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Global bases for quantum Borcherds–Bozec algebras

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Abstract
We provide an explicit construction of global bases for quantum Borcherds–Bozec algebras and their irreducible highest weight modules. Moreover, we give a new presentation for quantum Borcherds–Bozec algebras in terms of primitive generators.

Keywords Quantum Borcherds–Bozec algebra · Crystal basis · Global basis

Mathematics Subject Classification 17B37 · 17B67 · 16G20

1 Introduction

The *quantum Borcherds–Bozec algebras* were introduced by Bozec [1, 2] in terms of generators and relations when he solved a question asked by Lusztig in [15]. Namely, if we consider a quiver with loops, the Grothendieck group arising from Lusztig sheaves on representation varieties is generated by the elementary simple perverse sheaves $F_i^{(n)}$ with all vertices i and $n \in \mathbf{N}$. Bozec proved an analogue of Gabber–Kac theorem for the negative part $U_q^-(\mathfrak{g})$ of a quantum Borcherds–Bozec algebra, and showed that the above Grothendieck group is isomorphic to $U_q^-(\mathfrak{g})$, which gives a construction of its *canonical basis*.

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The canonical basis theory was first introduced by Lusztig in the simply-laced case in [13], arising from his geometric construction of the negative parts of quantum groups, and it has been generalized to symmetric Kac–Moody type in [14, 16]. On the other hand, Kashiwara constructed the crystal bases and global bases for quantum groups associated with symmetrizable Kac–Moody algebras in an algebraic way [11, 12]. In [5], Grojnowski and Lusztig proved that Kashiwara's global bases coincide with Lusztig's canonical bases. The canonical/crystal basis theory has become one of the most central themes in combinatorial and geometric representation theory of quantum groups because it provides us with very powerful geometric and combinatorial tools to investigate the structure of quantum groups and their integrable representations. In [7], Jeong, Kang, and Kashiwara developed the crystal basis theory for quantum Borcherds algebras, which were introduced in [8]. In [10], Kang and Schiffmann gave a construction of canonical basis for quantum Borcherds algebras and proved that, when all the diagonal entries of the corresponding Borcherds–Cartan matrices are non-zero, the canonical bases coincide with global bases.

Bozec's crystal basis theory for quantum Borcherds–Bozec algbras is based on *primitive generators* $\mathtt{a}_{il}$, $\mathtt{b}_{il}$ $(i,l) \in I^{\infty}$, not on the *Chevalley generators* e_{il}, f_{il} $(i,l) \in I^{\infty}$. The primitive generators have simpler commutation relations than Chevalley generators. Bozec defined the Kashiwara operators using primitive generators and proved several crucial theorems which are important steps for Kashiwara's grand-loop argument [2, Lemma 3.33, Lemma 3.34]. In this way, even though he did not check all the details, he was able to deduce that there exist unique crystal bases for quantum Borcherds–Bozec algbras and their integrable highest weight modules. Moreover, using Lusztig's and Nakajima's quiver varieties, he also gave a geometric construction of $\mathcal{B}(\infty)$, the crystal of the negative half $U_q^-(\mathfrak{g})$, and $\mathcal{B}(\lambda)$, the crystal of the integrable highest weight representation $V(\lambda)$, respectively.

The main goal of this paper is to construct the *global bases* for quantum Borcherds–Bozec algebras and their irreducible highest weight modules. As the first step, we give an explicit description of the radical $\mathscr{R}$ of Lusztig's bilinear form (Theorem 4). The higher order Serre relations we obtained have more general forms than those given in [1]. As a direct application, we give a new presentation of the quantum Borcherds–Bozec algebra $U_q(\mathfrak{g})$ in terms of primitive generators (Proposition 7).

Then we set up the frame work that can be found in [7, 12]. However, we still need more preparations. In the case of quantum Borcherds–Bozec algebras, for each I^{im}, there are infinitely many generators with higher degrees. Thus, compared with quantum Borcherds algebras, we need to take a much more complicated approach to the construction of global bases. To overcome these difficulties, we introduce a very natural and much expanded notion of balanced triples corresponding to the compositions or partitions of each higher degree of primitive generators (Proposition 22, Corollary 23). As can be expected, to prove our assertions, the imaginary indices with higher degrees should be treated with special care. In particular, the isotropic case (i.e., when $a_{ii} = 0$) requires very subtle and delicate treatments.

Now we can follow the steps given in [7, 12] and prove the existence and uniqueness of global bases (Theorem 26). The key ingredients of our proof are Proposition 22 and Corollary 23. We conjecture that our global bases coincide with (a variation of) Bozec's canonical bases.

This paper is organized as follows. In Sect. 2, we give an explicit description of the radical $\mathscr{R}$ of Lusztig's bilinear form $(\ ,\ )_L$ via higher order quantum Serre relations in quantum Borcherds–Bozec algebras. In Sect. 3, we give a new presentation of quantum Borcherds–Bozec algebras in terms of primitive generators as an application of higher order quantum Serre relations. In Sect. 4, we review the crystal basis theory for quantum Borcherds–Bozec algebras and give canonical characterizations of the crystal bases $(\mathcal{L}(\infty), \mathcal{B}(\infty))$ and $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$, respectively. We also define the quantum Boson algebra $\mathscr{B}_q(\mathfrak{g})$ for an arbitrary

Borcherds–Cartan datum. In Sect. 5, we define the $\mathbb{A}$-forms $U_{\mathbb{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ and $V(\lambda)^{\mathbb{A}}$ of $V(\lambda)$, respectively. We prove that $U_{\mathbb{A}}(\mathfrak{g})$ has the triangular decomposition and both $U_{\mathbb{A}}^-(\mathfrak{g})$ and $V(\lambda)^{\mathbb{A}}$ are stable under the Kashiwara operators. In Sect. 6, we prove the existence and uniqueness of global bases. As expected, most of this section is devoted to the proof of Proposition 22 and Corollary 23.

2 Higher order quantum serre relations

Let I be an index set possibly countably infinite. An integer-valued matrix $A = (a_{ij})_{i,j\in I}$ is called an *even symmetrizable Borcherds–Cartan matrix* if it satisfies the following conditions:

(i) $a_{ii} = 2, 0, -2, -4, \ldots,$
(ii) $a_{ij} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$,
(iii) there is a diagonal matrix $D = \mathrm{diag}(r_i \in \mathbf{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

Let $I^{\mathrm{re}} := \{i \in I \mid a_{ii} = 2\}$, $I^{\mathrm{im}} := \{i \in I \mid a_{ii} \leq 0\}$, and $I^{\mathrm{iso}} := \{i \in I \mid a_{ii} = 0\}$. The elements of I^{re} (resp. I^{im}, I^{iso}) are called *real indices* (resp. *imaginary indices*, *isotropic indices*).

A *Borcherds–Cartan datum* consists of

(a) an even symmetrizable Borcherds–Cartan matrix $A = (a_{ij})_{i,j\in I}$,
(b) a free abelian group $P^\vee = \left(\bigoplus_{i\in I} \mathbf{Z}h_i\right) \oplus \left(\bigoplus_{i\in I} \mathbf{Z}d_i\right)$, the *dual weight lattice*,
(c) $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$, the *Cartan subalgebra*,
(d) $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subseteq \mathbf{Z}\}$, the *weight lattice*,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, the set of *simple coroots*,
(f) $\Pi = \{\alpha_i \in P \mid i \in I\}$, the set of *simple roots*, which is linearly independent over $\mathbf{Q}$ and satisfies

$$\alpha_j(h_i) = a_{ij},\ \alpha_j(d_i) = \delta_{ij} \ \text{ for all } i, j \in I.$$

(g) for each $i \in I$, there is a $\Lambda_i \in P$, called the *fundamental weight*, defined by

$$\Lambda_i(h_j) = \delta_{ij},\ \Lambda_i(d_j) = 0 \ \text{ for all } i, j \in I.$$

We denote by P^+ the set $\{\lambda \in P \mid \lambda(h_i) \geq 0 \ \text{ for all } i \in I\}$ of *dominant integral weights*. The free abelian group $Q := \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ is called the *root lattice*. Set $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. For $\beta = \sum k_i\alpha_i \in Q_+$, we define its height to be $|\beta| := \sum k_i$.

There is a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \lambda) = r_i\lambda(h_i),\ (\Lambda_i, \lambda) = r_i\lambda(d_i) \ \text{ for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I,$$

and therefore we have

$$(\alpha_i, \alpha_j) = r_i a_{ij} = r_j a_{ji} \text{ for all } i, j \in I.$$

For $i \in I^{\mathrm{re}}$, we define the *simple reflection* $\omega_i \in GL(\mathfrak{h}^*)$ by

$$\omega_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$$

for all $\lambda \in \mathfrak{h}^*$. The subgroup W of $GL(\mathfrak{h}^*)$ generated by ω_i $(i \in I^{\mathrm{re}})$ is called the *Weyl group* of the Borcherds–Cartan datum given above. Note that the symmetric bilinear form $(\ ,\ )$ is W-invariant.

Let $I^\infty := (I^{\mathrm{re}} \times \{1\}) \cup (I^{\mathrm{im}} \times \mathbf{Z}_{>0})$. If $i \in I^{\mathrm{re}}$, we often write i for $(i, 1)$. Let q be an indeterminate and set

$$q_i = q^{r_i}, \quad q_{(i)} = q^{\frac{(\alpha_i,\alpha_i)}{2}}.$$

For each $i \in I^{\mathrm{re}}$ and $n \in \mathbf{Z}_{\geq 0}$, we define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i![n-k]_i!}.$$

Let $\mathscr{F} = \mathbf{Q}(q)\langle f_{il} \mid (i,l) \in I^\infty \rangle$ be the free associative algebra over $\mathbf{Q}(q)$ generated by the symbols f_{il} for $(i,l) \in I^\infty$. By setting $\deg f_{il} = -l\alpha_i$, $\mathscr{F}$ become a Q_--graded algebra. For a homogeneous element u in $\mathscr{F}$, we denote by $|u|$ the degree of u, and for any subset $A \subseteq Q_-$, set $\mathscr{F}_A = \{x \in \mathscr{F} \mid |x| \in A\}$.

We define a *twisted* multiplication on $\mathscr{F} \otimes \mathscr{F}$ by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(|x_2|,|y_1|)} x_1 y_1 \otimes x_2 y_2,$$

for all homogeneous $x_1, x_2, y_1, y_2 \in \mathscr{F}$, and equip $\mathscr{F}$ with a co-multiplication ϱ defined by

$$\varrho(f_{il}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} \otimes f_{in} \quad \text{for } (i,l) \in I^\infty.$$

Here, we understand $f_{i0} = 1$ and $f_{il} = 0$ for $l < 0$.

Proposition 1 [1, 2] *For any family $\nu = (\nu_{il})_{(i,l)\in I^\infty}$ of non-zero elements in $\mathbf{Q}(q)$, there exists a symmetric bilinear form $(\ ,\ )_L : \mathscr{F} \times \mathscr{F} \to \mathbf{Q}(q)$ such that*

(a) $(x, y)_L = 0$ *if* $|x| \neq |y|$,
(b) $(1, 1)_L = 1$,
(c) $(f_{il}, f_{il})_L = \nu_{il}$ *for all* $(i,l) \in I^\infty$,
(d) $(x, yz)_L = (\varrho(x), y \otimes z)_L$ *for all* $x, y, z \in \mathscr{F}$.

Here, $(x_1 \otimes x_2, y_1 \otimes y_2)_L = (x_1, y_1)_L (x_2, y_2)_L$ *for any* $x_1, x_2, y_1, y_2 \in \mathscr{F}$.

We denote by $\mathscr{R}$ the radical of $(\ ,\ )_L$.

Let $\mathcal{C}_l$ be the set of compositions $\mathbf{c}$ of l, and set $f_{i,\mathbf{c}} = f_{ic_1} \dots f_{ic_m}$ for every $i \in I^{\mathrm{im}}$ and every $\mathbf{c} = (c_1, \dots, c_m) \in \mathcal{C}_l$. Assume $\mathcal{C}_l = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r\}$, we see that $f_{i,\mathbf{c}_1}, \dots, f_{i,\mathbf{c}_r}$ form a basis of $\mathscr{F}_{-l\alpha_i}$. Hence, for any homogeneous element x in $\mathscr{F}$, $\varrho(x)$ can be written into the forms

$$\varrho(x) = x_{\mathbf{c}_1} \otimes f_{i,\mathbf{c}_1} + \cdots + x_{\mathbf{c}_r} \otimes f_{i,\mathbf{c}_r} + \text{ terms of bidegree not in } Q_- \times -l\alpha_i,$$
$$\varrho(x) = f_{i,\mathbf{c}_1} \otimes x'_{\mathbf{c}_1} + \cdots + f_{i,\mathbf{c}_r} \otimes x'_{\mathbf{c}_r} + \text{ terms of bidegree not in } -l\alpha_i \times Q_-.$$

We denote by $\varrho_{i,l}(x), \varrho^{i,l}(x) \colon \mathscr{F} \to \mathscr{F}^r$ the $\mathbf{Q}(q)$-linear maps:

$$\varrho_{i,l}(x) = (x_{\mathbf{c}_1}, \dots, x_{\mathbf{c}_r}), \ \varrho^{i,l}(x) = (x'_{\mathbf{c}_1}, \dots, x'_{\mathbf{c}_r}).$$

If x, y are homogeneous elements such that $\varrho_{i,k}(y) = 0$ for any $k > 0$, then we have

$$\varrho_{i,l}(xy) = q^{l(\alpha_i,|y|)} \varrho_{i,l}(x) y \ \text{ and } \ \varrho_{i,l}(yx) = y\varrho_{i,l}(x).$$

Here, $\varrho_{i,l}(x)y = (x_{\mathbf{c}_1} y, \dots, x_{\mathbf{c}_r} y)$ and $y\varrho_{i,l}(x) = (yx_{\mathbf{c}_1}, \dots, yx_{\mathbf{c}_r})$ if $\varrho_{i,l}(x) = (x_{\mathbf{c}_1}, \dots, x_{\mathbf{c}_r})$. Similarly, if $\varrho^{i,k}(y) = 0$ for any $k > 0$, we have

$$\varrho^{i,l}(xy) = \varrho^{i,l}(x) y \ \text{ and } \ \varrho^{i,l}(yx) = q^{l(\alpha_i,|y|)} y \varrho^{i,l}(x).$$

For $i \in I^{\mathrm{re}}$, we define the $\mathbf{Q}(q)$-linear maps $\varrho_i, \varrho^i : \mathscr{F} \to \mathscr{F}$ by

$$\varrho_i(1) = 0,\ \varrho_i(f_{j,k}) = \delta_{i,j}\delta_{k,1},\ \text{ and } \varrho_i(xy) = q^{(\alpha_i,|y|)}\varrho_i(x)y + x\varrho_i(y),$$
$$\varrho^i(1) = 0,\ \varrho^i(f_{j,k}) = \delta_{i,j}\delta_{k,1},\ \text{ and } \varrho^i(xy) = \varrho^i(x)y + q^{(\alpha_i,|x|)}x\varrho^i(y),$$

for all homogeneous x, y. Note that for any homogeneous $x \in \mathscr{F}$, we have

$$\varrho(x) = \varrho_i(x) \otimes f_i + \text{ terms of other bi-homogeneities},$$
$$\varrho(x) = f_i \otimes \varrho^i(x) + \text{ terms of other bi-homogeneities}.$$

The following lemma can be derived directly from the definitions of $\varrho_{i,l}$ and $\varrho^{i,l}$.

