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Complete Lagrangian self-shrinkers in R^4

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Abstract
The purpose of this paper is to study complete self-shrinkers of mean curvature flow in Euclidean spaces. In the paper, we give a complete classification for 2-dimensional complete Lagrangian self-shrinkers in Euclidean space $\mathbb{R}^4$ with constant squared norm of the second fundamental form.

Keywords Mean curvature flow · Self-shrinker · Lagrangian submanifold · The generalized maximum principle

Mathematics Subject Classification 53C24 · 53C40

1 Introduction

Let $X : M \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. A family of n-dimensional submanifolds $X(\cdot, t) : M \to \mathbb{R}^{n+p}$ is called a mean

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curvature flow if they satisfy $X(\cdot, 0) = X(\cdot)$ and

$$\frac{\partial X(p,t)}{\partial t} = \vec{H}(p,t), \quad (p,t) \in M \times [0,T), \tag{1.1}$$

where $\vec{H}(p,t)$ denotes the mean curvature vector of submanifold $M_t = X(M,t)$ at point $X(p,t)$. The mean curvature flow has been used to model various things in material sciences and physics such as cell, bubble growth and so on. The study of the mean curvature flow from the perspective of partial differential equations commenced with Huisken's paper [16] on the flow of convex hypersurfaces. One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. A key starting point for singularity analysis is Huisken's monotonicity formula, the monotonicity implies that the flow is asymptotically self-similar near a given type I singularity. Thus, it is modeled by self-shrinking solutions of the flow.

An n-dimensional submanifold $X : M \to \mathbb{R}^{n+p}$ in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$ is called a self-shrinker if it satisfies

$$\vec{H} + X^{\perp} = 0,$$

where $X^{\perp}$ denotes the normal part of the position vector X. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow-ups at a given singularity of the mean curvature flow.

For complete self-shrinkers with co-dimension 1, Abresch and Langer [1] classified closed self-shrinker curves in $\mathbb{R}^2$ and showed that the round circle is the only embedded self-shrinker. Huisken [15, 17], Colding and Minicozzi [11] have proved that if $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with mean curvature $H \geq 0$ and with polynomial volume growth, then $X : M \to \mathbb{R}^{n+1}$ is isometric to $\mathbb{R}^n$, or the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$. Halldorsson in [14] proved that there exist complete self-shrinking curves Γ in $\mathbb{R}^2$, which is contained in an annulus around the origin and whose image is dense in the annulus. Furthermore, Ding and Xin [12], Cheng and Zhou [10] proved that a complete self-shrinker has polynomial volume growth if and only if it is proper. Thus, the condition on polynomial volume growth in [15] and [11] is essential since these complete self-shrinking curves Γ of Halldorsson [14] are not proper and for any integer $n > 0$, $\Gamma \times \mathbb{R}^{n-1}$ is a complete self-shrinker without polynomial volume growth in $\mathbb{R}^{n+1}$.

As for the study on the rigidity of complete self-shrinkers, many important works have been done (cf. [4, 7–9, 12, 13, 22] and so on). In particular, Cheng and Peng in [8] proved that for an n-dimensional complete self-shrinker $X : M^n \to \mathbb{R}^{n+1}$ with $\inf H^2 > 0$, if the squared norm S of the second fundamental form is constant, then M^n is isometric to one of the following:

(1) $S^n(\sqrt{n})$,
(2) $S^m(\sqrt{m}) \times \mathbb{R}^{n-m} \subset \mathbb{R}^{n+1}$.

Furthermore, Ding and Xin [13] studied 2-dimensional complete self-shrinkers with polynomial volume growth and with constant squared norm S of the second fundamental form. They have proved that a 2-dimensional complete self-shrinker $X : M \to \mathbb{R}^3$ with polynomial volume growth is isometric to one of the following:

(1) $\mathbb{R}^2$,
(2) $S^1(1) \times \mathbb{R}$
(3) $S^2(\sqrt{2})$,

if S is constant. Recently, Cheng and Ogata [7] have removed both the assumption on polynomial volume growth in the above theorem of Ding and Xin [13] and the assumption $\inf H^2 > 0$ in the theorem of Cheng and Peng [8] for $n = 2$.
It is natural to ask the following problems:

Problem 1. To classify 2-dimensional complete self-shrinkers in $\mathbb{R}^4$ if the squared norm S of the second fundamental form is constant.

It is well-known that the unit sphere $S^2(1)$, the Clifford torus $S^1(1) \times S^1(1)$, the Euclidean plane $\mathbb{R}^2$ and the cylinder $S^1(1) \times \mathbb{R}^1$ are the canonical self-shrinkers in $\mathbb{R}^4$. Besides the standard examples, there are many examples of complete self-shrinkers in $\mathbb{R}^4$. For examples, compact minimal surfaces in the sphere $S^3(2)$ are compact self-shrinkers in $\mathbb{R}^4$. Further, Anciaux [2], Lee and Wang [21], Castro and Lerma [5] constructed many compact self-shrinkers in $\mathbb{R}^4$ (cf. Sect. 3). Except the canonical self-shrinkers in $\mathbb{R}^4$, the known examples of complete self-shrinkers in $\mathbb{R}^4$ do not have the constant squared norm S of the second fundamental form.

Since the above problem is very difficult, one may consider the special case of complete Lagrangian self-shrinkers in $\mathbb{R}^4$ first. Here we have identified $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and let us recall the definition of Lagrangian submanifolds. A submanifold $X : M \to \mathbb{R}^{2n}$ is called a Lagrangian submanifold if $J(T_pM) = T_p^{\perp}M$, for any $p \in M$, where J is the complex structure of $\mathbb{R}^{2n}$, T_pM and $T_p^{\perp}M$ denote the tangent space and the normal space at p.

It is known that the mean curvature flow preserves the Lagrangian property, which means that, if the initial submanifold $X : M \to \mathbb{R}^{2n}$ is Lagrangian, then the mean curvature flow $X(\cdot, t) : M \to \mathbb{R}^{2n}$ is also Lagrangian. Lagrangian submanifolds are a class of important submanifolds in geometry of submanifolds and they also have many applications in many other fields of differential geometry. For instance, the existence of special Lagrangian submanifolds in Calabi-Yau manifolds attracts a lot of attention since it plays a critical role in the T-duality formulation of Mirror symmetry of Strominger-Yau-Zaslow [28]. In particular, recently, the study on complete Lagrangian self-shrinkers of mean curvature flow has attracted much attention. Many important examples of compact Lagrangian self-shrinkers are constructed (see Sect. 3 and cf. [2, 5, 21]). It was proved by Smoczyk [26] that there are no Lagrangian self-shrinkers, which are topological spheres, in $\mathbb{R}^{2n}$. In [6], Castro and Lerma gave a classification of Hamiltonian stationary Lagrangian self-shrinkers in $\mathbb{R}^4$ and in [5], they proved that Clifford torus $S^1(1) \times S^1(1)$ is the only compact Lagrangian self-shrinker with $S \le 2$ in $\mathbb{R}^4$ if the Gaussian curvature does not change sign. Here, it is noticeable that compactness is important since the Gauss–Bonnet theorem is the key in their proof. In fact, Since $X : M^2 \to \mathbb{R}^4$ is compact, according to the Gauss–Bonnet theorem, we have

$$8\pi(1-g) = 2\int_M K dA = \int_M (H^2 - S) dA = \int_M (2 - S) dA.$$

Hence, $X : M^2 \to \mathbb{R}^4$ is a torus and $K \equiv 0$, $S \equiv 2$. Recently, Li and Wang [22] have removed the condition on Gaussian curvature. They proved that Clifford torus $S^1(1) \times S^1(1)$ is the only compact Lagrangian self-shrinker with $S \le 2$ in $\mathbb{R}^4$. Furthermore, they proved that Clifford torus $S^1(1) \times S^1(1)$ is the only compact Lagrangian self-shrinker with constant squared norm S of the second fundamental form in $\mathbb{R}^4$. The Gauss–Bonnet theorem is still the key in their proof. Since the Euclidean plane $\mathbb{R}^2$ and the cylinder $S^1(1) \times \mathbb{R}^1$ are complete and non-compact Lagrangian self-shrinkers with $S =$ constant in $\mathbb{R}^4$, we may ask the following problem:

Problem 2. Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. If the squared norm S of the second fundamental form is constant, is $X : M^2 \to \mathbb{R}^4$ isometric to one of the following

(1) $\mathbb{R}^2$,
(2) $S^1(1) \times \mathbb{R}^1$,
(3) $S^1(1) \times S^1(1)$?

It is our motivation to solve the above problem. In fact, we prove the following:

Theorem 1.1 *Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. If the squared norm S of the second fundamental form is constant, then $X : M^2 \to \mathbb{R}^4$ is isometric to one of*

(1) $\mathbb{R}^2$,
(2) $S^1(1) \times \mathbb{R}^1$,
(3) $S^1(1) \times S^1(1)$.

Remark 1.1 We should remark the condition that S is constant is essential. In fact, from examples of Lee-Wang in Sect. 3, we know

$$\frac{3m^2 + n^2}{n(m+n)} \le S \le \frac{m^2 + 3n^2}{m(n+m)},$$

for $m \le n$. By taking $n = m + 1$ and letting $m \to \infty$, we have

$$\frac{3m^2 + n^2}{n(m+n)} < 2, \quad \frac{m^2 + 3n^2}{m(n+m)} > 2$$

and

$$\lim_{m\to\infty} \frac{3m^2 + n^2}{n(m+n)} = \lim_{m\to\infty} \frac{m^2 + 3n^2}{m(n+m)} = 2.$$

Since we do not assume that Lagrangian self-shrinkers are compact, we can not use Gauss–Bonnet theorem. Hence, in this paper, in place of the powerful Gauss–Bonnet theorem, we use the generalized maximum principle and moving frame methods.

In order to prove our theorem, we need to compute the supremum and infimum of mean curvature about 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. Thus, a very precise computation is needed. Therefore, we must give a precise estimate of the squared norm of the second covariant derivative of the second fundamental form.

This paper is organized as follows.

In Sect. 2, in order to get a precise estimate of the squared norm of the second covariant derivative of the second fundamental form of 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$, we need to compute $\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2$ in two ways, which is a long computation.

In Sect. 3, we give several examples of 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$, which show that the condition of $S =$ constant is indispensable.

In Sect. 4, we prove our theorem. In order to do it, we make use of the generalized maximum principle. We choose a special frame fields at points, which we consider. We need to prove $h^*_{12} = \lambda = 0$. This assertion is the key in our proof. Thus, a precise and detailed computation is needed.

2 Preliminaries

Let $X : M \to \mathbb{R}^{2n}$ be an n-dimensional connected submanifold of the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^{2n}$ in $\mathbb{R}^{2n}$ with dual coframe field $\{\omega_A\}_{A=1}^{2n}$, such that, restricted to M, $e_1, \ldots, e_n$ are tangent to M^n. Here we have identified $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. For a Lagrangian submanifold $X : M \to \mathbb{R}^{2n}$, we choose an adapted Lagrangian frame field

$$e_1, e_2, \ldots, e_n \quad \text{and} \quad e_{1^*} = Je_1, e_{2^*} = Je_2, \ldots, e_{n^*} = Je_n.$$

From now on, we use the following conventions on the ranges of indices:

$$1 \leq i, j, k, l \leq n, \quad 1 \leq \alpha, \beta, \gamma \leq n$$

and $\sum_i$ means taking summation from 1 to n for i. Then we have

$$\begin{aligned} dX &= \sum_i \omega_i e_i, \\ de_i &= \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha^*} e_{\alpha^*}, \\ de_{\alpha^*} &= \sum_i \omega_{\alpha^* i} e_i + \sum_\beta \omega_{\alpha^*\beta^*} e_{\beta^*}, \end{aligned}$$

where ω_{ij} is the Levi–Civita connection of M, $\omega_{\alpha^*\beta^*}$ is the normal connection of $T^\perp M$.

By restricting these forms to M, we have

$$\omega_{\alpha^*} = 0 \quad \text{for} \quad 1 \leq \alpha \leq n \tag{2.1}$$

and the induced Riemannian metric of M is written as $ds_M^2 = \sum_i \omega_i^2$. Taking exterior derivatives of (2.1), we have

$$0 = d\omega_{\alpha^*} = \sum_i \omega_{\alpha^* i} \wedge \omega_i.$$

By Cartan's lemma, we have

$$\omega_{i\alpha^*} = \sum_j h_{ij}^{\alpha^*} \omega_j, \quad h_{ij}^{\alpha^*} = h_{ji}^{\alpha^*}.$$

Since $X : M \to \mathbb{R}^{2n}$ is a Lagrangian submanifold, we have

$$\begin{aligned} &h_{ij}^{p^*} = h_{ji}^{p^*} = h_{pj}^{i^*}, \quad \text{for any } i, j, p. \\ &h = \sum_{i,j,p} h_{ij}^{p^*} \omega_i \otimes \omega_j \otimes e_{p^*} \end{aligned} \tag{2.2}$$

and

$$\vec{H} = \sum_p H^{p^*} e_{p^*} = \sum_p \sum_i h_{ii}^{p^*} e_{p^*}$$

are called the second fundamental form and the mean curvature vector field of $X : M \to \mathbb{R}^{2n}$, respectively. Let $S = \sum_{i,j,p} (h_{ij}^{p^*})^2$ be the squared norm of the second fundamental form and $H = |\vec{H}|$ denote the mean curvature of $X : M \to \mathbb{R}^{2n}$. The induced structure equations of M are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji},$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} denotes components of the curvature tensor of M. Hence, the Gauss equations are given by

$$R_{ijkl} = \sum_p \left(h^{p^*}_{ik} h^{p^*}_{jl} - h^{p^*}_{il} h^{p^*}_{jk}\right), \quad R_{ik} = \sum_p H^{p^*} h^{p^*}_{ik} - \sum_{p,j} h^{p^*}_{ij} h^{p^*}_{jk}. \tag{2.3}$$

Letting $R_{p^*q^*ij}$ denote the curvature tensor of the normal connection $\omega_{p^*q^*}$ in the normal bundle of $X : M \to \mathbb{R}^{2n}$, then Ricci equations are given by

$$R_{p^*q^*kl} = \sum_i \left(h^{p^*}_{ik} h^{q^*}_{il} - h^{p^*}_{il} h^{q^*}_{ik}\right). \tag{2.4}$$

Defining the covariant derivative of $h^{p^*}_{ij}$ by

$$\sum_k h^{p^*}_{ijk}\omega_k = dh^{p^*}_{ij} + \sum_k h^{p^*}_{ik}\omega_{kj} + \sum_k h^{p^*}_{kj}\omega_{ki} + \sum_q h^{q^*}_{ij}\omega_{p^*q^*}, \tag{2.5}$$

we obtain the Codazzi equations

$$h^{p^*}_{ijk} = h^{p^*}_{ikj} = h^{i^*}_{pjk}. \tag{2.6}$$

By taking exterior differentiation of (2.5), and defining

$$\sum_l h^{p^*}_{ijkl}\omega_l = dh^{p^*}_{ijk} + \sum_l h^{p^*}_{ljk}\omega_{li} + \sum_l h^{p^*}_{ilk}\omega_{lj} + \sum_l h^{p^*}_{ijl}\omega_{lk} + \sum_q h^{q^*}_{ijk}\omega_{q^*p^*}, \tag{2.7}$$

we have the following Ricci identities:

$$h^{p^*}_{ijkl} - h^{p^*}_{ijlk} = \sum_m h^{p^*}_{mj} R_{mikl} + \sum_m h^{p^*}_{im} R_{mjkl} + \sum_q h^{q^*}_{ij} R_{q^*p^*kl}. \tag{2.8}$$

Defining

$$\begin{aligned}\sum_m h^{p^*}_{ijklm}\omega_m = dh^{p^*}_{ijkl} &+ \sum_m h^{p^*}_{mjkl}\omega_{mi} + \sum_m h^{p^*}_{imkl}\omega_{mj} + \sum_m h^{p^*}_{ijml}\omega_{mk}\\ &+ \sum_m h^{p^*}_{ijkm}\omega_{ml} + \sum_m h^{m^*}_{ijkl}\omega_{m^*p^*}\end{aligned} \tag{2.9}$$

and taking exterior differentiation of (2.7), we get

$$\begin{aligned}h^{p^*}_{ijkln} - h^{p^*}_{ijknl} = &\sum_m h^{p^*}_{mjk} R_{miln} + \sum_m h^{p^*}_{imk} R_{mjln} + \sum_m h^{p^*}_{ijm} R_{mkln}\\ &+ \sum_m h^{m^*}_{ijk} R_{m^*p^*ln}.\end{aligned} \tag{2.10}$$

For the mean curvature vector field $\vec{H} = \sum_p H^{p^*} e_{p^*}$, we define

$$\sum_i H^{p^*}_{,i} \omega_i = dH^{p^*} + \sum_q H^{q^*} \omega_{q^* p^*}, \tag{2.11}$$

$$\sum_j H^{p^*}_{,ij} \omega_j = dH^{p^*}_{,i} + \sum_j H^{p^*}_{,j} \omega_{ji} + \sum_q H^{q^*}_{,i} \omega_{q^* p^*}, \tag{2.12}$$

$$|\nabla^{\perp} \vec{H}|^2 = \sum_{i,p} (H^{p^*}_{,i})^2, \quad \Delta^{\perp} H^{p^*} = \sum_i H^{p^*}_{,ii}. \tag{2.13}$$

For a smooth function f, the $\mathcal{L}$-operator is defined by

$$\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle, \tag{2.14}$$

where Δ and ∇ denote the Laplacian and the gradient operator, respectively.

