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Bott–Chern and $\bar{\partial}$ harmonic forms on almost Hermitian 4-manifolds

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Abstract

We prove that on a compact almost Hermitian 4-manifold the space of $\bar{\partial}$-harmonic $(1,1)$-forms always has dimension $h^{1,1}_{\bar{\partial}} = b_- + 1$ or b_-, whilst the space of Bott–Chern harmonic $(1,1)$-forms always has dimension $h^{1,1}_{BC} = b_- + 1$. We also perform calculations of $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ on the Kodaira–Thurston manifold, thereby providing a full account of when $h^{p,q}_{BC}$ is or is not invariant of the choice of almost Hermitian metrics. Finally, we introduce a decomposition of the space of L^2 functions on all torus bundles over S^1, which has proven useful for solving linear PDEs, and we demonstrate its use in the calculation of $h^{p,q}_{\bar{\partial}}$.

Contents

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1 Introduction

On an almost complex manifold (X, J) endowed with an almost Hermitian metric, we can define the spaces

$$\mathcal{H}_{\bar\partial}^{p,q} := \ker \Delta_{\bar\partial}|_{\mathcal{A}^{p,q}}, \qquad \mathcal{H}_{BC}^{p,q} := \ker \Delta_{BC}|_{\mathcal{A}^{p,q}}$$

of $\bar\partial$-harmonic and Bott–Chern harmonic (p, q)-forms respectively. The corresponding laplacians are both elliptic differential operators and so the dimensions of these spaces, denoted by $h_{\bar\partial}^{p,q}$ and $h_{BC}^{p,q}$, are finite whenever X is compact.

In the case when J is integrable, i.e. J arises from a complex structure on X, $\mathcal{H}_{\bar\partial}^{p,q}$ is isomorphic to the Dolbeault cohomology

$$H_{\bar\partial}^{p,q} := \frac{\ker \bar\partial}{\mathrm{im}\, \bar\partial},$$

whilst $\mathcal{H}_{BC}^{p,q}$ is isomorphic to the Bott–Chern cohomology

$$H_{BC}^{p,q} := \frac{\ker \partial \cap \ker \bar\partial}{\mathrm{im}\, \partial\bar\partial}.$$

One consequence of this is that on complex manifolds $h_{\bar\partial}^{p,q}$ and $h_{BC}^{p,q}$ are both independent of the choice of Hermitian metric. However, when J is non-integrable it is no longer the case that $\bar\partial^2 = 0$ and so the cohomology groups are not well-defined. Indeed, in [6, 7] Zhang and myself proved that $h_{\bar\partial}^{0,1}$ can take different values on the same almost complex 4-manifold. In [13] Tardini and Tomassini proved the same was true for $h_{\bar\partial}^{1,1}$, although they were able to show that on compact almost Hermitian 4-manifolds $h_{\bar\partial}^{1,1}$ has a lower bound of b_-. In this paper we will give an upper bound of $b_- + 1$ and thereby obtain the result:

Theorem 3.1 *If (X, J, ω) is a compact almost Hermitian 4-manifold we have either $h_{\bar\partial}^{1,1} = b_-$ or $b_- + 1$.*

Following from Kodaira's classification of complex surfaces (see [1]), along with the work of Miyaoka [9], Todorov [14] and Siu [12], it is a widely-known result that when J is integrable on a compact 4-manifold, b_1 is even if and only if (M, J) is Kähler, i.e. if and only if (X, J) admits an Hermitian metric ω such that $d\omega = 0$. This result was later proven analytically by Buchdahl [2] and Lamari [8]. An equivalent result states that $h_{\bar\partial}^{1,1} = b_- + 1$ if and only if (X, J) is Kähler (see e.g. [4]). The Kähler criterion using b_1 does not appear to extend to non-integrable almost complex structures, however we still expect $h_{\bar\partial}^{1,1}$ to detect almost Kählerness (see Zhang's survey [16]). It was shown in [13], that when an almost Hermitian metric is *globally conformal to an almost Kähler metric* we have $h_{\bar\partial}^{1,1} = b_- + 1$ and when the metric is *strictly locally conformal to an almost Kähler metric* we have $h_{\bar\partial}^{1,1} = b_-$. We therefore ask the question

Question 3.3 *On a compact almost Hermitian 4-manifold, does the value of $h_{\bar\partial}^{1,1}$ give a full description of whether an almost Hermitian metric is conformally almost Kähler? Specifically, in the case when the metric is not locally conformally almost Kähler (and thus also not globally conformally almost Kähler) do we always have $h_{\bar\partial}^{1,1} = b_-$?* [1]

[1] In [11] Piovani and Tomassini give examples of non locally conformally almost Kähler structures on compact 4-manifolds, for which $h_{\bar\partial}^{1,1} = b_- + 1$, thereby answering this question in the negative.

Although the answer to this question is not yet known, we prove a similar result for the space of d-harmonic $(1, 1)$-forms, which we denote $h_d^{1,1}$.

Theorem 3.4 *On a compact almost Hermitian 4-manifold* (X, J, ω)*,* $h_d^{1,1} = b_- + 1$ *if* ω *is in the conformal class of an almost Kähler metric, otherwise* $h_d^{1,1} = b_-$.

At the end of Sect. 3 we calculate $h_{\bar{\partial}}^{1,1}$ for a family of almost Hermitian structures on a manifold M defined by $\Gamma \backslash Nil^4$, where Γ is a discrete subset acting on Nil^4 by left multiplication. We show that, at least for this example, Question 3.3 is answered positively.

In [10], Piovani and Tomassini prove that on a compact Hermitian 4-manifold $h_{BC}^{1,1}$ can only ever be b_- or $b_- + 1$. They ask whether both of these values can be attained by some choice of metric, a question we will answer in this paper with the following theorem.

Theorem 4.2 *Given any compact almost Hermitian 4-manifold* (X, J, ω)*, we have* $h_{BC}^{1,1} = b_- + 1$.

We also perform a calculation of $h_{BC}^{2,1}$ and $h_{BC}^{1,2}$ for a family of almost Hermitian structures on the Kodaira–Thurston manifold using the method developed in [6] for turning PDEs into a collection of ODE and number theory problems. From these results we conclude the following

Theorem 4.5 *On a compact almost Hermitian 4-manifold, when* $(p, q) = (0, 0)$*,* $(1, 0)$*,* $(0, 1)$*,* $(2, 0)$*,* $(1, 1)$*,* $(0, 2)$ *or* $(2, 2)$*,* $h_{BC}^{p,q}$ *is metric independent, but for* $(p, q) = (2, 1)$ *and* $(1, 2)$ *there exist examples for which* $h_{BC}^{p,q}$ *does vary with the metric.*

In addition to this, by building on some results of Chen and Zhang in [3] we will prove that $h_{BC}^{p,0} = h_{BC}^{0,p}$ are birational invariants for all values of p.

Theorem 4.6 *Let* $u : X \to Y$ *be a degree one pseudoholomorphic map between compact almost complex 4-manifolds. Then* $h_{BC}^{p,0}(X) = h_{BC}^{p,0}(Y)$ *for any* $p \in \{0, 1, 2\}$.

In the papers [6, 7], Weiyi Zhang and myself present a calculation of $h_{\bar{\partial}}^{0,1}$ on the Kodaira–Thurston manifold, achieved through the introduction of a method for decomposing smooth functions which proved useful for solving linear PDEs. The key idea was to view the manifold as a torus bundle over S^1, thereby allowing for a Fourier expansion on each fibre. Further information can then be gained by considering the behaviour of the Fourier coefficients when travelling around the base space. In the last two sections of this paper we will show how the techniques used to decompose functions on the Kodaira–Thurston manifold can be applied to any torus bundle M over S^1 given by $\mathbb{R}^{n+1}$ with points identified by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x} + \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t + \xi \\ A^{\xi}\mathbf{x} \end{pmatrix},$$

for all $\xi \in \mathbb{Z}$, $\in \mathbb{Z}^n$, with $A \in GL_n(\mathbb{Z})$. By partitioning $\mathbb{Z}^n$ into its orbits under the action of the group generated by A and separating the finite orbits from the infinite orbits, we produce the following decomposition of the space of L^2 functions on M. Here $\mathrm{Orb}_{\mathbf{y}}$ denotes the orbit containing the element $\mathbf{y} \in \mathbb{Z}^n$.

Theorem 5.8 *The space of* L^2 *functions on* M *decomposes in the following way.*

$$L^2(M) = \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = \infty}}} \mathcal{H}_{\mathbf{y}} \right) \oplus \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = N < \infty}}} \widehat{\bigoplus_{t_0 \in \mathbb{Z}}} \mathcal{H}_{t_0, \mathbf{y}} \right),$$

where

$$\mathcal{H}_{\mathbf{y}} = \left\{ \sum_{\xi \in \mathbb{Z}} f(t+\xi) e^{2\pi i \mathbf{y}\cdot A^{\xi}\mathbf{x}} \mid f \in L^2(\mathbb{R}) \right\}$$

and

$$\mathcal{H}_{t_0,\mathbf{y}} = \left\{ C e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y}\cdot A^{\xi}\mathbf{x} \right)} \mid C \in \mathbb{C} \right\}.$$

Here $\hat{\bigoplus}$ denotes the direct sum followed by the closure with respect to the L^2 norm.

Projection onto each of the components of this decomposition is described by the maps $\mathcal{F}_{\mathbf{y}} : L^2(M) \to L^2(\mathbb{R})$

$$\mathcal{F}_{\mathbf{y}}(f)(t) = \int_{[0,1]^n} f(t,\mathbf{x}) e^{-2\pi i \mathbf{y}\cdot\mathbf{x}} \, d\mathbf{x}$$

and $\mathcal{G}_{t_0,\mathbf{y}} : L^2(M) \to \mathbb{C}$

$$\mathcal{G}_{t_0,\mathbf{y}}(f) = \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(f)(t) e^{-\frac{2\pi i t_0 t}{N}} \, dt.$$

In [6, 7] this decomposition always leads to solving a combination of ODEs and lattice counting problems. In the last section we demonstrate how this decomposition could also reduce the PDEs deriving from the calculation of $h^{0,1}_{\bar{\partial}}$ to a recurrence relation problem.