Lemma 2

(a) If $i \in I^{re}$, then for any $x, y \in \mathscr{F}$, we have

$$(yf_i, x)_L = (f_i, f_i)_L(y, \varrho_i(x))_L,\ (f_i y, x)_L = (f_i, f_i)_L(y, \varrho^i(x))_L.$$

(b) If $i \in I^{im}$, let $x \in \mathscr{F}$ with $\varrho_{i,l}(x) = (x_{\mathbf{c}_1}, \cdots, x_{\mathbf{c}_r})$ and $\varrho^{i,l}(x) = (x'_{\mathbf{c}_1}, \cdots, x'_{\mathbf{c}_r})$. Then for any $x \in \mathscr{F}$, we have

$$(yf_{il}, x)_L = (f_{il}, f_{i,\mathbf{c}_1})_L(y, x_{\mathbf{c}_1})_L + \cdots + (f_{il}, f_{i,\mathbf{c}_r})_L(y, x_{\mathbf{c}_r})_L,$$
$$(f_{il}y, x)_L = (f_{il}, f_{i,\mathbf{c}_1})_L(y, x'_{\mathbf{c}_1})_L + \cdots + (f_{il}, f_{i,\mathbf{c}_r})_L(y, x'_{\mathbf{c}_r})_L.$$

(c) Let $x \in \mathscr{F}$ be a homogeneous element with $|x| \neq 0$, we have
(i) if $\varrho_{i,l}(x) \in \mathscr{R}$ for any $(i, l) \in I^{\infty}$, then $x \in \mathscr{R}$,
(ii) if $\varrho^{i,l}(x) \in \mathscr{R}$ for any $(i, l) \in I^{\infty}$, then $x \in \mathscr{R}$.
Here, if $i \in I^{im}$, $\varrho^{i,l}(x) \in \mathscr{R}$ means each component of $\varrho^{i,l}(x)$ belongs to $\mathscr{R}$.

For any $i \in I^{\mathrm{re}}$ and $n \in \mathbf{N}$, we set

$$f_i^{(n)} = \frac{f_i^n}{[n]_i!}.$$

By a similar argument in [16, 1.4.2], we can prove:

Lemma 3 *We have*

$$\varrho(f_i^{(n)}) = \sum_{p+p'=n} q_i^{-pp'} f_i^{(p)} \otimes f_i^{(p')} \tag{1}$$

for any $i \in I^{re}$ and $n \in \mathbf{N}$.

Theorem 4 *Assume that $i \in I^{re}$, $j \in I$ and $i \neq j$. Let $m \in \mathbf{Z}_{>0}$, $n \in \mathbf{Z}_{\geq 0}$ with $m > -a_{ij}n$. Then for any $\mathbf{c} \in \mathcal{C}_n$, the following element of $\mathscr{F}$*

$$\mathtt{F}_{i,j,m,n,\mathbf{c},\pm 1} = \sum_{r+s=m} (-1)^r q_i^{\pm r(-a_{ij}n-m+1)} f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s)} \tag{2}$$

belongs to $\mathscr{R}$. Here, we put $f_{j,\mathbf{c}} = f_j^n$ for $j \in I^{re}$.

Proof If $n = 0$, then

$$\mathtt{F}_{i,j,m,0,\mathbf{c},\pm 1} = \sum_{r+s=m} (-1)^r q_i^{\pm r(1-m)} f_i^{(r)} f_i^{(s)}.$$

Since $\sum_{r+s=m}(-1)^r q_i^{\pm r(1-m)} \begin{bmatrix} m \\ r \end{bmatrix}_i = 0$, we have $\mathtt{F}_{i,j,m,0,\mathbf{c},\pm 1} = 0$.

We shall only show it for $j \in I^{\text{im}}$, and the case $j \in I^{\text{re}}$ can be shown similarly. By Lemma 2 (c), it is enough to show each component of $\varrho^{i',k}(\mathtt{F}_{i,j,m,n,\mathbf{c},\pm 1})$ belongs to $\mathscr{R}$. If $i' \neq i, j$, there is nothing to show.

We first show it for $i' = j$. For $0 < k \leq n$ and $\mathbf{c} = (n_1, \ldots, n_t) \in \mathcal{C}_n$, we have

$$\begin{aligned} \varrho^{j,k}(f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s)}) &= \varrho^{j,k}(f_i^{(r)} f_{j,\mathbf{c}}) f_i^{(s)} = q^{-(r\alpha_i, k\alpha_j)} f_i^{(r)} \varrho^{j,k}(f_{j,\mathbf{c}}) f_i^{(s)} \\ &= q^{-(r\alpha_i, k\alpha_j)} f_i^{(r)} \left(\beta_{a_1,\ldots,a_t} f_{j,(n_1-a_1,\ldots,n_t-a_t)}\right)_{\substack{a_1 \leq n_1,\ldots,a_t \leq n_t \\ a_1+\cdots+a_t=k}} f_i^{(s)}, \end{aligned} \tag{3}$$

where

$$\beta_{a_1,\ldots,a_t} = q_{(j)}^{\sum_{h=1}^t a_h(a_h-n_h)} q_{(j)}^{2\sum_{1\leq p<q\leq t}(a_p-n_p)a_q}.$$

Note that $m > -a_{i,j}n \geq -a_{i,j}(n-k)$ and

$$q_i^{-r(-a_{ij}n-m+1)} q^{-(r\alpha_i, k\alpha_j)} = q_i^{-r[-a_{ij}(n-k)-m+1]}.$$

Therefore each component of $\varrho^{j,k}(\mathtt{F}_{i,j,m,n,\mathbf{c},-1})$ is a scalar multiple of $\mathtt{F}_{i,j,m,n-k,\mathbf{c}',-1}$ for some $\mathbf{c}' \in \mathcal{C}_{n-k}$.

We now show it for $i' = i$. Since $i \in I^{\text{re}}$, we have

$$\begin{aligned} \varrho^{i}(f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s)}) &= \varrho^{i}(f_i^{(r)} f_{j,\mathbf{c}}) f_i^{(s)} + q^{-(r\alpha_i+n\alpha_j,\alpha_i)} q_i^{1-s} f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s-1)} \\ &= q_i^{1-r} f_i^{(r-1)} f_{j,\mathbf{c}} f_i^{(s)} + q^{-(r\alpha_i+n\alpha_j,\alpha_i)} q_i^{1-s} f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s-1)}. \end{aligned} \tag{4}$$

Hence

$$\begin{aligned} \varrho^{i}(\mathtt{F}_{i,j,m,n,\mathbf{c},-1}) = &\sum_{r+s=m} (-1)^r q_i^{-r(-a_{ij}n-m+1)} q_i^{1-r} f_i^{(r-1)} f_{j,\mathbf{c}} f_i^{(s)} \\ &+ \sum_{r+s=m} (-1)^r q_i^{-r(-a_{ij}n-m+1)} q^{-(r\alpha_i+n\alpha_j,\alpha_i)} q_i^{1-s} f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s-1)}. \end{aligned} \tag{5}$$

Note that the coefficient of $f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s-1)}$ is

$$\begin{aligned} &q_i^{-(r+1)(-a_{ij}n-m+1)} q_i^{-r} - q_i^{-r(-a_{ij}n-m+1)} q^{-(r\alpha_i+n\alpha_j,\alpha_i)} q_i^{1-m+r} \\ &= q_i^{-(r+1)(-a_{ij}n-m+1)} q_i^{-r} - q_i^{-r(-a_{ij}n-m+1)} q_i^{-2r-na_{ij}} q_i^{1-m+r} \\ &= q_i^{-r(-a_{ij}n-m+2)} q_i^{a_{ij}n+m-1} (1 - q_i^{-2m-2na_{ij}+2}). \end{aligned} \tag{6}$$

Therefore

$$\begin{aligned} \varrho^{i}(\mathtt{F}_{i,j,m,n,\mathbf{c},-1}) &= (1 - q_i^{-2m-2na_{ij}+2}) q_i^{a_{ij}n+m-1} \\ &\quad \cdot \sum_{r+s=m-1} (-1)^r q_i^{-r(-a_{ij}n-m+2)} f_i^{(r)} f_{j,\mathbf{c}} f_i^{(s)} \\ &= \begin{cases} \beta \mathtt{F}_{i,j,m-1,n,\mathbf{c},-1} & \text{if } m > -a_{ij}n + 1, \\ 0 & \text{if } m = -a_{ij}n + 1. \end{cases} \end{aligned}$$

Here $\beta = (1 - q_i^{-2m-2na_{ij}+2}) q_i^{a_{ij}n+m-1}$ is a constant.

By the induction and Lemma 2(c), theorem follows. □

In particular, when $m = 1 - la_{ij}$, $n = l$ and $\mathbf{c} = (l)$, by Theorem 4, we conclude

$$\mathtt{F}_{i,j,m,n,\mathbf{c},\pm1} = \begin{cases} \sum_{r+s=1-la_{ij}} (-1)^r f_i^{(r)} f_j^l f_i^{(s)} & \text{if } j \in I^{\text{re}}, \\ \sum_{r+s=1-la_{ij}} (-1)^r f_i^{(r)} f_{jl} f_i^{(s)} & \text{if } j \in I^{\text{im}}, \end{cases}$$

belongs to $\mathscr{R}$.

Lemma 5 *Let $(i,k), (j,l) \in I^\infty$ such that $a_{ij} = 0$. Set $X = f_{ik} f_{jl} - f_{jl} f_{ik}$, Then $X \in \mathscr{R}$.*

Proof Note that if $i, j \in I^{\text{re}}$, then $X = f_i f_j - f_j f_i$. Since i and j cannot be equal, we have $X = -\mathtt{F}_{i,j,1,1,(1),\pm1} \in \mathscr{R}$.

If $i \in I^{\text{re}}$ and $j \in I^{\text{im}}$, we have $X = f_i f_{jl} - f_{jl} f_i = -\mathtt{F}_{i,j,1,l,\mathbf{c}=(l),\pm1} \in \mathscr{R}$.

We now assume that $i, j \in I^{\text{im}}$ and $i = j \in I^{\text{iso}}$. Note for any $0 < s \le k + l$, we have

$$\begin{aligned} \varrho^{i,s}(X) &= \varrho^{i,s}(f_{ik} f_{il} - f_{il} f_{ik}) \\ &= \left(f_{i,k-a_1} f_{i,l-a_2} - f_{i,l-a_2} f_{i,k-a_1}\right)_{\substack{a_1 \le k, a_2 \le l \\ a_1 + a_2 = s}}. \end{aligned}$$

By induction and Lemma 2, $X \in \mathscr{R}$.

Finally, if $i, j \in I^{\text{im}}$ and $i \ne j$, then for any $0 < s \le k$ and $0 < t \le l$, we have

$$\begin{aligned} \varrho^{i,s}(X) &= \varrho^{i,s}(f_{ik} f_{jl} - f_{jl} f_{ik}) = q_{(i)}^{-s(k-s)} (f_{i,k-s} f_{jl} - f_{jl} f_{i,k-s}), \\ \varrho^{j,t}(X) &= \varrho^{j,t}(f_{ik} f_{jl} - f_{jl} f_{ik}) = q_{(j)}^{-t(l-t)} (f_{ik} f_{j,l-t} - f_{j,l-t} f_{ik}). \end{aligned}$$

By induction and Lemma 2, $X \in \mathscr{R}$. □

3 Quantum Borcherds–Bozec algebras

From now on, we always assume that

$$\nu_{il} \in 1 + q\mathbf{Z}_{\ge 0}[[q]] \text{ for all } (i,l) \in I^\infty. \tag{7}$$

Under this assumption, the bilinear form $(\ ,\ )_L$ is non-degenerate on $\mathscr{F}(i) = \bigoplus_{l \ge 1} \mathscr{F}_{-l\alpha_i}$ for $i \in I^{\text{im}} \backslash I^{\text{iso}}$. Moreover, it has been showed in [1, Proposition 14] that the two-sided ideal $\mathscr{R}$ is generated by

$$\sum_{r+s=1-la_{ij}} (-1)^r f_i^{(r)} f_{jl} f_i^{(s)} \text{ for } i \in I^{\text{re}}, (j,l) \in I^\infty \text{ and } i \ne (j,l),$$

and $f_{ik} f_{jl} - f_{jl} f_{ik}$ for all $(i,k), (j,l) \in I^\infty$ with $a_{ij} = 0$

Given a Borcherds–Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$, we denote by $\widehat{U}$ the associative algebra over $\mathbf{Q}(q)$ with $\mathbf{1}$, generated by the elements q^h ($h \in P^\vee$) and e_{il}, f_{il} ($(i,l) \in I^\infty$) with defining relations

$$\begin{aligned} & q^0 = \mathbf{1}, \quad q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in P^\vee \\ & q^h e_{jl} q^{-h} = q^{l\alpha_j(h)} e_{jl}, \quad q^h f_{jl} q^{-h} = q^{-l\alpha_j(h)} f_{jl} \text{ for } h \in P^\vee, (j,l) \in I^\infty, \\ & \sum_{r+s=1-la_{ij}} (-1)^r e_i^{(r)} e_{jl} e_i^{(s)} = 0 \text{ for } i \in I^{\text{re}}, (j,l) \in I^\infty \text{ and } i \ne (j,l), \\ & \sum_{r+s=1-la_{ij}} (-1)^r f_i^{(r)} f_{jl} f_i^{(s)} = 0 \text{ for } i \in I^{\text{re}}, (j,l) \in I^\infty \text{ and } i \ne (j,l), \\ & e_{ik} e_{jl} - e_{jl} e_{ik} = f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \text{ for } a_{ij} = 0. \end{aligned} \tag{8}$$

We extend the grading by setting $|q^h| = 0$ and $|e_{il}| = l\alpha_i$.

The algebra $\widehat{U}$ is endowed with a co-multiplication $\Delta\colon \widehat{U} \to \widehat{U} \otimes \widehat{U}$ given by

$$\begin{aligned} \Delta(q^h) &= q^h \otimes q^h, \\ \Delta(e_{il}) &= \sum_{m+n=l} q_{(i)}^{mn} e_{im} \otimes K_i^{-m} e_{in}, \\ \Delta(f_{il}) &= \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}, \end{aligned} \tag{9}$$

where $K_i = q^{r_i h_i}$ $(i \in I)$.

Let $\widehat{U}^{\leq 0}$ be the subalgebra of $\widehat{U}$ generated by f_{il} and q^h, for all $(i, l) \in I^\infty$ and $h \in P^\vee$, and $\widehat{U}^+$ be the subalgebra generated by e_{il} for all $(i, l) \in I^\infty$. We extend $(\ ,\ )_L$ to a symmetric bilinear form $(\ ,\ )_L$ on $\widehat{U}^{\leq 0}$ and on $\widehat{U}^+$ by setting

$$\begin{aligned} &(q^h, 1)_L = 1,\ (q^h, f_{il})_L = 0, \\ &(q^h, K_j)_L = q^{-\alpha_j(h)}, \\ &(x, y)_L = (\omega(x), \omega(y))_L \ \text{ for all } x, y \in \widehat{U}^+, \end{aligned} \tag{10}$$

where $\omega\colon \widehat{U} \to \widehat{U}$ is the involution defined by

$$\omega(q^h) = q^{-h},\ \omega(e_{il}) = f_{il},\ \omega(f_{il}) = e_{il} \ \text{ for } h \in P^\vee,\ (i, l) \in I^\infty.$$

For any $x \in \widehat{U}$, we shall use the Sweedler's notation, and write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

Definition 1 Following the Drinfeld double process, we define the *quantum Borcherds–Bozec algebra* $U_q(\mathfrak{g})$ associated with a given Borcherds–Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ as the quotient of $\widehat{U}$ by the relations

$$\sum (a_{(1)}, b_{(2)})_L \omega(b_{(1)}) a_{(2)} = \sum (a_{(2)}, b_{(1)})_L a_{(1)} \omega(b_{(2)}) \ \text{ for all } a, b \in \widehat{U}^{\leq 0}. \tag{11}$$

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_{il} (resp. f_{il}) for $(i, l) \in I^\infty$, and $U_q^0(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by q^h for $h \in P^\vee$. We shall denote by U (resp. U^+ and U^-) for $U_q(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$) for simplicity. Then U has the following *triangular decomposition* $U \cong U^- \otimes U^0 \otimes U^+$.