Formulas in the following Lemma 2.1 may be found in several papers, for examples, [4, 8, 22, 23]. Since many calculations in their proof are used in this paper, we also provide the proofs for reader's convenience.

If $X : M^2 \to \mathbb{R}^4$ is a self-shrinker, then we have

$$H^{p^*} = -\langle X, e_{p^*} \rangle, \quad p = 1, 2. \tag{2.15}$$

From (2.15), we can get

$$H^{p^*}_{,i} = \nabla_i H^{p^*} = -\nabla_i \langle X, e_{p^*} \rangle = \sum_j h^{p^*}_{ij} \langle X, e_j \rangle. \tag{2.16}$$

Since

$$\nabla_i |X|^2 = 2\langle X, e_i \rangle, \tag{2.17}$$

we have the following equations from (2.15)

$$\begin{aligned} \nabla_j \nabla_i |X|^2 &= 2\langle e_i, e_j \rangle + 2\langle X, X_{ij} \rangle \\ &= 2\delta_{ij} + 2\langle X, \sum_p h^{p^*}_{ij} e_{p^*} \rangle \\ &= 2\delta_{ij} - 2\sum_p h^{p^*}_{ij} H^{p^*}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \nabla_j \nabla_i H^{p^*} &= \nabla_j (\sum_k h^{p^*}_{ik} \langle X, e_k \rangle) \\ &= \sum_k h^{p^*}_{ikj} \langle X, e_k \rangle + h^{p^*}_{ij} + \sum_k h^{p^*}_{ik} \sum_q h^{q^*}_{jk} \langle X, e_{q^*} \rangle \\ &= \sum_k h^{p^*}_{ikj} \langle X, e_k \rangle + h^{p^*}_{ij} - \sum_{k,q} h^{p^*}_{ik} h^{q^*}_{jk} H^{q^*}. \end{aligned} \tag{2.19}$$

By a direct calculation, from (2.15) and (2.19), we have

$$\mathcal{L} H^{p^*} = \sum_k H^{p^*}_{,kk} - \langle X, \sum_k H^{p^*}_{,k} e_k \rangle = H^{p^*} - \sum_{i,j,q} h^{p^*}_{ij} h^{q^*}_{ij} H^{q^*}. \tag{2.20}$$

From the definition of the self-shrinker, we get

$$\frac{1}{2} \mathcal{L} |X|^2 = 2 - H^2 - \langle X, \sum_i \langle X, e_i \rangle e_i \rangle = 2 - H^2 - |X^{\top}|^2 = 2 - |X|^2. \tag{2.21}$$

Since $X : M^2 \to \mathbb{R}^4$ is a 2-dimensional Lagrangian self-shrinker, we know

$$R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = R_{i^*j^*kl}, \tag{2.22}$$

where $K = \frac{1}{2}(H^2 - S)$ is the Gaussian curvature of $X : M^2 \to \mathbb{R}^4$.

According to (2.3), (2.6), (2.8), (2.22), we have

$$\begin{aligned}
\mathcal{L}h_{ij}^{p^*} &= \sum_k h_{ijkk}^{p^*} - \langle X, \sum_k \nabla_k h_{ij}^{p^*} e_k\rangle \\
&= \sum_k h_{ikjk}^{p^*} - \sum_k h_{ijk}^{p^*}\langle X, e_k\rangle \\
&= \sum_{m,k} h_{mk}^{p^*} R_{mijk} + \sum_{m,k} h_{im}^{p^*} R_{mkjk} \\
&\quad + \sum_{q,k} h_{ik}^{q^*} R_{q^*p^*jk} + \sum_k h_{ikkj}^{p^*} - \sum_k h_{ijk}^{p^*}\langle X, e_k\rangle \\
&= K\sum_{m,k} h_{mk}^{p^*}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) + K\sum_{m,k} h_{im}^{p^*}(\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{kj}) \\
&\quad + K\sum_{q,k} h_{ik}^{q^*}(\delta_{qj}\delta_{pk} - \delta_{qk}\delta_{pj}) + H_{,ij}^{p^*} - \sum_k h_{ijk}^{p^*}\langle X, e_k\rangle \\
&= K(h_{ij}^{p^*} - H^{p^*}\delta_{ij}) + K(2h_{ij}^{p^*} - h_{ij}^{p^*}) + K(h_{ij}^{p^*} - \sum_k h_{kk}^{i^*}\delta_{pj}) \\
&\quad + \sum_k h_{ijk}^{p^*}\langle X, e_k\rangle + h_{ij}^{p^*} - \sum_{q,k} h_{ik}^{p^*} h_{jk}^{q^*} H^{q^*} - \sum_k h_{ijk}^{p^*}\langle X, e_k\rangle \\
&= (3K+1)h_{ij}^{p^*} - K(H^{p^*}\delta_{ij} + H^{i^*}\delta_{pj}) - \sum_{q,k} h_{ik}^{p^*} h_{jk}^{q^*} H^{q^*}.
\end{aligned} \tag{2.23}$$

Hence, we get

$$\begin{aligned}
\frac{1}{2}\mathcal{L}S &= \frac{1}{2}\sum_k \nabla_k\nabla_k \sum_{i,j,p}(h_{ij}^{p^*})^2 - \frac{1}{2}\langle X, \sum_k \nabla_k S e_k\rangle \\
&= \sum_k \nabla_k(\sum_{i,j,p} h_{ijk}^{p^*} h_{ij}^{p^*}) - \langle X, \sum_{i,j,k,p} h_{ijk}^{p^*} h_{ij}^{p^*} e_k\rangle \\
&= \sum_{i,j,p} h_{ij}^{p^*}\mathcal{L}h_{ij}^{p^*} + \sum_{i,j,k,p}(h_{ijk}^{p^*})^2 \\
&= \sum_{i,j,k,p}(h_{ijk}^{p^*})^2 \\
&\quad + \sum_{i,j,p} h_{ij}^{p^*}\Big[(3K+1)h_{ij}^{p^*} - K(H^{p^*}\delta_{ij} + H^{i^*}\delta_{pj}) - \sum_{k,q} h_{ik}^{p^*} h_{jk}^{q^*} H^{q^*}\Big] \\
&= \sum_{i,j,k,p}(h_{ijk}^{p^*})^2 + (3K+1)S - K(H^2 + H^2) - \sum_{i,j,k,p,q} h_{ik}^{p^*} h_{ij}^{p^*} h_{jk}^{q^*} H^{q^*} \\
&= \sum_{i,j,k,p}(h_{ijk}^{p^*})^2 + S(1 - \frac{3}{2}S) + \frac{5}{2}H^2 S - H^4 - \sum_{i,j,k,p,q} h_{ik}^{p^*} h_{ij}^{p^*} h_{jk}^{q^*} H^{q^*}.
\end{aligned} \tag{2.24}$$

From (2.6) and (2.22), we get

$$\sum_{p,i} h^{p^*}_{ik} h^{p^*}_{ji} - \sum_{p} H^{p^*} h^{p^*}_{jk} = K(\delta_{kj} - 2\delta_{jk})$$

and

$$\sum_{p,i} h^{p^*}_{ik} h^{p^*}_{ji} = -K\delta_{jk} + \sum_{p} H^{p^*} h^{p^*}_{jk}.$$

Since

$$\sum_{j,k}(K\delta_{jk} - \sum_{p} H^{p^*} h^{p^*}_{jk}) \sum_{q} h^{q^*}_{jk} H^{q^*} = KH^2 - \sum_{j,k}\sum_{p}(H^{p^*} h^{p^*}_{jk}) \sum_{q}(H^{q^*} h^{q^*}_{jk}),$$

we obtain from (2.24)

$$\begin{aligned}
\frac{1}{2}\mathcal{L}S &= \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 + S(1 - \frac{3}{2}S) + \frac{5}{2}H^2 S - H^4 \\
&\quad + \sum_{j,k}(K\delta_{jk} - \sum_{p} H^{p^*} h^{p^*}_{jk}) \sum_{q} h^{q^*}_{jk} H^{q^*} \\
&= \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 + S(1 - \frac{3}{2}S) + 2H^2 S - \frac{1}{2}H^4 - \sum_{j,k,p,q} H^{p^*} h^{p^*}_{jk} H^{q^*} h^{q^*}_{jk}.
\end{aligned}$$

From (2.20), we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}H^2 &= \frac{1}{2}\mathcal{L}\sum_{p}(H^{p^*})^2 = \sum_{i,p}(\nabla_i H^{p^*})^2 + \sum_{p} H^{p^*}\mathcal{L}H^{p^*} \\
&= |\nabla^\perp \vec{H}|^2 + H^2 - \sum_{i,j,p,q} H^{p^*} h^{p^*}_{ij} H^{q^*} h^{q^*}_{ij}.
\end{aligned}$$

Thus, we conclude the following lemma

Lemma 2.1 *Let $X : M^2 \to \mathbb{R}^4$ is a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. We have*

$$\frac{1}{2}\mathcal{L}S = \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 + S(1 - \frac{3}{2}S) + 2H^2 S - \frac{1}{2}H^4 - \sum_{j,k,p,q} H^{p^*} h^{p^*}_{jk} H^{q^*} h^{q^*}_{jk}. \tag{2.25}$$

$$\frac{1}{2}\mathcal{L}H^2 = |\nabla^\perp \vec{H}|^2 + H^2 - \sum_{i,j,p,q} H^{p^*} h^{p^*}_{ij} \cdot H^{q^*} h^{q^*}_{ij}. \tag{2.26}$$

Next, we will prove the following lemma, by making use of a long calculation:

Lemma 2.2 *Let $X: M^2 \to \mathbb{R}^4$ is a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. Then*

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&\quad=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2+(10K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-5K|\nabla^{\perp}\vec{H}|^2+3\langle\nabla K,\nabla S\rangle\\
&\qquad-\frac{3K}{4}\langle\nabla S,\nabla|X|^2\rangle-\langle\nabla K,\nabla H^2\rangle-3\sum_{j,l,p}K_{,l}h^{p^*}_{lj}H^{p^*}_{,j}\\
&\qquad-2\sum_{i,j,k,l,p,q}h^{p^*}_{ijk}h^{p^*}_{ijl}h^{q^*}_{kl}H^{q^*}-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jlk}H^{q^*}\\
&\quad-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jl}H^{q^*}_{,k}
\end{aligned}\tag{2.27}$$

holds.

Proof We have the following equation from the Ricci identities (2.10).

$$\begin{aligned}
\mathcal{L}h^{p^*}_{ijk}&=\sum_l h^{p^*}_{ijkll}-\langle X,\sum_l\nabla_l h^{p^*}_{ijk}e_l\rangle\\
&=\sum_l h^{p^*}_{ijlkl}+\sum_{l,m}(h^{p^*}_{mj}R_{mikl}+h^{p^*}_{im}R_{mjkl}+h^{m^*}_{ij}R_{m^*p^*kl})_{,l}-\langle X,\sum_l\nabla_l h^{p^*}_{ijk}e_l\rangle\\
&=\sum_l h^{p^*}_{ijllk}+\sum_{l,m}h^{p^*}_{mjl}R_{mikl}+\sum_{l,m}h^{p^*}_{iml}R_{mjkl}+\sum_{l,m}h^{p^*}_{ijm}R_{mlkl}\\
&\quad+\sum_{l,m}h^{m^*}_{ijl}R_{m^*p^*kl}+\sum_{l,m}h^{p^*}_{mjl}R_{mikl}\\
&\quad+\sum_{l,m}h^{p^*}_{iml}R_{mjkl}+\sum_{l,m}h^{m^*}_{ijl}R_{m^*p^*kl}+\sum_{l,m}h^{p^*}_{mj}R_{mikl,l}\\
&\quad+\sum_{l,m}h^{p^*}_{im}R_{mjkl,l}+\sum_{l,m}h^{m^*}_{ij}R_{m^*p^*kl,l}-\langle X,\sum_l\nabla_l h^{p^*}_{ijk}e_l\rangle.
\end{aligned}\tag{2.28}$$

From (2.23), we have

$$\begin{aligned}
\sum_l h^{p^*}_{ijllk}&=\Big[(3K+1)h^{p^*}_{ij}-K(H^{p^*}\delta_{ij}+H^{i^*}\delta_{pj})\\
&\qquad-\sum_{l,q}h^{p^*}_{il}h^{q^*}_{jl}H^{q^*}+\langle X,\sum_l h^{p^*}_{ijl}e_l\rangle\Big]_{,k}\\
&=3K_{,k}h^{p^*}_{ij}+(3K+1)h^{p^*}_{ijk}-K_{,k}(H^{p^*}\delta_{ij}+H^{i^*}\delta_{pj})\\
&\quad-K(H^{p^*}_{,k}\delta_{ij}+H^{i^*}_{,k}\delta_{pj})\\
&\quad-\sum_{l,q}h^{p^*}_{ilk}h^{q^*}_{jl}H^{q^*}-\sum_{l,q}h^{p^*}_{il}h^{q^*}_{jlk}H^{q^*}-\sum_{l,q}h^{p^*}_{il}h^{q^*}_{jl}H^{q^*}_{,k}\\
&\quad+h^{p^*}_{ijk}+\langle X,\sum_{l,q}h^{p^*}_{ijl}h^{q^*}_{lk}e_{q^*}\rangle+\langle X,\sum_l h^{p^*}_{ijlk}e_l\rangle.
\end{aligned}\tag{2.29}$$

From (2.22), we obtain

$$
\begin{aligned}
(a)^{p^*}_{ijk} :=& \sum_{l,m} h^{p^*}_{mjl} R_{mikl} + \sum_{l,m} h^{p^*}_{iml} R_{mjkl} + \sum_{l,m} h^{p^*}_{ijm} R_{mlkl} + \sum_{l,m} h^{m^*}_{ijl} R_{m^* p^* kl} \\
&+ \sum_{l,m} h^{p^*}_{mjl} R_{mikl} + \sum_{l,m} h^{p^*}_{iml} R_{mjkl} + \sum_{l,m} h^{m^*}_{ijl} R_{m^* p^* kl} \\
=& 2K \sum_{l,m} \Big[h^{p^*}_{mjl} (\delta_{mk}\delta_{il} - \delta_{ml}\delta_{ik}) + h^{p^*}_{iml} (\delta_{mk}\delta_{jl} - \delta_{ml}\delta_{jk}) \\
&+ h^{m^*}_{ijl} (\delta_{mk}\delta_{pl} - \delta_{ml}\delta_{pk}) \Big] + \sum_{l,m} K h^{p^*}_{ijm} (\delta_{mk}\delta_{ll} - \delta_{ml}\delta_{lk}) \\
=& 2K \Big[h^{p^*}_{ijk} - H^{p^*}_{,j} \delta_{ik} + h^{p^*}_{ijk} - H^{p^*}_{,i} \delta_{jk} + h^{p^*}_{ijk} - H^{i^*}_{,j} \delta_{pk} + h^{p^*}_{ijk} - \frac{1}{2} h^{p^*}_{ijk} \Big] \\
=& 2K \Big[\frac{7}{2} h^{p^*}_{ijk} - H^{p^*}_{,j} \delta_{ik} - H^{p^*}_{,i} \delta_{jk} - H^{i^*}_{,j} \delta_{pk} \Big], \\
(b)^{p^*}_{ijk} :=& \sum_{l,m} h^{p^*}_{mj} R_{mikl,l} + \sum_{l,m} h^{p^*}_{im} R_{mjkl,l} + \sum_{l,m} h^{m^*}_{ij} R_{m^* p^* kl,l} \\
=& \sum_{l,m} K_{,l} \Big[h^{p^*}_{mj} (\delta_{mk}\delta_{il} - \delta_{ml}\delta_{ik}) + h^{p^*}_{im} (\delta_{mk}\delta_{jl} - \delta_{ml}\delta_{jk}) \\
&+ h^{m^*}_{ij} (\delta_{mk}\delta_{pl} - \delta_{ml}\delta_{pk}) \Big] \\
=& K_{,i} h^{p^*}_{jk} - \sum_{l} K_{,l} h^{p^*}_{lj} \delta_{ik} + K_{,j} h^{p^*}_{ik} - \sum_{l} K_{,l} h^{p^*}_{il} \delta_{jk} + K_{,p} h^{k^*}_{ij} - \sum_{l} K_{,l} h^{l^*}_{ij} \delta_{pk}
\end{aligned}
$$

