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SKT structures on nilmanifolds

Romina M. Arroyo[1] · Marina Nicolini[1]

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Abstract
The aim of this article is to study the existence of invariant SKT structures on nilmanifolds. More precisely, we give a negative answer to the question of whether there exist a k-step ($k > 2$) complex nilmanifold admitting an invariant SKT metric. We also provide a construction which serves as a tool to generate examples of invariant SKT structures on 2-step nilmanifolds in arbitrary dimensions.

1 Introduction

Let (M, J, g) be a Hermitian manifold with associated fundamental form ω. If ω is not closed, it means that the manifold is not Kähler, then the Levi–Civita connection does not preserve the complex structure. There are plenty of connections preserving both structures [11], but there is only one such that the torsion 3-form is totally skew-symmetric, the so-called *Bismut connection*. When the 3-torsion form is in addition closed, the Hermitian manifold (M, J, g) is said to be *strong Kähler with torsion* (SKT for short) or *pluriclosed*.

We are interested in the study of invariant SKT structures on nilmanifolds. Here, M is a compact quotient $\Gamma\backslash N$, of a simply-connected nilpotent Lie group N by a co-compact lattice Γ, and the Hermitian structure comes from a left-invariant Hermitian structure on the Lie group N.

Over recent years, invariant SKT structures on nilmanifolds have been studied by many authors, and remarkably, still not much is known about their existence. The classification in dimensions 4, 6 and 8 was obtained in [5, 8, 12], respectively. Regarding higher dimensions, a characterization of a class of SKT nilmanifolds was studied in [16], where the complex structure is nilpotent and the compatible metric is Kähler-like. On the other hand, to the best of our knowledge, the only non-existence results in arbitrary dimensions are given in [5].

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✉ Romina M. Arroyo
arroyo@famaf.unc.edu.ar

Marina Nicolini
mnicolini@famaf.unc.edu.ar

1 FAMAF and CIEM, Universidad Nacional de Córdoba, Av. Medina Allende s/n, Ciudad Universitaria, CP:X5000HUA Córdoba, Argentina

All known examples in the literature of nilmanifolds admitting an SKT structure are 2-step nilpotent. In [5, Theorem 1.2], it is stated that the latter exhaust all the nilpotent examples. Unfortunately, its proof has a gap (see [7]), leading to the following problem.

Problem 1.1 [3, 7, 9, 10] *Does a k-step complex nilmanifold ($k > 2$) admitting an invariant SKT structure exist?*

The first partial negative answer to Problem 1.1 was given in [16], where the authors work on Kähler-like structures on nilmanifolds assuming nilpotency on the complex structure. The latter turn out to be 2-step nilmanifolds and the complex structure is necessarily abelian. After that, in the recent work [9], a negative answer to the problem was obtained on complex nilmanifolds with the abelian assumption in the complex structure.

Our main result gives a complete answer to Problem 1.1. As an important consequence, [4, Theorem 2.3], [5, Theorem 1.1] and [6, Theorem 1.1] turn out to be valid. Moreover, the long-time behaviour of the pluriclosed flow of invariant SKT structures on nilmanifolds is now completely understood (see [1, Theorem A]).

Theorem 1.2 *Any nilmanifold admitting an invariant SKT structure is either a torus or 2-step nilpotent.*

According to Theorem 1.2, the next move is to understand invariant SKT structures on 2-step nilmanifolds. The really hard problem is to reach new examples in higher dimensions, and the lack of them motivated us to develop a method to construct families of invariant SKT structures on nilmanifolds in higher dimensions starting with low dimensional ones (see Sect. 5). This machinery provides explicit examples in every complex dimension. Moreover, as far as we know, we give the first examples of indecomposable SKT nilmanifolds with non-abelian complex structure in higher dimensions. Here, by a indecomposable SKT nilmanifold we mean a SKT nilmanifold which can not be decomposed as a product of two SKT nilmanifolds of lower dimensions.

We now give some insight into our main results. Any invariant SKT structure on a nilmanifold is determined by the following infinitesimal data, which we call an *SKT Lie algebra*: a nilpotent Lie algebra $\mathfrak{g}$, a complex structure J on $\mathfrak{g}$ and an inner product on $\mathfrak{g}$ satisfying a system of equations on $\mathfrak{g}$ due to the SKT condition. The key idea in the proof of Theorem 1.2 is to write $\mathfrak{g} = \operatorname{span}\{e_1, e_2\} \oplus \mathfrak{n}$, as the orthogonal sum of a subspace and an ideal $\mathfrak{n}$, where both spaces are J-invariant (see [13, Corollary 1.4]), and to prove that $(\mathfrak{n}, J|_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ is also SKT (see Sect. 4). Then, $\mathfrak{g}$ is determined by

$$A := \operatorname{ad}(e_1)_{\mathfrak{n}}, \quad B := \operatorname{ad}(e_2)_{\mathfrak{n}}, \quad X := [e_1, e_2], \quad \text{and} \quad [\cdot,\cdot]_{\mathfrak{n}},$$

where $\operatorname{ad}(e_i)_{\mathfrak{n}}$ denotes the projection of $\operatorname{ad}(e_i)|_{\mathfrak{n}}$ onto $\mathfrak{n}$, for $i = 1, 2$. If we apply induction on n to $\dim \mathfrak{g} = 2n$, then by induction hypothesis, the ideal $\mathfrak{n}$ is forced to be abelian or 2-step nilpotent (see Section 4.1). Therefore, $\mathfrak{n}$ can be decomposed as $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v} := \mathfrak{z}^{\perp}$ ($\mathfrak{n} = \mathfrak{z}$ when $\mathfrak{n}$ is abelian). Since $A, B \in \operatorname{Der}(\mathfrak{n})$, then

$$A = \begin{bmatrix} A_{\mathfrak{v}} & 0 \\ * & A_{\mathfrak{z}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathfrak{v}} & 0 \\ * & B_{\mathfrak{z}} \end{bmatrix}.$$

We first show that $A_{\mathfrak{z}} = 0$ and $B_{\mathfrak{z}} = 0$ by using the SKT condition (see Corollary 4.3 for an abelian $\mathfrak{n}$ and Lemma 4.2 for a 2-step nilpotent $\mathfrak{n}$). This fact together with the nilpotency of $\mathfrak{g}$ and the integrability of J are the ingredients to demonstrate that $A_{\mathfrak{v}} = 0$, $B_{\mathfrak{v}} = 0$ and $X \in \mathfrak{z}$, which proves that $\mathfrak{g}$ is at most 2-step nilpotent.