2 Preliminary results

In this section we will recall some important facts about almost Hermitian manifolds which will be useful for proving the results of this paper in later sections. Let (X, J) be an almost complex manifold. The existence of the almost complex structure J induces a decomposition of the space of complex valued k-forms $\mathcal{A}^k_{\mathbb{C}}$ into spaces of (p,q)-forms

$$\mathcal{A}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}.$$

This in turn leads to a decomposition of the exterior derivative $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ into the sum of 4 components

$$d = \mu + \partial + \bar{\partial} + \bar{\mu},$$

which change the bidegree of a (p,q)-forms by $(+2,-1)$, $(+1,0)$, $(0,+1)$ and $(-1,+2)$ respectively. We say that the almost complex structure J is integrable when $\mu = \bar{\mu} = 0$, in which case it arises from a complex structure on X. Given an almost Hermitian metric we define the d, $\bar{\partial}$ and Bott–Chern laplacians by

$$\begin{aligned}
\Delta_d &= dd^* + d^*d, \\
\Delta_{\bar{\partial}} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \\
\Delta_{BC} &= \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\partial + \bar{\partial}^*\bar{\partial},
\end{aligned}$$

along with the spaces of harmonic forms $\mathcal{H}^k_d = \ker \Delta_d|_{\mathcal{A}^k}$, $\mathcal{H}^{p,q}_{\bar\partial} = \ker \Delta_{\bar\partial}|_{\mathcal{A}^{p,q}}$ and $\mathcal{H}^{p,q}_{BC} = \ker \Delta_{BC}|_{\mathcal{A}^{p,q}}$. Here we define the adjoints of ∂ and $\bar\partial$ to be $\partial^* = - * \bar\partial *$ and $\bar\partial^* = - * \partial *$ where $*$ denotes the Hodge star operator. The dimensions of the spaces of $\bar\partial$ and Bott–Chern harmonic (p, q)-forms are denoted by $h^{p,q}_{\bar\partial}$ and $h^{p,q}_{BC}$.

On a compact manifold the property of a general differential form s being d, $\bar\partial$ or Bott–Chern harmonic can be equated to a collection of conditions as follows

$$\Delta_d s = 0 \iff \begin{cases} ds = 0, \\ d * s = 0, \end{cases}$$

$$\Delta_{\bar\partial} s = 0 \iff \begin{cases} \bar\partial s = 0, \\ \partial * s = 0, \end{cases} \qquad \Delta_{BC} s = 0 \iff \begin{cases} \partial s = 0, \\ \bar\partial s = 0, \\ \partial\bar\partial * s = 0. \end{cases} \quad . \tag{1}$$

At this point it should be noted that the existence of an almost Hermitian metric g is equivalent to the existence of a compatible $(1, 1)$-form ω called the fundamental form, one being derived from the other by the formula $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Consequently in this paper, as in many others, we will often refer to ω as if it were the corresponding almost Hermitian metric. If on a compact almost Hermitian 4-manifold, ω is Gauduchon, i.e. $\partial\bar\partial\omega = 0$, then a result of Tardini and Tomassini [13] tells us that $\mathcal{H}^{1,1}_{\bar\partial}$ can be characterised by

$$\mathcal{H}^{1,1}_{\bar\partial} = \{a\omega + \gamma \mid a \in \mathbb{C}, *\gamma = -\gamma, i d^c\gamma = a d\omega\}, \tag{2}$$

whilst a result of Piovani and Tomassini [10] tells us that $\mathcal{H}^{1,1}_{BC}$ can be characterised by

$$\mathcal{H}^{1,1}_{BC} = \{a\omega - \gamma \mid a \in \mathbb{C}, *\gamma = -\gamma, d\gamma = a d\omega\}. \tag{3}$$

Here we define $d^c := J^{-1} d J$ with J acting on a (p, q)-form as multiplication by i^{p-q}.

For any two conformal metrics $\omega = f\tilde\omega$ on a 4-manifold, the two resulting Hodge stars differ by $*_\omega = f^{2-p-q} *_{\tilde\omega}$ when acting on a (p, q)-form. From (1) we can see that this means $\mathcal{H}^{1,1}_{\bar\partial}$ and $\mathcal{H}^{1,1}_{BC}$ are conformally invariant. Therefore, since a result of Gauduchon [5] states that every conformal class contains a Gauduchon metric, we can apply (2) and (3) given any almost Hermitian metric by finding the Gauduchon metric to which it is conformal.

For any almost Hermitian metric ω we have the property that

$$d\omega = \alpha \wedge \omega$$

for some 1-form α. This comes as a consequence of the well known fact that the map $L^k : \mathcal{A}^{n-k} \to \mathcal{A}^{n+k}$ given by $s \mapsto s \wedge \omega^k$ is bijective. Furthermore, α is an exact form if and only if ω is *globally conformal to an almost Kähler metric*, while α is closed if and only if ω is *locally conformal to an almost Kähler metric*.

3 $\bar\partial$-harmonic (1, 1)-forms

From the characterisation (2) of $\mathcal{H}^{1,1}_{\bar\partial}$ we conclude the following.

Theorem 3.1 *If (X, J, ω) is a compact almost Hermitian 4-manifold we have either $h^{1,1}_{\bar\partial} = b_-$ or $b_- + 1$.*

Proof Since $h^{1,1}_{\bar\partial}$ is a conformal invariant we assume without loss of generality that ω is a Gauduchon metric.

From (1) we obtain the inclusion

$$\mathcal{H}_g^- \subseteq \mathcal{H}^{1,1}_{\bar\partial},$$

where $\mathcal{H}_g^-$ denotes the space of d-harmonic anti-self-dual (1, 1)-forms. When this inclusion is an equality then clearly we have $h^{1,1}_{\bar\partial} = b_-$. Suppose instead that $\mathcal{H}^{1,1}_{\bar\partial}$ has some element $a_0\omega + \gamma_0$ which is not in $\mathcal{H}_g^-$. Here a_0 is a constant and γ_0 is an anti-self-dual form satisfying $id^c\gamma_0 = a_0 d\omega$. Note that a_0 cannot be zero, as that would leave us with a d-harmonic anti-self-dual form. A general element of $\mathcal{H}^{1,1}_{\bar\partial}$ given by $a\omega + \gamma$ can then be rewritten as an element of $\mathcal{H}_g^-$ plus a multiple of the single additional element $a_0\omega + \gamma_0$

$$a\omega + \gamma = \frac{a}{a_0}\left(a_0\omega + \gamma_0\right) + \frac{1}{a_0}\left(a_0\gamma - a\gamma_0\right),$$

thus giving us $h^{1,1}_{\bar\partial} = b_- + 1$.

To see that the anti-self-dual form $a_0\gamma - a\gamma_0$ is indeed d-harmonic, first note that $d^c(a_0\gamma - a\gamma_0) = a_0 d^c\gamma - a d^c\gamma_0 = 0$. Then, since $d^c = J^{-1}dJ$ and J is the identity when acting on (1, 1)-forms, it follows that $d(a_0\gamma - a\gamma_0) = 0$. As our form is anti-self-dual we therefore also have $d * (a_0\gamma - a\gamma_0) = 0$. □

Corollary 3.2 *If (X, J, ω) is a compact almost Hermitian 4-manifold where we assume ω is Gauduchon, then $h^{1,1}_{\bar\partial} = b_- + 1$ if and only if there exists an anti-self-dual $(1, 1)$-form γ satisfying the equation*

$$id^c\gamma = d\omega. \tag{4}$$

Proof If such a γ exists then $\omega + \gamma$ is $\bar\partial$-harmonic, along with b_- many linearly independent elements of $\mathcal{H}_g^-$, therefore $h^{1,1}_{\bar\partial} = b_- + 1$.

Conversely, if $h^{1,1}_{\bar\partial} = b_- + 1$, then there must be some form in $\mathcal{H}^{1,1}_{\bar\partial}$ other than those contained in $\mathcal{H}_g^-$, i.e. a form which can be written as $a_0\omega + \gamma_0$ with $a_0 \neq 0$ such that $id^c\gamma_0 = a_0 d\omega$. Thus $\gamma = \frac{1}{a_0}\gamma_0$ gives us the desired solution. □

In [4], Draghici, Li and Zhang prove that, for integrable almost complex manifolds (X, J), $h^{1,1}_{\bar\partial}$ takes the value $b_- + 1$ when (X, J) is Kähler and otherwise takes the value b_-. Partially extending this result to non-integrable manifolds, in [13] it was proven that if a compact almost Hermitian 4-manifold (X, J, ω) is *globally conformally almost Kähler* then $h^{1,1}_{\bar\partial} = b_- + 1$, whereas if (X, J, ω) is *strictly locally conformally almost Kähler* then $h^{1,1}_{\bar\partial} = b_-$. We therefore ask the question:

Question 3.3 *On a compact almost Hermitian 4-manifold, does the value of $h^{1,1}_{\bar\partial}$ give a full description of whether an almost Hermitian metric is conformally almost Kähler? Specifically, in the case when the metric is not locally conformally almost Kähler (and thus also not globally conformally almost Kähler) do we have $h^{1,1}_{\bar\partial} = b_-$?*

Although the answer to this is not known, we can prove a similar result for the dimension of the space of d-harmonic (1, 1)-forms, which we will denote by $h^{1,1}_d$.

Theorem 3.4 *On a compact almost Hermitian 4-manifold (X, J, ω), $h^{1,1}_d = b_- + 1$ if ω is in the conformal class of an almost Kähler metric, otherwise $h^{1,1}_d = b_-$.*

Proof As in the proof of the previous theorem, we use the fact that $h_d^{1,1}$ is a conformal invariant and thereby assume ω is a Gauduchon metric. Furthermore, all almost Kähler metrics are Gauduchon, so the conformal class of ω contains an almost Kähler metric if and only if ω is almost Kähler itself.

On compact manifolds we know a differential form s is d-harmonic if and only if

$$ds = 0, \qquad d * s = 0.$$

From this we can see that the Hodge star maps d-harmonic forms to d-harmonic forms, meaning that if some $(1,1)$-form s is in $\mathcal{H}_d^{1,1}$ so too are its self-dual and anti-self-dual components, $\frac{1}{2}(s + *s)$ and $\frac{1}{2}(s - *s)$. Furthermore, we have the inclusion

$$\mathcal{H}_d^{1,1} \subseteq \mathcal{H}_{\bar\partial}^{1,1}$$

and so from (2) we know we can write any d-harmonic $(1,1)$-form as $a\omega + \gamma$ with $a \in \mathbb{C}$ a constant and γ an anti-self-dual form. But the self-dual component of this is only harmonic if $d\omega = 0$ or $a = 0$ and so either ω is almost Kähler and we have $h_d^{1,1} = b_- + 1$ or all d-harmonic $(1,1)$-forms are anti-self-dual and we have $h_d^{1,1} = b_-$. □

From this result we see that the above question is equivalent to asking whether $h_d^{1,1}$ and $h_{\bar\partial}^{1,1}$ are always equal on compact Hermitian 4-manifolds.

We conclude this section with a calculation of $h_{\bar\partial}^{1,1}$ for a large family of almost complex structures and compatible metrics. In doing so we will see that, at least for this family of almost Hermitian structures, Question 3.3 has a positive answer.