and

$$
\begin{aligned}
(c)^{p^*}_{ijk} :=& \sum_{l} \langle X, e_l \rangle h^{p^*}_{ijlk} - \sum_{l} \langle X, e_l \rangle h^{p^*}_{ijkl} \\
=& \sum_{l,m} \langle X, e_l \rangle \Big[h^{p^*}_{mj} R_{milk} + h^{p^*}_{im} R_{mjlk} + h^{m^*}_{ij} R_{m^* p^* lk} \Big] \\
=& K \sum_{l,m} \langle X, e_l \rangle \Big[h^{p^*}_{mj} (\delta_{ml}\delta_{ik} - \delta_{mk}\delta_{il}) + h^{p^*}_{im} (\delta_{ml}\delta_{jk} - \delta_{mk}\delta_{jl}) \\
&+ h^{m^*}_{ij} (\delta_{ml}\delta_{pk} - \delta_{mk}\delta_{pl}) \Big] \\
=& K \sum_{l} \langle X, e_l \rangle \Big[h^{p^*}_{lj} \delta_{ik} - h^{p^*}_{kj} \delta_{il} + h^{p^*}_{il} \delta_{jk} - h^{p^*}_{ik} \delta_{jl} + h^{l^*}_{ij} \delta_{pk} - h^{k^*}_{ij} \delta_{pl} \Big] \\
=& K \Big[\sum_{l} \langle X, e_l \rangle h^{p^*}_{lj} \delta_{ik} - \langle X, e_i \rangle h^{p^*}_{kj} + \sum_{l} \langle X, e_l \rangle h^{p^*}_{il} \delta_{jk} - \langle X, e_j \rangle h^{p^*}_{ik} \\
&+ \sum_{l} \langle X, e_l \rangle h^{l^*}_{ij} \delta_{pk} - \langle X, e_p \rangle h^{k^*}_{ij} \Big].
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&\sum_{i,j,k,p} h^{p^*}_{ijk}\cdot((a)^{p^*}_{ijk}+(b)^{p^*}_{ijk})\\
&\quad=2K\Big[\frac{7}{2}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-\sum_{j,p}H^{p^*}_{,j}H^{p^*}_{,j}-\sum_{i,p}H^{p^*}_{,i}H^{p^*}_{,i}-\sum_{j,i}H^{i^*}_{,j}H^{i^*}_{,j}\Big]\\
&\qquad+\sum_{i,j,k,p}K_{,i}h^{p^*}_{jk}h^{p^*}_{ijk}-\sum_{l,j,p}K_{,l}h^{p^*}_{lj}H^{p^*}_{,j}+\sum_{i,j,k,p}K_{,j}h^{p^*}_{ik}h^{p^*}_{ijk}\\
&\qquad-\sum_{l,i,p}K_{,l}h^{p^*}_{il}H^{p^*}_{,i}+\sum_{i,j,k,p}K_{,p}h^{k^*}_{ij}h^{p^*}_{ijk}-\sum_{l,i,j}K_{,l}h^{l^*}_{ij}H^{i^*}_{,j}\\
&\quad=2K\Big[\frac{7}{2}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-3|\nabla^{\perp}\vec{H}|^2\Big]+\frac{3}{2}\langle\nabla K,\nabla S\rangle-3\sum_{l,j,p}K_{,l}h^{p^*}_{lj}H^{p^*}_{,j}
\end{aligned}
\tag{2.30}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,p} h^{p^*}_{ijk}\cdot(c)^{p^*}_{ijk}\\
&\quad=K\Big[\sum_{l,j,p}\langle X,e_l\rangle h^{p^*}_{lj}H^{p^*}_{,j}+\sum_{l,i,p}\langle X,e_l\rangle h^{p^*}_{li}H^{p^*}_{,i}+\sum_{l,i,j}\langle X,e_l\rangle h^{l^*}_{ij}H^{i^*}_{,j}\\
&\qquad-\frac{1}{2}\sum_{i}\langle X,e_i\rangle\nabla_i S-\frac{1}{2}\sum_{i}\langle X,e_i\rangle\nabla_i S-\frac{1}{2}\sum_{i}\langle X,e_i\rangle\nabla_i S\Big]\\
&\quad=K\Big(3\sum_{l,j,p}\langle X,e_l\rangle h^{p^*}_{lj}H^{p^*}_{,j}-\frac{3}{4}\langle\nabla S,\nabla\langle X,X\rangle\rangle\Big)\\
&\quad=3K(|\nabla^{\perp}\vec{H}|^2-\frac{1}{4}\langle\nabla S,\nabla|X|^2\rangle).
\end{aligned}
\tag{2.31}
$$

From (2.29), we have

$$
\begin{aligned}
&\sum_{i,j,k,l,p} h^{p^*}_{ijk}(h^{p^*}_{ijll,k}-\langle X,h^{p^*}_{ijlk}e_l\rangle)\\
&=3\sum_{i,j,k,p}K_{,k}h^{p^*}_{ij}h^{p^*}_{ijk}+(3K+1)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-\sum_{k,p}K_{,k}(H^{p^*}H^{p^*}_{,k}+H^{p^*}H^{p^*}_{,k})\\
&\quad-\sum_{k,p}K(H^{p^*}_{,k}H^{p^*}_{,k}+H^{p^*}_{,k}H^{p^*}_{,k})-\sum_{i,j,k,l,p,q}h^{p^*}_{ijk}h^{p^*}_{ilk}h^{q^*}_{jl}H^{q^*}\\
&\quad-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jlk}H^{q^*}-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jl}H^{q^*}_{,k}\\
&\quad+\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-\sum_{i,j,k,l,p,q}h^{p^*}_{ijk}h^{p^*}_{ijl}h^{q^*}_{lk}H^{q^*}\\
&=\frac{3}{2}\langle\nabla K,\nabla S\rangle+(3K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-\langle\nabla K,\nabla H^2\rangle-2K|\nabla^{\perp}\vec{H}|^2\\
&\quad-2\sum_{i,j,k,l,p,q}h^{p^*}_{ijk}h^{p^*}_{ijl}h^{q^*}_{kl}H^{q^*}-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jlk}H^{q^*}-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jl}H^{q^*}_{,k}.
\end{aligned}
\tag{2.32}
$$

From the above equations, we get

$$\begin{aligned}
\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 &= \sum_{i,j,k,p} h^{p^*}_{ijk}\mathcal{L}h^{p^*}_{ijk} + \sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2 \\
&= \sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2 + 2K[\frac{7}{2}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 - 3|\nabla^\perp \vec{H}|^2] + \frac{3}{2}\langle \nabla K, \nabla S\rangle \\
&\quad - 3\sum_{l,j,p} K_{,l} h^{p^*}_{lj} H^{p^*}_{,j} + 3K\Big[|\nabla^\perp \vec{H}|^2 - \frac{1}{4}\langle \nabla S, \nabla |X|^2\rangle\Big] \\
&\quad + \frac{3}{2}\langle \nabla K, \nabla S\rangle + (3K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 - \langle \nabla K, \nabla H^2\rangle - 2K|\nabla^\perp \vec{H}|^2 \\
&\quad - 2\sum_{i,j,k,l,p,q} h^{p^*}_{ijk} h^{p^*}_{ijl} h^{q^*}_{kl} H^{q^*} - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jlk} H^{q^*} \\
&\quad - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jl} H^{q^*}_{,k} \\
&= \sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2 + (10K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 - 5K|\nabla^\perp \vec{H}|^2 \\
&\quad + 3\langle \nabla K, \nabla S\rangle - \frac{3K}{4}\langle \nabla S, \nabla |X|^2\rangle - \langle \nabla K, \nabla H^2\rangle \\
&\quad - 3\sum_{l,j,p} K_{,l} h^{p^*}_{lj} H^{p^*}_{,j} - 2\sum_{i,j,k,l,p,q} h^{p^*}_{ijk} h^{p^*}_{ijl} h^{q^*}_{kl} H^{q^*} \\
&\quad - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jlk} H^{q^*} - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jl} H^{q^*}_{,k}.
\end{aligned} \tag{2.33}$$

It completes the proof of the lemma. □

Lemma 2.3 *Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. If S is constant, we have*

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 \\
&\quad = \sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2 + (10K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 - 5K|\nabla^\perp \vec{H}|^2 \\
&\quad - \langle \nabla K, \nabla H^2\rangle - 3\sum_{l,j,p} K_{,l} h^{p^*}_{lj} H^{p^*}_{,j} - 2\sum_{i,j,k,l,p,q} h^{p^*}_{ijk} h^{p^*}_{ijl} h^{q^*}_{kl} H^{q^*} \\
&\quad - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jlk} H^{q^*} - \sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jl} H^{q^*}_{,k}
\end{aligned} \tag{2.34}$$

and

$$
\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&= (H^2-2S)(|\nabla^{\perp}\vec{H}|^2+H^2)+\frac{1}{2}|\nabla H^2|^2\\
&\quad+(3K+2-H^2+2S)\sum_{i,j,p,q}H^{p^*}h^{p^*}_{ij}H^{q^*}h^{q^*}_{ij}\\
&\quad-K(H^4+\sum_{j,k,p}H^{k^*}H^{j^*}H^{p^*}h^{p^*}_{jk})-\sum_{i,j,k,l,p,q,r}H^{r^*}H^{q^*}h^{r^*}_{jk}h^{p^*}_{jk}h^{p^*}_{il}h^{q^*}_{il}\\
&\quad+2\sum_{i,j,k,p,q}H^{p^*}_{,i}H^{q^*}h^{q^*}_{jk}h^{p^*}_{ijk}-\sum_{i,j,k,p,q,r}H^{p^*}H^{q^*}H^{r^*}h^{p^*}_{ik}h^{q^*}_{ji}h^{r^*}_{jk}\\
&\quad+\sum_{i,j,k}\Big(\sum_{q}(H^{q^*}_{,i}h^{q^*}_{jk}+H^{q^*}h^{q^*}_{ijk})\Big)\cdot\Big(\sum_{p}(H^{p^*}_{,i}h^{p^*}_{jk}+H^{p^*}h^{p^*}_{ijk})\Big).
\end{aligned}
\tag{2.35}
$$

Proof Since S is constant, we have the following equation from (2.33)

$$
\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2+(10K+2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2-5K|\nabla^{\perp}\vec{H}|^2-\langle\nabla K,\nabla H^2\rangle\\
&\quad-3\sum_{j,l,p}K_{,l}h^{p^*}_{lj}H^{p^*}_{,j}-2\sum_{i,j,k,l,p,q}h^{p^*}_{ijk}h^{p^*}_{ijl}h^{q^*}_{kl}H^{q^*}\\
&\quad-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jlk}H^{q^*}-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jl}H^{q^*}_{,k}.
\end{aligned}
$$

Now, we prove the formula (2.35). From (2.25) in Lemma 2.1, we obtain

$$
0=\frac{1}{2}\mathcal{L}S=\sum_{i,j,k,p}(h^{p^*}_{ijk})^2+S(1-\frac{3}{2}S)+2H^2S-\frac{1}{2}H^4-\sum_{j,k,p,q}H^{p^*}h^{p^*}_{jk}H^{q^*}h^{q^*}_{jk}.
\tag{2.36}
$$

Hence, we have

$$
\begin{aligned}
\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2&=-S\mathcal{L}H^2+\frac{1}{4}\mathcal{L}H^4+\frac{1}{2}\mathcal{L}\sum_{j,k,p,q}H^{p^*}h^{p^*}_{jk}H^{q^*}h^{q^*}_{jk}\\
&=-2S\Big(|\nabla^{\perp}\vec{H}|^2+H^2-\sum_{i,j,p,q}H^{p^*}h^{p^*}_{ij}H^{q^*}h^{q^*}_{ij}\Big)+\frac{1}{2}H^2\mathcal{L}H^2\\
&\quad+\frac{1}{2}|\nabla H^2|^2+\sum_{j,k}(\sum_{q}H^{q^*}h^{q^*}_{jk})\mathcal{L}(\sum_{p}H^{p^*}h^{p^*}_{jk})\\
&\quad+\sum_{i,j,k}\nabla_i(\sum_{q}H^{q^*}h^{q^*}_{jk})\cdot\nabla_i(\sum_{p}H^{p^*}h^{p^*}_{jk})\\
&=(H^2-2S)\Big(|\nabla^{\perp}\vec{H}|^2+H^2-\sum_{i,j,p,q}H^{p^*}h^{p^*}_{ij}H^{q^*}h^{q^*}_{ij}\Big)
\end{aligned}
$$

$$+\frac{1}{2}|\nabla H^2|^2+\sum_{j,k}(\sum_q H^{q^*}h^{q^*}_{jk})\mathcal{L}(\sum_p H^{p^*}h^{p^*}_{jk})$$
$$+\sum_{i,j,k,p,q}(H^{q^*}_{,i}h^{q^*}_{jk}+H^{q^*}h^{q^*}_{ijk})\cdot(H^{p^*}_{,i}h^{p^*}_{jk}+H^{p^*}h^{p^*}_{ijk}). \tag{2.37}$$

Since

$$\begin{aligned}\mathcal{L}\sum_p(H^{p^*}h^{p^*}_{jk})&=\sum_{i,p}\nabla_i\nabla_i(H^{p^*}h^{p^*}_{jk})-\langle X,\sum_{i,p}\nabla_i(H^{p^*}h^{p^*}_{jk})e_i\rangle\\&=\sum_{i,p}\nabla_i(H^{p^*}_{,i}h^{p^*}_{jk}+H^{p^*}h^{p^*}_{ijk})-\langle X,\sum_{i,p}(\nabla_iH^{p^*}h^{p^*}_{jk}+H^{p^*}\nabla_ih^{p^*}_{jk})e_i\rangle\\&=\sum_p h^{p^*}_{jk}\mathcal{L}H^{p^*}+\sum_p H^{p^*}\mathcal{L}h^{p^*}_{jk}+2\sum_{i,p}H^{p^*}_{,i}h^{p^*}_{ijk}\end{aligned} \tag{2.38}$$

and from (2.20) and (2.23),

$$\sum_p h^{p^*}_{jk}\mathcal{L}H^{p^*}=\sum_p h^{p^*}_{jk}H^{p^*}-\sum_{i,l,p,q}h^{p^*}_{jk}h^{p^*}_{il}h^{q^*}_{il}H^{q^*}, \tag{2.39}$$

$$\begin{aligned}\sum_p H^{p^*}\mathcal{L}h^{p^*}_{jk}&=(3K+1)\sum_p H^{p^*}h^{p^*}_{jk}-K\sum_p((H^{p^*})^2\delta_{kj}+H^{p^*}H^{k^*}\delta_{pj})\\&\quad-\sum_{i,p,q}h^{p^*}_{ik}H^{p^*}h^{q^*}_{ji}H^{q^*}\\&=(3K+1)\sum_p H^{p^*}h^{p^*}_{jk}-K(H^2\delta_{kj}+H^{k^*}H^{j^*})-\sum_{i,p,q}h^{p^*}_{ik}H^{p^*}h^{q^*}_{ji}H^{q^*},\end{aligned} \tag{2.40}$$

we get

$$\begin{aligned}\mathcal{L}\sum_p(H^{p^*}h^{p^*}_{jk})&=\sum_p h^{p^*}_{jk}H^{p^*}-\sum_{i,l,p,q}h^{p^*}_{jk}h^{p^*}_{il}h^{q^*}_{il}H^{q^*}+2\sum_{i,p}H^{p^*}_{,i}h^{p^*}_{ijk}\\&\quad+(3K+1)\sum_p H^{p^*}h^{p^*}_{jk}-K(H^2\delta_{kj}+H^{k^*}H^{j^*})-\sum_{i,p,q}h^{p^*}_{ik}H^{p^*}h^{q^*}_{ji}H^{q^*}\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}&\sum_{j,k}(\sum_q H^{q^*}h^{q^*}_{jk})\mathcal{L}(\sum_p H^{p^*}h^{p^*}_{jk})\\&=-\sum_{i,j,k,l,p,q,r}H^{r^*}h^{r^*}_{jk}h^{p^*}_{jk}h^{p^*}_{il}h^{q^*}_{il}H^{q^*}+2\sum_{i,j,k,p,q}H^{p^*}_{,i}h^{p^*}_{ijk}H^{q^*}h^{q^*}_{jk}\\&\quad+(3K+2)\sum_{j,k}\Big(\sum_q H^{q^*}h^{q^*}_{jk}\Big)\Big(\sum_p H^{p^*}h^{p^*}_{jk}\Big)\end{aligned}$$

$$-K(H^4+\sum_{j,k,p}H^{k^*}H^{j^*}H^{p^*}h^{p^*}_{jk})-\sum_{i,j,k,p,q,r}H^{p^*}H^{q^*}H^{r^*}h^{p^*}_{ik}h^{q^*}_{ji}h^{r^*}_{jk}. \tag{2.42}$$

From the above equations, we conclude

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&=(H^2-2S)(|\nabla^{\perp}\vec{H}|^2+H^2)+\frac{1}{2}|\nabla H^2|^2\\
&\quad+(3K+2-H^2+2S)\sum_{i,j,p,q}H^{p^*}h^{p^*}_{ij}H^{q^*}h^{q^*}_{ij}\\
&\quad-K(H^4+\sum_{j,k,p}H^{k^*}H^{j^*}H^{p^*}h^{p^*}_{jk})-\sum_{i,j,k,l,p,q,r}H^{r^*}H^{q^*}h^{r^*}_{jk}h^{p^*}_{jk}h^{p^*}_{il}h^{q^*}_{il}\\
&\quad+2\sum_{i,j,k,p,q}H^{p^*}_{,i}H^{q^*}h^{q^*}_{jk}h^{p^*}_{ijk}-\sum_{i,j,k,p,q,r}H^{p^*}H^{q^*}H^{r^*}h^{p^*}_{ik}h^{q^*}_{ji}h^{r^*}_{jk}\\
&\quad+\sum_{i,j,k}\Big(\sum_q(H^{q^*}_{,i}h^{q^*}_{jk}+H^{q^*}h^{q^*}_{ijk})\Big)\cdot\Big(\sum_p(H^{p^*}_{,i}h^{p^*}_{jk}+H^{p^*}h^{p^*}_{ijk})\Big).
\end{aligned}$$