Our second main result is a method that provides new explicit examples of SKT Lie algebras (see Sect. 5). We start with two SKT 2-step nilpotent Lie algebras $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_2, J_2, \langle\cdot,\cdot\rangle_2)$ of dimensions n_1 and n_2, respectively, satisfying

$$\mathfrak{n}_i = \mathfrak{v}_i \oplus \mathfrak{z}_i \quad \text{and} \quad \dim \mathfrak{z}_i > \dim [\mathfrak{n}_i, \mathfrak{n}_i], \quad i = 1, 2,$$

where $\mathfrak{z}_i$ is the center of $\mathfrak{n}_i$ and $\mathfrak{v}_i := \mathfrak{z}_i^{\perp}$, $i = 1, 2$, and we construct a new SKT Lie algebra of dimension $n_1 + n_2 + 2$ by setting $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \langle Z, W\rangle$ with Lie bracket given by

$$[\cdot,\cdot]|_{\mathfrak{n}_1\times\mathfrak{n}_1} = [\cdot,\cdot]_{\mathfrak{n}_1}, \quad [\cdot,\cdot]|_{\mathfrak{n}_2\times\mathfrak{n}_2} = [\cdot,\cdot]_{\mathfrak{n}_2}, \quad [Z, W] = X_{n_1} + Y_{n_2},$$

where $X_{n_1} \in \mathfrak{z}_1 \cap [\mathfrak{n}_1, \mathfrak{n}_1]^{\perp}$ and $Y_{n_2} \in \mathfrak{z}_2 \cap [\mathfrak{n}_2, \mathfrak{n}_2]^{\perp}$. The complex structure is defined as

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix},$$

and the inner product is the one that makes the above decomposition of $\mathfrak{g}$ orthogonal while extending $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. The SKT Lie algebra $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is indecomposable, in the sense that it is not a sum of two SKT Lie algebras, despite $\mathfrak{g}$ is decomposable (see Sect. 5.1).

The organization of this article is as follows. In Section 2 we review some basic facts about left-invariant SKT structures on Lie groups. In Sect. 3 we prove some useful results. Then, we apply these results in Sect. 4, which is devoted to the proof of Theorem 1.2. Finally, we present a construction in Sect. 5 and explicit examples are provided.

2 Preliminaries

Given (M^{2n}, J) a differentiable manifold of real dimension $2n$ endowed with a complex structure, a Riemannian metric g on M is said to be *Hermitian* if $g(J\cdot, J\cdot) = g(\cdot,\cdot)$. The pair (J, g) is called a *Hermitian structure* and $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is the fundamental 2-form associated to the pair. The *Bismut (or Strominger) connection* ∇^B on M is the unique Hermitian connection (that is, J and g are parallel) with totally skew-symmetric torsion. That is, the tensor

$$c(U, Y, Z) := g(U, T^B(Y, Z)) \tag{1}$$

is a 3-form, where $T^B(Y, Z) = \nabla^B_Y Z - \nabla^B_Z Y - [Y, Z]$ is the torsion of ∇^B (see [2, 14]). The metric g (or ω) is called *strong Kähler with torsion (SKT)* or *pluriclosed* if its fundamental 2-form satisfies $\partial\bar{\partial}\omega = 0$, or equivalently, the 3-form c is closed. In this case, (J, g) is called a *SKT-structure* and the triple (M, J, g) is said to be SKT.

We are interested in the study of *invariant SKT-structures* on Lie groups. Here, the universal cover $\tilde{M}$ of M is diffeomeophic to a simply-connected Lie group G and π^*J and π^*g are left-invariant tensors defining a Hermitian structure on G, where $\pi : G \to M$ denotes the universal covering map.

2.1 Nilpotent Lie groups and Lie algebras

Given a Lie group G with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, for each $X \in \mathfrak{g}$ we define the *adjoint map* as the linear map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, given by $\mathrm{ad}(X)(Y) = [X, Y]$ and we denote by $\mathfrak{z}(\mathfrak{g})$ the *center* of $\mathfrak{g}$, that is, $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \mathrm{ad}(X) = 0\}$.

For a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, we define its *descending central series* by:

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}], \text{ for } i \geq 1.$$

A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}_k = 0$. In addition, if $\mathfrak{g}_k = 0$ and $\mathfrak{g}_{k-1} \neq 0$, the Lie algebra is said to be *k-step nilpotent*. A Lie group G is *(k-step) nilpotent* if its Lie algebra is (k-step) nilpotent.

From now on, we simply denote by $\mathfrak{g}$ the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$.

2.2 Hermitian structures on Lie groups

Left-invariant Hermitian structures on simply-connected Lie groups (G, J, g) are completely determined by $(\mathfrak{g}, J(e), g(e))$, where e is the identity of G. Here, if we denote by $J := J(e)$ and $\langle \cdot,\cdot \rangle := g(e)$, then J is a linear endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\mathrm{Id}_{\mathfrak{g}}$ and the integrability condition

$$[J\cdot, J\cdot] = [\cdot,\cdot] + J[J\cdot,\cdot] + J[\cdot, J\cdot],$$

and $\langle \cdot,\cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an inner product on $\mathfrak{g}$ such that $\langle J\cdot, J\cdot \rangle = \langle \cdot,\cdot \rangle$.

From now on, we denote the Hermitian manifold (G, J, g) by $(\mathfrak{g}, J, \langle \cdot,\cdot \rangle)$.