Example 3.5 Let M be a compact manifold, given by identifying the points in $\mathbb{R}^4$ by the equivalence relations

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t \\ x + x_0 \\ y + y_0 \\ z + z_0 \end{pmatrix} \qquad \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t + t_0 \\ x + t_0 y + \frac{1}{2} t_0^2 z \\ y + t_0 z \\ z \end{pmatrix}$$

for all $t_0, x_0, y_0, z_0 \in \mathbb{Z}$. This is equivalent to the group Nil^4 modulo the discrete subgroup of elements with integer valued entries.

M has a smooth global frame given by

$$\epsilon_1 = \frac{\partial}{\partial t}, \qquad \epsilon_2 = \frac{\partial}{\partial x}, \qquad \epsilon_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \epsilon_4 = \frac{1}{2}t^2\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

along with the dual frame

$$\epsilon^1 = dt, \qquad \epsilon^2 = dx - t\,dy + \frac{1}{2}t^2 dz, \qquad \epsilon^3 = dy - t\,dz, \qquad \epsilon^4 = dz.$$

We can then define an almost complex structure J, such that $J\epsilon_1 = \epsilon_2$ and $J\epsilon_3 = \epsilon_4$. A pair of vector fields, spanning $T_p^{1,0}M$ at every point $p \in M$, can then be defined by

$$V_1 = \frac{1}{2}(\epsilon_0 - i\epsilon_1) \qquad \& \qquad V_2 = \frac{1}{2}(\epsilon_2 - i\epsilon_4),$$

along with their dual $(1,0)$-forms, given by

$$\phi^1 = \epsilon^1 + i\epsilon^2 \qquad \& \qquad \phi^2 = \epsilon^3 + i\epsilon^4.$$

These forms satisfy the structure equations

$$\begin{aligned} d\phi^1 &= -i\,\epsilon^1 \wedge \epsilon^3 \\ &= -\frac{i}{4}\left(\phi^{12} + \phi^{1\bar{2}} - \phi^{2\bar{1}} + \phi^{\bar{1}\bar{2}}\right), \end{aligned}$$

$$\begin{aligned} d\phi^2 &= -\epsilon^1 \wedge \epsilon^4 \\ &= \frac{i}{4}\left(\phi^{12} - \phi^{1\bar{2}} - \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right), \end{aligned}$$

with $\phi^{i\bar{j}}$ used here as shorthand for $\phi^1 \wedge \bar{\phi}^2$. From this we can see that J is a non-integrable almost complex structure, namely we have $\bar{\mu}\phi^1 = \bar{\mu}\phi^2 = -\frac{i}{4}\phi^{\bar{1}\bar{2}} \neq 0$.

Now it only remains to choose a family of almost Hermitian metrics

$$\omega_\lambda = i\left((1+\lambda^2)\phi^{1\bar{1}} - \lambda\phi^{1\bar{2}} - \lambda\phi^{2\bar{1}} + \phi^{2\bar{2}}\right)$$

varying over some $\lambda \in \mathbb{R}$, defined such that $V_1 + \lambda V_2$ and V_2 form a unitary basis on $T_p^{1,0}M$. Using the structure equations we can calculate

$$d\omega_\lambda = \frac{\lambda}{2}\left(\phi^{12\bar{1}} + \phi^{1\bar{1}\bar{2}}\right),$$

from which we can see firstly that ω_λ is an almost Kähler metric if and only if $\lambda = 0$ and secondly that $\partial\bar{\partial}\omega_\lambda = 0$, and thus ω_λ is Gauduchon for all λ.

Furthermore, we can write

$$d\omega_\lambda = \alpha_\lambda \wedge \omega_\lambda$$

with

$$\begin{aligned} \alpha_\lambda &= -\frac{\lambda}{2} i\left(\lambda\phi^1 - \phi^2 - \lambda\bar{\phi^1} + \bar{\phi^2}\right) \\ d\alpha_\lambda &= -\frac{\lambda^2}{8}\left(\phi^{12} + \phi^{1\bar{2}} - \phi^{2\bar{1}} + \phi^{\bar{1}\bar{2}}\right). \end{aligned}$$

ω_λ is therefore neither *globally* nor *locally conformally almost Kähler* except when $\lambda = 0$.

Finding $h^{1,1}_{\bar{\partial}}$ then amounts to asking whether there exists an anti-self-dual γ solving

$$i d^c\gamma = d\omega_\lambda.$$

Since J is the identity on (1, 1)-forms, this is equivalent to

$$i J^{-1} d\gamma = d\omega_\lambda.$$

If such a γ exists that would mean

$$\begin{aligned} J d\omega_\lambda &= \frac{i\lambda}{2}\left(\phi^{1\bar{1}\bar{2}} - \phi^{12\bar{1}}\right) \\ &= 2\lambda\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \\ &= 2\lambda dt \wedge (dx \wedge dy - t dx \wedge dz + \frac{1}{2}t^2 dy \wedge dz) \end{aligned}$$

is an exact 3-form. But consider the closed submanifold given by $z = 0$. The pullback onto this submanifold is $2\lambda dt \wedge dx \wedge dy$, which by Stokes' theorem cannot be exact, since its integral over the submanifold is non-zero, the only exception to this being when $\lambda = 0$.

Thus, in all the cases when ω_λ is not *globally almost Kähler*, there is no solution to (4) and so $h^{1,1}_{\bar{\partial}} = b_-$.

4 Bott–Chern harmonic forms

In this section we will give a collection of results which together will give a full description of when $h^{p,q}_{BC}$ is or is not metric independent for compact 4-manifolds.

For many values of (p,q) proving the metric invariance of $h^{p,q}_{BC}$ is a relatively trivial affair and so we will not spend too long on these cases.

Lemma 4.1 *On any compact almost Hermitian 4-manifold $h^{p,q}_{BC}$ is metric independent when (p,q) is equal to $(2,0)$, $(0,2)$, $(1,0)$, $(0,1)$, $(0,0)$ or $(2,2)$.*

Proof Bott–Chern harmonic $(0,0)$-forms are always just the constant functions, since Δ_{BC} is elliptic. Similarly Bott–Chern harmonic $(2,2)$-forms are just constant functions times the volume form so although $\mathcal{H}^{2,2}_{BC}$ might change with the metric, $h^{2,2}_{BC}$ does not.

For the remaining cases recall that a (p,q)-form s is Bott–Chern harmonic if and only if it satisfies the three conditions

$$\partial s = 0, \qquad \bar{\partial} s = 0, \qquad \partial\bar{\partial} * s = 0.$$

When $(p,q) = (2,0)$, $(0,2)$, $(1,0)$ or $(0,1)$ the third condition is always true leaving behind the first two conditions which do not depend on the metric. □

The more interesting cases are those when $(p,q) = (1,1)$, $(2,1)$ and $(1,2)$. We start with the case of $\mathcal{H}^{1,1}_{BC}$. From the characterisation (3) of $\mathcal{H}^{1,1}_{BC}$, in [10] it is deduced that $h^{1,1}_{BC}$ is either $b_- + 1$ or b_-, with the two cases corresponding, respectively, to the existence or non-existence of an anti-self-dual solution γ to the equation

$$d\gamma = d\omega. \tag{5}$$

Here ω is a Gauduchon metric conformal to the chosen Hermitian metric.

It turns out that solutions to the above equation can be found by making use of the Hodge decomposition

$$\mathcal{A}^k = d\mathcal{A}^{k-1} \oplus \mathcal{H}^k_d \oplus d^*\mathcal{A}^{k+1}.$$

Theorem 4.2 *Given any compact almost Hermitian 4-manifold (X, J, ω), we have $h^{1,1}_{BC} = b_- + 1$.*

Proof From the conformal invariance of $h^{1,1}_{BC}$ we may assume without losing generality that ω is Gauduchon. Then taking the Hodge decomposition we can write

$$\omega = d\alpha + h + d^*\beta$$

for some $\alpha \in \mathcal{A}^1$, $h \in \mathcal{H}^2_d$ and $\beta \in \mathcal{A}^3$. By defining a 2-form

$$\gamma = d * \beta + d^*\beta$$

we have

$$d\omega = dd^*\beta = d\gamma$$

and thus γ is a solution to (5).

It only remains to show that γ is anti-self-dual. Using the definition of d^* along with the fact that the square of the Hodge star when applied to a k-form is given by $*^2 = (-1)^k$, we can see that

$$\begin{aligned} *\gamma &= *d*\beta - **d*\beta \\ &= -d^*\beta - d*\beta \\ &= -\gamma. \end{aligned}$$

We therefore find that

$$\mathcal{H}^{1,1}_{BC} = \mathcal{H}^-_g \oplus \mathbb{C}\langle \omega - \gamma \rangle$$

and so $h^{1,1}_{BC}$ is always $b_- + 1$. □

We will now use the following example to show that $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ may, in general, depend on the choice of almost Hermitian metric.

4.1 Calculating $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ on the Kodaira–Thurston manifold

Briefly we recall the definition of the Kodaira–Thurston manifold $KT^4 = \Gamma \backslash G$ as the group $G = \mathbb{R} \times Nil^3$ modulo the subgroup Γ of elements with integer valued entries, acting on G by left multiplication. This is equivalent to $\mathbb{R}^4$ with points identified by the equivalence relation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t + t_0 \\ x + x_0 \\ y + y_0 \\ z + z_0 + t_0 y \end{pmatrix},$$

for all $t_0, x_0, y_0, z_0 \in \mathbb{Z}$.

$\mathbb{R} \times \mathrm{Nil}^3$ has a smooth global frame given by

$$\frac{\partial}{\partial t} \qquad \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y} + t\frac{\partial}{\partial z} \qquad \frac{\partial}{\partial z}$$

which descends to a global frame for KT^4 since all of the above vector fields are invariant under the action of Γ. We can define an almost complex structure acting on this frame by the matrix

$$J_{a,b} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{pmatrix},$$

with $a, b \in \mathbb{R}$, $b \neq 0$ and $c = -\frac{a^2+1}{b}$. A pair of vector fields

$$V_1 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial t}\right) \quad \& \quad V_2 = \frac{1}{2}\left(\left(\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right) - \frac{a-i}{b}\frac{\partial}{\partial z}\right)$$

can then be defined, spanning $T^{1,0}_p KT^4$ at every point $p \in KT^4$. Their dual $(1,0)$-forms are given by

$$\phi_1 = dx + i dt \quad \& \quad \phi_2 = (1 - ai)dy - ib(dz - t dy).$$

These forms satisfy the structure equations

$$d\phi^1 = 0, \qquad d\phi^2 = \frac{b}{4}\left(\phi^{12} + \phi^{1\bar2} + \phi^{2\bar1} - \phi^{\bar1\bar2}\right).$$

We can see that $J_{a,b}$ is a non-integrable almost complex structure, for all values of a and b, by noting that $\bar\mu\phi^2 = -\frac{b}{4}\phi^{\bar1\bar2} \neq 0$.