Proposition 6 [1, 2] *For any $i \in I^{im}$ and $l \geq 1$, there exist unique elements $\mathtt{b}_{il} \in U^-_{-l\alpha_i}$ and $\mathtt{a}_{il} = \omega(\mathtt{b}_{il})$ such that*

(1) $\mathbf{Q}(q)\langle f_{il} \mid l \geq 1\rangle = \mathbf{Q}(q)\langle \mathtt{b}_{il} \mid l \geq 1\rangle$ *and* $\mathbf{Q}(q)\langle e_{il} \mid l \geq 1\rangle = \mathbf{Q}(q)\langle \mathtt{a}_{il} \mid l \geq 1\rangle$,
(2) $(\mathtt{b}_{il}, z)_L = 0$ *for all* $z \in \mathbf{Q}(q)\langle f_{i1}, \ldots, f_{i,l-1}\rangle$,
$(\mathtt{a}_{il}, z)_L = 0$ *for all* $z \in \mathbf{Q}(q)\langle e_{i1}, \ldots, e_{i,l-1}\rangle$,
(3) $\mathtt{b}_{il} - f_{il} \in \mathbf{Q}(q)\langle f_{ik} \mid k < l\rangle$ *and* $\mathtt{a}_{il} - e_{il} \in \mathbf{Q}(q)\langle e_{ik} \mid k < l\rangle$,
(4) $\overline{\mathtt{b}}_{il} = \mathtt{b}_{il},\ \overline{\mathtt{a}}_{il} = \mathtt{a}_{il}$,
(5) $\varrho(\mathtt{b}_{il}) = \mathtt{b}_{il} \otimes 1 + 1 \otimes \mathtt{b}_{il},\ \varrho(\mathtt{a}_{il}) = \mathtt{a}_{il} \otimes 1 + 1 \otimes \mathtt{a}_{il}$,
(6) $\Delta(\mathtt{b}_{il}) = \mathtt{b}_{il} \otimes 1 + K_i^l \otimes \mathtt{b}_{il},\ \Delta(\mathtt{a}_{il}) = \mathtt{a}_{il} \otimes K_i^{-l} + 1 \otimes \mathtt{a}_{il}$,
(7) $S(\mathtt{b}_{il}) = -K_i^{-l} \mathtt{b}_{il},\ S(\mathtt{a}_{il}) = -\mathtt{a}_{il} K_i^l$.

Here, S is the antipode of U, and $^-\colon U^\pm \to U^\pm$ is the $\mathbf{Q}$-algebra involution defined by $\overline{e}_{il} = e_{il},\ \overline{f}_{il} = f_{il}$ and $\overline{q} = q^{-1}$.

Set $\tau_{il} = (\mathtt{a}_{il}, \mathtt{a}_{il})_L = (\mathtt{b}_{il}, \mathtt{b}_{il})_L$, we have the following commutation relations in $U_q(\mathfrak{g})$

$$\mathtt{a}_{il}\mathtt{b}_{jk} - \mathtt{b}_{jk}\mathtt{a}_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l}). \tag{12}$$

The elements $\mathtt{a}_{il}$'s and $\mathtt{b}_{il}$'s are called the *primitive generators*. Let $\mathcal{C}_l$ (resp. $\mathcal{P}_l$) be the set of compositions (resp. partitions) of l. For $i \in I^{\mathrm{im}}$, we define

$$\mathscr{C}_{i,l} = \begin{cases} \mathcal{C}_l & \text{if } i \in I^{\mathrm{im}} \backslash I^{\mathrm{iso}}, \\ \mathcal{P}_l & \text{if } i \in I^{\mathrm{iso}}. \end{cases}$$

and $\mathscr{C}_i = \bigsqcup_{l \geq 0} \mathscr{C}_{i,l}$. For $i \in I^{\mathrm{re}}$, we just put $\mathscr{C}_{i,l} = \{l\}$.

Assume that $i \in I^{\mathrm{im}}$. Let $\mathbf{c} = (c_1, \dots, c_t) \in \mathscr{C}_{i,l}$ and set

$$\mathtt{b}_{i,\mathbf{c}} = \mathtt{b}_{ic_1} \cdots \mathtt{b}_{ic_t}, \;\; \mathtt{a}_{i,\mathbf{c}} = \mathtt{a}_{ic_1} \dots \mathtt{a}_{ic_t} \text{ and } \tau_{i,\mathbf{c}} = \tau_{ic_1} \dots \tau_{ic_t}.$$

Note that $\{\mathtt{b}_{i,\mathbf{c}} \mid \mathbf{c} \in \mathscr{C}_{i,l}\}$ forms a basis of $U^-_{-l\alpha_i}$. For each $i \in I^{\mathrm{re}}$, we put $\mathtt{b}_i = f_i$, $\mathtt{a}_i = e_i$ and $\tau_i = \nu_i$.

Example 1 (1) Each $\lambda \in \mathcal{P}_l$ can be written as the form $\lambda = 1^{\lambda_1}2^{\lambda_2} \dots l^{\lambda_l}$, where λ_k are non-negative integers such that $\lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = l$. For $i \in I^{\mathrm{iso}}$, we have

$$\mathtt{b}_{il} = f_{il} - \sum_{\lambda \in \mathcal{P}_l \backslash (l)} \frac{1}{\prod_{k=1}^{l} \lambda_k!} \mathtt{b}_{i,\lambda}.$$

Note that assumption (7) implies $\nu_{il} \equiv 1 \pmod q$, hence we have $\tau_{il} \equiv \frac{1}{l} \pmod q$ by the following equation

$$\sum_{\lambda \in \mathcal{P}_l} \frac{1}{\prod_{k=1}^{l} k^{\lambda_k} \lambda_k!} = 1.$$

(2) Under the assumption (7), if $i \in I^{\mathrm{im}} \backslash I^{\mathrm{iso}}$, it was shown in [2, Lemma 3.32] that $\tau_{il} \equiv 1 \pmod q$ for all $l \geq 1$. Moreover, $\tau_{il} \in 1 + q\mathbf{Z}[[q]]$.

Let $\mathcal{U}$ be the associative algebra over $\mathbf{Q}(q)$ with $\mathbf{1}$ generated by the elements $\mathtt{t}_{il}, \mathtt{w}_{il}$ ($(i, l) \in I^\infty$) and q^h ($h \in P^\vee$) with defining relations

$$\begin{aligned}
& q^0 = \mathbf{1}, \quad q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in P^\vee \\
& q^h \mathtt{t}_{jl} q^{-h} = q^{l\alpha_j(h)} \mathtt{t}_{jl}, \;\; q^h \mathtt{w}_{jl} q^{-h} = q^{-l\alpha_j(h)} \mathtt{w}_{jl} \text{ for } h \in P^\vee, (j,l) \in I^\infty, \\
& \mathtt{t}_{il}\mathtt{w}_{jk} - \mathtt{w}_{jk}\mathtt{t}_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l}), \\
& \sum_{r+s=1-la_{ij}} (-1)^r \mathtt{t}_i^{(r)} \mathtt{t}_{jl} \mathtt{t}_i^{(s)} = 0 \text{ for } i \in I^{\mathrm{re}}, (j,l) \in I^\infty \text{ and } i \neq (j,l), \\
& \sum_{r+s=1-la_{ij}} (-1)^r \mathtt{w}_i^{(r)} \mathtt{w}_{jl} \mathtt{w}_i^{(s)} = 0 \text{ for } i \in I^{\mathrm{re}}, (j,l) \in I^\infty \text{ and } i \neq (j,l), \\
& \mathtt{t}_{ik}\mathtt{t}_{jl} - \mathtt{t}_{jl}\mathtt{t}_{ik} = \mathtt{w}_{ik}\mathtt{w}_{jl} - \mathtt{w}_{jl}\mathtt{w}_{ik} = 0 \text{ for } a_{ij} = 0.
\end{aligned} \tag{13}$$

Theorem 7 *There exists a $\mathbf{Q}(q)$-algebra isomorphism $\Phi : U \xrightarrow{\sim} \mathcal{U}$ mapping $\mathtt{a}_{il}$ to $\mathtt{t}_{il}$, $\mathtt{b}_{il}$ to $\mathtt{w}_{il}$, and q^h to q^h.*

Proof Recall that, for $j \in I^{\text{im}}$, e_{jl} (resp. f_{jl}) in U can be written as a homogeneous polynomials in $\mathtt{a}_{jk}$'s (resp. $\mathtt{b}_{jk}$'s) for $1 \le k \le l$. We may write

$$e_{jl} = \sum_{\mathbf{c} \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}} \mathtt{a}_{j,\mathbf{c}}, \ f_{jl} = \sum_{\mathbf{c} \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}} \mathtt{b}_{j,\mathbf{c}},$$

and let $\Phi : U \to \mathcal{U}$ be an algebra homomorphism sending $q^h \mapsto q^h$, $\mathtt{e}_i \mapsto \mathtt{t}_i$, $\mathtt{b}_i \mapsto \mathtt{w}_i$ if $i \in I^{\text{re}}$ and

$$\Phi(e_{jl}) = \sum_{\mathbf{c} \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}} \mathtt{t}_{j,\mathbf{c}},$$

$$\Phi(f_{jl}) = \sum_{\mathbf{c} \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}} \mathtt{w}_{j,\mathbf{c}}, \text{ if } j \in I^{\text{im}}.$$

We shall show Φ is well-defined. For each $\mathbf{c} \in \mathscr{C}_{j,l}$, we have $q^h \mathtt{t}_{j,\mathbf{c}} q^{-h} = q^{l\alpha_j(h)} \mathtt{t}_{j,\mathbf{c}}$ and $q^h \mathtt{w}_{j,\mathbf{c}} q^{-h} = q^{-l\alpha_j(h)} \mathtt{w}_{j,\mathbf{c}}$ in $\mathcal{U}$. Hence

$$\Phi(q^h e_{jl} q^{-h} - q^{l\alpha_j(h)} e_{jl}) = \Phi(q^h f_{jl} q^{-h} - q^{-l\alpha_j(h)} f_{jl}) = 0.$$

For the Serre-type relations, if $a_{ij} \neq 0$ we have

$$\Phi\left(\sum_{r+s=1-la_{ij}} (-1)^r e_i{}^{(r)} e_{jl} e_i^{(s)}\right) = \sum_{\mathbf{c} \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}} \sum_{r+s=1-la_{ij}} (-1)^r \mathtt{t}_i{}^{(r)} \mathtt{t}_{j,\mathbf{c}} \mathtt{t}_i^{(s)} = 0.$$

If $a_{ij} = 0$, we have

$$\Phi(e_{ik} e_{jl}) = \sum_{\mathbf{c} \in \mathscr{C}_{i,k}} \alpha_{\mathbf{c}} \mathtt{t}_{i,\mathbf{c}} \sum_{\mathbf{c}' \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}'} \mathtt{t}_{j,\mathbf{c}'} = \sum_{\mathbf{c}' \in \mathscr{C}_{j,l}} \alpha_{\mathbf{c}'} \mathtt{t}_{j,\mathbf{c}'} \sum_{\mathbf{c} \in \mathscr{C}_{i,k}} \alpha_{\mathbf{c}} \mathtt{t}_{i,\mathbf{c}} = \Phi(e_{jl} e_{ik}).$$

The other half Serre-type relations can be shown similarly.

For the commutation relations, we first claim that $\Phi(\mathtt{a}_{il}) = \mathtt{t}_{il}$ and $\Phi(\mathtt{b}_{il}) = \mathtt{w}_{il}$ for any $(i, l) \in I^\infty$. If $i \in I^{\text{re}}$, there is nothing to show. If $i \in I^{\text{im}}$, we could show it by induction on l. If $l = 1$, it is obvious since $e_{i1} = a_{i1}$. Assume the claim is true for all $k < l$. Since $\mathtt{a}_{il} = e_{il} - \sum_{\mathbf{c} \in \mathscr{C}_{i,l}, \mathbf{c} \neq (l)} \alpha_{\mathbf{c}} \mathtt{a}_{i,\mathbf{c}}$, we have $\Phi(\mathtt{a}_{il}) = \sum_{\mathbf{c} \in \mathscr{C}_{i,l}} \alpha_{\mathbf{c}} \mathtt{t}_{i,\mathbf{c}} - \sum_{\mathbf{c} \in \mathscr{C}_{i,l}, \mathbf{c} \neq (i)} \alpha_{\mathbf{c}} \mathtt{t}_{i,\mathbf{c}} = \mathtt{t}_{il}$. And $\Phi(\mathtt{b}_{il}) = \mathtt{w}_{il}$ can be shown similarly.

Now by the Drinfeld double process, the commutation relation in (11) is equivalent to the one in (12) (cf. [17, Lemma 3.2]). Moreover, $\Phi(\mathtt{a}_{il}\mathtt{b}_{jk} - \mathtt{b}_{jk}\mathtt{a}_{il} - \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l})) = 0$. This shows that Φ is well-defined.

Since Theorem 4 yields the following relations in U

$$\sum_{r+s=1-la_{ij}} (-1)^r \mathtt{a}_i{}^{(r)} \mathtt{a}_{jl} \mathtt{a}_i^{(s)} = 0 \ \text{ for } i \in I^{\text{re}}, \ (j, l) \in I^\infty \text{ and } i \neq (j, l),$$

$$\sum_{r+s=1-la_{ij}} (-1)^r \mathtt{b}_i{}^{(r)} \mathtt{b}_{jl} \mathtt{b}_i^{(s)} = 0 \ \text{ for } i \in I^{\text{re}}, \ (j, l) \in I^\infty \text{ and } i \neq (j, l).$$

We see that Φ has an obvious inverse $\Psi : \mathcal{U} \to U$ given by

$$\Psi(\mathtt{t}_{il}) = \mathtt{a}_{il}, \ \Psi(\mathtt{w}_{il}) = \mathtt{b}_{il}, \ \Psi(q^h) = q^h,$$

for $(i, l) \in I^\infty$ and $h \in P^\vee$. □

We note that Theorem 7 provides a new presentation of U with primitive generators. Now, Proposition 6 provides a Hopf algebra structure on $\mathcal{U}$:

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h,\\
&\Delta(\mathtt{t}_{il}) = \mathtt{t}_{il} \otimes K_i^{-l} + 1 \otimes \mathtt{t}_{il},\\
&\Delta(\mathtt{w}_{il}) = \mathtt{w}_{il} \otimes 1 + K_i^{l} \otimes \mathtt{w}_{il},\\
&S(\mathtt{t}_{il}) = -\mathtt{t}_{il}K_i^{l},\ S(\mathtt{w}_{il}) = -K_i^{-l}\mathtt{w}_{il},\ S(q^h) = q^{-h},\\
&\epsilon(\mathtt{t}_{il}) = \epsilon(\mathtt{w}_{il}) = 0,\ \epsilon(q^h) = 1,
\end{aligned} \tag{14}$$

where Δ, S, and ϵ denote the co-multiplication, antipode and counit, respectively.