2.3 SKT metrics on Lie groups

The torsion 3-form of the Bismut connection of a left-invariant Hermitian manifold $(\mathfrak{g}, J, \langle \cdot,\cdot \rangle)$ can be computed by (see [5, (3.2)])

$$c(U, Y, Z) = -\langle [JU, JY], Z \rangle - \langle [JY, JZ], U \rangle - \langle [JZ, JU], Y \rangle, \qquad U, Y, Z \in \mathfrak{g}, \tag{2}$$

and its exterior derivative is thus given by

$$\begin{aligned} dc(W, U, Y, Z) = {} & \langle [J[W, U], JY], Z \rangle + \langle [JY, JZ], [W, U] \rangle + \langle [JZ, J[W, U]], Y \rangle \\ & - \langle [J[W, Y], JU], Z \rangle - \langle [JU, JZ], [W, Y] \rangle - \langle [JZ, J[W, Y]], U \rangle \\ & + \langle [J[W, Z], JU], Y \rangle + \langle [JU, JY], [W, Z] \rangle + \langle [JY, J[W, Z]], U \rangle \\ & + \langle [J[U, Y], JW], Z \rangle + \langle [JW, JZ], [U, Y] \rangle + \langle [JZ, J[U, Y]], W \rangle \\ & - \langle [J[U, Z], JW], Y \rangle - \langle [JW, JY], [U, Z] \rangle - \langle [JY, J[U, Z]], W \rangle \\ & + \langle [J[Y, Z], JW], U \rangle + \langle [JW, JU], [Y, Z] \rangle + \langle [JU, J[Y, Z]], W \rangle. \end{aligned} \tag{3}$$

Then, the SKT condition $dc = 0$ can be written as a system of equations on $\mathfrak{g}$ involving the Lie bracket, the complex structure and the inner product.

From now on, we will say that $(\mathfrak{g}, J, \langle \cdot,\cdot \rangle)$ is SKT or an SKT Lie algebra if it is a Hermitian manifold with (3) vanishes.

3 SKT nilmanifolds

The aim of this section is to prove two helpful results for the following sections. We also recall a result from [5] and set up some notation.

Proposition 3.1 [5, Proposition 3.1] *If $(\mathfrak{g}, J, \langle \cdot,\cdot \rangle)$ is SKT with $\mathfrak{g}$ nilpotent, then $\mathfrak{z}(\mathfrak{g})$ is J-invariant.*

Notation 3.2 Let V be a vector space. If $T \in \mathfrak{gl}(V)$ and W is a subspace of V, then T_W denotes the projection of $T|_W$ onto W.

Proposition 3.3 *If* $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ *is SKT and* $\mathfrak{n}$ *is a* J*-invariant subalgebra of* $\mathfrak{g}$*, then* $(\mathfrak{n}, J_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ *is SKT.*

Proof Let $\iota : \mathfrak{n} \to \mathfrak{g}$ be the inclusion map. Then, $\iota^* d_{\mathfrak{g}} = d_{\mathfrak{n}} \iota^*$, since ι is a Lie algebra homomorphism. Moreover, $J_{\mathfrak{n}} = J|_{\mathfrak{n}} = J \circ \iota$ due to the J-invariance of $\mathfrak{n}$ and so $\iota \circ J_{\mathfrak{n}} = J_{\mathfrak{n}} = J \circ \iota$. Then, the corresponding pullbacks also commute and

$$d_{\mathfrak{n}} c_{\mathfrak{n}} = d_{\mathfrak{n}} J|_{\mathfrak{n}}^* d_{\mathfrak{n}} \omega|_{\mathfrak{n}} = d_{\mathfrak{n}} J|_{\mathfrak{n}}^* d_{\mathfrak{n}} \iota^* \omega_{\mathfrak{g}} = \iota^* d_{\mathfrak{g}} J^* d_{\mathfrak{g}} \omega_{\mathfrak{g}} = \iota^* d_{\mathfrak{g}} c_{\mathfrak{g}} = 0,$$

which concludes the proof. □

Lemma 3.4 *Let* $(\mathfrak{n}, J, \langle\cdot,\cdot\rangle)$ *be an SKT Lie algebra where* $\mathfrak{n}$ *is* 2*-step nilpotent. Then,* $Y \in \mathfrak{z}(\mathfrak{n})$ *if and only if* $[Y, JY] = 0$.

Proof Let $W, Y \in \mathfrak{n}$, then by (3),

$$\begin{aligned} dc(W, JW, Y, JY) = &+\langle [J[W, JW], JY], JY\rangle - \langle [JY, Y], [W, JW]\rangle \\ &-\langle [Y, J[W, JW]], Y\rangle + \langle [J[W, Y], W], JY\rangle - \langle [W, Y], [W, Y]\rangle \\ &+\langle [Y, J[W, Y]], JW\rangle - \langle [J[W, JY], W], Y\rangle \\ &-\langle [W, JY], [W, JY]\rangle + \langle [JY, J[W, JY]], JW\rangle \\ &+\langle [J[JW, Y], JW], JY\rangle - \langle [JW, Y], [JW, Y]\rangle \\ &-\langle [Y, J[JW, Y]], W\rangle - \langle [J[JW, JY], JW], Y\rangle \\ &-\langle [JW, JY], [JW, JY]\rangle - \langle [JY, J[JW, JY]], W\rangle \\ &+\langle [J[Y, JY], JW], JW\rangle - \langle [JW, W], [Y, JY]\rangle \\ &-\langle [W, J[Y, JY]], W\rangle. \end{aligned}$$

Since $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{z}(\mathfrak{n})$ is J-invariant by Proposition 3.1, we have that

$$\begin{aligned} 0 = dc(W, JW, Y, JY) = &-\langle [JY, Y], [W, JW]\rangle - \langle [W, Y], [W, Y]\rangle \\ &-\langle [W, JY], [W, JY]\rangle - \langle [JW, Y], [JW, Y]\rangle \\ &-\langle [JW, JY], [JW, JY]\rangle - \langle [JW, W], [Y, JY]\rangle. \end{aligned}$$

That means

$$-2\langle [Y, JY], [W, JW]\rangle = -\|[W, Y]\|^2 - \|[W, JY]\|^2 - \|[JW, Y]\|^2 - \|[JW, JY]\|^2.$$

Therefore, $[Y, JY] = 0$ if and only if $[W, Y] = 0$ for all $W \in \mathfrak{n}$, that is $Y \in \mathfrak{z}(\mathfrak{n})$. □