In the following examples, the metric we will be using is given by

$$\omega_\rho = i\left(\phi^{1\bar1} + \rho\phi^{2\bar2}\right),$$

for which V^1, $\bar V^1$, $\frac{1}{\sqrt\rho}V^2$ and $\frac{1}{\sqrt\rho}\bar V^2$ form an orthonormal basis. This is essentially the same metric as was used in [7] and in fact what follows is a more general, completed version of a calculation in [10].

Example 4.3 Let a general (2, 1)-form be given by $f\phi^{12\bar1} + g\phi^{12\bar2}$. Then from the conditions $\bar\partial s = 0$ and $\partial\bar\partial * s = 0$ we see that $s \in \mathcal{H}^{2,1}_{BC}$ if and only if the following PDEs hold.

$$\begin{cases} \rho V_1\bar V_1(f) + V_2\bar V_1(g) - \frac{b}{4}\rho V_1(f) + \frac{b}{4}\rho\bar V_1(f) - \frac{b}{4}\bar V_2(g) - \frac{b^2}{8}\rho f = 0, \\ \rho V_1\bar V_2(f) + V_2\bar V_2(g) + \frac{b}{4}\rho V_2(f) = 0, \\ \bar V_1(g) - \bar V_2(f) = 0. \end{cases} \tag{6}$$

Using the same method as in [7] we can perform a Fourier expansion with respect to x, y and z to simplify the above equations. We will write

$$f(t,x,y,z) = \sum_{l,m,n} \mathcal{F}_{l,m,n}(f)(t)e^{2\pi i(lx+my+nz)}$$

where

$$\mathcal{F}_{l,m,n}(f)(t) = \int_{[0,1]^3} f(t,x,y,z)e^{-2\pi i(lx+my+nz)}\,dx\,dy\,dz.$$

Applying a Fourier expansion to the second and third PDEs we obtain the ODE system

$$\frac{d}{dt}\begin{pmatrix}\mathcal{F}_{l,m,n}(f)\\ \mathcal{F}_{l,m,n}(g)\end{pmatrix} = 2\pi\left[\begin{pmatrix}0 & \frac{n}{\rho}\\ n & 0\end{pmatrix}t + \begin{pmatrix} l - \frac{b}{4\pi}i & \frac{1}{\rho}\left(m - n\frac{a-i}{b}\right) \\ m - n\frac{a+i}{b} & -l\end{pmatrix}\right]\begin{pmatrix}\mathcal{F}_{l,m,n}(f)\\ \mathcal{F}_{l,m,n}(g)\end{pmatrix} \tag{7}$$

for every $l, m, n \in \mathbb{Z}$. The ODE given by expanding our first PDE can be derived from the above ODE system and so adds no new information.

As was proven in [7], the solutions to (6) can be split into two cases:

Firstly if two smooth functions $\mathcal{F}_{l,m,n}(f), \mathcal{F}_{l,m,n}(g) \in C^\infty(\mathbb{R})$ satisfy the ODE (7) with $n \neq 0$ and $0 \le m < |n|$ then we have a solution to (6) given by

$$f = \sum_{\xi\in\mathbb{Z}} \mathcal{F}_{l,m,n}(f)(t+\xi)e^{2\pi i(lx+(m+n\xi)y+nz)}$$

$$g = \sum_{\xi\in\mathbb{Z}} \mathcal{F}_{l,m,n}(g)(t+\xi)e^{2\pi i(lx+(m+n\xi)y+nz)}$$

if and only if the two functions $\mathcal{F}_{l,m,n}(f)$ and $\mathcal{F}_{l,m,n}(g)$ are Schwartz.

Secondly, if $\mathcal{F}_{l,m,n}(f), \mathcal{F}_{l,m,n}(g) \in C^\infty(\mathbb{R})$ satisfy the ODE (7) with $n = 0$ then we have a solution to (6) given by

$$f = \mathcal{F}_{l,m,0}(f)e^{2\pi i(lx+my)}$$

$$g = \mathcal{F}_{l,m,0}(g)e^{2\pi i(lx+my)}$$

if and only if the two functions $\mathcal{F}_{l,m,n}(f)$ and $\mathcal{F}_{l,m,n}(g)$ are periodic with a period of 1.

Finding solutions in the first case amounts to solving a Stokes phenomenon problem. This can be tricky to do in general, but this problem has been solved for the ODE (7) in Theorem 3.1 of [6]. It turns out we have a solution for all $0 \leq m < |n|$ whenever $l = 0$ and n satisfies

$$64\pi^2 n^2 - 64\pi n u b^2 \sqrt{\rho} - b^4 \rho = 0$$

for some negative integer u. Or equivalently, if we set $d = \frac{b}{8\pi}$,

$$n^2 - 64\pi n u d^2 \sqrt{\rho} - 64\pi^2 d^4 \rho = 0.$$

Note that if d and ρ are both rational this case gives us no solutions as π is transcendental.

For the second case, since we are working with periodic functions, we can take another Fourier expansion with respect to t, writing

$$\mathcal{G}_{k,l,m,0}(f) = \int_0^1 \mathcal{F}_{l,m,0}(f)(t)e^{-2\pi i k t}dt.$$

Applying this expansion to (7) we obtain the equations

$$\rho\left(l - ik - \frac{b}{4\pi}i\right)\mathcal{G}_{k,l,m,0}(f) + m\mathcal{G}_{k,l,m,0}(g) = 0,$$
$$m\mathcal{G}_{k,l,m,0}(f) = (l + ik)\mathcal{G}_{k,l,m,0}(g).$$

This can be solved directly to find the solution

$$s = \phi^{12\bar{2}}$$

when $k = 0$, and the solution

$$s = ike^{2\pi i(kt+my)}\phi^{12\bar{1}} + me^{2\pi i(kt+my)}\phi^{12\bar{2}}$$

when $k \neq 0$ and $k, m \in \mathbb{Z}$ satisfy

$$\frac{m^2}{\rho} + (k + d)^2 = d^2.$$

Here we again set $d = \frac{b}{8\pi}$. Notice that when $d = 1$ and $\rho = 1$ we have 4 solutions given by $(k, m) = (-1, 1), (-1, -1)(-2, 0)$ and $(0, 0)$, however when we take $\rho = \frac{1}{2}$, leaving d unchanged, we only have the two solutions $(k, m) = (-1, 0)$ and $(0, 0)$. Therefore we conclude that on the Kodaira–Thurston manifold the value of $h^{2,1}_{BC}$ may depend on the choice of almost Hermitian metric.

Example 4.4 Now let a general (1, 2)-form be given by $f\phi^{1\bar{1}\bar{2}} + g\phi^{2\bar{1}\bar{2}}$. Then from the conditions $\partial s = 0$ and $\partial\bar{\partial} * s = 0$ we see that $s \in \mathcal{H}^{1,2}_{BC}$ if and only if the following PDEs hold.

$$\begin{cases} \rho V_1\bar{V}_1(f) + V_1\bar{V}_2(g) + \frac{b}{4}\rho V_1(f) - \frac{b}{4}\rho\bar{V}_1(f) - \frac{b}{4}\bar{V}_2(g) - \frac{b^2}{16}\rho f = 0, \\ \rho V_2\bar{V}_1(f) + V_2\bar{V}_2(g) + \frac{b}{4}\rho V_2(f) = 0, \\ V_1(g) - V_2(f) = 0. \end{cases} \tag{8}$$

Applying the same Fourier expansion as before, the second and third equations give us the ODE system

$$\frac{d}{dt}\begin{pmatrix}\mathcal{F}_{l,m,n}(f)\\ \mathcal{F}_{l,m,n}(g)\end{pmatrix} = 2\pi\left[\begin{pmatrix}0 & \frac{n}{\rho}\\ n & 0\end{pmatrix}t + \begin{pmatrix}-l+\frac{b}{4\pi}i & -\frac{1}{\rho}\left(m-n\frac{a+i}{b}\right)\\ -m+n\frac{a-i}{b} & l\end{pmatrix}\right]\begin{pmatrix}\mathcal{F}_{l,m,n}(f)\\ \mathcal{F}_{l,m,n}(g)\end{pmatrix}.$$

Again splitting the solutions into two cases we find that firstly we have a solution for all $n \neq 0$ and $0 \leq m < |n|$ which satisfy

$$n^2 - 64\pi nud^2\sqrt{\rho} - 64\pi^2 d^4\rho = 0$$

for some negative integer u. Secondly, for $n = 0$ we have solutions

$$s = \phi^{2\bar{1}\bar{2}}$$

and

$$s = ike^{2\pi i(kt+my)}\phi^{1\bar{1}\bar{2}} - me^{2\pi i(kt+my)}\phi^{2\bar{1}\bar{2}}$$

for all $k, m \in \mathbb{Z}$, with $k \neq 0$, satisfying

$$\frac{m^2}{\rho} + (k-d)^2 = d^2.$$

From the above we see that for this family of almost Hermitian structures we have $h^{1,2}_{BC} = h^{2,1}_{BC}$ (although this need not always be the case). Thus the value of $h^{1,2}_{BC}$ may also depend on the choice of almost Hermitian metric.

Furthermore, when $\rho = 1$, the calculation of Theorem 4.1 in [6] tells us that $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ here are both equal to $h^{0,1}_{\bar{\partial}}$ defined using the same family of almost complex structures $J_{a,b}$. In particular, $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ can both be made arbitrarily large by varying the value of b.

We can now bring the results of this section together into the following theorem.

Theorem 4.5 *On a compact almost Hermitian 4-manifold, when* $(p, q) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)$ *or* $(2, 2)$, $h^{p,q}_{BC}$ *is metric independent, but for* $(p, q) = (2, 1)$ *and* $(1, 2)$ *there exist examples for which* $h^{p,q}_{BC}$ *does vary with the metric.*

4.2 Birational invariance of $h^{p,0}_{BC}$

It is known from Theorem 5.5 in [3] that $h^{p,0}_{\bar{\partial}}$ is birationally invariant on compact 4-manifolds for any $p \in \{0, 1, 2\}$. This means that if we have a sequence of almost complex 4-manifolds $X_0, X_1, X_2 \ldots, X_{k+1}$ along with a sequence of degree one pseudoholomorphic maps $u_0, \ldots u_k$ such that $u_{2i-1} : X_{2i-1} \to X_{2i}$ and $u_{2i} : X_{2i+1} \to X_{2i}$ then $h^{p,0}_{\bar{\partial}}(X) = h^{p,0}_{\bar{\partial}}(Y)$. It turns out this result can be extended to show that the numbers $h^{p,0}_{BC}$ are also birational invariants.