□

Lemma 2.4 *Let $X: M^2\to\mathbb{R}^4$ be a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. Then we have*

$$\begin{aligned}
&\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2\\
&=4(h^{1^*}_{1122})^2+6(h^{1^*}_{2211})^2+6(h^{2^*}_{1122})^2+4(h^{2^*}_{2211})^2+4(h^{1^*}_{2222})^2+4(h^{2^*}_{1111})^2\\
&\quad+(h^{1^*}_{1111})^2+(h^{2^*}_{2222})^2+(h^{1^*}_{1112})^2+(h^{2^*}_{2221})^2\\
&\geq 2(h^{1^*}_{1122}-h^{1^*}_{2211})^2+2(h^{2^*}_{1122}-h^{2^*}_{2211})^2+\frac{1}{2}(h^{1^*}_{2222}-h^{2^*}_{2221})^2.
\end{aligned} \tag{2.43}$$

Proof Since

$$\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2=\sum_{i,j,k,l}(h^{1^*}_{ijkl})^2+\sum_{i,j,k,l}(h^{2^*}_{ijkl})^2, \tag{2.44}$$

$$\begin{aligned}
\sum_{i,j,k,l}(h^{1^*}_{ijkl})^2&=(h^{1^*}_{1111})^2+(h^{1^*}_{1112})^2+(h^{1^*}_{2221})^2+(h^{1^*}_{2222})^2\\
&\quad+3(h^{1^*}_{1121})^2+3(h^{1^*}_{1122})^2+3(h^{1^*}_{2211})^2+3(h^{1^*}_{2212})^2\\
&=(h^{1^*}_{1111})^2+3\Big[(h^{1^*}_{1122})^2+(h^{1^*}_{2211})^2\Big]+3(h^{2^*}_{1122})^2\\
&\quad+(h^{1^*}_{1112})^2+(h^{2^*}_{2211})^2+3(h^{2^*}_{1111})^2+(h^{1^*}_{2222})^2,
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
\sum_{i,j,k,l}(h^{2^*}_{ijkl})^2&=(h^{2^*}_{1111})^2+(h^{2^*}_{1112})^2+(h^{2^*}_{2222})^2+(h^{2^*}_{2221})^2\\
&\quad+3(h^{2^*}_{1121})^2+3(h^{2^*}_{1122})^2+3(h^{2^*}_{2211})^2+3(h^{2^*}_{2212})^2\\
&=(h^{2^*}_{1111})^2+(h^{2^*}_{2222})^2+(h^{2^*}_{2221})^2+(h^{1^*}_{1122})^2
\end{aligned}$$

$$+3(h^{1^*}_{2211})^2+3(h^{2^*}_{1122})^2+3(h^{2^*}_{2211})^2+3(h^{1^*}_{2222})^2, \tag{2.46}$$

we get

$$\begin{aligned}
&\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2\\
&=4(h^{1^*}_{1122})^2+6(h^{1^*}_{2211})^2+6(h^{2^*}_{1122})^2+(h^{1^*}_{1111})^2+(h^{1^*}_{1112})^2+3(h^{2^*}_{1111})^2\\
&\quad+(h^{1^*}_{2222})^2+(h^{2^*}_{1111})^2+(h^{2^*}_{2222})^2+(h^{2^*}_{2221})^2+3(h^{1^*}_{2222})^2+4(h^{2^*}_{2211})^2\\
&=4(h^{1^*}_{1122})^2+6(h^{1^*}_{2211})^2+6(h^{2^*}_{1122})^2+4(h^{2^*}_{2211})^2+4(h^{1^*}_{2222})^2+4(h^{2^*}_{1111})^2\\
&\quad+(h^{1^*}_{1111})^2+(h^{2^*}_{2222})^2+(h^{1^*}_{1112})^2+(h^{2^*}_{2221})^2\\
&\geq 2(h^{1^*}_{1122}-h^{1^*}_{2211})^2+2(h^{2^*}_{1122}-h^{2^*}_{2211})^2+\frac{1}{2}(h^{1^*}_{2222}-h^{2^*}_{2221})^2\\
&\quad+2(h^{1^*}_{1122}+h^{1^*}_{2211})^2+2(h^{2^*}_{1122}+h^{2^*}_{2211})^2+\frac{1}{2}(h^{1^*}_{2222}+h^{2^*}_{2221})^2\\
&\geq 2(h^{1^*}_{1122}-h^{1^*}_{2211})^2+2(h^{2^*}_{1122}-h^{2^*}_{2211})^2+\frac{1}{2}(h^{1^*}_{2222}-h^{2^*}_{2221})^2.
\end{aligned} \tag{2.47}$$

□

If $\vec{H}\neq 0$ at p, we can choose a local orthogonal frame $\{e_1,e_2\}$ such that

$$e_{1^*}=\frac{\vec{H}}{|\vec{H}|},\quad H^{1^*}=H=|\vec{H}|,\quad H^{2^*}=h^{2^*}_{11}+h^{2^*}_{22}=0. \tag{2.48}$$

Defining $\lambda=h^{1^*}_{12}$, $\lambda_1=h^{1^*}_{11}$ and $\lambda_2=h^{1^*}_{22}$, we have $h^{2^*}_{22}=-\lambda$.

Lemma 2.5 *Let $X:M^2\to\mathbb{R}^4$ be a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. If S is constant, $\vec{H}(p)\neq 0$ and $\sum_{i,j,k,p}(h^{p^*}_{ijk})^2(p)=0$, then we have, at p,*

$$\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2 \tag{2.49}$$

and

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&=H^2\Big[H^2-2S+\frac{1}{2}H^4-H^2\lambda^2-KH^2-K^2\Big]\\
&\quad+H^2(S+2-\frac{3}{2}H^2-\lambda_1^2-\lambda_2^2-2\lambda^2)(\lambda_1^2+\lambda_2^2+2\lambda^2).
\end{aligned} \tag{2.50}$$

Proof Since $\vec{H}\neq 0$ at p, we can choose a local orthogonal frame $\{e_1,e_2\}$ such that

$$e_{1^*}=\frac{\vec{H}}{|\vec{H}|},\quad H^{1^*}=H=|\vec{H}|,\quad H^{2^*}=h^{2^*}_{11}+h^{2^*}_{22}=0. \tag{2.51}$$

By the definition of $\lambda=h^{1^*}_{12}$, $\lambda_1=h^{1^*}_{11}$ and $\lambda_2=h^{1^*}_{22}$, we have $h^{2^*}_{22}=-\lambda$. Since S is constant and $h^{p^*}_{ijk}=0$ at p, we obtain from (2.34) of Lemma 2.3,

$$\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2. \tag{2.52}$$

Furthermore, by making use of

$$S=\lambda_1^2+3\lambda_2^2+4\lambda^2,\quad H\lambda_1=K+\lambda_1^2+\lambda_2^2+2\lambda^2, \tag{2.53}$$

from (2.35) in Lemma 2.3, we have the following equations, at p,

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&=(H^2-2S)H^2+(\frac{1}{2}H^2+\frac{1}{2}S+2)H^2(\lambda_1^2+\lambda_2^2+2\lambda^2)\\
&\quad-K(H^4+H^3\lambda_1)-H^2\Big\{(\lambda_1^2+\lambda_2^2+2\lambda^2)^2+\lambda^2H^2\Big\}\\
&\quad-H^3(\lambda_1^3+\lambda_2^3+3H\lambda^2)\\
&=H^2\Big[H^2-2S+(\frac{1}{2}H^2+\frac{1}{2}S+2-\lambda_1^2-\lambda_2^2-2\lambda^2)(\lambda_1^2+\lambda_2^2+2\lambda^2)\\
&\quad-K(H^2+K+\lambda_1^2+\lambda_2^2+2\lambda^2)-H^2(\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2+4\lambda^2)\Big]\\
&=H^2\Big[H^2-2S+(S+2-\lambda_1^2-\lambda_2^2-2\lambda^2)(\lambda_1^2+\lambda_2^2+2\lambda^2)-K(H^2+K)\\
&\quad-H^2(\lambda_1^2+\lambda_2^2+2\lambda^2)+\frac{1}{2}H^4-H^2\lambda^2-\frac{1}{2}H^2(\lambda_1^2+\lambda_2^2+2\lambda^2)\Big]\\
&=H^2\Big[H^2-2S+\frac{1}{2}H^4-H^2\lambda^2-KH^2-K^2\Big]\\
&\quad+H^2(S+2-\frac{3}{2}H^2-\lambda_1^2-\lambda_2^2-2\lambda^2)(\lambda_1^2+\lambda_2^2+2\lambda^2).
\end{aligned} \tag{2.54}$$

This finishes the proof. □

In order to prove our results, we need the following important generalized maximum principle for $\mathcal{L}$-operator on self-shrinkers which was proved by Cheng and Peng in [8]:

Lemma 2.6 (Generalized maximum principle for $\mathcal{L}$-operator) *Let $X: M^n\to\mathbb{R}^{n+p}$ $(p\geq 1)$ be a complete self-shrinker with Ricci curvature bounded from below. Let f be any C^2-function bounded from above on this self-shrinker. Then, there exists a sequence of points $\{p_m\}\subset M^n$, such that*

$$\lim_{m\to\infty}f(X(p_m))=\sup f,\quad \lim_{m\to\infty}|\nabla f|(X(p_m))=0,\quad \limsup_{m\to\infty}\mathcal{L}f(X(p_m))\leq 0.$$

3 Examples of Lagrangian self-shrinkers in $\mathbb{R}^4$

It is known that the Euclidean plane $\mathbb{R}^2$, the cylinder $S^1(1)\times\mathbb{R}^1$ and the Clifford torus $S^1(1)\times S^1(1)$ are the canonical Lagrangian self-shrinkers in $\mathbb{R}^4$. Apart from the standard examples, there are many other examples of complete Lagrangian self-shrinkers in $\mathbb{R}^4$.

Example 3.1 Let $\Gamma_1(s)=(x_1(s),y_1(s))^T$, $0\leq s<L_1$ and $\Gamma_2(t)=(x_2(t),y_2(t))^T$, $0\leq t<L_2$ be two self-shrinker curves in $\mathbb{R}^2$ with arc length as parameters, respectively. We

consider Riemannian product $\Gamma_1(s) \times \Gamma_2(t)$ of $\Gamma_1(s)$ and $\Gamma_2(t)$ defined by

$$X(s,t) = \begin{pmatrix} x_1(s) \\ x_2(t) \\ y_1(s) \\ y_2(t) \end{pmatrix}.$$

We can prove $\Gamma_1(s) \times \Gamma_2(t)$ is a Lagrangian self-shrinker in $\mathbb{R}^4$ and the Gaussian curvature K of $\Gamma_1(s) \times \Gamma_2(t)$ satisfies $K \equiv 0$.

In [1], Abresch and Langer classfied closed self-shrinking curves. For two closed self-shrinking curves $\Gamma_1(s)$ and $\Gamma_2(t)$ of Abresch and Langer in $\mathbb{R}^2$, $\Gamma_1(s) \times \Gamma_2(t)$ is a compact Lagrangian self-shrinker in $\mathbb{R}^4$, which is called Abresch-Langer torus. It is known that complete and non-compact self-shrinking curves exist in $\mathbb{R}^2$ (see [14]). Consequently, there are many complete and non-compact Lagrangian self-shrinkers with zero Gaussian curvature in $\mathbb{R}^4$.

Example 3.2 For a closed curve $\gamma(t) = (x_1(t), x_2(t))^T$, $t \in I$, such that its curvature κ_γ satisfy

$$\kappa_\gamma = E\frac{e^{\frac{|\gamma|^2}{2}}}{|\gamma|^2}(|\gamma|^2 - 1), \quad E^2 = |\gamma|^4(1 - (\frac{d|\gamma|}{dt})^2)e^{|\gamma|^2},$$

where E is a positive constant. In [2], Anciaux proved that

$$X(s,t) = \begin{pmatrix} x_1(t)\cos s \\ x_1(t)\sin s \\ x_2(t)\cos s \\ x_2(t)\sin s \end{pmatrix}$$

defines a compact Lagrangian self-shrinker in $\mathbb{R}^4$, which is called Anciaux torus, and the squared norm S of the second fundamental form satisfies

$$S = E^2\frac{e^{\frac{|\gamma|^6}{2}}}{|\gamma|^2}(|\gamma|^4 - 2|\gamma|^2 + 4).$$

Example 3.3 For positive integers m, n with $(m, n) = 1$, define $X^{m,n} : \mathbb{R}^2 \to \mathbb{R}^4$ by

$$X^{m,n}(s,t) = \sqrt{m+n}\begin{pmatrix} \frac{\cos s}{\sqrt{n}}\cos\sqrt{\frac{n}{m}}t \\ \frac{\sin s}{\sqrt{m}}\cos\sqrt{\frac{m}{n}}t \\ \frac{\cos s}{\sqrt{n}}\sin\sqrt{\frac{n}{m}}t \\ \frac{\sin s}{\sqrt{m}}\sin\sqrt{\frac{m}{n}} \end{pmatrix}.$$

Lee and Wang [21] proved $X^{m,n} : \mathbb{R}^2 \to \mathbb{R}^4$ is a Lagrangian self-shrinker in $\mathbb{R}^4$. It is not difficult to prove that the squared norm S and the Gauss curvature K of $X^{m,n} : \mathbb{R}^2 \to \mathbb{R}^4$, for $m \leq n$, satisfy

$$\frac{3m^2+n^2}{n(m+n)} \leq S \leq \frac{m^2+3n^2}{m(n+m)},$$
$$-\frac{n(n-m)}{m(m+n)} \leq K \leq \frac{m(n-m)}{n(n+m)}.$$

4 Proofs of the main results

First of all, we prove the following:

Theorem 4.1 *Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. If the squared norm S of the second fundamental form is constant, then $S \leq 2$.*

Proof Since S is constant, from the Gauss equations, we know that the Ricci curvature of $X : M^2 \to \mathbb{R}^4$ is bounded from below. We can apply the generalized maximum principle for $\mathcal{L}$-operator to the function $-|X|^2$. Thus, there exists a sequence $\{p_m\}$ in M^2 such that

$$\lim_{m\to\infty} |X|^2(p_m) = \inf |X|^2, \quad \lim_{m\to\infty} |\nabla |X|^2(p_m)| = 0, \quad \liminf_{m\to\infty} \mathcal{L}|X|^2(p_m) \geq 0.$$

Since $|\nabla |X|^2|^2 = 4\sum_{i=1}^{2} \langle X, e_i\rangle^2$ and

$$\frac{1}{2}\mathcal{L}|X|^2 = 2 - |X|^2,$$

we have

$$\lim_{m\to\infty} \sum_j \langle X, e_j\rangle^2(p_m) = 0 \quad \text{and} \quad 2 - \inf |X|^2 \geq 0. \tag{4.1}$$

Since $X : M^2 \to \mathbb{R}^4$ is a self-shrinker, we know

$$H^{p^*}_{,i} = \sum_k h^{p^*}_{ik} \langle X, e_k\rangle, \quad i, p = 1, 2. \tag{4.2}$$

From the definition of the self-shrinker, (4.1) and (4.2), we get

$$\inf |X|^2 = \lim_{m\to\infty} H^2(p_m) \leq 2, \quad \lim_{m\to\infty} |\nabla^{\perp} \vec{H}|^2(p_m) = 0. \tag{4.3}$$

Since $S = \sum_{i,j,p^*} (h^{p^*}_{ij})^2$ is constant, from (2.25) in Lemma 2.1, we know $\{h^{p^*}_{ij}(p_m)\}$ and $\{h^{p^*}_{ijl}(p_m)\}$ are bounded sequences for any i, j, l, p. Thus, we can assume

$$\lim_{m\to\infty} h^{p^*}_{ijl}(p_m) = \bar{h}^{p^*}_{ijl}, \quad \lim_{m\to\infty} h^{p^*}_{ij}(p_m) = \bar{h}^{p^*}_{ij},$$

for $i, j, l, p = 1, 2$.

Therefore, we have

$$\bar{h}^{p^*}_{11j} + \bar{h}^{p^*}_{22j} = 0, \quad \text{for } j, p = 1, 2 \tag{4.4}$$

and

$$\sum_{i,j,p} h^{p^*}_{ij} h^{p^*}_{ijk} = 0, \quad \text{for } k = 1, 2,$$

because of S constant. Since $X : M^2 \to \mathbb{R}^4$ is a Lagrangian self-shrinker,

$$h^{1^*}_{11} h^{1^*}_{11k} + 3h^{1^*}_{12} h^{1^*}_{12k} + 3h^{2^*}_{12} h^{2^*}_{12k} + h^{2^*}_{22} h^{2^*}_{22k} = 0, \quad \text{for } k = 1, 2$$

holds. Thus, we conclude

$$\bar{h}^{1^*}_{11} \bar{h}^{1^*}_{11k} + 3\bar{h}^{1^*}_{12} \bar{h}^{1^*}_{12k} + 3\bar{h}^{2^*}_{12} \bar{h}^{2^*}_{12k} + \bar{h}^{2^*}_{22} \bar{h}^{2^*}_{22k} = 0, \quad \text{for } k = 1, 2. \tag{4.5}$$

If $\lim_{m\to\infty} H^2(p_m) = 0$, we get

$$\bar{h}^{1^*}_{11} + \bar{h}^{1^*}_{22} = 0, \quad \bar{h}^{2^*}_{11} + \bar{h}^{2^*}_{22} = 0.$$

Consequently, from (4.4) and (4.5), we have the following equations for $k = 1, 2$,

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ 4\bar{h}^{1^*}_{11}\bar{h}^{1^*}_{11k} + 4\bar{h}^{2^*}_{11}\bar{h}^{2^*}_{11k} = 0. \end{cases}$$

Hence, we obtain $S = 0$ or $\bar{h}^{p^*}_{ijk} = 0$ for any i, j, k and p. According to (2.25) in Lemma 2.1, we have $S = 0$ or $S = \frac{2}{3}$.