4 Crystal bases and polarization

Definition 2 For $i \in I^{\mathrm{im}}$ and $\mathbf{c} \in \mathscr{C}_i$, we define the linear maps $\delta_{i,\mathbf{c}}, \delta^{i,\mathbf{c}}\colon U^- \to U^-$ by

$$\varrho(x) = \sum_{\mathbf{c}\in\mathscr{C}_i} \delta_{i,\mathbf{c}}(x) \otimes \mathtt{b}_{i,\mathbf{c}} + \text{ terms of bidegree not in } Q_- \times -\mathbf{N}\alpha_i,$$

$$\varrho(x) = \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}} \otimes \delta^{i,\mathbf{c}}(x) + \text{ terms of bidegree not in } -\mathbf{N}\alpha_i \times Q_-,$$

where x is a homogeneous element in U^-.

Let $i \in I^{\mathrm{im}}$, $l > 0$, then for any homogeneous $x, y, z \in U^-$ and $\mathbf{c} = (c_1, \ldots, c_t) \in \mathscr{C}_i$, we have the following equations

$$\gamma^{i,l}(xy) = \gamma^{i,l}(x)y + q^{l(\alpha_i,|x|)}x\gamma^{i,l}(y), \tag{15}$$

$$\gamma^{i,l}(\mathtt{b}_{i,\mathbf{c}}) = \sum_{k:c_k=l} q_{(i)}^{-2l\sum_{j<k}c_j}\mathtt{b}_{i,\mathbf{c}\setminus c_k}, \tag{16}$$

$$[\mathtt{a}_{il}, z] = \tau_{il}\left(\gamma_{i,l}(z)K_i^{l} - K_i^{-l}\gamma^{i,l}(z)\right), \tag{17}$$

where $\mathbf{c}\setminus c_k = (c_1, \ldots, \widehat{c_k}, \ldots, c_r)$ means removing c_k from $\mathbf{c}$. We will denote the operator $\gamma^{i,l}$ by $e'_{i,l}$ in the following.

Recall from [2] that every $u \in U^-$ can be written uniquely as

$$u = \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}}u_{\mathbf{c}},$$

where $e'_{i,l}\, u_{\mathbf{c}} = 0$ for all $l \geq 1$ and $\mathbf{c} \in \mathscr{C}_i$. Moreover, if u is homogeneous, then every $u_{\mathbf{c}}$ is homogeneous. Then the Kashiwara operators are defined by

$$\widetilde{e}_{il}u = \begin{cases} \sum_{\mathbf{c}:c_1=l} \mathtt{b}_{i,\mathbf{c}\setminus c_1} u_{\mathbf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \sum_{\mathbf{c}\in\mathscr{C}_i} \sqrt{\frac{m_l(\mathbf{c})}{l}}\, \mathtt{b}_{i,\mathbf{c}\setminus l} u_{\mathbf{c}} & \text{if } i \in I^{\text{iso}}, \end{cases}$$

$$\widetilde{f}_{il}u = \begin{cases} \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,(l,c)} u_{\mathbf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \sum_{\mathbf{c}\in\mathscr{C}_i} \sqrt{\frac{l}{m_l(\mathbf{c})+1}}\, \mathtt{b}_{i,\mathbf{c}\cup l} u_{\mathbf{c}} & \text{if } i \in I^{\text{iso}}, \end{cases}$$

where $m_l(\mathbf{c}) = \#\{k \mid c_k = l\}$.

Remark 1 Note that the square roots appear in the above definition. So we need to consider an extension $\mathbf{F}$ of $\mathbf{Q}$ that contains all the necessary square roots (see [2, Remark 3.12]).

Let $\mathbb{A}_0 = \{f \in \mathbf{F}(q) \mid f \text{ is regular at } q = 0\}$, and let $\mathcal{L}(\infty)$ be the $\mathbb{A}_0$-submodule of U^- generated by the elements $\widetilde{f}_{i_1,l_1} \dots \widetilde{f}_{i_r,l_r}\mathbf{1}$ for $r \geq 0$ and $(i_k, l_k) \in I^\infty$, where the Kashiwara operators $\widetilde{f}_i$ for $i \in I^{\text{re}}$ have been defined in [12]. Set

$$\mathcal{B}(\infty) = \{\widetilde{f}_{i_1,l_1} \dots \widetilde{f}_{i_r,l_r}\mathbf{1} \bmod q\mathcal{L}(\infty) \mid r \geq 0, (i_k, l_k) \in I^\infty\} \subseteq \mathcal{L}(\infty)/q\mathcal{L}(\infty),$$

then $(\mathcal{L}(\infty), \mathcal{B}(\infty))$ is the crystal base of U^-.

By [2, Lemma 3.33] and [12, Proposition 5.1.2], we have the following proposition.

Proposition 8

(i) $(\mathcal{L}(\infty), \mathcal{L}(\infty))_L \subseteq \mathbb{A}_0$.
Let $(\ ,\ )_L^0$ *denote the* $\mathbf{F}$*-valued inner product on* $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$ *induced by* $(\ ,\ )_L|_{q=0}$ *on* $\mathcal{L}(\infty)$.
(ii) $(\widetilde{e}_{il}u, v)_L^0 = (u, \widetilde{f}_{il}v)_L^0$ *for* $u, v \in \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ *and* $(i, l) \in I^\infty$.
(iii) $\mathcal{B}(\infty)$ *is an orthonormal base of* $(\ ,\ )_L^0$. *In particular,* $(\ ,\ )_L^0$ *is positive definite.*
(iv) $\mathcal{L}(\infty) = \{u \in U^- \mid (u, \mathcal{L}(\infty))_L \subseteq \mathbb{A}_0\} = \{u \in U^- \mid (u, u)_L \in \mathbb{A}_0\}$.

Let $\lambda \in P^+$, and let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Then we have a $U_q^-(\mathfrak{g})$-module isomorphism (cf. [3, 9])

$$V(\lambda) \simeq U_q^-(\mathfrak{g}) \Bigg/ \left(\sum_{i\in I^{\text{re}}} U_q^-(\mathfrak{g}) f_i^{\lambda(h_i)+1} + \sum_{\substack{i\in I^{\text{im}},\lambda(h_i)=0 \\ (i,l)\in I^\infty}} U_q^-(\mathfrak{g}) f_{il} \right). \tag{18}$$

Recall from [2] that, for any $i \in I^{\text{im}}$ and $\lambda \in P^+$, $v \in V(\lambda)_\mu$ has a decomposition of the following form

$$v = \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}} v_{\mathbf{c}},$$

where $v_{\mathbf{c}} \in V(\lambda)_{\mu+|\mathbf{c}|\alpha_i}$ and $e_{il}v_{\mathbf{c}} = 0$ for all $l \geq 1$ and $\mathbf{c} \in \mathscr{C}_i$. Moreover, if we omit the terms $\mathtt{b}_{i,\mathbf{c}}v_{\mathbf{c}}$ with $|\mathbf{c}| \neq 0$ and $(\mu + |\mathbf{c}|\alpha_i, \alpha_i) = 0$, which are equal to zero trivially, then the decomposition of v is unique.

Define the Kashiwara operators on $V(\lambda)$ by

$$\widetilde{e}_{il}v = \begin{cases} \sum_{\mathbf{c}:c_1=l} \mathrm{b}_{i,\mathbf{c}\setminus c_1} v_{\mathbf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \sum_{\mathbf{c}\in\mathscr{C}_i} \sqrt{\frac{m_l(\mathbf{c})}{l}}\, \mathrm{b}_{i,\mathbf{c}\setminus l} v_{\mathbf{c}} & \text{if } i \in I^{\text{iso}}, \end{cases}$$

$$\widetilde{f}_{il}v = \begin{cases} \sum_{\mathbf{c}\in\mathscr{C}_i} \mathrm{b}_{i,(l,c)} v_{\mathbf{c}} & \text{if } i \notin I^{\text{iso}}, \\ \sum_{\mathbf{c}\in\mathscr{C}_i} \sqrt{\frac{l}{m_l(\mathbf{c})+1}}\, \mathrm{b}_{i,\mathbf{c}\cup l} v_{\mathbf{c}} & \text{if } i \in I^{\text{iso}}. \end{cases}$$

Let $\mathcal{L}(\lambda) = \sum_{\iota_1,\ldots,\iota_s \in I^\infty} \mathbb{A}_0 \widetilde{f}_{\iota_1} \ldots \widetilde{f}_{\iota_s} v_\lambda$ be an $\mathbb{A}_0$-submodule of $V(\lambda)$ and let

$$\mathcal{B}(\lambda) = \{\widetilde{f}_{\iota_1} \ldots \widetilde{f}_{\iota_s} v_\lambda \mid \iota_k \in I^\infty\}\setminus\{0\} \subseteq \mathcal{L}(\lambda)/q\mathcal{L}(\lambda),$$

then $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is the crystal base of $V(\lambda)$.

There exists a unique symmetric bilinear form $\{-,-\}$ on $V(\lambda)$ such that

$$\begin{aligned} &\{v_\lambda, v_\lambda\} = 1, \\ &\{q^h v, v'\} = \{v, q^h v'\}, \\ &\{\mathrm{b}_{il} v, v'\} = -\{v, K_i^l \mathrm{a}_{il} v'\} \ \text{ if } i \in I^{\text{im}}, \\ &\{\mathrm{b}_i v, v'\} = \frac{1}{q_i^2 - 1}\{v, K_i \mathrm{a}_i v'\} \ \text{ if } i \in I^{\text{re}}, \end{aligned}$$

for every $v, v' \in V(\lambda)$ and $(i,l) \in I^\infty$.

Similarly, by [2, Lemma 3.34] and [12, Proposition 5.1.1], we have the following proposition.

Proposition 9

(i) $\{\mathcal{L}(\lambda), \mathcal{L}(\lambda)\} \subseteq \mathbb{A}_0$.
Let $\{\ ,\ \}_0$ *denote the* $\mathbf{F}$*-valued inner product on* $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ *induced by* $\{\ ,\ \}|_{q=0}$ *on* $\mathcal{L}(\lambda)$.
(ii) $\{\widetilde{e}_{il}u, v\}_0 = \{u, \widetilde{f}_{il}v\}_0$ *for* $u, v \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ *and* $(i,l) \in I^\infty$.
(iii) $\mathcal{B}(\lambda)$ *is an orthonormal base of* $\{\ ,\ \}_0$. *In particular,* $\{\ ,\ \}_0$ *is positive definite.*
(iv) $\mathcal{L}(\lambda) = \{v \in V(\lambda) \mid \{v, \mathcal{L}(\lambda)\} \subseteq \mathbb{A}_0\} = \{v \in V(\lambda) \mid \{v, v\} \in \mathbb{A}_0\}$.

The following proposition follows from Kashiwara's grand-loop argument, which describes the relations between $\mathcal{B}(\infty)$ and $\mathcal{B}(\lambda)$.

Proposition 10 *Let* $\pi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ *be the* $U_q^-(\mathfrak{g})$*-module homomorphism given by* $P \mapsto Pv_\lambda$*, then we have*

(i) $\pi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda)$*, hence* π_λ *induces the surjective homomorphism*

$$\overline{\pi}_\lambda : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda).$$

(ii) $\{b \in \mathcal{B}(\infty) \mid \overline{\pi}_\lambda(b) \neq 0\}$ *is isomorphic to* $\mathcal{B}(\lambda)$ *under the map* $\overline{\pi}_\lambda$.
(iii) *If* $b \in \mathcal{B}(\infty)$ *satisfies* $\overline{\pi}_\lambda(b) \neq 0$*, then* $\widetilde{e}_{il}\overline{\pi}_\lambda(b) = \overline{\pi}_\lambda(\widetilde{e}_{il}b)$.
(iv) $\widetilde{f}_{il} \circ \overline{\pi}_\lambda = \overline{\pi}_\lambda \circ \widetilde{f}_{il}$.

Let $(i,l) \in I^\infty$ and let $P \in U^-$, then there exist unique $Q, R \in U^-$ such that

$$[\mathtt{a}_{il}, P] = \tau_{il}(K_i^l Q - K_i^{-l} R).$$

Note that $e'_{i,l}(P) = R$ by (17). If we set $e''_{i,l}(P) = Q$, then we have

$$e'_{i,l}\mathtt{b}_{jk} = \delta_{ij}\delta_{kl} + q_i^{-kla_{ij}}\mathtt{b}_{jk}e'_{i,l},$$
$$e''_{i,l}\mathtt{b}_{jk} = \delta_{ij}\delta_{kl} + q_i^{kla_{ij}}\mathtt{b}_{jk}e''_{i,l},$$

and

$$e'_{i,l}e''_{j,k} = q_i^{kla_{ij}}e''_{j,k}e'_{i,l}.$$

Definition 3 Let $\mathscr{B}_q(\mathfrak{g})$ be the algebra over $\mathbf{F}(q)$ generated by $e'_{i,l}$, $\mathtt{b}_{il}$ $(i,l) \in I^\infty$ with defining relations

$$e'_{i,l}\mathtt{b}_{jk} = \delta_{ij}\delta_{kl} + q_i^{-kla_{ij}}\mathtt{b}_{jk}e'_{i,l},$$
$$\sum_{r=0}^{1-la_{ij}} (-1)^r \begin{bmatrix} 1-la_{ij} \\ r \end{bmatrix}_i e_i'^{\,1-la_{ij}-r} e'_{j,l} e_i'^{\,r} = 0 \ \text{ for } i \in I^{\mathrm{re}} \text{ and } i \neq (j,l),$$
$$\sum_{r=0}^{1-la_{ij}} (-1)^r \begin{bmatrix} 1-la_{ij} \\ r \end{bmatrix}_i \mathtt{b}_i^{1-la_{ij}-r} \mathtt{b}_{j,l} \mathtt{b}_i^{r} = 0 \ \text{ for } i \in I^{\mathrm{re}} \text{ and } i \neq (j,l),$$
$$e'_{i,k}e'_{j,l} - e'_{j,l}e'_{i,k} = \mathtt{b}_{ik}\mathtt{b}_{jl} - \mathtt{b}_{jl}\mathtt{b}_{ik} = 0 \ \text{ for } a_{ij} = 0.$$

We call $\mathscr{B}_q(\mathfrak{g})$ the *quantum boson algebra* associated with $\mathfrak{g}$. One can show that $\mathscr{B}_q(\mathfrak{g})$ is a left $U_q^-(\mathfrak{g})$-module by the standard argument in [12]. Furthermore, we have

$$U_q^-(\mathfrak{g}) \cong \mathscr{B}_q(\mathfrak{g}) \Big/ \sum_{(i,l)\in I^\infty} \mathscr{B}_q(\mathfrak{g})e'_{i,l}.$$

Lemma 11 *For all $P, Q \in U^-$ and $(i,l) \in I^\infty$, we have*

$$(P\mathtt{b}_{il}, Q)_L = \tau_{il}(P, K_i^l e''_{i,l} Q K_i^{-l})_L.$$

Proof By (17), we have $K_i^l e''_{i,l} Q = \delta_{i,l}(Q) K_i^l$ and hence $K_i^l e''_{i,l} Q K_i^{-l} = \delta_{i,l}(Q)$. Thus we obtain

$$(P\mathtt{b}_{il}, Q)_L = \tau_{il}(P, \delta_{i,l}(Q))_L = \tau_{il}(P, K_i^l e''_{i,l} Q K_i^{-l})_L$$

as desired. □

Let $*\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the $\mathbf{F}(q)$-linear anti-involution given by

$$e_{il}^* = e_{il}, \quad f_{il}^* = f_{il}, \quad (q^h)^* = q^{-h}.$$

Note that $** = id$ and $*- = -*$ on $U^\pm$, and $\mathtt{a}_{il}, \mathtt{b}_{il}$ are stable under $*$ for any $(i,l) \in I^\infty$.