Theorem 4.6 *Let* $u : X \to Y$ *be a degree one pseudoholomorphic map between compact almost complex 4-manifolds. Then* $h^{p,0}_{BC}(X) = h^{p,0}_{BC}(Y)$ *for any* $p \in \{0, 1, 2\}$.

Proof From [3] we know that the pullback with respect to u describes a bijection

$$u^* : \mathcal{H}^{p,0}_{\bar{\partial}}(Y) \to \mathcal{H}^{p,0}_{\bar{\partial}}(X).$$

Restricting this to the forms $s \in \mathcal{H}^{p,0}_{\bar\partial}(Y)$ which satisfy $\partial s = 0$ gives us

$$u^* : \mathcal{H}^{p,0}_{BC}(Y) \to \mathcal{H}^{p,0}_{BC}(X).$$

The injectivity of this map follows directly from the injectivity of u^* acting on $\mathcal{H}^{p,0}_{\bar\partial}(Y)$, so it only remains to prove surjectivity.

Since u^* is invertible when acting on $\mathcal{H}^{p,0}_{\bar\partial}(Y)$ we know that for any $s \in \mathcal{H}^{p,0}_{BC}(X)$ there is some $t \in \mathcal{H}^{p,0}_{\bar\partial}(Y)$ such that $u^* t = s$. By Theorem 1.5 in [15] we know there is a finite set $Y_1 \subset Y$ such that the restriction

$$u : X \backslash u^{-1}(Y_1) \to Y \backslash Y_1$$

is a diffeomorphism. This means we have

$$t\big|_{X\backslash u^{-1}(Y_1)} = (u^{-1})^* s\big|_{Y\backslash Y_1}$$

and so $\partial t = 0$ on $Y \backslash Y_1$. But since t is smooth and $\overline{Y\backslash Y_1} = Y$, we must have $\partial t = 0$ on all of Y, thus $t \in \mathcal{H}^{p,0}_{BC}(Y)$ and $u^*\big|_{\mathcal{H}^{p,0}_{BC}(Y)}$ is surjective. □

Corollary 4.7 *$h^{0,p}_{BC}$ is a birational invariant on compact almost complex 4-manifolds for any $p = 0, 1$ or 2.*

Proof Recall that $s \in \mathcal{H}^{p,q}_{BC}$ if and only if the following conditions hold

$$\bar\partial s = 0 \qquad \partial s = 0 \qquad \partial\bar\partial * s = 0.$$

If s is either a $(p, 0)$-form or a $(0, p)$-form for any $p = 0, 1$ or 2 then the third condition is always true for reasons of bidegree. The remaining two conditions, when taken together, are unchanged by a conjugation of s. The corollary therefore follows simply from the fact that $\mathcal{H}^{0,p}_{BC} = \overline{\mathcal{H}^{p,0}_{BC}}$. □

5 Harmonic analysis on Torus bundles over S^1

In this section we introduce a technique which may be used to simplify or solve certain linear PDEs on torus bundles over S^1. Special cases of this technique have already proven useful in the calculation of $h^{p,q}_{\bar\partial}$ on the Kodaira–Thurston manifold [6]. We will start by first describing a decomposition of smooth functions. Then, by considering a specific example of calculating $h^{0,1}_{\bar\partial}$ on a torus bundle with Euclidean geometry, we will see how PDEs can be simplified through the application of this decomposition. In our example it will simplify to a recurrence relation.

5.1 Decomposition of functions

Let M be any n-torus bundle over S^1. This can be described as the mapping torus of an n-torus determined by a matrix $A \in GL_n(\mathbb{Z})$. In other words, M is given by $\mathbb{R}^{n+1}$ with points identified by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x} + \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t + \xi \\ A^{\xi}\mathbf{x} \end{pmatrix} \tag{9}$$

for all $\xi \in \mathbb{Z}, \in \mathbb{Z}^n$.

When t is fixed, $\mathbf{x}$ describes a point on a torus. This means any smooth function $f \in C^\infty(M)$, when viewed as a function on $\mathbb{R}^{n+1}$ satisfying

$$f(t, \mathbf{x}) = f(t, \mathbf{x} +) \quad \text{and} \quad f(t, \mathbf{x}) = f(t + \xi, A^\xi \mathbf{x}) \tag{10}$$

can be decomposed into the Fourier series

$$f(t, \mathbf{x}) = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$$

where we define

$$\mathcal{F}_{\mathbf{x}_0}(f)(t) = \int_{[0,1]^n} f(t, \mathbf{x}) e^{-2\pi i \mathbf{x}_0 \cdot \mathbf{x}} \, d\mathbf{x}.$$

Here we have to be careful: notice that we have no guarantee that the summands $\mathcal{F}_{\mathbf{x}_0}(f) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$ will satisfy the same condition (10) as f, and so the summands are not themselves smooth functions on M. In particular, it is the second condition of (10) that may fail. We do however have the following result.

Proposition 5.1 *A function $f \in C^\infty(\mathbb{R}^{n+1})$ satisfies (10) if and only if it can be written as the Fourier series*

$$f(t, \mathbf{x}) = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$$

such that

$$\mathcal{F}_{(A^T)^\xi \mathbf{x}_0}(f)(t) = \mathcal{F}_{\mathbf{x}_0}(f)(t + \xi)$$

for all $\xi \in \mathbb{Z}$.

Proof It is clear that f has a Fourier expansion if and only if it satisfies the first condition of (10). Taking the expansion of the second condition we see that

$$\sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t + \xi) e^{2\pi i \mathbf{x}_0 \cdot A^\xi \mathbf{x}}$$

or equivalently

$$\sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{(A^T)^{-\xi} \mathbf{x}_0}(f)(t + \xi) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}.$$

By the uniqueness of Fourier coefficients, this is identical to requiring

$$\mathcal{F}_{(A^T)^\xi \mathbf{x}_0}(t) = \mathcal{F}_{\mathbf{x}_0}(f)(t + \xi).$$

□

This proposition suggests that by grouping together terms in the expansion, we may obtain a decomposition of f into smooth functions on M.

Definition 5.2 Let $\mathrm{Orb}_\mathbf{y}$ denote the orbit of the point $\mathbf{y} \in \mathbb{Z}^n$ being acted on by the group generated by the transpose matrix A^T. That is to say we have

$$\mathrm{Orb}_\mathbf{y} = \{(A^T)^\xi \mathbf{y} \mid \xi \in \mathbb{Z}\}.$$

We use these orbits to partition $\mathbb{Z}^n$ and define $\mathcal{O}$ to be the set of all such orbits.

Proposition 5.3 *Any $f \in C^\infty(M)$ can be written as the series*

$$\sum_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}|=\infty}} \left(\sum_{\xi \in \mathbb{Z}} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) + \sum_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}|=N<\infty}} \left(\sum_{\xi=0}^{N-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right)$$

and we have

$$\left(\sum_{\xi \in \mathbb{Z}} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) \in C^\infty(M)$$

$$\left(\sum_{\xi=0}^{N-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) \in C^\infty(M)$$

in the cases where $\mathbf{y} \in \mathbb{Z}^n$ satisfies $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, respectively $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$.

Proof By partitioning $\mathbb{Z}^n$ into the orbits $\mathrm{Orb}_{\mathbf{y}}$ we can write

$$\sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O}} \sum_{\mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}.$$

Then by Proposition 5.1, if we have $\mathbf{x}_0 = (A^T)^\xi \mathbf{y}$ for some $\xi \in \mathbb{Z}$, then we can write

$$\mathcal{F}_{\mathbf{x}_0}(f)(t) = \mathcal{F}_{\mathbf{y}}(f)(t+\xi)$$

and thus

$$\sum_{\mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\xi} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}}$$

with ξ ranging over different values depending on the size of $\mathrm{Orb}_{\mathbf{y}}$. □

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = N$ for some $N < \infty$ the function $\mathcal{F}_{\mathbf{y}}(f)$ is periodic with period N, and so we can further decompose it as follows

Proposition 5.4 *Given $f \in C^\infty(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$, we can write*

$$\mathcal{F}_{\mathbf{y}}(f)(t) = \sum_{t_0 \in \mathbb{Z}} \mathcal{G}_{t_0,\mathbf{y}}(f) e^{\frac{2\pi i t_0 t}{N}}$$

where $\mathcal{G}_{t_0,\mathbf{y}} \in \mathbb{C}$ is defined by

$$\mathcal{G}_{t_0,\mathbf{y}}(f) = \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(f)(t) e^{-\frac{2\pi i t_0 t}{N}} \, dt.$$

Proof This is simply the Fourier expansion of the periodic function $\mathcal{F}_{\mathbf{y}}(f)(t)$. □

Corollary 5.5 *In the decomposition of f in Proposition 5.3, the summand*

$$\left(\sum_{\xi=0}^{N-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) \in C^\infty(M)$$

can be further decomposed into

$$\sum_{t_0 \in \mathbb{Z}} \left(\mathcal{G}_{t_0,\mathbf{y}}(f) e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^{\xi} \mathbf{x} \right)} \right)$$

such that each term

$$\left(\mathcal{G}_{t_0,\mathbf{y}}(f) e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^{\xi} \mathbf{x} \right)} \right)$$

is itself a smooth function on M.

Proof This result is achieved by substituting the expression for $\mathcal{F}_{\mathbf{y}}(f)$ in Proposition 5.4 into the summand. That the terms of the decomposition are themselves smooth functions on M can be verified through the use of Proposition 5.1. □

In the case when $\left|\mathrm{Orb}_{\mathbf{y}}\right| = \infty$ there does not seem to be any further useful decomposition of $\mathcal{F}$, however there are additional properties which $\mathcal{F}$ must satisfy.

Proposition 5.6 *For any $f \in C^{\infty}(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $\left|\mathrm{Orb}_{\mathbf{y}}\right| = \infty$, we require that all derivatives of $\mathcal{F}_{\mathbf{y}}(f)(t)$ tend to zero as $t \to \pm\infty$ faster than any power of $\left|(A^T)^{\xi}\mathbf{y}\right|$ grows as $\xi \to \pm\infty$. Specifically, for any compact set $K \subset \mathbb{R}$ we require*

$$\sup_{\substack{t \in K \\ \xi \in \mathbb{Z}}} \left| ||(A^T)^{\xi}\mathbf{y}||^p \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) \right| < \infty$$

for all $p, q \in \mathbb{N}$.

Proof Let f be a function in $C^{\infty}(\mathbb{R}^{n+1})$ satisfying (10), i.e. $f \in C^{\infty}(M)$. First, note that since f is smooth on $\mathbb{R}^{n+1}$, all its derivatives must be bounded over any compact $\tilde{K} \subset \mathbb{R}^{n+1}$. If we then take $\tilde{K} = [0,1]^n \times K$, we see that the Fourier coefficients $\mathcal{F}_{\mathbf{x}_0}$ of all the derivatives of f must be bounded for t ranging over K. Importantly, this bound is independent of $\mathbf{x}_0 \in \mathbb{Z}^n$.