If $\lim_{m\to\infty} H^2(p_m) = \bar{H}^2 \neq 0$, without loss of the generality, at each point p_m, we choose e_1, e_2 such that

$$\vec{H} = H^{1^*} e_{1^*}.$$

Then we have

$$\bar{h}^{1^*}_{11} + \bar{h}^{1^*}_{22} = \bar{H}, \quad \bar{h}^{2^*}_{11} + \bar{h}^{2^*}_{22} = 0$$

and

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ (\bar{h}^{1^*}_{11} - 3\bar{h}^{1^*}_{22})\bar{h}^{1^*}_{11k} + 4\bar{h}^{2^*}_{11}\bar{h}^{2^*}_{11k} = 0. \end{cases} \tag{4.6}$$

If $\bar{h}^{1^*}_{11} = 3\bar{h}^{1^*}_{22}$ and $\bar{h}^{2^*}_{11} = 0$, we know

$$\lim_{m\to\infty} H^2(p_m) = (\bar{\lambda}_1 + \bar{\lambda}_2)^2 = 16\bar{\lambda}_2^2 \leq 2 \ \text{ and } \ S = 12\bar{\lambda}_2^2 \leq \frac{3}{2}.$$

If $\bar{h}^{1^*}_{11} \neq 3\bar{h}^{1^*}_{22}$ or $\bar{h}^{2^*}_{11} \neq 0$, we have $\bar{h}^{p^*}_{ijk} = 0$ for any i, j, k, p from (4.6). Thus, from (2.25) in Lemma 2.1, we get

$$\begin{aligned} 0 &= S(1 - \frac{3}{2}S) + 2\bar{H}^2 S - \frac{1}{2}\bar{H}^4 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2) \\ &= S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 - \frac{1}{2}\bar{H}^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2 - 2\bar{H}^2\bar{\lambda}^2. \end{aligned}$$

Then we conclude

$$S \leq 2.$$

This completes the proof of Theorem 4.1. □

Since S is constant, from the result of Cheng and Peng in [8], we know that $S = 0$ or $S = 1$ if $S \leq 1$. Thus, we only need to prove the following

Theorem 4.2 *There are no 2-dimensional complete Lagrangian self-shrinkers $X : M^2 \to \mathbb{R}^4$ with constant squared norm S of the second fundamental form and $1 < S < 2$.*

The following lemma is key in this paper.

Lemma 4.1 *If $X : M^2 \to \mathbb{R}^4$ is a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$ with S =constant and $1 \le S \le 2$, there exists a sequence $\{p_m\}$ in M such that*

$$\lim_{m\to\infty} H^2(p_m) = \sup H^2, \quad \lim_{m\to\infty} h^{p^*}_{ijl}(p_m) = \bar{h}^{p^*}_{ijl}, \quad \lim_{m\to\infty} h^{p^*}_{ij}(p_m) = \bar{h}^{p^*}_{ij},$$

for $i, j, l, p = 1, 2$, and one can choose an orthonormal frame e_1, e_2 at p_m such that $\bar{\lambda} = \bar{h}^{1^*}_{12} = 0$.

Proof From (2.25) and (2.26) in Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2}\mathcal{L}H^2 &= |\nabla^\perp \vec{H}|^2 + H^2 - \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 - S(1 - \frac{3}{2}S) - 2H^2 S + \frac{1}{2}H^4 \\ &= |\nabla^\perp \vec{H}|^2 - \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 + \frac{1}{2}(H^2 - S)(H^2 - 3S + 2). \end{aligned}$$

If, at $p \in M$, $H = 0$, we have $H^2 < S$. If $H \neq 0$ at $p \in M$, we choose e_1, e_2 such that

$$\vec{H} = H^{1^*} e_{1^*}.$$

From $2ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$, we obtain

$$S = \lambda_1^2 + 3\lambda_2^2 + 4\lambda^2, \quad H^2 = (\lambda_1 + \lambda_2)^2 \le \frac{4}{3}(\lambda_1^2 + 3\lambda_2^2) \le \frac{4}{3}S,$$

where we denote $\lambda_1 = h^{1^*}_{11}$, $\lambda_2 = h^{1^*}_{22}$ and $\lambda = h^{1^*}_{12}$. Hence, we have on M

$$H^2 \le \frac{4}{3}S$$

and the equality holds if and only if $\lambda_1 = 3\lambda_2$ and $\lambda = 0$. Thus, by applying the generalized maximum principle of Cheng and Peng [8] to H^2, there exists a sequence $\{p_m\}$ in M^2 such that

$$\lim_{m\to\infty} H^2(p_m) = \sup H^2, \quad \lim_{m\to\infty} |\nabla H^2(p_m)| = 0, \quad \limsup_{m\to\infty} \mathcal{L}H^2(p_m) \le 0.$$

Since $X : M^2 \to \mathbb{R}^4$ is a self-shrinker, we have

$$H^{p^*}_{,i} = \sum_k h^{p^*}_{ik} \langle X, e_k \rangle, \quad i, p = 1, 2. \tag{4.7}$$

According to $1 \le S \le 2$, we know $\sup H^2 > 0$. Hence, without loss of the generality, at each point p_m, we can assume $H(p_m) \neq 0$ and choose e_1, e_2 such that

$$\vec{H} = H^{1^*} e_{1^*}.$$

From (2.25) in Lemma 2.1, Lemma 2.3 and the definition of S, we know that $\{h^{p^*}_{ij}(p_m)\}$, $\{h^{p^*}_{ijl}(p_m)\}$ and $\{h^{p^*}_{ijkl}(p_m)\}$, for any i, j, k, l, p, are bounded sequences. We can assume

$$\lim_{m\to\infty} h^{p^*}_{ijl}(p_m) = \bar{h}^{p^*}_{ijl}, \quad \lim_{m\to\infty} h^{p^*}_{ij}(p_m) = \bar{h}^{p^*}_{ij}, \quad \lim_{m\to\infty} h^{p^*}_{ijkl}(p_m) = \bar{h}^{p^*}_{ijkl},$$

for $i, j, k, l, p = 1, 2$

and get

$$\begin{cases} \lim_{m\to\infty} H^2(p_m) = \sup H^2 = \bar{H}^2, \quad \lim_{m\to\infty} |\nabla H^2(p_m)| = 0, \\ 0 \geq \lim_{m\to\infty} |\nabla^\perp \vec{H}|^2(p_m) - \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 + \frac{1}{2}(\bar{H}^2 - S)(\bar{H}^2 - 3S + 2). \end{cases} \tag{4.8}$$

From $\lim_{m\to\infty} |\nabla H^2(p_m)| = 0$ and $|\nabla H^2|^2 = 4\sum_i (\sum_{p^*} H^{p^*} H^{p^*}_{,i})^2$, we have

$$\bar{H}^{1^*}_{,k} = 0. \tag{4.9}$$

From (4.7), we obtain

$$\begin{cases} \bar{\lambda}_1 \lim_{m\to\infty} \langle X, e_1\rangle(p_m) + \bar{\lambda} \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0, \\ \bar{\lambda} \lim_{m\to\infty} \langle X, e_1\rangle(p_m) + \bar{\lambda}_2 \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0. \end{cases} \tag{4.10}$$

We will then prove $\bar{\lambda} = 0$.

Let us assume $\bar{\lambda} \neq 0$, we will get a contradiction. The proof is divided into three cases.

Case 1: $\bar{\lambda}_2 = 0$.

Since $\bar{H}^2 \neq 0$, we have $\bar{\lambda}_1 \neq 0$. From (4.10), we get

$$\lim_{m\to\infty} \langle X, e_1\rangle(p_m) = \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0.$$

Thus, for $k = 1, 2$, from (4.5) and (4.7),

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ \bar{\lambda}_1 \bar{h}^{1^*}_{11k} + 4\bar{\lambda}\bar{h}^{2^*}_{11k} = 0. \end{cases} \tag{4.11}$$

We can draw a conclusion, for any i, j, k, p,

$$\bar{h}^{p^*}_{ijk} = 0.$$

From (4.8), we know $S \leq \bar{H}^2$, which is in contradiction to $S = \bar{H}^2 + 4\bar{\lambda}^2 > \bar{H}^2$.

Case 2: $\bar{\lambda}_1 = 0$.

In this case, we have

$$\bar{\lambda}_2 \neq 0, \quad \bar{H}^2 = \bar{\lambda}_2^2, \quad S = 3\bar{\lambda}_2^2 + 4\bar{\lambda}^2 = 3\bar{H}^2 + 4\bar{\lambda}^2.$$

From (4.10), we obtain

$$\lim_{m\to\infty} \langle X, e_1\rangle(p_m) = \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0.$$

Therefore, we infer

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ 3\bar{\lambda}_2 \bar{h}^{1^*}_{22k} + 4\bar{\lambda}\bar{h}^{2^*}_{11k} = 0. \end{cases} \tag{4.12}$$

By solving the above system of equations, we have for any i, j, k, p,

$$\bar{h}^{p^*}_{ijk} = 0.$$

From (4.8), we know

$$(\bar{H}^2 - S)(\bar{H}^2 - 3S + 2) = (2\bar{H}^2 + 4\bar{\lambda}^2)(2S - 2 + 2\bar{H}^2 + 4\bar{\lambda}^2) \leq 0,$$

it is impossible since $S \geq 1$.

Case 3: $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$.
From (4.10), we have

$$(\bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}^2) \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0. \tag{4.13}$$

If $\bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}^2$, we get, for $k = 1, 2$, in view of (4.5) and (4.9),

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ (\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{11k} + 3\bar{\lambda}\bar{h}^{2^*}_{11k} - \bar{\lambda}\bar{h}^{2^*}_{22k} = 0. \end{cases}$$

By solving the above system of equations, we have

$$(\bar{\lambda}_1 + 3\bar{\lambda}_2)^2\bar{h}^{1^*}_{111} = -4\bar{\lambda}^2\bar{h}^{2^*}_{222}.$$

Hence, we obtain

$$\begin{aligned} \bar{h}^{1^*}_{111} &= \frac{-4\bar{\lambda}^2}{(\bar{\lambda}_1 + 3\bar{\lambda}_2)^2}\bar{h}^{2^*}_{222}, \\ \bar{h}^{1^*}_{221} &= -\bar{h}^{1^*}_{111}, \\ \bar{h}^{1^*}_{222} &= -\bar{h}^{2^*}_{111} = -\frac{\bar{\lambda}(\bar{\lambda}_1 - 3\bar{\lambda}_2)}{(\bar{\lambda}_1 + 3\bar{\lambda}_2)^2}\bar{h}^{2^*}_{222}. \end{aligned}$$

Since

$$\lim_{m\to\infty} |\nabla^\perp \vec{H}|^2(p_m) = (\bar{h}^{2^*}_{112} + \bar{h}^{2^*}_{222})^2 = \frac{(10\bar{\lambda}^2 + \bar{\lambda}_1^2 + 9\bar{\lambda}_2^2)^2}{(\bar{\lambda}_1 + 3\bar{\lambda}_2)^4}(\bar{h}^{2^*}_{222})^2$$

and

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 7(\bar{h}^{1^*}_{111})^2 + 8(\bar{h}^{2^*}_{111})^2 + (\bar{h}^{2^*}_{222})^2 = \frac{(10\bar{\lambda}^2 + \bar{\lambda}_1^2 + 9\bar{\lambda}_2^2)^2}{(\bar{\lambda}_1 + 3\bar{\lambda}_2)^4}(\bar{h}^{2^*}_{222})^2,$$

we get the following inequality from (4.8)

$$(\bar{H}^2 - S)(\bar{H}^2 - 3S + 2) \leq 0,$$

that is,

$$S \leq \bar{H}^2 \leq 3S - 2.$$

It is impossible because of $S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 + 4\bar{\lambda}^2 > \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2 = \bar{H}^2$. Hence, we obtain $\bar{\lambda}_1\bar{\lambda}_2 \neq \bar{\lambda}^2$.

From (4.10) and (4.13), we have

$$\lim_{m\to\infty} \langle X, e_2\rangle(p_m) = \lim_{m\to\infty} \langle X, e_1\rangle(p_m) = 0.$$

Thus, we know from (4.7)

$$\bar{H}^{p^*}_{,k} = 0,$$

for any k, $p = 1, 2$. Hence we infer

$$\begin{cases} \bar{h}^{p^*}_{11k} + \bar{h}^{p^*}_{22k} = 0, \\ (\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{11k} + 4\bar{\lambda}\bar{h}^{2^*}_{11k} = 0. \end{cases}$$

Through the above system, we have

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0.$$

From (4.8) and (2.25) in Lemma 2.1, we get

$$\begin{aligned} & S \le \bar{H}^2 \le 3S - 2, \\ & S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2) = 0. \end{aligned} \tag{4.14}$$

From Lemma 2.5 and taking limit,

$$\begin{aligned} 0 \le \sum_{i,j,k,l,p} (\bar{h}^{p^*}_{ijkl})^2 &= \frac{1}{2} \lim_{m\to\infty} \mathcal{L} \sum_{i,j,k,p} (h^{p^*}_{ijk})^2(p_m) \\ &= \bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] - \bar{\lambda}^2\bar{H}^4 \\ &\quad + \bar{H}^2(S + 2 - \frac{3}{2}\bar{H}^2 - \bar{\lambda}_1^2 - \bar{\lambda}_2^2 - 2\bar{\lambda}^2)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2) \\ &< \bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] \\ &\quad + \bar{H}^2(S + 2 - \frac{3}{2}\bar{H}^2 - \bar{\lambda}_1^2 - \bar{\lambda}_2^2 - 2\bar{\lambda}^2)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2). \end{aligned}$$

According to (4.14), we have

$$\begin{aligned} 0 &< \bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] \\ &\quad + \left(S + 2 - \frac{3}{2}\bar{H}^2 - \frac{1}{\bar{H}^2}\Big(S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4\Big)\right) \\ &\quad \times \left(S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4\right) \\ &= \frac{1}{4\bar{H}^2}\left(\bar{H}^8 - 2S\bar{H}^6 - 6S(S-1)\bar{H}^4 + 2S(2-3S)^2\bar{H}^2 - (2-3S)^2S^2\right) \le 0. \end{aligned}$$

This is a contradiction. In fact, we consider a function $f(t)$ defined by

$$f(t) = t^4 - 2St^3 - 6S(S-1)t^2 + 2S(2-3S)^2t - (2-3S)^2S^2, \tag{4.15}$$

for $S \le t \le 3S - 2$. Thus, we have

$$f^{'}(t) = 4t^3 - 6St^2 - 12S(S-1)t + 2S(2-3S)^2, \quad f^{''}(t) = 12(t^2 - St - S(S-1)), \tag{4.16}$$

$f^{''}(t) < 0$ for $t \in (S, \frac{S+\sqrt{S^2+4S(S-1)}}{2})$, $f^{''}(t) > 0$ for $t \in (\frac{S+\sqrt{S^2+4S(S-1)}}{2}, 3S-2)$. Hence, $f^{'}(t)$ is a decreasing function for $t \in (S, \frac{S+\sqrt{S^2+4S(S-1)}}{2})$ and $f^{'}(t)$ is an increasing function for $t \in (\frac{S+\sqrt{S^2+4S(S-1)}}{2}, 3S-2)$. According to

$$f(S) = 2(S-1)(S-2)S^2 \leq 0, \quad f(3S-2) = 2(3S-2)^2(S-1)(S-2) \leq 0, \tag{4.17}$$

we conclude $f(t) \leq 0$ for $t \in (S, 3S-2)$ because of $f'(S) = 4S(S-1)(S-2) \leq 0$. Hence, we obtain $\bar{\lambda} = 0$ and

$$\bar{h}^{1^*}_{ij} = \bar{\lambda}_i \delta_{ij}, \quad \bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2, \quad S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2.$$

□

Since the proof of Theorem 4.2 is very long, we will divide the proof into three steps. In the first step, we prove the following:

Proposition 4.1 *If $X : M^2 \to \mathbb{R}^4$ is a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$ with S =constant and $1 < S < 2$, there exists a sequence $\{p_m\}$ in M and at p_m, we can choose an orthonormal e_1, e_2 such that*

$$\lim_{m\to\infty} h^{p^*}_{ijl}(p_m) = \bar{h}^{p^*}_{ijl}, \quad \lim_{m\to\infty} h^{p^*}_{ij}(p_m) = \bar{h}^{p^*}_{ij},$$

for $i, j, l, p = 1, 2$, $\bar{\lambda} = 0$ and the following holds, either

$$\begin{cases} \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0, \quad \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\ S < \sup H^2 = \bar{H}^2 \leq 3S-2 \quad \text{and} \quad S < \sup H^2 < \frac{4}{3}S, \end{cases} \tag{4.18}$$

or

$$\begin{cases} \bar{\lambda}_1 = 3\bar{\lambda}_2, \quad \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \quad \bar{h}^{p^*}_{11k} + \bar{h}^{p^*}_{22k} = 0 \\ \sup H^2 = \frac{4}{3}S, \quad S \geq \frac{6}{5}, \end{cases} \tag{4.19}$$

for $k, p = 1, 2$, where we denote $\bar{\lambda}_1 = \bar{h}^{1^*}_{11}$, $\bar{\lambda}_2 = \bar{h}^{1^*}_{22}$ and $\bar{\lambda} = \bar{h}^{1^*}_{12}$.