Lemma 12 *For any $P, Q \in U^-$, we have*

$$(P^*, Q^*)_L = (P, Q)_L.$$

Proof Note that $e''_{i,l}(Q^*) = K_i^{-l}(e'_{i,l}Q)^* K_i^l$ and $e'_{i,l}(Q^*) = K_i^l(e''_{i,l}Q)^* K_i^{-l}$. We shall prove this lemma by induction on $|P|$. If $P = 1$, our assertion is clear. By Lemma 11 and the inductive hypothesis, we have

$$\begin{aligned}((P\mathrm{b}_{il})^*, Q^*)_L &= (\mathrm{b}_{il}P^*, Q^*)_L = \tau_{il}(P^*, e'_{i,l}(Q^*))_L \\ &= \tau_{il}(P^*, K_i^l(e''_{i,l}Q)^* K_i^{-l})_L \\ &= \tau_{il}(P, K_i^l e''_{i,l} Q K_i^{-l})_L \\ &= (P\mathrm{b}_{il}, Q)_L,\end{aligned}$$

which proves our claim. □

The following corollary is an immediate consequence of Lemma 12 and Proposition 8.

Corollary 13 $\mathcal{L}(\infty)^* = \mathcal{L}(\infty)$.

Proposition 14 *Let $P, Q \in U_q^-(\mathfrak{g})_{-\beta}$ for $\beta \in Q_+$. If $\lambda \gg 0$, we have*

$$\{Pv_\lambda, Qv_\lambda\} \equiv c(P, Q)_L \pmod{q\mathbb{A}_0}$$

for some $c \in \mathbb{A}_0 \backslash q\mathbb{A}_0$.

Proof We use the induction on $|\beta|$. If $i \in I^{\mathrm{im}}$, we have

$$\begin{aligned}\{\mathrm{b}_{il}Pv_\lambda, Qv_\lambda\} &= -\{Pv_\lambda, K_i^l \mathrm{a}_{il} Qv_\lambda\} \\ &= -\{Pv_\lambda, K_i^l(Q\mathrm{a}_{il} + \tau_{il}(K_i^l e''_{i,l}Q - K_i^{-l}e'_{i,l}Q))v_\lambda\} \\ &= -\tau_{il}\{Pv_\lambda, K_i^{2l}e''_{i,l}Qv_\lambda - e'_{i,l}Qv_\lambda\} \\ &= -\tau_{il}\{Pv_\lambda, q_i^{2l(\lambda-\beta)(h_i)}e''_{i,l}Qv_\lambda\} + \tau_{il}\{Pv_\lambda, e'_{i,l}Qv_\lambda\},\end{aligned}$$

where $P \in U^-_{-\beta}$ and $Q \in U^-_{-\beta-l\alpha_i}$. Hence

$$\{\mathrm{b}_{il}Pv_\lambda, Qv_\lambda\} \equiv \tau_{il}\{Pv_\lambda, e'_{il}Qv_\lambda\} \equiv c\tau_{il}(P, e'_{i,l}Q)_L = c(\mathrm{b}_{il}P, Q)_L \pmod{q\mathbb{A}_0}.$$

if $i \in I^{\mathrm{re}}$, we have

$$\begin{aligned}\{\mathrm{b}_i Pv_\lambda, Qv_\lambda\} &= \frac{1}{q_i^2-1}\{Pv_\lambda, K_i\mathrm{a}_i Qv_\lambda\} \\ &= \frac{1}{q_i^2-1}\{Pv_\lambda, K_i\tau_i(K_ie''_iQ - K_i^{-1}e'_iQ)v_\lambda\} \\ &= \frac{1}{q_i^2-1}\tau_i\{Pv_\lambda, q_i^{2(\lambda-\beta)(h_i)}e''_{il}Qv_\lambda\} + \frac{1}{q_i^2-1}\tau_i\{Pv_\lambda, e'_iQv_\lambda\},\end{aligned}$$

where $P \in U^-_{-\beta}$ and $Q \in U^-_{-\beta-\alpha_i}$. Hence

$$\begin{aligned}\{\mathrm{b}_iPv_\lambda, Qv_\lambda\} &\equiv \frac{1}{q_i^2-1}\tau_i\{Pv_\lambda, e'_iQv_\lambda\} \equiv \frac{1}{q_i^2-1}c\tau_i(P, e'_iQ)_L \\ &= \frac{1}{q_i^2-1}c(\mathrm{b}_iP, Q)_L \pmod{q\mathbb{A}_0},\end{aligned}$$

which completes the proof. □

Corollary 15 *If $\lambda \gg 0$ and $Pv_\lambda \in \mathcal{L}(\lambda)$, then $P^*v_\lambda \in \mathcal{L}(\lambda)$.*

Proof If $Pv_\lambda \in \mathcal{L}(\lambda)$, then $\{Pv_\lambda, Pv_\lambda\} \in \mathbb{A}_0$ by Proposition 9. Since $\{Pv_\lambda, Pv_\lambda\} \equiv c(P, P)_L \pmod{q\mathbb{A}_0}$ for some $c \in \mathbb{A}_0 \setminus q\,\mathbb{A}_0$, we have $(P, P)_L \in \mathbb{A}_0$. Hence $P \in \mathcal{L}(\infty)$ and $P^* \in \mathcal{L}(\infty)$ by Proposition 8 and Corollary 13. Now Proposition 10 yields $\pi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda)$. Thus we get $P^* v_\lambda \in \mathcal{L}(\lambda)$ by applying π_λ. □

5 $\mathbb{A}$-form of $U_q^-(\mathfrak{g})$

Let $\mathbb{A} = \mathbf{F}[q, q^{-1}]$ and $\mathbb{A}_\infty = \{f \in \mathbf{F}(q) \mid f \text{ is regular at } q = \infty\}$. We denote by $U_{\mathbb{A}}^-(\mathfrak{g})$ the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $\mathtt{b}_i^{(n)}$ $(i \in I^{\mathrm{re}}, n \geq 0)$ and $\mathtt{b}_{il}$ $(i \in I^{\mathrm{im}}, l \geq 1)$.

For each $i \in I^{\mathrm{re}}$, set

$$A_i = \mathtt{a}_i/\tau_i(q_i - q_i^{-1}), \tag{19}$$

which yields the following commutation relation

$$A_i \mathtt{b}_i - \mathtt{b}_i A_i = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \tag{20}$$

For $i \in I^{\mathrm{im}}$ and $l \geq 1$, set $A_{il} = \mathtt{a}_{il}/\tau_{il}$. Then we have

$$A_{il}\mathtt{b}_{il} - \mathtt{b}_{il}A_{il} = K_i^l - K_i^{-l}. \tag{21}$$

Let $U_{\mathbb{A}}(\mathfrak{g})$ be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $A_i^{(n)}, \mathtt{b}_i^{(n)}$ $(i \in I^{\mathrm{re}}, n \geq 0)$, $A_{il}, \mathtt{b}_{il}$ $(i \in I^{\mathrm{im}}, l \geq 1)$ and q^h $(h \in P^\vee)$, $\left\{ \begin{matrix} K_i q_i^n \\ m \end{matrix} \right\}_i$ $(i \in I^{\mathrm{re}}, m \in \mathbf{Z}_{\geq 0}, n \in \mathbf{Z})$, where

$$\left\{ \begin{matrix} K_i q_i^n \\ m \end{matrix} \right\}_i = \frac{1}{[m]_i!} \prod_{s=1}^{m} \frac{K_i q_i^{n+1-s} - K_i^{-1} q_i^{-n-1+s}}{q_i - q_i^{-1}}. \tag{22}$$

Let $U_{\mathbb{A}}^+(\mathfrak{g})$ (resp. $U_{\mathbb{A}}^0(\mathfrak{g})$) be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $A_i^{(n)}$ $(i \in I^{\mathrm{re}})$ and A_{il} $(i \in I^{\mathrm{im}}, l \geq 1)$ (resp. q^h, $\left\{ \begin{matrix} K_i q_i^n \\ m \end{matrix} \right\}_i$ for $h \in P^\vee, m \in \mathbf{Z}_{\geq 0}, n \in \mathbf{Z}$ and $i \in I^{\mathrm{re}}$). Then using the commutations relations (20), (21) and the definition (22), one can prove that $U_{\mathbb{A}}(\mathfrak{g})$ has the triangular decomposition (see also [12, Section 1], [6, Exercise 3.6])

$$U_{\mathbb{A}}(\mathfrak{g}) \cong U_{\mathbb{A}}^-(\mathfrak{g}) \otimes U_{\mathbb{A}}^0(\mathfrak{g}) \otimes U_{\mathbb{A}}^+(\mathfrak{g}).$$

Let $\lambda \in P^+$ and consider an $\mathbf{F}$-linear automorphism $^-: V(\lambda) \to V(\lambda)$ given by $Pv_\lambda \mapsto \overline{P}v_\lambda$ for $P \in U_q(\mathfrak{g})$. Set $\mathcal{L}(\lambda)^- = \overline{\mathcal{L}(\lambda)}$. Then $\mathcal{L}(\lambda)$ (resp. $\mathcal{L}(\lambda)^-$) is a free $\mathbb{A}_0$-lattice (resp. free $\mathbb{A}_\infty$-lattice) of $V(\lambda)$.

Since

$$\left\{ \begin{matrix} K_i q_i^n \\ m \end{matrix} \right\}_i v_\lambda = \begin{bmatrix} \lambda(h_i) + n \\ m \end{bmatrix}_i v_\lambda \in \mathbf{Z}[q, q^{-1}]v_\lambda,$$

we get $U_{\mathbb{A}}^0(\mathfrak{g})v_\lambda = \mathbb{A}v_\lambda$. This leads us to give the following definition

$$V(\lambda)^{\mathbb{A}} := U_{\mathbb{A}}(\mathfrak{g})v_\lambda = U_{\mathbb{A}}^-(\mathfrak{g})v_\lambda.$$

Note that $\overline{\mathtt{b}}_{il} = \mathtt{b}_{il}$ for all $(i, l) \in I^\infty$. Hence $U_{\mathbb{A}}^-(\mathfrak{g})$ and $V(\lambda)^{\mathbb{A}}$ are stable under $-$. Also, since $U_{\mathbb{A}}^-(\mathfrak{g})$ is graded by Q_-, we have $V(\lambda)^{\mathbb{A}} = \bigoplus_{\mu \leq \lambda} V(\lambda)_\mu^{\mathbb{A}}$, where $V(\lambda)_\mu^{\mathbb{A}} = V(\lambda)^{\mathbb{A}} \cap V(\lambda)_\mu$.

Fix $i \in I$. In [2], Bozec proved that every $u \in U_q^-(\mathfrak{g})$ has the following decomposition.

$$u = \begin{cases} \sum_{n\geq 0} \mathtt{b}_i^{(n)} u_n & \text{with } i \in I^{\mathrm{re}} \text{ and } e_i' u_n = 0 \text{ for all } n \geq 0, \\ \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}} u_{\mathbf{c}} & \text{with } i \in I^{\mathrm{im}} \text{ and } e_{il}' u_{\mathbf{c}} = 0 \text{ for all } l > 0, \mathbf{c} \in \mathscr{C}_i. \end{cases} \tag{23}$$

Lemma 16 *For each $i \in I$ and $u \in U_q^-(\mathfrak{g})$, consider the decomposition (23). If $u \in U_{\mathbb{A}}^-(\mathfrak{g})$, then all $u_n, u_{\mathbf{c}} \in U_{\mathbb{A}}^-(\mathfrak{g})$.*

Proof We first prove that $e_{i,l}' U_{\mathbb{A}}^-(\mathfrak{g}) \subseteq U_{\mathbb{A}}^-(\mathfrak{g})$ for all $(i, l) \in I^\infty$.

Since $e_{i,l}' \mathtt{b}_{jk} = \delta_{ij}\delta_{kl} + q_i^{-kla_{ij}} \mathtt{b}_{jk} e_{i,l}'$, we have

$$e_i' \mathtt{b}_i = 1 + q_i^{-2} \mathtt{b}_i e_i' \quad \text{for } i \in I^{\mathrm{re}}.$$

It follows that

$$e_i' \mathtt{b}_i^{(n)} = q_i^{1-n} \mathtt{b}_i^{(n-1)} + q_i^{-2n} \mathtt{b}_i^{(n)} e_i'.$$

Furthermore, by a direct calculation, we have

$$e_i'^n \mathtt{b}_i^{(m)} = \sum_{k=0}^{n} q_i^{-2nm+(m+n)k-k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_i \mathtt{b}_i^{(m-k)} e_i'^{n-k},$$

where $\mathtt{b}_i^{(r)} = 0$ if $r < 0$. These imply our assertion.

For $i \in I^{\mathrm{re}}$, let

$$P = \sum_{n\geq 0} (-1)^n q_i^{-n(n-1)/2} \mathtt{b}_i^{(n)} e_i'^n.$$

Then we obtain (cf. [12, Section 3.2]):

(i) $P\mathtt{b}_i = 0, \ \ e_i' P = 0$,
(ii) $\sum_{n\geq 0} q_i^{n(n-1)/2} \mathtt{b}_i^{(n)} P e_i'^n = 1$,
(iii) $P e_i'^n u = q_i^{-n(n-1)/2} u_n$ for $u \in U_q^-(\mathfrak{g})$.

Hence, if $u \in U_{\mathbb{A}}^-(\mathfrak{g})$, then $u_n \in U_{\mathbb{A}}^-(\mathfrak{g})$ for all $n \geq 0$.

For $i \in I^{\mathrm{im}}$, we use a similar argument in [1, Proposition 3.11]. Assume that $u \in U_{\mathbb{A}}^-(\mathfrak{g})$ has the form $u = m\mathtt{b}_{i,\mathbf{c}} m'$ for some $\mathbf{c} \in \mathscr{C}_i$ and homogeneous elements $m, m' \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g})$, where $\mathcal{K}_i = \bigcap_{l>0} \ker e_{i,l}'$. We shall show that u can be written into the form

$$u = \sum_{\mathbf{c}'\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}'} u_{\mathbf{c}'} \quad \text{with } u_{\mathbf{c}'} \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g}).$$

If $|\mathbf{c}| = 0$, then $u = mm' \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g})$. Otherwise, we have

$$u = (m\mathtt{b}_{ic_1} - q^{c_1(|m|,\alpha_i)} \mathtt{b}_{ic_1} m) \mathtt{b}_{i,\mathbf{c}\setminus c_1} m' + q^{c_1(|m|,\alpha_i)} \mathtt{b}_{ic_1} m \mathtt{b}_{i,\mathbf{c}\setminus c_1} m',$$

where $m\mathtt{b}_{ic_1} - q^{c_1(|m|,\alpha_i)} \mathtt{b}_{ic_1} m \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g})$. Now our claim follows by using the induction on $|\mathbf{c}|$.