4 Proof of Theorem 1.2

Let $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ be a $2n$-dimensional real nilpotent Lie algebra endowed with a Hermitian structure. Using [13, Corollary 1.4], there exists an orthonormal basis $\{e^1, \ldots, e^{2n}\}$ of $\mathfrak{g}^*$ satisfying that $Je^1 = e^2$ and

$$de^i = \sum_{j,k<i} c^i_{jk} e^{jk}, \tag{4}$$

where $-c^i_{jk}$ denote the structural constants of the Lie bracket on $\mathfrak{g}$. In particular, $de^1 = 0$, $de^2 = 0$ and $\mathfrak{n} := \operatorname{span}\{e_1, e_2\}^\perp$ is a J-invariant ideal of $\mathfrak{g}$. Then, the Lie bracket of $\mathfrak{g}$ is determined by

$$A := \operatorname{ad}(e_1)_{\mathfrak{n}}, \quad B := \operatorname{ad}(e_2)_{\mathfrak{n}}, \quad X := [e_1, e_2], \quad \text{and} \quad [\cdot,\cdot]_{\mathfrak{n}}. \tag{5}$$

In particular, $(\mathfrak{n}, J_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ is a $(2n-2)$-dimensional real nilpotent Lie algebra endowed with a Hermitian structure. Moreover, if $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT, then $(\mathfrak{n}, J_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ turns out to be SKT by Proposition 3.3.

Remark 4.1 The integrability condition implies that

$$[Je_1, JY] = [e_1, Y] + J[Je_1, Y] + J[e_1, JY], \ \forall Y \in \mathfrak{n}, \text{ i.e. } [J, A] = J[B, J].$$

From now on, we will denote by $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ the Hermitian manifold such that $\mathfrak{g}_{A,B,X,\mathfrak{n}}$ is the nilpotent Lie algebra defined as in (5), $Je_1 = e_2$, $J\mathfrak{n} \subseteq \mathfrak{n}$, and $\langle\cdot,\cdot\rangle$ satisfies that $\langle e_1, e_2\rangle = 0$ and $\{e_1, e_2\} \perp \mathfrak{n}$.

4.1 2-step nilpotent ideal of codimension 2

The aim of this section is to prove Theorem 1.2 for $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ in the case that $\mathfrak{n}$ is 2-step nilpotent.

Assume that $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ is an SKT Lie algebra and $\mathfrak{n}$ is 2-step nilpotent. By Proposition 3.3, $(\mathfrak{n}, J_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ is SKT. Hence, we can decompose $\mathfrak{n}$ as

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z},$$

where $\mathfrak{z} := \mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$ and $\mathfrak{v} := \mathfrak{z}^{\perp}$. Note that $\mathfrak{z}$ and $\mathfrak{v}$ are invariant by J and $[\mathfrak{v}, \mathfrak{v}]_{\mathfrak{n}} \subseteq \mathfrak{z}$. According to the above decomposition, $J_{\mathfrak{n}}$ is determined by $J_{\mathfrak{v}}$ and $J_{\mathfrak{z}}$ (see Notation 3.2). In addition, since $A, B \in \mathrm{Der}(\mathfrak{n})$, then

$$A = \left[A_{\mathfrak{v}} \ 0 * \ A_{\mathfrak{z}}\right], \qquad B = \left[B_{\mathfrak{v}} \ 0 * \ B_{\mathfrak{z}}\right].$$

Lemma 4.2 *If $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ is SKT, then $A_{\mathfrak{z}} = 0$ and $B_{\mathfrak{z}} = 0$.*

Proof According to (2), for $Z \in \mathfrak{z}$ we have that

$$\begin{aligned} dc(e_1, e_2, Z, JZ) =& \\ &= -c(X, Z, JZ) + c(AZ, e_2, JZ) - c(AJZ, e_2, Z) - c(BZ, e_1, JZ) + c(BJZ, e_1, Z) \\ &= \langle (JAJA + AJAJ + JBJB + BJBJ)Z, Z\rangle - |AZ|^2 - |AJZ|^2 - |BZ|^2 - |BJZ|^2 \\ &= \langle (J[A, B] + [A, B]J + 2(BJA - AJB - A^2 - B^2))Z, Z\rangle - |AZ|^2 \\ &\quad - |AJZ|^2 - |BZ|^2 - |BJZ|^2. \end{aligned}$$

The last equation follows from Remark 4.1. On the other hand, since $[A, B] = \mathrm{ad}(X)$, for all $Z \in \mathfrak{z}$ we have that $[A, B]Z$ and $[A, B]JZ$ vanish. Hence, the SKT condition yields

$$\begin{aligned} 0 =& \langle (BJA - AJB)Z, Z\rangle - \langle A^2 Z, Z\rangle - \langle B^2 Z, Z\rangle \\ &- \frac{1}{2}(|AZ|^2 + |AJZ|^2 + |BZ|^2 + |BJZ|^2), \end{aligned} \tag{6}$$

for every $Z \in \mathfrak{z}$. If we sum over any orthonormal basis of $\mathfrak{z}$, then (6) gives us

$$0 = \mathrm{tr}(B_{\mathfrak{z}} J_{\mathfrak{z}} A_{\mathfrak{z}} - A_{\mathfrak{z}} J_{\mathfrak{z}} B_{\mathfrak{z}}) - \mathrm{tr}\, A_{\mathfrak{z}}^2 - \mathrm{tr}\, B_{\mathfrak{z}}^2 - |A_{\mathfrak{z}}|^2 - |B_{\mathfrak{z}}|^2. \tag{7}$$

By the Jacobi condition, $A_{\mathfrak{z}}$ and $B_{\mathfrak{z}}$ commute and since they are nilpotent, (7) implies that

$$0 = |A_{\mathfrak{z}}|^2 + |B_{\mathfrak{z}}|^2,$$

then $A_{\mathfrak{z}} = B_{\mathfrak{z}} = 0$. □

Corollary 4.3 *If* $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ *is SKT and* $\mathfrak{n}$ *is abelian, then* $\mathfrak{g}_{A,B,X,\mathfrak{n}}$ *is at most* 2*-step nilpotent.*

Proof The proof follows immediately from Lemma 4.2 since $\mathfrak{n} = \mathfrak{z}$ when it is abelian. □

Remark 4.4 It follows from $A, B \in \mathrm{Der}(\mathfrak{n})$, $A_{\mathfrak{z}} = B_{\mathfrak{z}} = 0$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}$, that

$$[AY, Z] = -[Y, AZ], \qquad [BY, Z] = -[Y, BZ], \qquad \forall Y, Z \in \mathfrak{n}. \tag{8}$$

Lemma 4.5 *For any* $Y \in \mathfrak{v}$, $[AY, BY] = 0$.