Proof By making use of the same assertion as in the proof of Lemma 4.1, there exists a sequence $\{p_m\}$ in M^2 such that

$$\lim_{m\to\infty} h^{p^*}_{ijl}(p_m) = \bar{h}^{p^*}_{ijl}, \quad \lim_{m\to\infty} h^{p^*}_{ij}(p_m) = \bar{h}^{p^*}_{ij}, \quad \lim_{m\to\infty} h^{p^*}_{ijkl}(p_m) = \bar{h}^{p^*}_{ijkl},$$

for $i, j, k, l, p = 1, 2$ and

$$\begin{cases} \lim_{m\to\infty} H^2(p_m) = \sup H^2 = \bar{H}^2, \quad \lim_{m\to\infty} |\nabla H^2(p_m)| = 0, \\ 0 \geq \lim_{m\to\infty} |\nabla^\perp \vec{H}|^2(p_m) - \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 + \frac{1}{2}(\bar{H}^2 - S)(\bar{H}^2 - 3S + 2), \end{cases} \tag{4.20}$$

with $\bar{\lambda} = 0$. From $\lim_{k\to\infty} |\nabla H^2(p_m)| = 0$ and $|\nabla H^2|^2 = 4\sum_i(\sum_{p^*} H^{p^*} H^{p^*}_{,i})^2$, we have

$$\bar{H}^{1^*}_{,k} = 0. \tag{4.21}$$

From (4.7) and $\bar{\lambda} = 0$, we have

$$\bar{\lambda}_1 \lim_{m\to\infty} \langle X, e_1\rangle(p_m) = 0, \quad \bar{\lambda}_2 \lim_{m\to\infty} \langle X, e_2\rangle(p_m) = 0,$$

it means that,

$$\bar{\lambda}_i \lim_{m\to\infty} \langle X, e_i\rangle(p_m) = 0.$$

According to $S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 > 1$ and $\sup H^2 = (\bar{\lambda}_1 + \bar{\lambda}_2)^2$, if $\bar{\lambda}_2 = 0$, we have

$$\bar{\lambda}_1 \neq 0, \quad S = \sup H^2, \quad \bar{H}^{2^*}_{,k} = 0$$

because of $H^{p^*}_{,i} = \sum_k h^{p^*}_{ik}\langle X, e_k\rangle$. Hence, by using the same calculations as in (4.6), we have

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ \bar{\lambda}_1 \bar{h}^{1^*}_{11k} = 0. \end{cases} \tag{4.22}$$

Then we obtain

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0.$$

From $S = \sup H^2$ and (2.25), we get $S = 1$ or $S = 0$. It is impossible. If $\bar{\lambda}_1 = 0$, we have

$$\bar{\lambda}_2 \neq 0, \quad S = 3\sup H^2, \quad \bar{H}^{2*}_{,1} = 0.$$

In this way, by using the same calculations as in (4.6), we get

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{111} + \bar{h}^{2^*}_{221} = 0, \\ 3\bar{\lambda}_2 \bar{h}^{1^*}_{22k} = 0. \end{cases}$$

So, we know

$$\bar{h}^{p^*}_{ijk} = 0, \quad \text{except } \ i = j = k = p^* = 2$$

and

$$|\nabla^{\perp}\vec{H}|^2 = \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 \ \text{ and } \ \frac{1}{2}(H^2 - S)(H^2 - 3S + 2) \leq 0.$$

Hence $S \leq \dfrac{3}{4}$. This is also impossible.

We get $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$.

Because of

$$H^{1^*}_{,i} = \sum_k h^{1^*}_{ik}\langle X, e_k\rangle, \quad H^{2^*}_{,i} = \sum_k h^{2^*}_{ik}\langle X, e_k\rangle,$$

for $i = 1, 2$, we obtain $\lim_{m\to\infty}\langle X, e_i\rangle(p_m) = 0$ from $\bar{H}^{1^*}_{,i} = 0$ and $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$. Thus, we have $\bar{H}^{2^*}_{,i} = 0$, then we get from (4.5), for $i = 1, 2$,

$$\begin{cases} \bar{h}^{1^*}_{11i} + \bar{h}^{1^*}_{22i} = 0, \\ \bar{h}^{2^*}_{11i} + \bar{h}^{2^*}_{22i} = 0, \\ (\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{11i} = 0. \end{cases} \tag{4.23}$$

If $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$, we have

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0.$$

Therefore, from (4.8) and (2.25), we get

$$S < \sup H^2 \leq 3S - 2.$$

If $\bar{\lambda}_1 = 3\bar{\lambda}_2$, we have $\sup H^2 = \frac{4}{3}S$ and $\lim_{m\to\infty} |\nabla^{\perp}\vec{H}|^2(p_m) = 0$. From (2.25), we know

$$\begin{aligned}\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 &= -S(1-\frac{1}{2}S) + (S - \sup H^2)^2 - \frac{1}{2}(\sup H^2)^2 + \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \\ &= (\frac{5}{6}S - 1)S.\end{aligned} \tag{4.24}$$

Hence, we get $S \geq \frac{6}{5}$. When $S = \frac{6}{5}$, from (4.24), we have $\sum_{i,j,k,p}(\bar{h}^{p^*}_{ijk})^2 = 0$. It finishes the proof of Proposition 4.1. □

In the step 2, we prove the following:

Proposition 4.2 *Under the assumptions of Proposition 4.1, the formula (4.19) in Proposition 4.1 does not occur.*

Proof If the formula (4.19) holds, we have

$$\bar{\lambda} = 0, \quad \bar{\lambda}_1 = 3\bar{\lambda}_2, \quad S \geq \frac{6}{5}, \quad \sup H^2 = \bar{H}^2 = \frac{4}{3}S$$

and

$$\bar{H}^{p^*}_{,k} = \bar{h}^{p^*}_{11k} + \bar{h}^{p^*}_{22k} = 0, \quad \text{for } k, p = 1, 2. \tag{4.25}$$

From (2.6) and (4.25), we have

$$\begin{aligned}
&\sum_{i,j}(\bar{h}^{1^*}_{ij1})^2 = (\bar{h}^{1^*}_{111})^2 + (\bar{h}^{1^*}_{221})^2 + 2(\bar{h}^{1^*}_{121})^2 = 2(\bar{h}^{1^*}_{111})^2 + 2(\bar{h}^{1^*}_{112})^2 \\
&\sum_{i,j}(\bar{h}^{2^*}_{ij1})^2 = (\bar{h}^{2^*}_{111})^2 + (\bar{h}^{2^*}_{221})^2 + 2(\bar{h}^{2^*}_{121})^2 = 2(\bar{h}^{1^*}_{111})^2 + 2(\bar{h}^{1^*}_{112})^2 \\
&\sum_{i,j,p}(\bar{h}^{p^*}_{ij1})^2 = \sum_{p}\left[(\bar{h}^{p^*}_{111})^2 + (\bar{h}^{p^*}_{221})^2 + 2(\bar{h}^{p^*}_{121})^2\right] = \sum_{p}\left[2(\bar{h}^{p^*}_{111})^2 + 2(\bar{h}^{p^*}_{112})^2\right] \\
&\sum_{i,j,p}(\bar{h}^{p^*}_{ij2})^2 = \sum_{p}\left[(\bar{h}^{p^*}_{112})^2 + (\bar{h}^{p^*}_{222})^2 + 2(\bar{h}^{p^*}_{122})^2\right] = \sum_{p}\left[2(\bar{h}^{p^*}_{111})^2 + 2(\bar{h}^{p^*}_{112})^2\right] \\
&\sum_{i,j,p}(\bar{h}^{p^*}_{ijk})^2 = \sum_{i,j,p}(\bar{h}^{p^*}_{ij1})^2 + \sum_{i,j,p}(\bar{h}^{p^*}_{ij2})^2 = 8(\bar{h}^{1*}_{111})^2 + 8(\bar{h}^{1*}_{112})^2.
\end{aligned}$$

From (2.25) in Lemma 2.1 and $\bar{H}^2 = \frac{4}{3}S$, we get

$$\sum_{i,j,p^*}(\bar{h}^{p^*}_{ijk})^2 = (\frac{5}{6}S - 1)S.$$

Thus, we obtain

$$\sum_{i,j}(\bar{h}_{ij1}^{1^*})^2=\sum_{i,j}(\bar{h}_{ij1}^{2^*})^2=\frac{1}{4}(\frac{5}{6}S-1)S$$

and

$$\sum_{i,j,p^*}(\bar{h}_{ij1}^{p^*})^2=\sum_{i,j,p^*}(\bar{h}_{ij2}^{p^*})^2=\frac{1}{2}(\frac{5}{6}S-1)S.$$

Since

$$\bar{H}\bar{\lambda}_1=\frac{\bar{H}^2-S}{2}+\bar{\lambda}_1^2+\bar{\lambda}_2^2=\frac{\bar{H}^2}{2}+\frac{S}{3}=S,$$
$$\bar{H}(\bar{\lambda}_1^3+\bar{\lambda}_2^3)=\bar{H}^2(\bar{\lambda}_1^2-\bar{\lambda}_1\bar{\lambda}_2+\bar{\lambda}_2^2)=\frac{7S^2}{9},$$

according to (2.35) in Lemma 2.3, we get

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty}\mathcal{L}\sum_{i,j,k,p^*}(h_{ijk}^{p^*})^2(p_m)\\
&=(\bar{H}^2-2S)\bar{H}^2+(\frac{1}{2}\bar{H}^2+\frac{S}{2}+2)\bar{H}^2(\bar{\lambda}_1^2+\bar{\lambda}_2^2)\\
&\quad-\frac{\bar{H}^2-S}{2}(\bar{H}^4+\bar{H}^3\bar{\lambda}_1)-\bar{H}^2(\bar{\lambda}_1^2+\bar{\lambda}_2^2)^2-\bar{H}^3(\bar{\lambda}_1^3+\bar{\lambda}_2^3)+\bar{H}^2\sum_{i,j,k}(\bar{h}_{ijk}^{1^*})^2\\
&=(\bar{H}^2-2S)\bar{H}^2+(\frac{1}{2}\bar{H}^2+\frac{S}{2}+2)\bar{H}^2\frac{5S}{6}\\
&\quad-\frac{\bar{H}^2-S}{2}\Big(\bar{H}^4+\bar{H}^2S\Big)-\bar{H}^2(\frac{5S}{6})^2-\bar{H}^2\frac{7S^2}{9}+\bar{H}^2\sum_{i,j,k}(\bar{h}_{ijk}^{1^*})^2\\
&=S^2(\frac{2}{3}-\frac{17}{27}S)<0.
\end{aligned}\tag{4.26}$$

On the other hand, from (2.34) in Lemma 2.3, we have

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty}\mathcal{L}\sum_{i,j,k,p^*}(h_{ijk}^{p^*})^2(p_m)\\
&=\sum_{i,j,k,l,p^*}(\bar{h}_{ijkl}^{p^*})^2+(5\bar{H}^2-5S+2)\sum_{i,j,k,p^*}(\bar{h}_{ijk}^{p^*})^2\\
&\quad-2\bar{H}\sum_{i,j,k,p^*}(\bar{h}_{ijk}^{p^*})^2\bar{h}_{kk}^{1^*}-\bar{H}\sum_{i,j,k}(\bar{h}_{ijk}^{1*})^2\bar{h}_{kk}^{1^*}\\
&=\sum_{i,j,k,l,p^*}(\bar{h}_{ijkl}^{p^*})^2+2\sum_{i,j,k,p^*}(\bar{h}_{ijk}^{p^*})^2\geq 0.
\end{aligned}\tag{4.27}$$

Hence, we conclude that (4.26) is in contradiction to (4.27). It completes the proof of Proposition 4.2. □

In the step 3, we prove the following:

Proposition 4.3 *Under the assumptions of Proposition 4.1, the formula (4.18) in Proposition 4.1 does not occur either.*

In this case, we have $\sum_{i,j,k,p}(\bar{h}^{p^*}_{ijk})^2 = 0$, $\bar{\lambda} = 0$ and $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$.

Since $\bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2$ and S=$\bar{\lambda}_1^2 + 3\bar{\lambda}_2^2$, we get

$$\bar{\lambda}_1 = \frac{3\bar{H} \pm \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} \mp \sqrt{4S - 3\bar{H}^2}}{4}.$$

Lemma 4.2 *Under the assumptions of Proposition 4.1, if*

$$\begin{cases} \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0, \ \ \bar{\lambda} = 0, \ \ \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\ S < \sup H^2 = \bar{H}^2 \leq 3S - 2 \ \ \textit{and} \ \ S < \sup H^2 < \frac{4}{3}S \end{cases} \tag{4.28}$$

is satisfied, then

$$\bar{\lambda}_1 = \frac{3\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}.$$

do not occur.

Proof If

$$\bar{\lambda}_1 = \frac{3\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}$$

hold, we have

$$\bar{\lambda}_1^2 + \bar{\lambda}_2^2 = \frac{\bar{H}^2 + 2S + \sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{4}. \tag{4.29}$$

Due to $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$, we know $\bar{H}^2 < \frac{4}{3}S$ and $\frac{4S}{3} \leq 3S - 2$ if and only if $S \geq \frac{6}{5}$. According to (2.25) and $\sum_{i,j,k,p}(\bar{h}^{p^*}_{ijk})^2 = 0$, we have

$$S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}(\bar{H}^2)^2 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) = 0. \tag{4.30}$$

We get from (4.29)

$$S(1 - \frac{3}{2}S) + \frac{3}{2}S\bar{H}^2 - \frac{3}{4}\bar{H}^4 - \frac{\bar{H}^2\sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{4} = 0. \tag{4.31}$$

We consider function

$$f(x) = S(1 - \frac{3}{2}S) + \frac{3}{2}Sx - \frac{3}{4}x^2 - \frac{x\sqrt{(4S - 3x)x}}{4}$$

for $S < x < \frac{4}{3}S$. We know that

$$f(S) = S(1 - S) < 0$$

since $1 < S < 2$,

$$
\begin{aligned}
f'(x) = \frac{df(x)}{dx} &= \frac{3}{2}S - \frac{3}{2}x - \frac{\sqrt{(4S-3x)x}}{4} - \frac{x(2S-3x)}{4\sqrt{(4S-3x)x}} \\
&= \frac{3}{2}(S-x) - \frac{3x(S-x)}{2\sqrt{(4S-3x)x}} \\
&= \frac{3}{2}(S-x)\Big(1 - \frac{x}{\sqrt{(4S-3x)x}}\Big) > 0
\end{aligned}
\tag{4.32}
$$

since $S < x$ and $x > \sqrt{(4S-3x)x}$. Thus, $f(x)$ is an increasing function of x.

If $S \geq \frac{6}{5}$, then $\frac{4}{3}S \leq 3S - 2$. Hence, we have $S < \bar{H}^2 < \frac{4}{3}S$.

Since

$$
f(\frac{4}{3}S) = S(1 - \frac{3}{2}S) + \frac{3}{2}S\frac{4}{3}S - \frac{3}{4}(\frac{4}{3}S)^2 = S(1 - \frac{5}{6}S) \leq 0,
$$

we conclude $f(x) < 0$ for any $x \in (S, \frac{4}{3}S)$, which is in contradiction to (4.31). Thus, we must have $S < \frac{6}{5}$. In this case, $\frac{4S}{3} > 3S - 2$ and

$$
\begin{aligned}
f(3S-2) &= S(1 - \frac{3}{2}S) + \frac{3}{2}S(3S-2) - \frac{3}{4}(3S-2)^2 \\
&\quad - \frac{(3S-2)\sqrt{(4S-3(3S-2))(3S-2)}}{4} \\
&= (3S-2)\Big(\frac{3}{2} - \frac{5}{4}S - \frac{\sqrt{(6-5S)(3S-2)}}{4}\Big) \\
&= (3S-2)\frac{\sqrt{6-5S}}{4}\Big(\sqrt{6-5S} - \sqrt{3S-2}\Big) < 0.
\end{aligned}
$$

Therefore, it is also impossible. It finishes the proof of Lemma 4.2.

□

Lemma 4.3 *Under the assumptions of Proposition 4.1, if*

$$
\begin{cases}
\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0, \ \ \bar{\lambda} = 0, \ \ \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\
S < \sup H^2 = \bar{H}^2 \leq 3S - 2 \ \ \textit{and} \ \ S < \sup H^2 < \frac{4}{3}S,
\end{cases}
\tag{4.33}
$$

is satisfied, then we have

$$
\bar{\lambda}_1 = \frac{3\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}.
$$

and $S \geq \frac{6}{5}$.