The Fourier coefficients of the derivatives of f can take the form of $M(\mathbf{x}_0)\frac{d^q}{dt^q}\left(\mathcal{F}_{\mathbf{x}_0}(f)(t)\right)$ for any monomial M and any $q \in \mathbb{N}$. This means for all monomials M and all $q \in \mathbb{N}$ we require

$$\sup_{\substack{t \in K \\ \mathbf{x}_0 \in \mathbb{Z}^n}} \left| M(\mathbf{x}_0) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{x}_0}(f)(t) \right| < \infty$$

and thus if we restrict our attention to $\mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}$ we require

$$\sup_{\substack{t \in K \\ \mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}}} \left| M(\mathbf{x}_0) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{x}_0}(f)(t) \right| = \sup_{\substack{t \in K \\ \xi \in \mathbb{Z}}} \left| M((A^T)^{\xi}\mathbf{y}) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) \right| < \infty.$$

$M(\mathbf{x}_0)$ can then be chosen to be $||\mathbf{x}_0||^p$ for arbitrarily large $p \in \mathbb{N}$, giving us the desired result. □

Corollary 5.7 *For any $f \in C^{\infty}(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $\left|\mathrm{Orb}_{\mathbf{y}}\right| = \infty$, we require that $\mathcal{F}_{\mathbf{y}}(f)(t) \in \mathcal{S}(\mathbb{R})$. Here $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions*

$$\mathcal{S}(\mathbb{R}) = \left\{ h(t) \in C^{\infty}(\mathbb{R}) \;\middle|\; \sup_{t \in \mathbb{R}} \left| t^p \frac{d^q}{dt^q} h(t) \right| < \infty, \text{ for all } p, q \in \mathbb{N} \right\}.$$

Proof If $\left|\text{Orb}_{\mathbf{y}}\right| = \infty$ then $||(A^T)^{\xi}\mathbf{y}||$ must blow up as $\xi \to \pm\infty$, since an infinite orbit cannot repeat the same point twice. Furthermore, since the number of lattice points within a bounded region of $\mathbb{Z}^n$ grows like R^2 with the radius R of the region, it must be the case that $||(A^T)^{\xi}\mathbf{y}||$ blows up at least as fast as $\left|\xi\right|^{\frac{1}{2}}$. Substituting this speed of growth into the above proposition gives the definition of $\mathcal{S}(\mathbb{R})$.

Note that if $||(A^T)^{\xi}\mathbf{y}||$ blows up faster than polynomially, then the Proposition yields an even stricter condition on $\mathcal{F}_{\mathbf{y}}$ than Schwartz. □

Theorem 5.8 *The space of L^2 functions on M decomposes in the following way.*

$$L^2(M) = \left(\widehat{\bigoplus_{\substack{\text{Orb}_{\mathbf{y}} \in \mathcal{O} \\ \left|\text{Orb}_{\mathbf{y}}\right| = \infty}}} \mathcal{H}_{\mathbf{y}} \right) \oplus \left(\widehat{\bigoplus_{\substack{\text{Orb}_{\mathbf{y}} \in \mathcal{O} \\ \left|\text{Orb}_{\mathbf{y}}\right| = N < \infty}}} \widehat{\bigoplus_{t_0 \in \mathbb{Z}}} \mathcal{H}_{t_0,\mathbf{y}} \right),$$

where

$$\mathcal{H}_{\mathbf{y}} = \left\{ \sum_{\xi \in \mathbb{Z}} f(t+\xi) e^{2\pi i \mathbf{y} \cdot A^{\xi} \mathbf{x}} \;\middle|\; f \in L^2(\mathbb{R}) \right\}$$

and

$$\mathcal{H}_{t_0,\mathbf{y}} = \left\{ C e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^{\xi} \mathbf{x} \right)} \;\middle|\; C \in \mathbb{C} \right\}.$$

Here $\hat{\bigoplus}$ denotes the direct sum followed by the closure with respect to the L^2 norm.

Proof From Propositions 5.3 and 5.4 we see that any smooth function can be decomposed in the way described above and so, since $L^2(M)$ is the closure of $C^{\infty}(M)$ with respect to the L^2 norm, we obtain the desired result. □

It will be useful now to consider what the orbits of $\mathbf{y} \in \mathbb{Z}^n$ actually look like. In particular, when exactly is $\left|\text{Orb}_{\mathbf{y}}\right| < \infty$. First, we define the generalised eigenvectors of A.

Definition 5.9 Let $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be the eigenvalues of $A \in GL_n(\mathbb{Z})$ with values repeated for geometric multiplicity. Then any n linearly independent vectors $\mathbf{v}_{i,j} \in \mathbb{C}^n$ with $i = 1, \ldots, k$ and $j = 1, \ldots, m_i$ such that

$$(A - \lambda_i)^j \mathbf{v}_{i,j} = 0 \quad \text{but} \quad (A - \lambda_i)^{j-1} \mathbf{v}_{i,j} \neq 0$$

are called generalised eigenvectors of A. Note that when $j = 1$ we just have the standard eigenvectors of A. Furthermore, we can make a choice of $\mathbf{v}_{i,j}$ so that when i is fixed, the sequence $\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \ldots, \mathbf{v}_{i,m_i}$ forms a Jordan chain of length m_i. This means for all $j \neq 1$ we have

$$(A - \lambda_i)\mathbf{v}_{i,j} = \mathbf{v}_{i,j-1} \tag{11}$$

and for $j = 1$ we have

$$(A - \lambda_i)\mathbf{v}_{i,1} = 0. \tag{12}$$

These $\mathbf{v}_{i,j}$ can be used to describe when the orbit of the group generated by A^T acting on $\mathbf{y} \in \mathbb{Z}^n$ is finite.

Proposition 5.10 *Let* $\mathbf{v}_{i,j}$ *be the generalised eigenvectors of* $A \in GL_n(\mathbb{Z})$ *as defined above, with corresponding eigenvalues* $\lambda_1, \ldots, \lambda_k$. *Given* $\mathbf{y} \in \mathbb{Z}^n$, *if* $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$ *it must be the case that* $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ *except for when* i *and* j *are chosen such that* $\lambda_i^N = 1$ *and* $j = m_i$.

Proof If $\mathrm{Orb}_{\mathbf{y}}$ is a finite subset of $\mathbb{Z}^n$, then $(A^T)^\xi \mathbf{y}$ must be bounded over $\xi \in \mathbb{Z}$. This means $\mathbf{v}_{i,j} \cdot ((A^T)^\xi \mathbf{y}) = (A^\xi \mathbf{v}_{i,j}) \cdot \mathbf{y}$ must be bounded over $\xi \in \mathbb{Z}$ for all $\mathbf{v}_{i,j}$.

From (12) we know that $A\mathbf{v}_{i,1} = \lambda \mathbf{v}_{i,1}$ and thus

$$A^\xi \mathbf{v}_{i,1} \cdot \mathbf{y} = \lambda^\xi \mathbf{v}_{i,1} \cdot \mathbf{y}.$$

But if $|\lambda_i| > 1$ then λ_i^ξ will blow up as $\xi \to \infty$ and if $|\lambda_i| < 1$ then it will blow up as $\xi \to -\infty$. From this we conclude that $|\mathrm{Orb}_{\mathbf{y}}| < \infty$ only if $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ for all i such that $|\lambda_i| \neq 1$. Rewriting (11) as $A\mathbf{v}_{i,j} = \lambda \mathbf{v}_{i,j} + \mathbf{v}_{i,j-1}$ and using $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ we can apply the above argument again to prove the same result for $\mathbf{v}_{i,2}$. In fact, continuing by induction, we see that $|\mathrm{Orb}_{\mathbf{y}}|$ is finite only if $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all i and j such that $|\lambda_i| \neq 1$.

Now, consider the case when $|\lambda_i| = 1$. From (11) we can see that when $m_i \geq 2$ then

$$A^\xi \mathbf{v}_{i,2} = \lambda^\xi \mathbf{v}_{i,2} + \xi \lambda^{\xi-1} \mathbf{v}_{i,1}.$$

This means $A^\xi \mathbf{v}_{i,2} \cdot \mathbf{y}$ will blow up as $\xi \to \pm\infty$ unless $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$. Similarly, if $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ then the same argument works to show $A^\xi \mathbf{v}_{i,3} \cdot \mathbf{y}$ will blow up unless $\mathbf{v}_{i,2} \cdot \mathbf{y} = 0$, provided $m_i \geq 3$. Repeating this procedure, we find that $|\mathrm{Orb}_{\mathbf{y}}| < \infty$ implies that $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all i and j such that $|\lambda_i| = 1$ and $j < m_i$

Finally, it remains to consider the case of $\mathbf{v}_{i,m_i}$. If $|\mathrm{Orb}_{\mathbf{y}}| = N$ then we know that $(A^T)^N \mathbf{y} = \mathbf{y}$, and also we have shown that $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all $j \neq m_i$. The following must therefore hold:

$$\begin{aligned} \mathbf{v}_{i,m_i} \cdot \mathbf{y} &= \mathbf{v}_{i,m_i} \cdot (A^T)^N \mathbf{y} \\ &= A^N \mathbf{v}_{i,m_i} \cdot \mathbf{y} \\ &= \lambda_i^N \mathbf{v}_{i,m_i} \cdot \mathbf{y}. \end{aligned}$$

Thus $|\mathrm{Orb}_{\mathbf{y}}| = N$ requires that for all i, either $\mathbf{v}_{i,m_i} \cdot \mathbf{y} = 0$ or $\lambda_i^N = 1$. □

Corollary 5.11 *Whenever* $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$, *it holds that*

$$A\mathbf{v}_{i,j} \cdot \mathbf{y} = \begin{cases} e^{2\pi i \theta_i} \mathbf{v}_{i,j} \cdot \mathbf{y} & \text{if } \lambda_i^N = 1 \text{ and } j = m_i \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_i \in \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$ *is some rational number depending on* i *satisfying* $N\theta_i \in \mathbb{Z}$.

5.2 Properties of the decomposition

We would now like to consider some of the properties of this decomposition, which will be useful when considering the example in the following section. But in order to do this we must first construct a special frame on M.

Definition 5.12 Given any invertible matrix $A \in GL_n(\mathbb{Z})$, then for some choice of matrix logarithm $\ln A$ we can define the power $A^t := e^{t \ln A}$ for all $t \in \mathbb{R}$. Note that such a logarithm always exists, but may be complex valued.

