tau_0, \xi_0)$ do not belong to $WF_{s_0}(U)$ for every $s_0 \geq 1$. This can be obtained ether via FBI transform, performing the classical deformation argument of the integral path with respect to the t-variable, or noticing that the operator Q is elliptic for $\tau \neq 0$. Since $WF_{s_0}(\delta(t)) = \{(x,0,0,\tau) : x \in \mathbb{R}^n \text{ and } \tau \in \mathbb{R}\setminus\{0\}\}$ we have that $WF_{s_0}(U) \cap WF_{s_0}(\delta(t)) = \emptyset$, or equivalently that the normal to the manifold $t = 0$ does not intersect the $WF_{s_0}(U)$, then the product of U and $\delta(t)$ is well defined. This allow us to consider $u(x)$ as $U(t,x) \times \delta(t)$ in the sense of distributions. More in general we can define the map $\pi : \{U \in \mathscr{E}'(\mathbb{R}^{n+1}) : WF_{s_0}(U) \cap WF_{s_0}(\delta(t)) = \emptyset\} \to \mathscr{E}'(\mathbb{R}^n)$ in the following way $u(\phi_0) = \pi(U)(\phi_0) = U(\phi_1\delta(t))$ for all $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ where $\phi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ and $\pi(\phi_1) = \phi_0$. Following the same strategy used in [20] we have that $WF_{s_0}(u) = WF_{s_0}(\pi(U))$ which is contained in $\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\setminus\{0\} : \exists\tau \in \mathbb{R} \text{ with } (x,0,\xi,\tau) \in WF_{s_0}(U)\}$.
This concludes the proof of Theorem 1.1.

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