Proof According to Lemma 4.2, we must have

$$
\bar{\lambda}_1 = \frac{3\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}.
$$

Thus,

$$\bar{\lambda}_1^2 + \bar{\lambda}_2^2 = \frac{\bar{H}^2 + 2S - \sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{4}. \tag{4.34}$$

If $S < \frac{6}{5}$ holds, then we get $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$ and $\frac{4S}{3} > 3S - 2$. According to (2.25), we have

$$S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}(\bar{H}^2)^2 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) = 0.$$

we obtain from (4.34)

$$S(1 - \frac{3}{2}S) + \frac{3}{2}S\bar{H}^2 - \frac{3}{4}\bar{H}^4 + \frac{\bar{H}^2\sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{4} = 0. \tag{4.35}$$

Now we consider function

$$f_1(x) = S(1 - \frac{3}{2}S) + \frac{3}{2}Sx - \frac{3}{4}x^2 + \frac{x\sqrt{(4S - 3x)x}}{4}$$

for $S < x \leq 3S - 2$. Since

$$\begin{aligned} f_1'(x) = \frac{df_1(x)}{dx} &= \frac{3}{2}S - \frac{3}{2}x + \frac{\sqrt{(4S - 3x)x}}{4} + \frac{x(2S - 3x)}{4\sqrt{(4S - 3x)x}} \\ &= \frac{3}{2}(S - x) + \frac{3x(S - x)}{2\sqrt{(4S - 3x)x}} \\ &= \frac{3}{2}(S - x)\Big(1 + \frac{x}{\sqrt{(4S - 3x)x}}\Big) < 0 \end{aligned}$$

for $S < x$, $f_1(x)$ is a decreasing function of x on $(S, 3S - 2)$.

$$\begin{aligned} f_1(3S - 2) &= S(1 - \frac{3}{2}S) + \frac{3}{2}S(3S - 2) - \frac{3}{4}(3S - 2)^2 \\ &\quad + \frac{(3S - 2)\sqrt{(4S - 3(3S - 2))(3S - 2)}}{4} \\ &= (3S - 2)\left(\frac{3}{2} - \frac{5}{4}S + \frac{\sqrt{(6 - 5S)(3S - 2)}}{4}\right) \\ &= (3S - 2)\frac{\sqrt{6 - 5S}}{4}\Big(\sqrt{6 - 5S} + \sqrt{3S - 2}\Big) > 0 \end{aligned}$$

since $S < \frac{6}{5}$. Thus $f_1(x) > 0$ for any $x \in (S, 3S - 2]$, which is in contradiction to (4.35). □

Lemma 4.4 *Under the assumptions of Proposition 4.1, if*

$$\begin{cases} \sum_{i,j,k,p} (\bar{h}_{ijk}^{p^*})^2 = 0, \quad \bar{\lambda} = 0, \quad \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\ S < \sup H^2 = \bar{H}^2 \leq 3S - 2 \quad \text{and} \quad S < \sup H^2 < \frac{4}{3}S, \end{cases} \tag{4.36}$$

are satisfied, then we have $1.89 \leq S < 2$.

Proof According to Lemmas 4.2 and 4.3, we know $2 > S \geq \frac{6}{5}$ and

$$\bar{\lambda}_1 = \frac{3\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}.$$

In this case, $\frac{4S}{3} \leq 3S - 2$. Hence, we have

$$\frac{6}{5} \leq S < 2, \quad S < \bar{H}^2 < \frac{4}{3}S, \quad \sum_{i,j,k,p^*} (\bar{h}^{p^*}_{ijk})^2 = 0$$

and

$$\bar{\lambda}_1 = \frac{3\bar{H} - \sqrt{4S - 3\bar{H}^2}}{4}, \quad \bar{\lambda}_2 = \frac{\bar{H} + \sqrt{4S - 3\bar{H}^2}}{4}.$$

From (2.35) of Lemma 2.3 in Sect. 2, we get

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty} \mathcal{L} \sum_{i,j,k,p} (h^{p^*}_{ijk})^2(p_m) \\
&= (\bar{H}^2 - 2S)\bar{H}^2 + (\frac{1}{2}\bar{H}^2 + \frac{S}{2} + 2)\bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \\
&\quad - \frac{\bar{H}^2 - S}{2}(\bar{H}^4 + H^3\bar{\lambda}_1) - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)^2 - \bar{H}^3(\bar{\lambda}_1^3 + \bar{\lambda}_2^3).
\end{aligned}$$

Since

$$H\lambda_1 = \frac{H^2 - S}{2} + \lambda_1^2 + \lambda_2^2, \ \ \lambda_1^3 + \lambda_2^3 = H(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2),$$

we have

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty} \mathcal{L} \sum_{i,j,k,p} (h^{p^*}_{ijk})^2(p_m) \\
&= \bar{H}^2\Big[\bar{H}^2 - 2S - \frac{(\bar{H}^2 - S)^2}{4} - \frac{\bar{H}^2(\bar{H}^2 - S)}{2} + \frac{\bar{H}^4}{2}\Big] \\
&\quad + \bar{H}^2\Big(S - \frac{3\bar{H}^2}{2} + 2 - \bar{\lambda}_1^2 - \bar{\lambda}_2^2\Big)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2).
\end{aligned}$$

By making use of

$$\bar{\lambda}_1^2 + \bar{\lambda}_2^2 = \frac{\bar{H}^2 + 2S - \sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{4},$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty} \mathcal{L} \sum_{i,j,k,p} (h^{p^*}_{ijk})^2(p_m) \\
&= \bar{H}^2\Big[-\frac{\bar{H}^4}{2} + \frac{3\bar{H}^2}{2} - S + \frac{(\bar{H}^2 - 1)\sqrt{(4S - 3\bar{H}^2)\bar{H}^2}}{2}\Big].
\end{aligned} \tag{4.37}$$

On the other hand, from (2.34) and (2.35)

$$\frac{1}{2}\lim_{m\to\infty}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2(p_m)=\lim_{m\to\infty}\sum_{i,j,k,l,p}(\bar{h}^{p^*}_{ijkl})^2(p_m)$$
$$\geq 2(\bar{h}^{1^*}_{1122}-\bar{h}^{1^*}_{2211})^2+2(\bar{h}^{2^*}_{1122}-\bar{h}^{2^*}_{2211})^2+\frac{1}{2}(\bar{h}^{1^*}_{2222}-\bar{h}^{2^*}_{2221})^2.$$

From Gauss equation and Ricci identities, we have

$$\begin{aligned}
h^{1^*}_{2222}-h^{2^*}_{2221}&=h^{2^*}_{2212}-h^{2^*}_{2221}\\
&=\sum_m h^{2^*}_{m2}R_{m212}+\sum_m h^{2^*}_{2m}R_{m212}+\sum_m h^{m^*}_{22}R_{m212}\\
&=(h^{2^*}_{12}+h^{2^*}_{21}+h^{1^*}_{22})R_{1212}\\
&=3\lambda_2K(\delta_{11}\delta_{22}-\delta_{12}\delta_{21})\\
&=3\lambda_2K,\\
h^{1^*}_{1122}-h^{1^*}_{2211}&=h^{1^*}_{1212}-h^{1^*}_{1221}\\
&=\sum_m h^{1^*}_{m2}R_{m112}+\sum_m h^{1^*}_{1m}R_{m212}+\sum_m h^{m^*}_{12}R_{m112}\\
&=\lambda_2R_{2112}+\lambda_1R_{1212}+\lambda_2R_{2112}\\
&=(\lambda_1-2\lambda_2)K.
\end{aligned}$$

From the above equations, we obtain

$$\begin{aligned}
&2(h^{1^*}_{1122}-h^{1^*}_{2211})^2+2(h^{2^*}_{1122}-h^{2^*}_{2211})^2+\frac{1}{2}(h^{1^*}_{2222}-h^{2^*}_{2221})^2\\
&=2(\lambda_1-2\lambda_2)^2K^2+\frac{1}{2}(3\lambda_2K)^2\\
&=K^2[2\lambda_1^2-8\lambda_1\lambda_2+\frac{25}{2}\lambda_2^2]\\
&=K^2[2S-4(H^2-\lambda_1^2-\lambda_2^2)+\frac{13}{2}\lambda_2^2]\\
&=K^2(6S-4H^2-\frac{3}{2}\lambda_2^2).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}\lim_{m\to\infty}\mathcal{L}\sum_{i,j,k,p^*}(h^{p^*}_{ijk})^2(p_m)=\lim_{m\to\infty}\sum_{i,j,k,l,p^*}(\bar{h}^{p^*}_{ijkl})^2(p_m)\\
&\geq 2(\bar{h}^{1^*}_{1122}-\bar{h}^{1^*}_{2211})^2+2(\bar{h}^{2^*}_{1122}-\bar{h}^{2^*}_{2211})^2+\frac{1}{2}(\bar{h}^{1^*}_{2222}-\bar{h}^{2^*}_{2221})^2\\
&\geq\frac{(\bar{H}^2-S)^2}{4}\Big(6S-4\bar{H}^2-\frac{3\bar{\lambda}_2^2}{2}\Big)\\
&=\frac{(\bar{H}^2-S)^2}{4}\Big[(6-\frac{3}{8})S-(4-\frac{3}{16})\bar{H}^2-\frac{3\sqrt{(4S-3\bar{H}^2)\bar{H}^2}}{16}\Big]
\end{aligned}$$

Hence, we obtain, in view of (4.37) and (4.38)

$$\bar{H}^2\Big[-\frac{\bar{H}^4}{2}+\frac{3\bar{H}^2}{2}-S+\frac{(\bar{H}^2-1)\sqrt{(4S-3\bar{H}^2)\bar{H}^2}}{2}\Big]$$
$$\geq\frac{(\bar{H}^2-S)^2}{4}\Big[(6-\frac{3}{8})S-(4-\frac{3}{16})\bar{H}^2-\frac{3\sqrt{(4S-3\bar{H}^2)\bar{H}^2}}{16}\Big].$$

From (2.25) and $\sum_{i,j,k,p}(\bar{h}^{p^*}_{ijk})^2=0$, we know

$$\frac{\bar{H}^2\sqrt{(4S-3\bar{H}^2)\bar{H}^2}}{4}=-S(1-\frac{3}{2}S)-\frac{3}{2}S\bar{H}^2+\frac{3}{4}\bar{H}^4.$$

Therefore, we conclude

$$\bar{H}^6-3\bar{H}^4S+3\bar{H}^2S^2-3S^2+2S$$
$$\geq\frac{(\bar{H}^2-S)^2}{4}\Big[(6+\frac{3}{4})S-(4+\frac{3}{8})\bar{H}^2-\frac{3S(3S-2)}{8\bar{H}^2}\Big]. \tag{4.38}$$

Since $-\frac{3}{4}x-\frac{3S(3S-2)}{8x}$ is a decreasing function of x, for $S<x<\frac{4S}{3}$, we have

$$-\frac{3}{4}\bar{H}^2-\frac{3S(3S-2)}{8\bar{H}^2}>-S-\frac{9(3S-2)}{32}.$$

Hence, we get

$$\bar{H}^6-3\bar{H}^4S+3\bar{H}^2S^2-3S^2+2S$$
$$\geq\frac{(\bar{H}^2-S)^2}{4}\Big[(5-\frac{3}{32})S-(4-\frac{3}{8})\bar{H}^2+\frac{9}{16}\Big]. \tag{4.39}$$

We consider a function $g=g(x)$ of x defined by

$$g(x)=x^3-3x^2S+3xS^2-3S^2+2S-\frac{(x-S)^2}{4}\Big[(5-\frac{3}{32})S-(4-\frac{3}{8})x+\frac{9}{16}\Big].$$
$$g'(x)=3x^2-6xS+3S^2+\frac{(x-S)^2}{4}(4-\frac{3}{8})$$
$$-\frac{(x-S)}{2}\Big[(5-\frac{3}{32})S-(4-\frac{3}{8})x+\frac{9}{16}\Big]$$
$$=(x-S)\Big[(6-\frac{9}{32})x-(6+\frac{23}{64})S-\frac{9}{32}\Big].$$

Hence, $g(x)$ attains its minimum at $(6-\frac{9}{32})x-(6+\frac{23}{64})S-\frac{9}{32}=0$.

$$g(S)=S(S-1)(S-2)<0,\quad g(\frac{4S}{3})=(1+\frac{121}{36\cdot 96})S^3-\frac{193}{64}S^2+2S<0$$

if $\frac{6}{5}\leq S<1.89$. We have $g(x)<0$ for $S<x<\frac{4S}{3}$, which is in contradiction to (4.39). Hence, S satisfies

$$1.89\leq S<2.$$

□

Lemma 4.5 *Under the assumptions of Proposition 4.1, if*

$$\begin{cases} \sum_{i,j,k,p} (\bar{h}_{ijk}^{p^*})^2 = 0, \quad \bar{\lambda} = 0, \quad \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\ S < \sup H^2 = \bar{H}^2 \leq 3S - 2 \quad \text{and} \quad S < \sup H^2 < \frac{4}{3}S, \end{cases}$$

is satisfied, then we have

$$S < \bar{H}^2 \leq S + \frac{1}{5}S.$$

Proof Since $\bar{H}^2 = \sup H^2$, if $S + \frac{S}{5} < \bar{H}^2 < S + \frac{S}{3}$, we consider a function $f_2 = f_2(x)$ of x defined by

$$f_2(x) = S(1 - \frac{3}{2}S) + \frac{3}{2}Sx - \frac{3}{4}x^2 + \frac{x\sqrt{(4S-3x)x}}{4}$$

for $\frac{6}{5}S < x \leq \frac{4}{3}S$. We know that

$$f_2(\frac{6S}{5}) = S(1 - \frac{39 - 6\sqrt{3}}{50}S) < 0$$

since $1.89 < S < 2$,

$$\begin{aligned} f_2'(x) = \frac{df(x)}{dx} &= \frac{3}{2}S - \frac{3}{2}x + \frac{\sqrt{(4S-3x)x}}{4} + \frac{x(2S-3x)}{4\sqrt{(4S-3x)x}} \\ &= \frac{3}{2}(S-x) + \frac{3x(S-x)}{2\sqrt{(4S-3x)x}} \\ &= \frac{3}{2}(S-x)\Big(1 + \frac{x}{\sqrt{(4S-3x)x}}\Big) < 0. \end{aligned}$$

$f_2(x)$ is a decreasing function of x and we can not have

$$S(1 - \frac{3}{2}S) + \frac{3}{2}S\bar{H}^2 - \frac{3}{4}\bar{H}^4 + \frac{\bar{H}^2\sqrt{(4S-3\bar{H}^2)\bar{H}^2}}{4} = 0.$$

Hence, we must have

$$S < \bar{H}^2 < \frac{6}{5}S.$$

□

Proof of Proposition 4.3 According to Lemma 4.4 and Lemma 4.5, we have

$$\begin{cases} \sum_{i,j,k,p} (\bar{h}_{ijk}^{p^*})^2 = 0, \quad \bar{\lambda} = 0, \quad \bar{\lambda}_1\bar{\lambda}_2 \neq 0, \\ S < \sup H^2 < \frac{6}{5}S, \quad 1.89 < S < 2. \end{cases}$$

We obtain from (4.38)

$$\begin{aligned} &\bar{H}^6 - 3\bar{H}^4S + 3\bar{H}^2S^2 - 3S^2 + 2S \\ &\geq \frac{(\bar{H}^2 - S)^2}{4}\Big[(6 + \frac{3}{4})S - (4 + \frac{3}{8})\bar{H}^2 - \frac{3S(3S-2)}{8\bar{H}^2}\Big]. \end{aligned}$$

Since $-\frac{3}{4}\bar{H}^2 - \frac{3S(3S-2)}{8\bar{H}^2}$ is a decreasing function of $\bar{H}^2$, for $S < \bar{H}^2 < \frac{6S}{5}$, we have

$$-\frac{3}{4}\bar{H}^2 - \frac{3S(3S-2)}{8\bar{H}^2} > -\frac{9}{10}S - \frac{5(3S-2)}{16}.$$

$$\begin{aligned}&\bar{H}^6 - 3\bar{H}^4 S + 3\bar{H}^2 S^2 - 3S^2 + 2S \\ &\geq \frac{(\bar{H}^2 - S)^2}{4}\Big[(5 - \frac{7}{80})S - (4 - \frac{3}{8})\bar{H}^2 + \frac{5}{8}\Big].\end{aligned} \tag{4.40}$$

We consider function

$$\begin{aligned}f_3(x) &= x^3 - 3x^2 S + 3xS^2 - 3S^2 + 2S - \frac{(x-S)^2}{4}\Big[(5 - \frac{7}{80})S - (4 - \frac{3}{8})x + \frac{5}{8}\Big].\\ f_3'(x) &= 3x^2 - 6xS + 3S^2 + \frac{(x-S)^2}{4}(4 - \frac{3}{8}) \\ &\quad - \frac{(x-S)}{2}\Big[(5 - \frac{7}{80})S - (4 - \frac{3}{8})x + \frac{5}{8}\Big] \\ &= (x-S)\Big[(6 - \frac{9}{32})x - (6 + \frac{29}{80})S - \frac{5}{16}\Big].\end{aligned}$$

Hence, $f_3(x)$ attains its minimum at $(6 - \frac{9}{32})x - (6 + \frac{29}{80})S - \frac{5}{16} = 0$.

$$f_3(S) = S(S-1)(S-2) < 0, \quad f_3(\frac{6S}{5}) = (1 + \frac{1}{125} - \frac{9}{1600})S^3 - (3 + \frac{1}{160})S^2 + 2S < 0$$

if $\frac{6}{5} \leq S < 2$. This is in contradiction to (4.40). Therefore, we conclude that the formula (4.18) in Proposition 4.1 does not occur either. □

Proof of Theorem 4.2 According to Propositions 4.1, 4.2 and 4.3, we know that there are no 2-dimensional complete Lagrangian self-shrinkers $X : M^2 \to \mathbb{R}^4$ with constant squared norm S of the second fundamental form and $1 < S < 2$. □

Theorem 4.3 *Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional Lagrangian self-shrinker in $\mathbb{R}^4$. If $S \equiv 2$ or $S \equiv 1$, then the mean curvature H satisfies $H \neq 0$ on M^2.*

Proof If there exists a point $p \in M^2$ such that $H = 0$ at p, then we know $H^{1^*} = H^{2^*} = 0$. Thus, at p, we have

$$H = 0, \quad H^{1^*} = \lambda_1 + \lambda_2 = 0, \quad \lambda = h^{1^*}_{12} = -h^{2^*}_{22}.$$