Proof Given $Y \in \mathfrak{v}$, then $[AY, BY] = -[BAY, Y]$ from (8). Since $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and $A_{\mathfrak{z}} = B_{\mathfrak{z}} = 0$, it follows that

$$[BAY, Y] = [B_{\mathfrak{v}} A_{\mathfrak{v}} Y, Y].$$

From the Jacobi condition, we know that $[A, B]_{\mathfrak{v}} = 0$ and therefore $[A_{\mathfrak{v}}, B_{\mathfrak{v}}] = 0$. Hence,

$$\begin{aligned}[AY, BY] &= -[BAY, Y] = -[B_{\mathfrak{v}} A_{\mathfrak{v}} Y, Y] = -[A_{\mathfrak{v}} B_{\mathfrak{v}} Y, Y] \\ &= -[ABY, Y] = [BY, AY] = -[AY, BY],\end{aligned}$$

and the assertion follows. □

Lemma 4.6 *If* $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ *is SKT, then* $A_{\mathfrak{v}} = B_{\mathfrak{v}} = 0$ *and* $X \in \mathfrak{z}$.

Proof Since $A_{\mathfrak{v}}$ and $B_{\mathfrak{v}}$ are nilpotent and commute, we can take a non-zero $Y \in \mathfrak{v}$ such that $A_{\mathfrak{v}} Y = B_{\mathfrak{v}} Y = 0$, or equivalently, $AY \in \mathfrak{z}$ and $BY \in \mathfrak{z}$. We now proceed by showing that JY satisfies the same conditions. Recall that from Lemma 3.4, it is sufficient to prove that $0 = [AJY, JAJY] = [BJY, JBJY]$. By Remark 4.1,

$$\begin{aligned}[AJY, JAJY] &= [AJY, (BJ - JB - A)Y] \\ &= [AJY, BJY] - [AJY, JBY] - [AJY, AY],\end{aligned} \tag{9}$$

which vanishes by Lemma 4.5 and Proposition 3.1 applied to $(\mathfrak{n}, J_{\mathfrak{n}})$. Hence, $A_{\mathfrak{v}} JY = 0$, and it analogously follows that $B_{\mathfrak{v}} JY = 0$.

Furthermore, setting $\mathfrak{b} := \mathrm{Ker}(A_{\mathfrak{v}}) \cap \mathrm{Ker}(B_{\mathfrak{v}}) \neq 0$, we showed that $\mathfrak{b}$ is J-invariant. If we prove that $\mathfrak{b} = \mathfrak{v}$, the assertion follows.

On the contrary, suppose that $\mathfrak{a} := (\mathrm{Ker}(A_{\mathfrak{v}}) \cap \mathrm{Ker}(B_{\mathfrak{v}}))^{\perp} \neq \{0\}$ and $A_{\mathfrak{a}}$ and $B_{\mathfrak{a}}$ are defined according to Notation 3.2. It can be easily seen that $A_{\mathfrak{a}}$ and $B_{\mathfrak{a}}$ are nilpotent and commute, therefore, there exists $0 \neq W \in \mathfrak{a}$ such that $A_{\mathfrak{a}} W = B_{\mathfrak{a}} W = 0$. In other words, $AW, BW \in \mathfrak{b} \oplus \mathfrak{z}$.

In the same way as we proceed after equation (9), we can show that

$$[AJW, JAJW] = [JW, AJBW] + [JW, A^2 W].$$

From the fact that $A(\mathfrak{b} \oplus \mathfrak{z}) \subseteq \mathfrak{z}$ and $\mathfrak{b} \oplus \mathfrak{z}$ is J-invariant, it follows that $AJW \in \mathfrak{z}$. In the same manner we can prove that $BJW \in \mathfrak{z}$. Therefore, $JW \in \mathfrak{b}$ which leads to a contradiction since $\mathfrak{b}$ is J-invariant and $W \in \mathfrak{a}$. We conclude that $A_{\mathfrak{v}} = B_{\mathfrak{v}} = 0$, so

$$A = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}.$$

The fact that X lies in $\mathfrak{z}$ follows immediately from the Jacobi condition, i.e. $[A, B] = \mathrm{ad}(X)$. □

An immediate consequence of the Lie algebra structure of $\mathfrak{g}_{A,B,X,\mathfrak{n}}$ given in (5) and the above lemma is the following result.

Corollary 4.7 *If* $(\mathfrak{g}_{A,B,X,\mathfrak{n}}, J, \langle\cdot,\cdot\rangle)$ *is SKT, then* $\mathfrak{g}_{A,B,X,\mathfrak{n}}$ *is at most* 2*-step nilpotent.*

4.2 General case

In the above section we proved Theorem 1.2 for two particular cases. We are now in position to prove Theorem 1.2 in the general case, which is the main result of this article.

Theorem 4.8 *If* $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ *is SKT with* $\mathfrak{g}$ *nilpotent, then* $\mathfrak{g}$ *is at most* 2*-step nilpotent.*

Proof The proof is by induction on n, where $\dim \mathfrak{g} = 2n$. It is clear that the assertion is true for $n = 1$. Suppose that it holds for every SKT nilpotent Lie algebra of dimension $2k$, with $k < n$.

By the discussion at the beginning of Sect. 4 there exists $A, B \in \mathfrak{gl}(2(n-1), \mathbb{R})$, $X \in \mathbb{R}^{2(n-1)}$ and $\mathfrak{n}$ ideal of $\mathfrak{g}$ of dimension $2(n-1)$ such that

$$\mathfrak{g} = \mathfrak{g}_{A,B,X,\mathfrak{n}}.$$

By Proposition 3.3, $(\mathfrak{n}, J_{\mathfrak{n}}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}})$ is SKT and of course nilpotent. Then, by hypothesis, $\mathfrak{n}$ is at most 2-step nilpotent. We are now under the hypothesis of Corollary 4.3 or Corollary 4.7, and this implies that $\mathfrak{g}$ is at most 2-step nilpotent. □

5 Construction of examples

In this section we present a method to construct examples of SKT Lie algebras of arbitrary dimensions. The idea is to start with two SKT Lie algebras of dimension n_1 and n_2 that satisfy certain condition, and to construct a new SKT Lie algebra of dimension $n_1 + n_2 + 2$. With this method and some already known examples, we can provide an example of an SKT Lie algebra of any even-dimension.