Throughout this paper, the choice of $\ln A$ will always be made such that

$$A^t \mathbf{v}_{i,j} \cdot \mathbf{y} = \begin{cases} e^{2\pi i \theta_i t} \mathbf{v}_{i,j} \cdot \mathbf{y} & \text{if } \lambda_i^N = 1 \text{ and } j = m_i \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_i \in \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$.

Using the generalised eigenvectors of A given by $\mathbf{v}_{i,j}$, a smooth frame for the complexified tangent bundle of M can be given by

$$\epsilon_0 = \frac{\partial}{\partial t} \qquad \epsilon_{i,j} = A^t \mathbf{v}_{i,j} \cdot \nabla_{\mathbf{x}}.$$

Here we are using $\nabla_{\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$ to denote the gradient excluding the variable t. We verify that this is indeed a well defined frame on M in following proposition.

Proposition 5.13 *Viewing M as a torus bundle over S^1, any smooth frame of the complexified tangent bundle on a single fibre may be extended to a smooth frame on all of M.*

Proof We can assume, without loss of generality, that we are starting with a frame on the $t = 0$ fibre, where t is parametrising the base space S^1, as in the definition of M (9).

Let $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}^n$ be smooth maps sending each point $\mathbf{x} \in \mathbb{T}^n$ to n linearly independent vectors. Then the collection $\{\mathbf{a}_i \cdot \nabla_{\mathbf{x}}\}_{i=1,\ldots,n}$ defines a general frame on the $t = 0$ fibre. A frame for $T_{\mathbb{C}}M$ is then given by $u_0 = \frac{\partial}{\partial t}$ and $u_i = A^t \mathbf{a}_i \cdot \nabla_{\mathbf{x}}$ with $i = 1, \ldots, n$.

These are indeed all well defined vector fields on M, in particular they do not conflict with the identification of points given in (9). To check the first identification, simply note that the maps $\mathbf{a}_i(\mathbf{x})$ are defined on the torus. For the second we consider the map

$$\phi_\xi : \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} t + \xi \\ A^\xi \mathbf{x} \end{pmatrix}$$

with $\xi \in \mathbb{Z}$ and try to show that u_i are invariant under the pushforward. Certainly this is true of $\frac{\partial}{\partial t}$, and we also know that, for $i = 1, \ldots, n$, we have

$$\begin{aligned} (\phi_\xi)_*(\mathbf{e}_i \cdot \nabla_{\mathbf{x}}) &= (\phi_\xi)_* \frac{\partial}{\partial x_i} \\ &= A^\xi \mathbf{e}_i \cdot \nabla_{\mathbf{x}} \end{aligned}$$

with $\mathbf{e}_i$ signifying the standard basis vector $(0, \ldots, 1, \ldots, 0)$ with a 1 in the ith position. Therefore

$$\begin{aligned} (\phi_\xi)_* u_i(t) &= (\phi_\xi)_*(A^t \mathbf{a}_i \cdot \nabla_{\mathbf{x}}) \\ &= A^{t+\xi} \mathbf{a}_i \cdot \nabla_{\mathbf{x}} \\ &= u_i(t + \xi). \end{aligned}$$

□

It should be noted that if A has a real-valued logarithm and we choose $\mathbf{a}_i$ to be maps into $\mathbb{R}^n$, then the construction in the above proof will give us a smooth frame on the standard, non-complexified tangent bundle.

Proposition 5.14 *Given any $\mathbf{y} \in \mathbb{Z}^n$ and any $f \in C^\infty(M)$, $\mathcal{F}_{\mathbf{y}}$ has the properties* i)

$$\mathcal{F}_{\mathbf{y}}(\epsilon_0 f)(t) = \epsilon_0\, \mathcal{F}_{\mathbf{y}}(f)(t),$$

ii)

$$\mathcal{F}_{\mathbf{y}}(\epsilon_{i,j} f)(t) = 2\pi i\, A^t \mathbf{v}_{i,j} \cdot \mathbf{y}\, \mathcal{F}_{\mathbf{y}}(f).$$

Proof Since $\mathcal{F}_{\mathbf{y}}(f)$ is just one of the Fourier coefficients of f in the standard expansion

$$f(t, \mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{y}}(f)(t) e^{2\pi i \mathbf{y} \cdot \mathbf{x}}$$

this proposition is simply restating results from classical Fourier analysis. □

Proposition 5.15 *Given any* $\mathbf{y} \in \mathbb{Z}^n$ *such that* $\left|\mathrm{Orb}_{\mathbf{y}}\right| = N < \infty$ *and any* $f \in C^{\infty}(M)$, $\mathcal{G}_{t_0,\mathbf{y}}$ *has the properties*

i)

$$\mathcal{G}_{t_0,\mathbf{y}}(\epsilon_0 f)(t) = 2\pi i \frac{t_0}{N}\, \mathcal{G}_{t_0,\mathbf{y}}(f),$$

ii)

$$\mathcal{G}_{t_0,\mathbf{y}}(\epsilon_{i,j} f) = \begin{cases} 2\pi i \mathbf{v}_{i,j} \cdot \mathbf{y}\, \mathcal{G}_{t_0 + N\theta_i, \mathbf{y}}(f) & \text{if } \lambda_i^N = 1 \text{ and } j = m_i \\ 0 & \text{otherwise} \end{cases},$$

with θ_i *defined as in Corollary 5.11.*

Proof For part $i)$, we make use of the result $i)$ in the previous proposition along with the definition of $\mathcal{G}_{t_0,\mathbf{y}}$ to write

$$\begin{aligned} \mathcal{G}_{t_0,\mathbf{y}}(\epsilon_0 f)(t) &= \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(\epsilon_0 f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= \frac{1}{N} \int_0^N \left(\epsilon_0 \mathcal{F}_{\mathbf{y}}(f)\right) e^{-\frac{2\pi i t_0 t}{N}}\, dt. \end{aligned}$$

Then, since $\mathcal{F}_{\mathbf{y}}(f)(t)$ is periodic with period N, we can make use of integration by parts to get

$$\begin{aligned} \frac{1}{N} \int_0^N \left(\epsilon_0 \mathcal{F}_{\mathbf{y}}(f)\right) e^{-\frac{2\pi i t_0 t}{N}}\, dt &= -\frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(f) \left(\epsilon_0 e^{-\frac{2\pi i t_0 t}{N}}\right) dt \\ &= 2\pi i \frac{t_0}{N} \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= 2\pi i \frac{t_0}{N} \mathcal{G}_{t_0,\mathbf{y}}(f). \end{aligned}$$

For part $ii)$, we make use of the result $ii)$ in the previous proposition to write

$$\begin{aligned} \mathcal{G}_{t_0,\mathbf{y}}(\epsilon_{i,j} f) &= \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(\epsilon_{i,j} f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= \frac{1}{N} \int_0^N 2\pi i A^t \mathbf{v}_{i,j} \cdot \mathbf{y} \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt. \end{aligned}$$

Then, because of the way A^t was defined in Definition 5.12, we get

$$\frac{1}{N}\int_0^N 2\pi i A^t \mathbf{v}_{i,j}\cdot \mathbf{y}\mathcal{F}_{\mathbf{y}}(f)e^{-\frac{2\pi i t_0 t}{N}}\,dt = 0$$

unless $\lambda_i^N = 1$ and $j = m_i$, in which case

$$\begin{aligned}\frac{1}{N}\int_0^N 2\pi i A^t \mathbf{v}_{i,j}\cdot \mathbf{y}\mathcal{F}_{\mathbf{y}}(f)e^{-\frac{2\pi i t_0 t}{N}}\,dt &= 2\pi i \mathbf{v}_{i,j}\cdot \mathbf{y}\frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(f)e^{-\frac{2\pi i (t_0+N\theta_i)t}{N}}\,dt\\ &= 2\pi i \mathbf{v}_{i,j}\cdot \mathbf{y}\,\mathcal{G}_{t_0+N\theta_i,\mathbf{y}}(f).\end{aligned}$$

□

6 Calculating $h_{\bar\partial}^{0,1}$ using harmonic analysis

It should be noted the Kodaira–Thurston manifold KT^4 can be viewed as a torus bundle over S^1 with $A = \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$. The calculations done on KT^4 in Sect. 4.1 can be thought of as an application of the above results, with the case when $n \neq 0$ corresponding to an infinite orbit and the case when $n = 0$ corresponding to a finite orbit of length 1.

In this section we will see what it looks like to use our decomposition to perform calculations on a manifold for which we have a finite orbit with length greater that 1. Specifically, we will show how the decomposition might be used in the calculation of $h_{\bar\partial}^{0,1}$ on an almost Hermitian manifold defined by setting

$$A = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

and identifying points in $\mathbb{R}^4$ by

$$\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix} \sim \begin{pmatrix}t\\ \mathbf{x}+\mathbf{x}_0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}t\\ \mathbf{x}\end{pmatrix} \sim \begin{pmatrix}t+\xi\\ A^{\xi}\mathbf{x}\end{pmatrix}$$

for all $\mathbf{x}_0 \in \mathbb{Z}^3$ and all $\xi \in \mathbb{Z}$. The matrix A has eigenvalues of 1, $e^{-\frac{2}{3}\pi i}$ and $e^{\frac{2}{3}\pi i}$ corresponding to eigenvectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}e^{\frac{2}{3}\pi i}\\ e^{-\frac{2}{3}\pi i}\\ 1\end{pmatrix}$ and $\begin{pmatrix}e^{-\frac{2}{3}\pi i}\\ e^{\frac{2}{3}\pi i}\\ 1\end{pmatrix}$. We therefore define a smooth frame on the complexified tangent bundle by

$$\epsilon_0 = \frac{\partial}{\partial t} \qquad \epsilon_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\nabla_{\mathbf{x}} \qquad \epsilon_2 = e^{-\frac{2}{3}\pi i t}\begin{pmatrix}e^{\frac{2}{3}\pi i}\\ e^{-\frac{2}{3}\pi i}\\ 1\end{pmatrix}\cdot\nabla_{\mathbf{x}} \qquad \epsilon_3 = e^{\frac{2}{3}\pi i t}\begin{pmatrix}e^{-\frac{2}{3}\pi i}\\ e^{\frac{2}{3}\pi i}\\ 1\end{pmatrix}\cdot\nabla_{\mathbf{x}}$$

where we define $\nabla_{\mathbf{x}} := \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$. The dual frame is given by

$$\epsilon^0 = \mathrm{d}t, \qquad \epsilon^1 = \frac{1}{3}\left(\mathrm{d}x + \mathrm{d}y + \mathrm{d}z\right),$$

$$\epsilon^2 = \frac{e^{\frac{2}{3}\pi i t}}{3}\left(e^{-\frac{2}{3}\pi i}\mathrm{d}x + e^{\frac{2}{3}\pi i}\mathrm{d}y + \mathrm{d}z\right), \qquad \epsilon^3 = \frac{e^{-\frac{2}{3}\pi i t}}{3}\left(e^{\frac{2}{3}\pi i}\mathrm{d}x + e^{-\frac{2}{3}\pi i}\mathrm{d}y + \mathrm{d}z\right).$$