We next show that if $u \in U_{\mathbb{A}}^-(\mathfrak{g})$, then u can be written into the form

$$u = \sum_{\mathbf{c}\in\mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}} u_{\mathbf{c}} \quad \text{with } u_{\mathbf{c}} \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g}).$$

We will use the induction on $-|u|$.

Assume that u is a monomial in $U_{\mathbb{A}}^-(\mathfrak{g})$. Then there exists some monomial $u' \in U_{\mathbb{A}}^-(\mathfrak{g})$ such that $u = \mathtt{b}_j^{(n)} u'$ for some $j \in I^{\mathrm{re}}$ or $u = \mathtt{b}_{jl} u'$ for some $j \in I^{\mathrm{im}}$. By induction hypothesis, $u' = \sum_{\mathbf{c} \in \mathscr{C}_i} \mathtt{b}_{i,\mathbf{c}} u_{\mathbf{c}}$ with $u_{\mathbf{c}} \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g})$. If $j \neq i$, then $u = \sum_{\mathbf{c} \in \mathscr{C}_i} \mathtt{b}_j^{(n)} \mathtt{b}_{i,\mathbf{c}}\, u_{\mathbf{c}}$ or $u = \sum_{\mathbf{c} \in \mathscr{C}_i} \mathtt{b}_{jl} \mathtt{b}_{i,\mathbf{c}} u_{\mathbf{c}}$ is of the form $m \mathtt{b}_{i,\mathbf{c}} m'$ with $m, m' \in \mathcal{K}_i \cap U_{\mathbb{A}}^-(\mathfrak{g})$. If $i = j$, then $u = \sum_{\mathbf{c} \in \mathscr{C}_i} \mathtt{b}_{i,(l,\mathbf{c})} u_{\mathbf{c}}$ is already in the form we wanted.

Thus, our assertion follows from the uniquesness of the decomposition. □

Define

$$(\mathtt{b}_i^n U_q^-(\mathfrak{g}))^{\mathbb{A}} := \mathtt{b}_i^n U_q^-(\mathfrak{g}) \cap U_{\mathbb{A}}^-(\mathfrak{g}) \quad \text{for } i \in I^{\mathrm{re}} \text{ and } n \geq 1,$$
$$(\mathtt{b}_{i,\mathbf{c}} U_q^-(\mathfrak{g}))^{\mathbb{A}} := \mathtt{b}_{i,\mathbf{c}} U_q^-(\mathfrak{g}) \cap U_{\mathbb{A}}^-(\mathfrak{g}) \quad \text{for } i \in I^{\mathrm{im}} \text{ and } \mathbf{c} \in \mathscr{C}_i \backslash \{0\}.$$

By the above lemma, $U_{\mathbb{A}}^-(\mathfrak{g})$ is stable under the Kashiwara operators $\widetilde{e}_{il}$ and $\widetilde{f}_{il}$. Moreover, we have the following corollary.

Corollary 17

(i) *For* $i \in I^{re}$, $(b_i^n U_q^-(\mathfrak{g}))^{\mathbb{A}} = \sum_{k \geq n} b_i^{(k)} U_{\mathbb{A}}^-(\mathfrak{g}) = \bigoplus_{k \geq n} b_i^{(k)} (U_{\mathbb{A}}^-(\mathfrak{g}) \cap \ker e_i')$.
For $i \in I^{im}$, $(b_{i,\mathbf{c}} U_q^-(\mathfrak{g}))^{\mathbb{A}} = b_{i,\mathbf{c}} U_{\mathbb{A}}^-(\mathfrak{g}) = \bigoplus_{\mathbf{c}' \in \mathscr{C}_i} b_{i,(\mathbf{c},\mathbf{c}')} (U_{\mathbb{A}}^-(\mathfrak{g}) \cap \mathcal{K}_i)$.

(ii) *For* $i \in I^{re}$, $(b_i^n V(\lambda))^{\mathbb{A}} := (b_i^n U_q^-(\mathfrak{g}))^{\mathbb{A}} v_\lambda = \sum_{k \geq n} b_i^{(k)} V(\lambda)^{\mathbb{A}}$.
For $i \in I^{im}$, $(b_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}} := (b_{i,\mathbf{c}} U_q^-(\mathfrak{g}))^{\mathbb{A}} v_\lambda = b_{i,\mathbf{c}} V(\lambda)^{\mathbb{A}}$.

6 Existence of global bases

Let V be a finite-dimensional vector space over $\mathbf{F}(q)$, M be an $\mathbb{A}$-submodule of V, and L_0 (resp. L_∞) be a free $\mathbb{A}_0$-submodule (resp. free $\mathbb{A}_\infty$-submodule) of V such that $V \cong \mathbf{F}(q) \otimes_{\mathbb{A}_0} L_0 \cong \mathbf{F}(q) \otimes_{\mathbb{A}_\infty} L_\infty$.

Definition 4 The triple $(V^{\mathbb{A}}, L_0, L_\infty)$ is called an *balanced triple* for V if the canonical linear map $V^{\mathbb{A}} \cap L_0 \cap L_\infty \longrightarrow L_0 \big/ q L_0$ is an isomorphism.

Lemma 18 [12, Lemma 7.1.1] *Let* V, M, L_0, L_∞ *be as above.*

(i) *Assume that the canonical map* $M \cap L_0 \cap L_\infty \to M \cap L_0 / M \cap q L_0$ *is an isomorphism. Then*

$$M \cap L_0 \cong \mathbf{F}[q] \otimes_{\mathbf{F}} (M \cap L_0 \cap L_\infty),$$
$$M \cap L_\infty \cong \mathbf{F}[q^{-1}] \otimes_{\mathbf{F}} (M \cap L_0 \cap L_\infty),$$
$$M \cong \mathbb{A} \otimes_{\mathbf{F}} (M \cap L_0 \cap L_\infty),$$
$$M \cap L_0 \cap L_\infty \cong (M \cap L_\infty)/(M \cap q^{-1} L_\infty),$$
$$M \cap L_0 \cap L_\infty \cong (\mathbf{F}(q) \otimes_{\mathbb{A}} M) \cap L_0 / (\mathbf{F}(q) \otimes_{\mathbb{A}} M) \cap q L_0.$$

(ii) *Let* E *be an* $\mathbf{F}$*-vector space and* $\varphi \colon E \to M \cap L_0 \cap L_\infty$ *a homomorphism. Assume that* $M = \mathbb{A}\varphi(E)$ *and* $E \to L_0 / q L_0$, $E \to L_\infty / q^{-1} L_\infty$ *are injective. Then* $E \to M \cap L_0 \cap L_\infty \to M \cap L_0 / M \cap q L_0$ *is an isomorphism.*

Lemma 19 [12, Lemma 7.1.2] *Let* V, M, L_0, L_∞ *be as above and let* N *be an* $\mathbb{A}$*-submodule of* M*. Assume that*

(1) $N \cap L_0 \cap L_\infty \cong N \cap L_0 / N \cap qL_0$.
(2) *There exist an* $\mathbf{F}$*-vector space* F *and a homomorphism* $\varphi\colon F \to M \cap (L_0 + N) \cap (L_\infty + N)$ *such that*
(a) $M = \mathbb{A}\varphi(F) + N$,
(b) *the induced homomorphisms* $\varphi_0\colon F \to (L_0 + N)/(qL_0 + N)$ *and* $\varphi_\infty\colon F \to (L_\infty + N)/(q^{-1}L_\infty + N)$ *are injective.*

Then the following statements hold.

(i) $M \cap L_0 \cap L_\infty \to M \cap L_0 / M \cap qL_0$ *is an isomorphism.*
(ii) $M \cap L_0 / M \cap qL_0 \cong F \oplus (N \cap L_0 / N \cap qL_0)$.

For $r \geq 0$, set

$$Q_+(r) = \{\alpha \in Q_+ \mid |\alpha| \leq r\}.$$

We will prove the following inductive statements on $r \geq 0$.
A(r): For any $\alpha \in Q_+(r)$, we have the following canonical isomorphism

$$U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^- \xrightarrow{\sim} \frac{U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty)}{U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty)} \xrightarrow{\sim} \mathcal{L}(\infty)_{-\alpha}/q\mathcal{L}(\infty)_{-\alpha}.$$

We denote by G_∞ the inverse of this isomorphism.
B(r): For any $\alpha \in Q_+(r)$ and $\lambda \in P^+$, we have the following canonical isomorphism

$$V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^- \xrightarrow{\sim} \frac{V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda)}{V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda)} \xrightarrow{\sim} \mathcal{L}(\lambda)_{\lambda-\alpha}/q\mathcal{L}(\lambda)_{\lambda-\alpha}.$$

We denote by G_λ the inverse of this isomorphism.
C(r): For $\alpha \in Q_+(r)$, $(i,l) \in I^\infty$, and $n \geq 0$, assume that $b \in \widetilde{f}_{il}^n \mathcal{B}(\infty)_{-\alpha+ln\alpha_i}$. Then we have

$$G_\infty(b) \in \mathfrak{b}_{il}^n U_q^-(\mathfrak{g}).$$

If $r = 0$, our assertions are obvious. Now we assume that A($r-1$), B($r-1$) and C($r-1$) are true. Then Lemma 18 and Proposition 10 imply the following result.

Lemma 20 *For* $\alpha \in Q_+(r-1)$, *we have*

$$U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty) = \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha}} \mathbf{F}[q]G_\infty(b), \quad U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} = \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha}} \mathbb{A}G_\infty(b),$$
$$V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) = \bigoplus_{b \in \mathcal{B}(\lambda)_{\lambda-\alpha}} \mathbf{F}[q]G_\lambda(b), \quad V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} = \bigoplus_{b \in \mathcal{B}(\lambda)_{\lambda-\alpha}} \mathbb{A}G_\lambda(b),$$

and

$$G_\infty(b)v_\lambda = G_\lambda(\overline{\pi}_\lambda(b)).$$

Lemma 21 *For* $\alpha \in Q_+(r-1)$, $b \in \mathcal{B}(\infty)_{-\alpha}$ *(resp.* $b \in \mathcal{B}(\lambda)_{\lambda-\alpha}$*), we have* $\overline{G_\infty(b)} = G_\infty(b)$ *(resp.* $\overline{G_\lambda(b)} = G_\lambda(b)$*).*

Proof Let $Q = (G_\infty(b) - \overline{G_\infty(b)})/(q - q^{-1})$. Then we have $Q \in U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-$ since $1/(q-q^{-1}) \in q\mathbb{A}_0$. □

Let $i \in I^{\mathrm{iso}}$, $\lambda \in P^+$ and $\alpha \in Q_+$. For each partition $\mathbf{c} = (1^{l_1} 2^{l_2} \cdots k^{l_k} \cdots)$, we define

$$(\mathtt{b}_{i,\mathbf{c}} * U_q^-(\mathfrak{g}))^{\mathbb{A}}_{-\alpha} := \sum_{k\geq 1} (\mathtt{b}_{i,k}^{l_k} U_q^-(\mathfrak{g}))^{\mathbb{A}}_{-\alpha} = \sum_{k\geq 1} \mathtt{b}_{i,k}^{l_k} (U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha+kl_k\alpha_i}),$$

$$(\mathtt{b}_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} := \sum_{k\geq 1} (\mathtt{b}_{i,k}^{l_k} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = \sum_{k\geq 1} \mathtt{b}_{i,k}^{l_k} (V(\lambda)^{\mathbb{A}}_{\lambda-\alpha+kl_k\alpha_i}).$$

Here $(\mathtt{b}_{i,k}^{l_k} U_q^-(\mathfrak{g}))^{\mathbb{A}}_{-\alpha} = (\mathtt{b}_{i,k}^{l_k} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = 0$ if $l_k = 0$.

Proposition 22 *Let $\alpha \in Q_+(r)$ and $\lambda \in P^+$.*

(i) For $i \in I^{re}$ and $n \geq 1$, we have

$$(b_i^n V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^- \xrightarrow{\sim} \frac{(b_i^n V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda)}{(b_i^n V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda)} \cong \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha}\cap\widetilde{f}_i^n\mathcal{B}(\lambda)} \mathbf{F}b.$$

(ii) For $i \in I^{im}\backslash I^{iso}$ and any composition $\mathbf{c}$ with $|\mathbf{c}| \neq 0$, we have

$$(b_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^- \xrightarrow{\sim} \frac{(b_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda)}{(b_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda)} \cong \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha}\cap\widetilde{f}_{i,\mathbf{c}}\mathcal{B}(\lambda)} \mathbf{F}b.$$

(iii) For $i \in I^{iso}$ and any partition $\mathbf{c} = 1^{l_1}2^{l_2}\cdots k^{l_k}\cdots$, we have

$$(b_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^- \xrightarrow{\sim} \frac{(b_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda)}{(b_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda)}$$
$$\cong \bigoplus_{b\in B(\lambda)_{\lambda-\alpha}\cap(\widetilde{f}_{i,\mathbf{c}}*\mathcal{B}(\lambda))} \mathbf{F}b,$$

where $\widetilde{f}_{i,\mathbf{c}} * \mathcal{B}(\lambda) := \bigcup_{k\geq 1} \widetilde{f}_{i,k}^{l_k}\mathcal{B}(\lambda)$.

Proof Our assertion (i) has been proved in [12, Proposition 7.4.1].

Assume $i \in I^{\mathrm{im}}\backslash I^{\mathrm{iso}}$. Let $\mathbf{c} \in \mathscr{C}_i$ such that $|\mathbf{c}| = n > 0$. Recall that $(\mathtt{b}_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = \mathtt{b}_{i,\mathbf{c}}(V(\lambda)^{\mathbb{A}}_{\lambda-\alpha+n\alpha_i})$. If $(\lambda-\alpha+n\alpha_i)(h_i) = 0$, then $(\mathtt{b}_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = 0$, and hence our assertion is trivial. Thus we may assume that $(\lambda-\alpha+n\alpha_i)(h_i) > 0$. In this case, for any $b \in \mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}$, we have $\widetilde{f}_{i,\mathbf{c}} b \neq 0$.

By B$(r-1)$, we have $V(\lambda)^{\mathbb{A}}_{\lambda-\alpha+n\alpha_i} = \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \mathbb{A}G_\lambda(b)$. Hence

$$(\mathtt{b}_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = \sum_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \mathbb{A}\mathtt{b}_{i,\mathbf{c}} G_\lambda(b).$$

Let $F = \sum_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \mathbf{F}\mathtt{b}_{i,\mathbf{c}} G_\lambda(b)$. We first show that F is a direct sum. Assume that

$$\sum_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \beta_b \mathtt{b}_{i,\mathbf{c}} G_\lambda(b) = 0 \quad \text{for some } \beta_b \in \mathbf{F}.$$

Since $\widetilde{f}_{i,\mathbf{c}} G_\lambda(b) = \mathtt{b}_{i,\mathbf{c}} G_\lambda(b)$ and $G_\lambda(b) \equiv b \pmod{q\mathcal{L}(\lambda)}$ for any $b \in \mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}$, we obtain

$$\sum_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \beta_b \widetilde{f}_{i,\mathbf{c}} b = 0.$$

By applying $\widetilde{e}_{i,\widetilde{\mathbf{c}}}$, we get $\sum_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \beta_b b = 0$, which implies $\beta_b = 0$ for all $b \in \mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}$.