5.1 A new construction

For $i = 1, 2$, let $(\mathfrak{n}_i, J_i, \langle\cdot,\cdot\rangle_i)$ be an *indecomposable* 2-step nilpotent SKT Lie algebra. It is to say, $\mathfrak{n}_i$ can not be decomposed as an orthogonal sum of J-invariant ideals, or equivalently, it is not a sum of SKT Lie algebras of lower dimensions (see Proposition 3.3). Suppose in addition that for each $i = 1, 2$,

$$\mathfrak{n}_i = \mathfrak{v}_i \oplus \mathfrak{z}_i \quad \text{and} \quad \dim \mathfrak{z}_i > \dim\,[\mathfrak{n}_i, \mathfrak{n}_i].$$

Set $n_i := \dim \mathfrak{n}_i$, for $i = 1, 2$, and let $\{X_1, \dots, X_{n_1}\}$ and $\{Y_1, \dots, Y_{n_2}\}$ be orthonormal bases of $(\mathfrak{n}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_2, \langle\cdot,\cdot\rangle_2)$, respectively. There is no loss of generality in assuming that

$$X_{n_1} \in \mathfrak{z}_1 \cap [\mathfrak{n}_1, \mathfrak{n}_1]^{\perp}, \quad Y_{n_2} \in \mathfrak{z}_2 \cap [\mathfrak{n}_2, \mathfrak{n}_2]^{\perp}.$$

Let $\mathfrak{g}$ be the Lie algebra with underlying vector space $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathbb{R}^2$. Take $Z, W \in \mathfrak{g}$ such that $\{X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}, Z, W\}$ is a basis of $\mathfrak{g}$, and consider $\langle\cdot,\cdot\rangle$ which makes it an orthonormal basis. It is obvious that $\langle\cdot,\cdot\rangle|_{\mathfrak{n}_1\times\mathfrak{n}_1} = \langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle|_{\mathfrak{n}_2\times\mathfrak{n}_2} = \langle\cdot,\cdot\rangle_2$.

Let the Lie bracket on $\mathfrak{g}$ be determined by

$$[\cdot,\cdot]|_{\mathfrak{n}_1\times\mathfrak{n}_1} = [\cdot,\cdot]_{\mathfrak{n}_1}, \quad [\cdot,\cdot]|_{\mathfrak{n}_2\times\mathfrak{n}_2} = [\cdot,\cdot]_{\mathfrak{n}_2}, \quad [Z, W] = X_{n_1} + Y_{n_2}, \tag{10}$$

and the complex structure defined as

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}$$

In particular $JZ = W$ and $[Z, JZ] \in ([\mathfrak{n}_1, \mathfrak{n}_1] \oplus [\mathfrak{n}_2, \mathfrak{n}_2])^{\perp}$.

In order to prove that $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT, we only have to check that

$$\begin{aligned} 0 = dc(Z, JZ, W_1, W_2) &= dc(Z, W_1, W_2, W_3) \\ &= dc(JZ, W_1, W_2, W_3), \quad W_1, W_2, W_3 \in \mathfrak{n}_1 \cup \mathfrak{n}_2. \end{aligned}$$

Indeed, by (3)

$$\begin{aligned} dc(Z, JZ, W_1, W_2) = &\langle [J[Z, JZ], JW_1], W_2\rangle + \langle [JW_1, JW_2], [Z, JZ]\rangle \\ &+ \langle [JW_2, J[Z, JZ]], W_1\rangle + \langle [J[W_1, W_2], JZ], JZ\rangle \\ &+ \langle [JZ, JJZ], [W_1, W_2]\rangle + \langle [JJZ, J[W_1, W_2]], Z\rangle, \end{aligned}$$

which vanishes because J preserve $\mathfrak{z}_1 \oplus \mathfrak{z}_2$, $\mathfrak{g}$ is 2-step nilpotent and $[Z, JZ]$ is orthogonal to $[\mathfrak{n}_1, \mathfrak{n}_1] \oplus [\mathfrak{n}_2, \mathfrak{n}_2]$. On the other hand, it is immediate from (3) and the Lie algebra structure of $\mathfrak{g}$, that $dc(Z, W_1, W_2, W_3)$ and $dc(JZ, W_1, W_2, W_3)$ vanish.

It only remains to see that $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is indecomposable. Suppose that there exists an orthogonal J-invariant decomposition of ideals

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b},$$

where $\mathfrak{a}$ is indecomposable. If $\mathfrak{n}_1 \cap \mathfrak{a} \neq \{0\}$, then it is a J-invariant ideal contained in $\mathfrak{a}$ indecomposable. Therefore, $\mathfrak{a} = \mathfrak{n}_1$ and $\mathfrak{b} = \mathfrak{n}_2 \oplus \text{span}\{Z, JZ\}$. This contradicts the fact that $\mathfrak{b}$ is an ideal, since $[Z, JZ] = X_{n_1} + Y_{n_2} \in \mathfrak{n}_1 \oplus \mathfrak{n}_2$. If $\mathfrak{n}_2 \cap \mathfrak{a} \neq \{0\}$, we can proceed analogously and to get a contradiction. Finally, if $\mathfrak{n}_i \cap \mathfrak{a} = \{0\}$ for $i = 1, 2$, it follows that $(\mathfrak{n}_1 \oplus \mathfrak{n}_2) \cap \mathfrak{a} = \{0\}$ by using that it is an ideal of $\mathfrak{a}$ and $\mathfrak{a}$ is indecomposable. Then $\mathfrak{a}$ has to be zero. Indeed, if $A \in \mathfrak{a}$, $A = N_1 + N_2 + \alpha Z + \beta JZ$, with $\alpha \neq 0$ or $\beta \neq 0$. Then, $[A, Z] \in \mathfrak{a}$ and $[A, JZ] \in \mathfrak{a}$, which means that $X_{n_1} + Y_{n_2} \in \mathfrak{a}$, and we obtain a contradiction.