Let an almost complex structure J be defined by the mapping

$$\epsilon_0 \mapsto \frac{1}{2}(\epsilon_2 + \epsilon_3) \qquad \text{and} \qquad \epsilon_1 \mapsto -\frac{i}{2}(\epsilon_2 - \epsilon_3).$$

We can then find a pair of vectors fields spanning $T_p^{1,0}M$ at all points $p \in M$

$$V_1 = \frac{1}{2}\left(\epsilon_0 - \frac{i}{2}(\epsilon_2 + \epsilon_3)\right) \qquad V_2 = \frac{1}{2}\left(\epsilon_1 - \frac{1}{2}(\epsilon_2 - \epsilon_3)\right)$$

with dual (1, 0)-forms given by

$$\phi^1 = \epsilon^0 + i(\epsilon^2 + \epsilon^3) \qquad \phi^2 = \epsilon^1 - \epsilon^2 + \epsilon^3$$

which satisfy the structure equations

$$\begin{aligned} \mathrm{d}\phi^1 &= \frac{\pi}{6}\left(\phi^{12} - \phi^{1\bar{2}} - \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right) \\ \mathrm{d}\phi^2 &= \frac{\pi}{3}\phi^{1\bar{1}}. \end{aligned}$$

The metric can be chosen so that V_1 and V_2 form a unitary basis.

6.1 Deriving the equations

Let a general (0, 1)-form be written as $s = f\bar{\phi}^1 + g\bar{\phi}^2$. The two requirements

$$\bar{\partial}s = 0 \qquad \text{and} \qquad \partial * s = 0$$

which are equivalent to s being $\bar{\partial}$-harmonic, give rise to the two PDEs

$$\begin{aligned} &-(\bar{V}_2 - \frac{\pi}{6})f + \bar{V}_1 g = 0, \\ &V_1 f + V_2 g = 0. \end{aligned}$$

We will now try taking a Fourier expansion. If f is a smooth function on M we know from Proposition 5.1 that we can write it as

$$f(t, x, y, z) = \sum_{l,m,n\in\mathbb{Z}} \mathcal{F}_{l,m,n}(f)(t)e^{2\pi i(lx+my+nz)},$$

with $\mathcal{F}_{l,m,n}(f)$ satisfying the property

$$\mathcal{F}_{m,n,l}(t) = \mathcal{F}_{l,m,n}(f)(t + \xi),$$

for all $\xi \in \mathbb{Z}$. This gives us two cases: if $l = m = n$ then $\mathcal{F}_{l,m,n}(f)$ is periodic with period length 1, and otherwise it is periodic with period length 3. So in both cases we can further expand the function with respect to the variable t.

6.1.1 The orbits of length 1

Here we have a standard Fourier expansion in all 4 variables, with

$$\mathcal{F}_{n,n,n}(f) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,n,n,n}(f)(t)e^{2\pi ikt}.$$

In this case by Proposition 5.15 we see that $\mathcal{G}_{k,n,n,n}$ satisfies the properties

$$\begin{aligned}
\mathcal{G}_{k,n,n,n}(\epsilon_0 f) &= 2\pi ik\mathcal{G}_{k,n,n,n}(f),\\
\mathcal{G}_{k,n,n,n}(\epsilon_1 f) &= 2\pi i(n+n+n)\mathcal{G}_{k,n,n,n}(f) = 6\pi in\mathcal{G}_{k,n,n,n}(f),\\
\mathcal{G}_{k,n,n,n}(\epsilon_2 f) &= 2\pi i(e^{\frac{2}{3}\pi i}n + e^{-\frac{2}{3}\pi i}n + n)\mathcal{G}_{k-1,n,n,n}(f) = 0,\\
\mathcal{G}_{k,n,n,n}(\epsilon_3 f) &= 2\pi i(e^{-\frac{2}{3}\pi i}n + e^{\frac{2}{3}\pi i}n + n)\mathcal{G}_{k+1,n,n,n}(f) = 0,
\end{aligned}$$

which we can use to rewrite our two PDEs into the form

$$\begin{pmatrix} k & 3n \\ -3n - \frac{i}{6} & k \end{pmatrix}\begin{pmatrix} \mathcal{G}(f) \\ \mathcal{G}(g) \end{pmatrix} = 0.$$

This has non-trivial solutions if and only if the matrix has zero determinant, i.e.

$$k^2 + 9n^2 + 3n\frac{i}{6} = 0$$

which is only the case when $k = n = 0$. Corresponding to this case we have the solution

$$f = 0 \qquad g = const.$$

6.1.2 The orbits of length 3

In the case where l, m and n are not all equal, $\mathcal{F}_{l,m,n}(f)$ is still periodic, but with period 3 and so our expansion now looks like

$$\mathcal{F}_{l,m,n}(f)(t) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,l,m,n}(f)e^{\frac{2\pi ikt}{3}}.$$

For the sake of notational simplicity we define

$$\alpha_1 = l + m + n \qquad \alpha_2 = e^{\frac{2}{3}\pi i}l + e^{-\frac{2}{3}\pi i}m + n \qquad \alpha_3 = e^{-\frac{2}{3}\pi i}l + e^{\frac{2}{3}\pi i}m + n$$

and also we will use $\mathcal{G}_k(f)$ to denote $\mathcal{G}_{k,l,m,n}(f)$. Then by Proposition 5.15 we can say that $\mathcal{G}_k(f)$ satisfies

$$\begin{aligned}
\mathcal{G}_k(\epsilon_0 f) &= 2\pi ik\mathcal{G}_k(f),\\
\mathcal{G}_k(\epsilon_1 f) &= 2\pi i\alpha_1\mathcal{G}_k(f),\\
\mathcal{G}_k(\epsilon_2 f) &= 2\pi i\alpha_2\mathcal{G}_{k-1}(f),\\
\mathcal{G}_k(\epsilon_3 f) &= 2\pi i\alpha_3\mathcal{G}_{k+1}(f).
\end{aligned}$$

Applying these properties to our two PDEs gives us a pair of equations

$$
\begin{aligned}
&\frac{\alpha_3}{2}\mathcal{G}_{k-1}(f) - (\alpha_1 + \frac{i}{6})\mathcal{G}_k(f) - \frac{\alpha_2}{2}\mathcal{G}_{k+1}(f) \\
&+ \frac{i\alpha_3}{2}\mathcal{G}_{k-1}(g) + \frac{k}{3}\mathcal{G}_k(g) + \frac{i\alpha_2}{2}\mathcal{G}_{k+1}(g) = 0, \\
&- \frac{i\alpha_3}{2}\mathcal{G}_{k-1}(f) + \frac{k}{3}\mathcal{G}_k(f) - \frac{i\alpha_2}{2}\mathcal{G}_{k+1}(f) \\
&+ \frac{\alpha_3}{2}\mathcal{G}_{k-1}(g) + \alpha_1\mathcal{G}_k(g) - \frac{\alpha_2}{2}\mathcal{G}_{k+1}(g) = 0.
\end{aligned}
$$

By choosing to cancel either the terms $\mathcal{G}_{k-1}(f)$ & $\mathcal{G}_{k-1}(g)$ or the terms $\mathcal{G}_{k+1}(f)$ & $\mathcal{G}_{k+1}(g)$ we can simplify to the pair of equations

$$
\begin{aligned}
&\left(\frac{k}{3} + \frac{1}{6} - i\alpha_1\right)\mathcal{G}_k(f) - i\alpha_2\mathcal{G}_{k+1}(f) + i\left(\frac{k}{3} - i\alpha_1\right)\mathcal{G}_k(g) + \alpha_2\mathcal{G}_{k+1}(g) = 0, \\
&-i\alpha_3\mathcal{G}_{k-1}(f) + \left(\frac{k}{3} - \frac{1}{6} + i\alpha_1\right)\mathcal{G}_k(f) + \alpha_3\mathcal{G}_{k-1}(g) - i\left(\frac{k}{3} + i\alpha_1\right)\mathcal{G}_k(g) = 0.
\end{aligned}
$$

Evaluating the second of these at $k+1$ instead of k we can cancel either the $\mathcal{G}_{k+1}(f)$ term or the $\mathcal{G}_{k+1}(g)$ term. In this way we can write our equations as the recurrence relation

$$
\begin{pmatrix} \mathcal{G}_{k+1}(f) \\ \mathcal{G}_{k+1}(g) \end{pmatrix} = \frac{6}{(4k+3+12i\alpha_1)} B_k \begin{pmatrix} \mathcal{G}_k(f) \\ \mathcal{G}_k(g) \end{pmatrix}
$$

where

$$
B_k = \begin{pmatrix} -i\left[\left(\frac{k}{3}+\frac{1}{6}\right)\left(\frac{k}{3}+\frac{1}{3}\right) + \alpha_1^2 - \frac{1}{6}i\alpha_1 - \alpha_2\alpha_3\right] & \left[\frac{k}{3}\left(\frac{k}{3}+\frac{1}{3}\right) + \alpha_1^2 - \frac{1}{3}i\alpha_1 - \alpha_2\alpha_3\right] \\ -\left[\left(\frac{k}{3}+\frac{1}{6}\right)^2 + \alpha_1^2 + \alpha_2\alpha_3\right] & -i\left[\frac{k}{3}\left(\frac{k}{3}+\frac{1}{6}\right) + \alpha_1^2 - \frac{1}{6}i\alpha_1 + \alpha_2\alpha_3\right] \end{pmatrix},
$$

and so the values of $\mathcal{G}_k(f)$ and $\mathcal{G}_k(g)$ for all $k \in \mathbb{Z}$ are determined by a choice for $\mathcal{G}_0(f)$ and $\mathcal{G}_0(g)$. Since we are looking for smooth solutions f and g, we require that $\mathcal{F}_{l,m,n}(f)(t) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,l,m,n}(f)e^{\frac{2\pi ikt}{3}}$ be smooth, and likewise for $\mathcal{F}_{l,m,n}(g)(t)$. This is equivalent to asking that the sequences $\mathcal{G}_k(f)$ and $\mathcal{G}_k(g)$ are Schwartz, i.e. they are contained in

$$
\mathcal{S}(\mathbb{Z}) = \left\{ (a_k)_{k\in\mathbb{Z}} \;\middle|\; \sup_{k\in\mathbb{Z}} \left|k^p a_k\right| < \infty \textit{ for all } p \in \mathbb{N} \right\}.
$$

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