Let $N = 0$, $M = (\mathrm{b}_{i,\mathbf{c}} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$, $L_0 = \mathcal{L}(\lambda)_{\lambda-\alpha}$ and $L_\infty = \mathcal{L}(\lambda)^-_{\lambda-\alpha}$. Set $\varphi\colon F \to M \cap L_0 \cap L_\infty$ be the $\mathbf{F}$-linear map given by

$$\mathrm{b}_{i,\mathbf{c}} G_\lambda(b) \longmapsto \mathrm{b}_{i,\mathbf{c}} G_\lambda(b) = \widetilde{f}_{i,\mathbf{c}} G_\lambda(b).$$

Then, it is easy to check F, N, M, L_0, L_∞ and φ satisfy the conditions in Lemma 19, and hence we get

$$M \cap L_0 \cap L_\infty \xrightarrow{\sim} M \cap L_0 / M \cap q L_0 \cong \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+n\alpha_i}} \mathbf{F} \widetilde{f}_{i,\mathbf{c}} b = \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha}\cap \widetilde{f}_{i,\mathbf{c}}\mathcal{B}(\lambda)} \mathbf{F} b$$

as desired.

Now, we shall prove (iii). Let $i \in I^{\mathrm{iso}}$. If l_k is sufficient large, then $(\mathrm{b}_{ik}^{l_k} U_q^-(\mathfrak{g}))^{\mathbb{A}}_{-\alpha} = 0$. Hence we can use descending induction on $N = \sum_{k\geq 1} l_k$. Without loss of generality, we may assume that $l_1 \neq 0$ and $(\lambda - \alpha)(h_i) > 0$. Then, by A$(r-1)$ and B$(r-1)$, we have

$$(\mathrm{b}_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = \mathrm{b}_{i,1}^{l_1} V(\lambda)^{\mathbb{A}}_{\lambda-\alpha+l_1\alpha_i} + \sum_{k\geq 2} (\mathrm{b}_{i,k}^{l_k} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$$

and

$$V(\lambda)^{\mathbb{A}}_{\lambda-\alpha+l_1\alpha_i} = \bigoplus_{b\in\mathcal{B}(\lambda)_{\lambda-\alpha+l_1\alpha_i}} \mathbb{A} G_\lambda(b) = \bigoplus_{\substack{b\in\mathcal{B}(\infty)_{-\alpha+l_1\alpha_i} \\ \overline{\pi}_\lambda(b)\neq 0}} \mathbb{A} G_\infty(b) v_\lambda.$$

Let $b \in \mathcal{B}(\infty)_{-\alpha+l_1\alpha_i}$ with $\widetilde{e}_{i1} b \neq 0$. Then $b \in \widetilde{f}_{i1}\mathcal{B}(\infty) \cap \mathcal{B}(\infty)_{-\alpha+l_1\alpha_i}$, which implies $G_\infty(b) \in \mathrm{b}_{i1} U_q^-(\mathfrak{g}) \cap U_{\mathbb{A}}^-(\mathfrak{g})$ by C$(r-1)$. Hence $\mathrm{b}_{i1}^{l_1} G_\infty(b) v_\lambda \in (\mathrm{b}_{i1}^{l_1+1} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$. For $k \neq 1$ with $l_k \neq 0$, if $b \in \widetilde{f}_{ik}^{l_k}\mathcal{B}(\infty)$, then $G_\infty(b) \in \mathrm{b}_{ik}^{l_k} U_q^-(\mathfrak{g}) \cap U_{\mathbb{A}}^-(\mathfrak{g})$ by C$(r-1)$. Therefore $\mathrm{b}_{i,1}^{l_1} G_\infty(b) v_\lambda \in (\mathrm{b}_{i,k}^{l_k} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$. Hence we have

$$(\mathrm{b}_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} = \sum_{b\in S} \mathbb{A} \mathrm{b}_{i,1}^{l_1} G_\infty(b) v_\lambda + (\mathrm{b}_{i,\mathbf{c}\cup\{1\}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha},$$

where

$$\begin{aligned} S &= \left\{ b \in \mathcal{B}(\infty)_{-\alpha+l_1\alpha_i} \mid \overline{\pi}_\lambda(b) \neq 0,\ \widetilde{e}_{i,1} b = 0,\ b \notin \bigcup_{k\geq 2} \widetilde{f}_{i,k}^{l_k}\mathcal{B}(\infty) \right\} \\ &\xrightarrow[\sim]{\overline{\pi}_\lambda} \left\{ b \in \mathcal{B}(\lambda)_{\lambda-\alpha+l_1\alpha_i} \mid \widetilde{e}_{i,1} b = 0,\ b \notin \bigcup_{k\geq 2} \widetilde{f}_{i,k}^{l_k}\mathcal{B}(\lambda) \right\} \\ &\xrightarrow[\sim]{\widetilde{f}_{i,1}^{l_1}} \mathcal{B}(\lambda)_{\lambda-\alpha} \cap \left(\widetilde{f}_{i,1}^{l_1}\mathcal{B}(\lambda) \backslash (\widetilde{f}_{i,\mathbf{c}\cup\{1\}} * \mathcal{B}(\lambda)) \right). \end{aligned} \tag{24}$$

The last isomorphism follows from the fact that $\widetilde{f}_{il}\widetilde{e}_{il'} = \widetilde{e}_{il'}\widetilde{f}_{il}$ and $\widetilde{f}_{il}\widetilde{f}_{il'} = \widetilde{f}_{il'}\widetilde{f}_{il}$ for any $l, l' \geq 1$ with $l \neq l'$.

Let $V = V(\lambda)_{\lambda-\alpha}$, $M = (\mathrm{b}_{i,\mathbf{c}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$, $N = (\mathrm{b}_{i,\mathbf{c}\cup\{1\}} * V(\lambda))^{\mathbb{A}}_{\lambda-\alpha}$, $L_0 = \mathcal{L}(\lambda)_{\lambda-\alpha}$, $L_\infty = \mathcal{L}(\lambda)^-_{\lambda-\alpha}$ and

$$F = \sum_{b\in S} \mathbf{F} \mathrm{b}_{i1}^{l_1} G_\infty(b) v_\lambda.$$

For $b \in S$, we have $b = G_\infty(b) + q\mathcal{L}(\infty)$. Assume that $G_\infty(b)$ has the decomposition

$$G_\infty(b) = \sum_{\mathbf{c} \in \mathscr{C}_i} \mathrm{b}_{i,\mathbf{c}} u_{\mathbf{c}} \in U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha+l_1\alpha_i} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-.$$

Then we have

$$\widetilde{f}_{i1}\widetilde{e}_{i1} G_\infty(b) = G_\infty(b) - \sum_{\mathbf{c} \in \mathscr{C}_i; 1 \notin \mathbf{c}} \mathrm{b}_{i,\mathbf{c}} u_{\mathbf{c}} = G_\infty(b) - u_b \in q\mathcal{L}(\infty).$$

Hence we obtain

(i) $\mathrm{b}_{i1}^{l_1} G_\infty(b) \equiv \mathrm{b}_{i1}^{l_1} u_b \bmod (\mathrm{b}_{i1}^{l_1+1} U_q^-(\mathfrak{g}))^{\mathbb{A}}$, which implies

$$\mathrm{b}_{i1}^{l_1} G_\infty(b) v_\lambda \in M \cap (N + L_0) \cap (N + L_\infty). \tag{25}$$

(ii) $\widetilde{f}_{i1}^{l_1} b = \beta_b \mathrm{b}_{i1}^{l_1} u_b + q\mathcal{L}(\infty)$ for some $\beta_b \in \mathbf{F}^*$, which implies

$$\overline{\pi}_\lambda(\widetilde{f}_{i1}^{l_1} b) = \beta_b \mathrm{b}_{i1}^{l_1} u_b v_\lambda + q\mathcal{L}(\lambda). \tag{26}$$

Set

$$H := (N + L_0)/(N + qL_0) \cong \frac{L_0/qL_0}{N \cap L_0/N \cap qL_0}.$$

By induction hypothesis, we have $H \cong \bigoplus_{b \in \mathcal{B}(\lambda)_{\lambda-\alpha} \setminus (\widetilde{f}_{i,\mathbf{c}\cup\{1\}} * \mathcal{B}(\lambda))} \mathbf{F}b$. Hence (24), (25) and (26) imply that the following canonical maps are injective:

$$\begin{aligned} \varphi_0 \colon F &\xrightarrow{\varphi} M \cap (L_0 + N) \cap (L_\infty + N) \to \frac{N + L_0}{N + qL_0} \xrightarrow{\sim} H \\ b_{i1}^{l_1} G_\infty(b) v_\lambda &\mapsto \mathrm{b}_{i1}^{l_1} G_\infty(b) v_\lambda \mapsto \mathrm{b}_{i1}^{l_1} u_b v_\lambda + (N + qL_0) \mapsto \frac{1}{\beta_b} \overline{\pi}_\lambda(\widetilde{f}_{i1}^{l_1} b) \end{aligned} \tag{27}$$

By taking $-$, the following canonical map is injective

$$\varphi_\infty \colon F \xrightarrow{\varphi} M \cap (L_0 + N) \cap (L_\infty + N) \to \frac{N + L_\infty}{N + q^{-1} L_\infty}.$$

Note that $M = \mathbb{A}\varphi(F) + N$. Hence Lemma 19 yields

$$M \cap L_0 \cap L_\infty \xrightarrow{\sim} (M \cap L_0)/(M \cap qL_0) \cong F \oplus (N \cap L_0/N \cap qL_0),$$

where

$$\begin{aligned} F \oplus (N \cap L_0/N \cap qL_0) &\cong \bigoplus_{S \sqcup (\mathcal{B}(\lambda)_{\lambda-\alpha} \cap \widetilde{f}_{i,\mathbf{c}\cup\{1\}} * \mathcal{B}(\lambda))} \mathbf{F}b \\ &= \bigoplus_{\mathcal{B}(\lambda)_{\lambda-\alpha} \cap (\widetilde{f}_{i,\mathbf{c}} * \mathcal{B}(\lambda))} \mathbf{F}b. \end{aligned}$$

This completes the proof. □

Corollary 23 *Let* $\alpha \in Q_+(r)$.

(i) For $i \in I^{re}$ and $n \geq 1$, we have

$$(b_i^n U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^- \xrightarrow{\sim} \frac{(b_i^n U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty)}{(b_i^n U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap q\mathcal{L}(\infty)}$$
$$\cong \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap \tilde{f}_i^n \mathcal{B}(\infty)} \mathbf{F} b.$$

(ii) For $i \in I^{im} \backslash I^{iso}$ and $|\mathbf{c}| \neq 0$, we have

$$(b_{i,\mathbf{c}} U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^- \xrightarrow{\sim} \frac{(b_{i,\mathbf{c}} U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty)}{(b_{i,\mathbf{c}} U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap q\mathcal{L}(\infty)}$$
$$\cong \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap \tilde{f}_{i,\mathbf{c}} \mathcal{B}(\infty)} \mathbf{F} b.$$

(iii) For $i \in I^{iso}$ and any partition $\mathbf{c} = (1^{l_1} 2^{l_2} \cdots k^{l_k} \cdots)$, we have

$$(b_{i,\mathbf{c}} * U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^- \xrightarrow{\sim} \frac{(b_{i,\mathbf{c}} * U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty)}{(b_{i,\mathbf{c}} * U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap q\mathcal{L}(\infty)}$$
$$\cong \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap (\tilde{f}_{i,\mathbf{c}} * \mathcal{B}(\infty))} \mathbf{F} b,$$

where $\tilde{f}_{i,\mathbf{c}} * \mathcal{B}(\infty) = \bigcup_{k \geq 1} \tilde{f}_{ik}^{l_k} \mathcal{B}(\infty)$.

Proof We shall prove (iii) only. The proof of (i) and (ii) are similar. For $\lambda \gg 0$, we have

$$U_q^-(\mathfrak{g})_{-\alpha} \xrightarrow{\sim} V(\lambda)_{\lambda-\alpha}, \quad (\mathrm{b}_{i,\mathbf{c}} * U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \xrightarrow{\sim} (\mathrm{b}_{i,\mathbf{c}} * V(\lambda))_{\lambda-\alpha}^{\mathbb{A}},$$

$$\mathcal{L}(\infty)_{-\alpha} \xrightarrow{\sim} \mathcal{L}(\lambda)_{\lambda-\alpha}, \quad \mathcal{L}(\infty)_{-\alpha}^- \xrightarrow{\sim} \mathcal{L}(\lambda)_{\lambda-\alpha}^-,$$

and

$$\bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap (\tilde{f}_{i,\mathbf{c}} * \mathcal{B}(\infty))} \mathbf{F} b \xrightarrow{\sim} \bigoplus_{b \in \mathcal{B}(\lambda)_{\lambda-\alpha} \cap (\tilde{f}_{i,\mathbf{c}} * \mathcal{B}(\lambda))} \mathbf{F} b.$$

Hence our assertion follows immediately. □

For $\alpha \in Q_+(r)$ and $(i, l) \in I^\infty$, let us denote by G_{il} the inverse of the isomorphism

$$(\mathrm{b}_{il} U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^- \xrightarrow{\sim} \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap \tilde{f}_{il} \mathcal{B}(\infty)} \mathbf{F} b.$$

Then Corollary 23 implies $(\mathrm{b}_{il}^n U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} = \bigoplus_{b \in \mathcal{B}(\infty)_{-\alpha} \cap \tilde{f}_{il}^n \mathcal{B}(\infty)} \mathbb{A} G_{il}(b)$ for any $n \geq 1$.

Lemma 24 Let $(i, l), (j, s) \in I^\infty$, $\alpha \in Q_+(r)$ and $b \in \tilde{f}_{il} \mathcal{B}(\infty) \cap \tilde{f}_{js} \mathcal{B}(\infty) \cap \mathcal{B}(\infty)_{-\alpha}$. Then we have

$$G_{il}(b) = G_{js}(b).$$

Proof Let us write $b = \widetilde{f}_{il} \dots \widetilde{f}_{km} \cdot 1$, where $(k, m) \in I^{\infty}$. If $k \in I^{\mathrm{re}}$, then our claim was proved in [12]. So we will assume that $k \in I^{\mathrm{im}}$. Take $\lambda \in P^+$ with $\lambda(h_k) = 0$ and $\lambda(h_j) \gg 0$ for all $j \in I \setminus \{k\}$. Then (18) yields

$$V(\lambda)_{\lambda-\alpha} \simeq U_q^-(\mathfrak{g})_{-\alpha} \Big/ \sum_{n\geq 1} U_q^-(\mathfrak{g})_{-\alpha+n\alpha_k} \mathfrak{b}_{kn}.$$

The same argument in [12, Lemma 7.5.1] shows that

$$Q = G_{il}(b) - G_{js}(b) \in \left(\sum_{n\geq 1} U_q^-(\mathfrak{g})_{-\alpha+n\alpha_k} \mathfrak{b}_{kn} \right) \cap U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-.$$

Then Corollary 13 implies

$$Q^* \in \left(\sum_{n\geq 1} \mathfrak{b}_{kn} U_q^-(\mathfrak{g})_{-\alpha+n\alpha_k} \right) \cap U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-.$$

If $k \in I^{\mathrm{im}} \setminus I^{\mathrm{iso}}$, we assume that $Q^* = \mathfrak{b}_{k1}u_1 + \cdots + \mathfrak{b}_{kt}u_t$. Since $\overline{Q^*} = \overline{Q}^* = Q^*$, we have $Q^* = \mathfrak{b}_{k1}\overline{u}_1 + \cdots + \mathfrak{b}_{kt}\overline{u}_t$. Note that for each $1 \leq j \leq t$, $\mathfrak{b}_{kj}u_j = \widetilde{f}_{kj}\widetilde{e}_{kj}Q^* \in \mathfrak{b}_{kj}U_q^-(\mathfrak{g}) \cap U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty)$ and $u_j = \overline{u}_j = \widetilde{e}_{kj}Q^*$. Hence $b_{kj}u_j \in (\mathfrak{b}_{kj}U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} \cap q\mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-$ and Corollary 23 (ii) implies $Q^* = 0$.