Remark 5.1 Observe that a quick computation shows that the SKT Lie algebra $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ obtained by the above construction satisfies $\dim \mathfrak{z}(\mathfrak{g}) > \dim[\mathfrak{g}, \mathfrak{g}]$. To the obtained example, we can apply the construction again in order to get higher dimensional examples.

Remark 5.2 Setting $[Z, W] = rX_{n_1} + sY_{n_2}$, for $s, t \in \mathbb{R}\backslash\{0\}$, instead of $[Z, W] = X_{n_1} + Y_{n_2}$ in (10), we obtain a family of examples of SKT Lie algebras. An interesting question is whether they are pairwise non-equivalent.

Remark 5.3 In the previous construction, if both J_1 and J_2 are abelian, then J results abelian, and if one of them is not, then J is not abelian. Recall that a complex structure J on $\mathfrak{g}$ is called *abelian* if $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

5.2 Known examples

In this section, we present some known examples of SKT Lie algebras to set up some notation.

Example 5.4 [12] Consider the 4-dimensional Lie algebra $\mathfrak{n}_1$ with basis $\{e_1, \ldots, e_4\}$ and Lie bracket determined by

$$de^3 = -e^{12}.$$

Let $\langle\cdot,\cdot\rangle_1$ be the inner product such that the basis is orthonormal, and the abelian complex structure J_1 is defined by,

$$J_1 e_1 = e_2, \quad J_1 e_3 = e_4.$$

The Hermitian manifold $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ has the following torsion 3-form of the Bismut connection

$$c = -e^{123},$$

which turns out to be closed and therefore $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ is an SKT Lie algebra. Note that if $\mathfrak{z}_1$ is the center of $\mathfrak{n}_1$, then $\mathfrak{z}_1 \cap [\mathfrak{n}_1, \mathfrak{n}_1]^\perp = \mathrm{span}\{e_4\}$.

Example 5.5 [8, 15] Let $\mathfrak{n}_2$ be the 6-dimensional Lie algebra with basis $\{f_1, \ldots, f_6\}$ and Lie bracket determined by

$$df^5 = -f^{12} + f^{14} - f^{23} - f^{34}.$$

Let $\langle\cdot,\cdot\rangle_2$ be the inner product such that the basis is orthonormal, and the abelian complex structure J_2 is defined by,

$$J_2 f_1 = f_2, \quad J_2 f_3 = f_4, \quad J_2 f_5 = f_6.$$

Then, the torsion 3-form of the Bismut connection of $(\mathfrak{n}_2, J_2, \langle\cdot,\cdot\rangle_2)$ is

$$c = -f^{125} + f^{145} - f^{235} - f^{345},$$

and it is closed, so $(\mathfrak{n}_2, J_2, \langle\cdot,\cdot\rangle_2)$ is SKT. Observe that if $\mathfrak{z}_2$ is the center of $\mathfrak{n}_2$, then $\mathfrak{z}_2 \cap [\mathfrak{n}_2, \mathfrak{n}_2]^\perp = \mathrm{span}\{f_6\}$.

Example 5.6 [5] Consider the 8-dimensional Lie algebra $\mathfrak{n}_3$ with basis $\{v_1, \ldots, v_8\}$ and Lie bracket determined by

$$dv^5 = -2v^{12} + v^{14} - v^{34}, \quad dv^6 = -v^{13}, \quad dv^7 = -v^{12} + v^{34}.$$

Let $\langle\cdot,\cdot\rangle_3$ be the inner product such that the basis is orthonormal, and the non-abelian complex structure J_3 defined by,

$$J_3 v_1 = v_2, \quad J_3 v_3 = v_4, \quad J_3 v_5 = v_6, \quad J_3 v_7 = v_8.$$

The Hermitian manifold $(\mathfrak{n}_3, J_3, \langle\cdot,\cdot\rangle_3)$ has the following torsion 3-form of the Bismut connection

$$c = -2v^{125} - v^{127} - v^{235} - v^{246} - v^{345} + v^{347},$$

which is closed, and therefore $(\mathfrak{n}_3, J_3, \langle\cdot,\cdot\rangle_3)$ is SKT. Note that if $\mathfrak{z}_3$ is the center of $\mathfrak{n}_3$, then $\mathfrak{z}_3 \cap [\mathfrak{n}_3, \mathfrak{n}_3]^\perp = \mathrm{span}\{v_8\}$.

5.3 Applications

The aim of this section is to apply the construction given in Sect. 5.1. We provide two new examples of SKT Lie algebras by using Examples 5.4, 5.5 and 5.6.

Example 5.7 Let $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_2, J_2, \langle\cdot,\cdot\rangle_2)$ be the indecomposable SKT Lie algebras defined in Examples 5.4 and 5.5, respectively. According to the method presented in Section 5.1, we can construct a $(4+6+2)$-dimensional SKT Lie algebra $\mathfrak{g}$ with orthonormal basis $\{e_1, \dots, e_4, f_1, \dots, f_6, w_1, w_2\}$, Lie bracket determined by,

$$de^3 = -e^{12}, \quad df^5 = -f^{12} + f^{14} - f^{23} - f^{34}, \quad de^4 = -w^{12}, \quad df^6 = -w^{12}$$

and complex structure

$$J|_{\mathfrak{n}_1} = J_1, \quad J|_{\mathfrak{n}_2} = J_2, \quad Jw_1 = w_2.$$

Indeed, the resulting torsion 3-form of the Bismut connection is

$$c = -e^{123} - 2v^{125} - v^{127} - v^{235} - v^{246} - v^{345} + v^{347} - e^4 \wedge w^{12} - f^6 \wedge w^{12},$$

which is closed and therefore $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT. An easy computation shows that J is abelian, which is consistent with Remark 5.3.

Example 5.8 Let $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_3, J_3, \langle\cdot,\cdot\rangle_3)$ be the indecomposable SKT Lie algebras defined in Examples 5.4 and 5.6, respectively. As we did in the previous example, we construct a $(4+8+2)$-dimensional SKT Lie algebra $\mathfrak{g}$ with orthonormal basis $\{e_1, \dots, e_4, v_1, \dots, v_8, w_1, w_2\}$, Lie bracket determined by,

$$de^3 = -e^{12}, \quad dv^5 = -2v^{12} + v^{14} - v^{34}, \quad dv^6 = -v^{13},$$
$$dv^7 = -v^{12} + v^{34}, \quad de^4 = -w^{12}, \quad dv^8 = -w^{12},$$

and complex structure:

$$J|_{\mathfrak{n}_1} = J_1, \quad J|_{\mathfrak{n}_3} = J_3, \quad Jw_1 = w_2.$$

Indeed, the resulting torsion 3-form of the Bismut connection is

$$c = -e^{123} - 2v^{125} - v^{127} - v^{235} - v^{246} - v^{345} + v^{347} - e^4 \wedge w^{12} - v^8 \wedge w^{12},$$

which is closed and therefore $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is SKT. Note that J is not abelian since J_3 is not abelian.