If $k \in I^{\mathrm{iso}}$, since $Q^* \in (\sum_{n\geq 1} \mathfrak{b}_{kn} U_q^-(\mathfrak{g})_{-\alpha+n\alpha_k}) \cap U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha}$, the decomposition of Q^* can be expressed as the form $Q^* = \mathfrak{b}_{k1}u_1 + \cdots + \mathfrak{b}_{kt}u_t$ with

$$u_j = \sum_{\substack{\mathbf{c}\in\mathscr{C}_k \text{ and} \\ \mathbf{c} \text{ contains no } j+1,\dots,s}} \mathfrak{b}_{k,\mathbf{c}} u_{\mathbf{c}}.$$

For every $1 \leq j \leq t$, we have

$$\mathfrak{b}_{kj}u_j = \widetilde{f}_{kj}\widetilde{e}_{kj}(Q^* - \sum_{j<p\leq s} b_{kp}u_p) \in (\mathfrak{b}_{kj}U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}}.$$

Hence $Q^* \in (\mathfrak{b}_{k,(1^1 2^1 \dots t^1)} * U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}}$, and Corollary 23 (iii) implies $Q^* = 0$. □

Thus we can define

$$G\colon \mathcal{L}(\infty)_{-\alpha}/q\mathcal{L}(\infty)_{-\alpha} \to U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^-$$

by

$$b \mapsto G_{il}(b) \;\text{ for } b \in \widetilde{f}_{il}\mathcal{B}(\infty) \cap \mathcal{B}(\infty)_{-\alpha},\, (i,l) \in I^{\infty}.$$

Then we have $b = G(b) + q\mathcal{L}(\infty)$ and

$$(\mathfrak{b}_{il}^n U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}} = \bigoplus_{b\in\mathcal{B}(\infty)_{-\alpha}\cap\widetilde{f}_{il}^n\mathcal{B}(\infty)} \mathbb{A}G(b) \tag{28}$$

for any $n \geq 1$. Since $U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} = \sum_{(i,l)\in I^{\infty}} (\mathfrak{b}_{il}U_q^-(\mathfrak{g}))_{-\alpha}^{\mathbb{A}}$, we obtain

$$U_{\mathbb{A}}^-(\mathfrak{g})_{-\alpha} = \sum_{b\in\mathcal{B}(\infty)_{-\alpha}} \mathbb{A}G(b).$$

Let $E = \mathcal{L}(\infty)_{-\alpha}/q\mathcal{L}(\infty)_{-\alpha}$ and $M = U_{\mathbb{A}}^{-}(\mathfrak{g})_{-\alpha}$. Then by Lemma 18(ii), we deduce that

$$\mathcal{L}(\infty)_{-\alpha}/q\mathcal{L}(\infty)_{-\alpha} \xrightarrow{G} U_{\mathbb{A}}^{-}(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^{-} \to \frac{U_{\mathbb{A}}^{-}(\mathfrak{g})_{-\alpha} \cap \mathcal{L}(\infty)}{U_{\mathbb{A}}^{-}(\mathfrak{g})_{-\alpha} \cap q\mathcal{L}(\infty)}$$

is an isomorphism, which proves A(r). Now C(r) follows from (28). Finally, we shall prove B(r).

Lemma 25 *Let $\alpha \in Q_{+}(r)$, $b \in \mathcal{B}(\infty)_{-\alpha}$ and $\lambda \in P^{+}$. If $\overline{\pi}_{\lambda}(b) = 0$, then $G(b)v_{\lambda} = 0$.*

Proof Take $(i, l) \in I^{\infty}$ with $\tilde{e}_{il} b \neq 0$. Then $G(b)v_{\lambda} \in (\mathfrak{b}_{il} V(\lambda))^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^{-}$ by Proposition 22. □

By this lemma, we have $V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} = \sum_{\substack{b\in\mathcal{B}(\infty)_{-\alpha} \\ \overline{\pi}_{\lambda}(b)\neq 0}} \mathbb{A}G(b)v_{\lambda}$. Let

$$E = \sum_{\substack{b\in\mathcal{B}(\infty)_{-\alpha} \\ \overline{\pi}_{\lambda}(b)\neq 0}} \mathbf{F}G(b)v_{\lambda} \subseteq V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^{-}.$$

Since $\{b \in \mathcal{B}(\infty)_{-\alpha} \mid \overline{\pi}_{\lambda}(b) \neq 0\} \xrightarrow{\sim} \mathcal{B}(\lambda)_{\lambda-\alpha}$, we have

$$E \xrightarrow{\sim} \mathcal{L}(\lambda)_{\lambda-\alpha}/q\mathcal{L}(\lambda)_{\lambda-\alpha}$$

given by

$$G(b)v_{\lambda} \longmapsto G(b)v_{\lambda} + q\mathcal{L}(\lambda) = \overline{\pi}_{\lambda}(b).$$

By Lemma 18(ii), we get

$$E \xrightarrow{\sim} V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^{-} \xrightarrow{\sim} \frac{V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap \mathcal{L}(\lambda)}{V(\lambda)^{\mathbb{A}}_{\lambda-\alpha} \cap q\mathcal{L}(\lambda)} \cong \mathcal{L}(\lambda)_{\lambda-\alpha}/q\mathcal{L}(\lambda)_{\lambda-\alpha},$$

which proves B(r).

To summarize, we obtain the main goal of this paper.

Theorem 26 *There exist canonical isomorphisms*

$$U_{\mathbb{A}}^{-}(\mathfrak{g}) \cap \mathcal{L}(\infty) \cap \mathcal{L}(\infty)^{-} \xrightarrow{\sim} \mathcal{L}(\infty)/q\mathcal{L}(\infty),$$
$$V(\lambda)^{\mathbb{A}} \cap \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda)^{-} \xrightarrow{\sim} \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \quad (\lambda \in P^{+}).$$

Definition 5 (a) $\mathbf{B} := \{G(b) \mid b \in \mathcal{B}(\infty)\}$ is called the *global basis* of $U_{\mathbb{A}}^{-}(\mathfrak{g})$ corresponding to $\mathcal{B}(\infty)$.
(b) $\mathbf{B}^{\lambda} := \{G^{\lambda}(b) \mid b \in \mathcal{B}(\lambda)\}$ is called the *global basis* of $V(\lambda)^{\mathbb{A}}$ corresponding to $\mathcal{B}(\lambda)$.

Remark 2 The global bases $\mathbf{B}$ and $\mathbf{B}^{\lambda}$ are unique because they are stable under the bar involution.

We conjecture that our global bases coincide with (a variation of) Bozec's canonical bases. The following proposition would be a key ingredient of the proof.

Proposition 27 *Let $i \in I^{im} \setminus I^{iso}$, $\alpha \in Q_{+}$, $l \geq 0$ and $\lambda \in P^{+}$. Define*

$$\begin{aligned} B_{\alpha,i,\geq l} &:= \bigcup_{|\mathbf{c}|=l} \tilde{f}_{i,\mathbf{c}}(\mathcal{B}(\infty)_{-\alpha}), \quad B_{\alpha,i,l} := B_{\alpha,i,\geq l} \setminus B_{\alpha,i,\geq l+1}, \\ B^{\lambda}_{\alpha,i,\geq l} &:= \bigcup_{|\mathbf{c}|=l} \tilde{f}_{i,\mathbf{c}}(\mathcal{B}(\lambda)_{\lambda-\alpha}), \quad B^{\lambda}_{\alpha,i,l} = B^{\lambda}_{\alpha,i,\geq l} \setminus B^{\lambda}_{\alpha,i,\geq l+1}. \end{aligned} \tag{29}$$

(a) For any $b \in B_{\alpha,i,l}$, there exist an element $b_0 \in B_{\alpha-l\alpha_i,i,0}$ and a composition $\mathbf{c}$ *of l such that*

$$\mathbb{b}_{i,\mathbf{c}}G(b_0) - G(b) \in \bigoplus_{b' \in B_{\alpha,i,\geq l+1}} \mathbb{A}G(b').$$

(b) For any $b \in B^{\lambda}_{\alpha,i,l}$, there exist an element $b_0 \in B^{\lambda}_{\alpha-l\alpha_i,i,0}$ and a composition $\mathbf{c}$ *of l such that*

$$\mathbb{b}_{i,\mathbf{c}}G^{\lambda}(b_0) - G^{\lambda}(b) \in \bigoplus_{b' \in B^{\lambda}_{\alpha,i,\geq l+1}} \mathbb{A}G^{\lambda}(b').$$

Proof We will prove (a) only. The proof of (b) is similar. Recall that

$$U^{-}_{\mathbb{A}}(\mathfrak{g}) = \bigoplus_{\mathbf{c}} \mathbb{b}_{1,\mathbf{c}}\mathcal{K}_i, \quad \text{where } \mathcal{K} := \bigcap_{l>0} \ker e'_{il}.$$

Let $P_i : U^{-}_{\mathbb{A}}(\mathfrak{g}) \longrightarrow \mathcal{K}_i$ be the projection. Then, for $u \in U_{\mathbb{A}}$, we have

$$\widetilde{f}_{il}(u) = \mathbb{b}_{i,l}P_i(u) \mod \left(\sum_{|\mathbf{c}'|\geq l+1} \mathbb{b}_{i,\mathbf{c}'}U^{-}_{\mathbb{A}}(\mathfrak{g})\right). \tag{30}$$

By the crystal basis theory, for any $b \in B_{\alpha,i,l}$, there exist an element $b_0 \in B_{\alpha-l\alpha_i,i,0}$ and a composition $\mathbf{c}$ of l such that $\widetilde{f}_{i,\mathbf{c}}b_0 = b$. Thus we have

$$\begin{aligned}\mathbb{b}_{i,\mathbf{c}}G(b_0) =& aG(b) + a_1G(b_1) + \cdots + a_rG(b_r) \\ & \mod \left(\sum_{|\mathbf{c}'|\geq l+1} \mathbb{b}_{i,\mathbf{c}'}U^{-}_{\mathbb{A}}(\mathfrak{g})\right),\end{aligned} \tag{31}$$

where $a, a_1, \ldots, a_r \in \mathbb{A}$, $b_1, \ldots, b_r \in B_{\alpha-l\alpha_i,i,0}$.

Since the left-hand side and right-hand side of (31) are invariant under the bar involution, we have

$$\overline{a} = a,\ \overline{a}_1 = a_1, \ldots, \overline{a}_r = a_r.$$

We know

$$\widetilde{f}_{i,\mathbf{c}}\,G(b_0) = G(b) \mod q\mathcal{L}(\infty). \tag{32}$$

On the other hand, as in (31),we have

$$\begin{aligned}\widetilde{f}_{i,\mathbf{c}}G(b_0) &= \mathbb{b}_{i,\mathbf{c}}(P_iG(b_0) \mod \left(\sum_{|\mathbf{c}'|\geq l+1} \mathbb{b}_{i,\mathbf{c}'}U^{-}_{\mathbb{A}}(\mathfrak{g})\right) \\ &= \mathbb{b}_{i,\mathbf{c}}G(b_0) \mod \left(\sum_{|\mathbf{c}'|\geq l+1} \mathbb{b}_{i,\mathbf{c}'}U^{-}_{\mathbb{A}}(\mathfrak{g})\right) \\ &= (aG(b) + a_1G(b_1) + \cdots + a_rG(b_r) \mod \left(\sum_{|\mathbf{c}'|\geq l+1} \mathbb{b}_{i,\mathbf{c}'}U^{-}_{\mathbb{A}}(\mathfrak{g})\right).\end{aligned} \tag{33}$$

Comparing with (32), we get

$$a = 1, a_1 = \cdots = a_r = 0 \qquad \mod q\mathcal{L}(\infty).$$

Since $a, a_1, \ldots, a_r$ are all bar-invariant, we conclude $a = 1, a_1 = \cdots = a_r = 0$ in $\mathbb{A}$.

Thus we have finished our proof. □

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References

1. Bozec, T.: Quivers with loops and perverse sheaves. Math. Ann. **362**, 773–797 (2015)
2. Bozec, T.: Quivers with loops and generalized crystals. Compos. Math. **152**, 1999–2040 (2016)
3. Bozec, T., Schiffmann, O., Vasserot, E.: On the number of points of nilpotent quiver varieties over finite fields. Ann. Sci. Ec. Norm. Super. **53**(6), 1501–1544 (2020)
4. Fan, Z., Kang, S.-J., Kim, Y.R., Tong, B.: Classical limit of quantum Borcherds-Bozec algebras. J. Pure Appl. Algebra **225**, 106502 (2021)
5. Grojnowski, I., Lusztig, G.: A comparison of bases of quantized enveloping algebras. Contemp. Math. **153**, 11–19 (1993)
6. Hong, J., Kang, S.-J.: Introduction to Quantum Groups and Crystal Bases. Graduate Studies in Mathematics, vol. 42. American Mathematical Society, Providence, RI (2002)
7. Jeong, K., Kang, S.-J., Kashiwara, M.: Crystal Bases for quantum generalized Kac-Moody algebras. Proc. Lond. Math. Soc. **3**(90), 395–438 (2005)
8. Kang, S.-J.: Quantum deformations of generalized Kac-Moody algebras and their modules. J. Algebra **3**(175), 1041–1066 (1995)
9. Kang, S.-J., Kim, Y.R.: Quantum Borcherds-Bozec algebras and their integrable representations. J. Pure Appl. Algebra **224**, 106388 (2020)
10. Kang, S.-J., Schiffmann, O.: Canonical bases for quantum generalized Kax-Moody algebras. Adv. Math. **200**, 455–478 (2006)
11. Kashiwara, M.: Crystallizing the q-analogue of universal enveloping algebras. Commun. Math. Phys. **133**, 249–260 (1990)
12. Kashiwara, M.: On crystal bases of the q-analogue of universal enveloping algebras. Duke Math. J. **63**, 465–516 (1991)
13. Lusztig, G.: Canonical bases arising from quantized enveloping algebras. J. Am. Math. Soc. **3**, 447–498 (1990)
14. Lusztig, G.: Quivers, perverse sheaves, and quantized enveloping algebras. J. Am. Math. Soc. **4**, 365–421 (1991)
15. Lusztig, G.: Tight monomials in quantized enveloping algebras, Quantum deformations of algebras and their representations. In: Joseph A. et al. (ed.) Isr. Math. Conf. Proc., vol. 7, pp. 117–132. American Mathematical Society (1993)
16. Lusztig, G.: Introduction to Quantum Groups. Modern Birkhäuser Classics, Springer, New York (2010)
17. Sevenhant, B., Van den Bergh, M.: On the double of the Hall algebra of a quiver. J. Algebra **221**, 135–160 (1999)