Remark 5.9 It is worth pointing out that at least one example of an indecomposable SKT Lie algebra on any dimension can be reached by applying the construction repeatedly to Examples 5.4, 5.5 and 5.6 (see Remark 5.1). For instance, in order to obtain an example of dimension $4+6m$, with $m \in \mathbb{N}$, the only needed SKT Lie algebra is $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ of Example 5.4. In fact, applying the construction to $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$, an SKT Lie algebra of dimension 10 is obtained. Using the new SKT Lie algebra and again $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$, an SKT Lie algebra of dimension 16 is constructed, and go on. Analogously, examples of dimensions $6+6m$ and $8+6m$, with $m \in \mathbb{N}$, can be obtained from $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_2, J_2, \langle\cdot,\cdot\rangle_2)$ given in Examples 5.4 and 5.5, and $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_3, J_3, \langle\cdot,\cdot\rangle_3)$ given in Examples 5.4 and 5.6, respectively.

5.4 More examples of SKT Lie algebras with non-abelian complex structures

Example 5.10 [8, 15] Let $\mathfrak{n}$ be the 6-dimensional Lie algebra with basis $\{e_1, \dots, e_6\}$ and Lie bracket determined by

$$de^5 = -e^{12} - e^{14} - e^{34}, \quad de^6 = e^{13}.$$

Let $\langle \cdot, \cdot \rangle$ be the inner product such that the basis is orthonormal, and the non-abelian complex structure J defined by,

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6.$$

The torsion 3-form of the Bismut connection of $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ is

$$c = -e^{125} + e^{235} + e^{246} - e^{345},$$

and it is closed. Therefore $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ is SKT. Note that if $\mathfrak{z}$ is the center of $\mathfrak{n}$, then $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}] = \operatorname{span}\{e_5, e_6\}$.

Example 5.11 Consider the 10-dimensional Lie algebra $\mathfrak{n}$ with basis $\{e_1, \dots, e_{10}\}$ and Lie bracket determined by

$$\begin{aligned} de^7 &= -e^{12} + e^{24} - e^{34} - 2e^{36}, \quad de^8 = -e^{14} - \tfrac{5}{2}e^{34} + 2e^{35} - 2e^{56}, \\ de^9 &= -e^{12} + e^{16} - e^{25} + e^{36} - e^{45} - e^{56}. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ be the inner product such that the basis is orthonormal, and the non-abelian complex structure J defined by,

$$Je_{2i-1} = e_{2i}, \quad \forall i \in \{1, \dots, 5\}.$$

The Hermitian manifold $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ has the following torsion 3-form of the Bismut connection

$$\begin{aligned} c = &-e^{127} - e^{129} + e^{137} + e^{169} + e^{238} - e^{259} - e^{347} - \tfrac{5}{2}e^{348} \\ &+e^{369} + 2e^{457} - e^{459} + 2e^{468} - 2e^{568} - e^{569}, \end{aligned}$$

which turns out to be closed and therefore $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ is SKT. Observe that if $\mathfrak{z}$ is the center of $\mathfrak{n}$, then $\mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]^{\perp} = \operatorname{span}\{e_{10}\}$.

Example 5.12 Consider the 12-dimensional Lie algebra $\mathfrak{n}$ with basis $\{e_1, \dots, e_{12}\}$ and Lie bracket determined by

$$\begin{gathered} de^7 = -e^{12} + e^{24}, \quad de^8 = -e^{14} + 2e^{16} - 2e^{25}, \quad de^9 = -e^{12} - e^{34} - e^{56}, \\ de^{10} = -e^{34}, \quad de^{11} = -e^{12} + e^{36} - e^{45} - 3e^{56}. \end{gathered}$$

Let $\langle \cdot, \cdot \rangle$ be the inner product such that the basis is orthonormal, and the non-abelian complex structure J defined by,

$$Je_{2i-1} = e_{2i}, \quad \forall i \in \{1, \dots, 6\}.$$

The the Bismut connection of $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ has the following torsion 3-form

$$\begin{aligned} c = &-e^{127} - e^{129} - e^{12\,11} + e^{137} + 2e^{168} + e^{238} - 2e^{258} \\ &-e^{349} - e^{34\,10} + e^{36\,11} - e^{45\,11} - e^{569} - 3e^{56\,11}, \end{aligned}$$

which is closed, and so $(\mathfrak{n}, J, \langle \cdot, \cdot \rangle)$ is SKT. Note that if $\mathfrak{z}$ is the center of $\mathfrak{n}$, then $\mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]^{\perp} = \operatorname{span}\{e_{12}\}$.

Proposition 5.13 *For each natural $n \geq 3$, there exists at least one $2n$-dimensional SKT Lie algebra with non-abelian complex structure.*

Proof For $n = 3, 4, 5, 6$, see Examples 5.10, 5.6, 5.11 and 5.12. In order to obtain examples of higher dimensions, the construction described above can be repeatedly applied, starting with one SKT Lie algebra with J non-abelian. For instance, in order to obtain an example of dimension 14, the construction can be applied to the SKT Lie algebras $(\mathfrak{n}_1, J_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{n}_3, J_3, \langle\cdot,\cdot\rangle_3)$ from Examples 5.4 and 5.6, respectively. Then, using the new SKT Lie algebra and Example 5.4, a new SKT Lie algebra of dimension 20 is obtained, and with an inductive argument, SKT Lie algebras of dimension $8 + 6m$, with $m \in \mathbb{N}$ are reached. Analogously, examples of dimensions $10 + 6m$ and $12 + 6m$, with $m \in \mathbb{N}$, are obtained from Examples 5.4 and 5.11, and Examples 5.4 and 5.12, respectively. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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