From

$$H^{p^*}_{,i} = \sum_k h^{p^*}_{ik}\langle X, e_k\rangle, \quad \text{for } i, p = 1, 2,$$

we have

$$\begin{aligned}h^{1^*}_{111} + h^{1^*}_{221} &= H^{1^*}_{,1} = \lambda_1\langle X, e_1\rangle + \lambda\langle X, e_2\rangle,\\ h^{1^*}_{112} + h^{1^*}_{222} &= H^{1^*}_{,2} = \lambda\langle X, e_1\rangle - \lambda_1\langle X, e_2\rangle,\\ h^{2^*}_{112} + h^{2^*}_{222} &= H^{2^*}_{,2} = -\lambda_1\langle X, e_1\rangle - \lambda\langle X, e_2\rangle\end{aligned}$$

and

$$0 = \frac{1}{2}\nabla_i S = \lambda_1(h^{1^*}_{11i} - 3h^{1^*}_{22i}) + \lambda(3h^{2^*}_{11i} - h^{2^*}_{22i}), \quad \text{for } i = 1, 2,$$

it means that,

$$\lambda_1(h^{1^*}_{11i} - 3h^{1^*}_{22i}) + \lambda(3h^{2^*}_{11i} - h^{2^*}_{22i}) = 0, \quad \text{for } i = 1, 2$$

since S is constant. Thus, we get a system of linear equations

$$\begin{pmatrix} -\lambda & -3\lambda_1 & 3\lambda & \lambda_1 & 0 \\ 0 & -\lambda & -3\bar{\lambda}_1 & 3\lambda & \lambda_1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h^{2^*}_{222} \\ h^{1^*}_{222} \\ h^{1^*}_{122} \\ h^{1^*}_{112} \\ h^{1^*}_{111} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda_1\langle X, e_1\rangle + \lambda\langle X, e_2\rangle \\ \lambda\langle X, e_1\rangle - \lambda_1\langle X, e_2\rangle \\ -\lambda_1\langle X, e_1\rangle - \lambda\langle X, e_2\rangle \end{pmatrix}. \tag{4.41}$$

From $S = 4\lambda^2 + 4\lambda_1^2$, we know $\lambda_1 = -\lambda_2$ and $2\lambda_1^2 + 2\lambda^2 = \frac{1}{2}S \neq 0$. Hence, by solving the above system, we get

$$\begin{aligned} h^{2^*}_{222} &= \frac{-5\lambda_1}{4}\langle X, e_1\rangle - \frac{3\lambda}{4}\langle X, e_2\rangle, \\ h^{1^*}_{222} &= \frac{3\lambda}{4}\langle X, e_1\rangle - \frac{\lambda_1}{4}\langle X, e_2\rangle, \\ h^{1^*}_{221} &= \frac{\lambda_1}{4}\langle X, e_1\rangle - \frac{\lambda}{4}\langle X, e_2\rangle, \\ h^{1^*}_{211} &= \frac{\lambda}{4}\langle X, e_1\rangle - \frac{3\lambda_1}{4}\langle X, e_2\rangle, \\ h^{1^*}_{111} &= \frac{3\lambda_1}{4}\langle X, e_1\rangle + \frac{5\lambda}{4}\langle X, e_2\rangle. \end{aligned}$$

With a direct calculation, we obtain

$$\begin{aligned} |\nabla^\perp \vec{H}|^2 &= \sum_{i,p}(H^{p^*}_{,i})^2 \\ &= \sum_i (h^{1^*}_{11i} + h^{1^*}_{22i})^2 + \sum_i (h^{2^*}_{11i} + h^{2^*}_{22i})^2 \\ &= 2(\lambda^2 + \lambda_1^2)\langle X, e_1\rangle^2 + 2(\lambda^2 + \lambda_1^2)\langle X, e_2\rangle^2 = 2(\lambda^2 + \lambda_1^2)|X|^2 \end{aligned}$$

and

$$\begin{aligned} &\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 \\ &= \sum_p (h^{p^*}_{111})^2 + \sum_p (h^{p^*}_{222})^2 + 3\sum_p (h^{p^*}_{221})^2 + 3\sum_p (h^{p^*}_{112})^2 \\ &= \sum_p (h^{1^*}_{11p})^2 + (h^{1^*}_{222})^2 + (h^{2^*}_{222})^2 + 3\sum_p (h^{1^*}_{22p})^2 + 3(h^{1^*}_{112})^2 + 3(h^{2^*}_{112})^2 \\ &= \frac{5}{2}\Big\{(\lambda_1^2 + \lambda^2)\langle X, e_1\rangle^2 + (\lambda_1^2 + \lambda^2)\langle X, e_2\rangle^2\Big\} \\ &= \frac{5}{2}(\lambda_1^2 + \lambda^2)|X|^2 = \frac{5}{4}|\nabla^\perp \vec{H}|^2. \end{aligned} \tag{4.42}$$

From the Ricci identity (2.8), we have

$$\begin{aligned} h^{2^*}_{1122} - h^{2^*}_{2211} &= 3\lambda K, \\ h^{1^*}_{1112} - h^{1^*}_{1121} &= -3\lambda K, \\ h^{2^*}_{1112} - h^{2^*}_{1121} &= 3\lambda_1 K, \\ h^{2^*}_{2212} - h^{2^*}_{2221} &= -3\lambda_1 K. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\sum_{i,j,k,l,p} (h^{p^*}_{ijkl})^2 \\ &= 4(h^{1^*}_{1122})^2 + 6(h^{1^*}_{2211})^2 + 6(h^{2^*}_{1122})^2 + 4(h^{2^*}_{2211})^2 + 4(h^{1^*}_{2222})^2 + 4(h^{2^*}_{1111})^2 \\ &\quad + (h^{1^*}_{1111})^2 + (h^{2^*}_{2222})^2 + (h^{1^*}_{1112})^2 + (h^{2^*}_{2221})^2 \\ &= 2(h^{1^*}_{1122} - h^{1^*}_{2211})^2 + 2(h^{1^*}_{1122} + h^{1^*}_{2211})^2 \\ &\quad + 2(h^{2^*}_{1122} - h^{2^*}_{2211})^2 + 2(h^{2^*}_{1122} + h^{2^*}_{2211})^2 \\ &\quad + \frac{1}{2}(h^{1^*}_{2222} - h^{2^*}_{2221})^2 + \frac{1}{2}(h^{1^*}_{2222} + h^{2^*}_{2221})^2 \\ &\quad + \frac{1}{2}(h^{1^*}_{1121} + h^{1^*}_{1112})^2 + \frac{1}{2}(h^{1^*}_{1121} - h^{1^*}_{1112})^2 \\ &\quad + \frac{1}{2}(h^{1^*}_{1111} + h^{1^*}_{2211})^2 + \frac{1}{2}(h^{1^*}_{1111} - h^{1^*}_{2211})^2 \\ &\quad + \frac{1}{2}(h^{2^*}_{2222} - h^{2^*}_{1122})^2 + \frac{1}{2}(h^{2^*}_{2222} + h^{2^*}_{1122})^2 \\ &\quad + \frac{1}{2}(h^{1^*}_{2211} - h^{1^*}_{2222})^2 + \frac{1}{2}(h^{1^*}_{2211} + h^{1^*}_{2222})^2 \\ &\quad + (h^{2^*}_{1122})^2 + 3(h^{2^*}_{1111})^2 + 2(h^{1^*}_{2222})^2 \\ &\geq 18\lambda_1^2 K^2 + \frac{9}{2}\lambda_1^2 K^2 + 18\lambda^2 K^2 + \frac{9}{2}\lambda^2 K^2 \\ &= \frac{45}{2}(\lambda_1^2 + \lambda^2)K^2 \\ &= \frac{45}{32} S^3 \end{aligned} \tag{4.43}$$

because of $S = 4(\lambda_1^2 + \lambda^2)$ and $K = -\frac{1}{2}S$.

On the other hand, since S is constant and $H = 0$ at p, from (2.35) in Lemma 2.3 and (4.42), we obtain, at p,

$$\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2 = -\frac{3S}{2}|\nabla^\perp \vec{H}|^2 = -\frac{3}{5}S^2(3S-2). \tag{4.44}$$

According to $h^{p^*}_{11k} + h^{p^*}_{22k} = H^{p^*}_{,k}$ and $H^{p^*} = 0$ for $p, k = 1, 2$, by a direct calculation, we have

$$\sum_{i,j,k,l,p,q} h^{p^*}_{il} h^{p^*}_{ijk} h^{q^*}_{jl} H^{q^*}_{,k} = \frac{S}{4}|\nabla^\perp \vec{H}|^2.$$

From (2.34) in Lemma 2.3, we get

$$
\begin{aligned}
&\frac{1}{2}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2\\
&=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2-(5S-2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2+\frac{5S}{2}|\nabla^{\perp}\vec{H}|^2\\
&\quad-\sum_{i,j,k,l,p,q}h^{p^*}_{il}h^{p^*}_{ijk}h^{q^*}_{jl}H^{q^*}_{,k}\\
&=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2-(5S-2)\sum_{i,j,k,p}(h^{p^*}_{ijk})^2+\frac{9S}{4}|\nabla^{\perp}\vec{H}|^2\\
&=\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2-(5S-2)\frac{S(3S-2)}{2}+\frac{9}{10}S^2(3S-2).
\end{aligned}
$$

Thus, we have from (4.44)

$$
\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2-(5S-2)\frac{S(3S-2)}{2}+\frac{9}{10}S^2(3S-2)=-\frac{3}{5}S^2(3S-2),
$$

namely,

$$
\sum_{i,j,k,l,p}(h^{p^*}_{ijkl})^2=S(S-1)(3S-2),
$$

which is in contradiction to (4.43) for $S\equiv 2$ or $S\equiv 1$. Hence, we conclude that $H\neq 0$ on M^2. □

Proposition 4.4 *Let $X: M^2\to\mathbb{R}^4$ be a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. If the squared norm S of the second fundamental form satisfies $S\equiv 1$ or $S\equiv 2$, then* $\sup H^2=S$.

Proof In terms of Lemma 4.1, there exists a sequence $\{p_m\}$ in M^2 such that

$$
\lim_{m\to\infty}H^2(p_m)=\sup H^2,\quad \lim_{m\to\infty}|\nabla H^2(p_m)|=0,\quad \limsup_{m\to\infty}\mathcal{L}H^2(p_m)\leq 0
$$

and

$$
\bar{\lambda}=0,\quad \bar{h}^{1^*}_{ij}=\bar{\lambda}_i\delta_{ij}.
$$

(1) Case for $S\equiv 2$. By making use of the same assertion as in the proof of Proposition 4.1, we have for $k=1,2$,

$$
\begin{cases}
\bar{h}^{1^*}_{11k}+\bar{h}^{1^*}_{22k}=0,\\
\bar{h}^{2^*}_{11k}+\bar{h}^{2^*}_{22k}=0,\\
(\bar{\lambda}_1-3\bar{\lambda}_2)\bar{h}^{1^*}_{11k}=0
\end{cases}
\tag{4.45}
$$

with $\bar{\lambda}_1\bar{\lambda}_2\neq 0$.

If $\bar{\lambda}_1=3\bar{\lambda}_2$, we get

$$
\lim_{m\to\infty}H^2(p_m)=\bar{H}^2=(\bar{\lambda}_1+\bar{\lambda}_2)^2=16\bar{\lambda}_2^2=\frac{4S}{3}.
$$

By making use of the same assertion as in the proof of Proposition 4.2, we can know that this is impossible.

Thus, we get $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$. In this case, we obtain $\bar{h}^{p^*}_{ijk} = 0$ for any i, j, k, p from (4.45). Hence, we have from (2.25) in Lemma 2.1,

$$\begin{aligned} 0 &= S(1 - \frac{3}{2}S) + 2\bar{H}^2 S - \frac{1}{2}\bar{H}^4 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \\ &= -(S - \bar{H}^2)^2 - \frac{1}{2}\bar{H}^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2. \end{aligned} \tag{4.46}$$

We conclude

$$\sup H^2 = \bar{H}^2 = S = 2.$$

(2) Case for $S \equiv 1$. Since $S = 1$, we have $\sup H^2 > 0$. From $\lim_{m\to\infty} |\nabla H^2(p_m)| = 0$ and $|\nabla H^2|^2 = 4\sum_i (\sum_{p^*} H^{p^*} H^{p^*}_{,i})^2$, we get

$$\bar{H}^{1^*}_{,i} = 0.$$

From (4.7) and (4.10), we have

$$\bar{\lambda}_i \lim_{m\to\infty} \langle X, e_i\rangle(p_m) = 0.$$

Next, we take the following three cases into consideration.

(a) If $\bar{\lambda}_1 = 0$, in this case, $\bar{\lambda}_2 \neq 0$, $3\bar{H}^2 = S = 1$. Since $\bar{H}^{1^*}_{,i} = 0$ and $S = 1$, we get

$$\bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \quad \bar{h}^{1^*}_{22k} = 0, \quad k = 1, 2.$$

Therefore,

$$\bar{h}^{1^*}_{111} = \bar{h}^{1^*}_{112} = \bar{h}^{1^*}_{122} = \bar{h}^{1^*}_{222} = 0$$

and

$$|\nabla^\perp \vec{H}|^2 = \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2.$$

From $\limsup_{m\to\infty} \mathcal{L}|H|^2(p_m) \leq 0$ and (4.8), we obtain

$$\frac{1}{2}(\bar{H}^2 - 1)^2 \leq 0,$$

it means that, $\bar{H}^2 = 1$. It is a contradiction.

(b) If $\bar{\lambda}_2 = 0$, in this case, $\bar{\lambda}_1 \neq 0$, $\sup H^2 = \bar{H}^2 = S = 1$.

(c) If $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$, in this case, for $k = 1, 2$,

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ (\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{11k} = 0. \end{cases}$$

If $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$, from the above equations, we know

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 0, \quad i, j, k, p = 1, 2.$$

From (4.8), we get

$$0 \geq \frac{1}{2}(\bar{H}^2 - 1)^2.$$

Hence, we have

$$\sup \bar{H}^2 = 1 = S.$$

If $\bar{\lambda}_1 = 3\bar{\lambda}_2$, we have $\bar{H}^2 = \frac{4}{3}S = \frac{4}{3}$ and $1 = S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 = 12\bar{\lambda}_2^2$. From (2.25), we get

$$\sum_{i,j,k,p} (\bar{h}_{ijk}^{p^*})^2 = -\frac{1}{6} < 0.$$

It is impossible. From the above arguments, we conclude that, for $S = 2$ or $S = 1$,

$$\sup H^2 = S.$$

□

Theorem 4.4 *Let $X : M^2 \to \mathbb{R}^4$ be a 2-dimensional complete Lagrangian self-shrinker in $\mathbb{R}^4$. If the squared norm S of the second fundamental form satisfies $S \equiv 1$ or $S \equiv 2$, then $H^2 = S$ is constant.*

Proof We can apply the generalized maximum principle for $\mathcal{L}$-operator to the function $-H^2$. Thus, there exists a sequence $\{p_m\}$ in M^2 such that

$$\lim_{m\to\infty} H^2(p_m) = \inf H^2, \quad \lim_{m\to\infty} |\nabla H^2(p_m)| = 0, \quad \liminf_{m\to\infty} \mathcal{L}H^2(p_m) \geq 0.$$

By making use of the similar assertion as in the proof of Lemma 4.1, we have

$$\begin{cases} \lim\limits_{m\to\infty} H^2(p_m) = \inf H^2 = \bar{H}^2, \quad \lim\limits_{m\to\infty} |\nabla H^2(p_m)| = 0, \\ \lim\limits_{m\to\infty} |\nabla^{\perp}\vec{H}|^2(p_m) - \sum\limits_{i,j,k,p} (\bar{h}_{ijk}^{p^*})^2 + \frac{1}{2}(\bar{H}^2 - S)(\bar{H}^2 - 3S + 2) \geq 0. \end{cases} \tag{4.47}$$

By taking the limit and making use of the same assertion as in Theorem 4.3, we can prove $\inf H^2 \neq 0$. Hence, without loss of the generality, at each point p_m, we choose e_1, e_2 such that

$$\vec{H} = H^{1^*} e_{1^*}$$

and we can assume

$$\lim_{m\to\infty} h_{ijl}^{p^*}(p_m) = \bar{h}_{ijl}^{p^*}, \quad \lim_{m\to\infty} h_{ij}^{p^*}(p_m) = \bar{h}_{ij}^{p^*}, \quad \lim_{m\to\infty} h_{ijkl}^{p^*}(p_m) = \bar{h}_{ijkl}^{p^*},$$

for $i, j, k, l, p = 1, 2$. From $\lim_{k\to\infty} |\nabla H^2(p_m)| = 0$ and $|\nabla H^2|^2 = 4\sum_i (\sum_{p^*} H^{p^*} H_{,i}^{p^*})^2$, we have

$$\bar{H}_{,k}^{1^*} = 0. \tag{4.48}$$

From (4.7) and (4.48), we obtain

$$\begin{cases} \bar{\lambda}_1 \lim\limits_{m\to\infty} \langle X, e_1\rangle(p_m) + \bar{\lambda} \lim\limits_{m\to\infty} \langle X, e_2\rangle(p_m) = 0, \\ \bar{\lambda} \lim\limits_{m\to\infty} \langle X, e_1\rangle(p_m) + \bar{\lambda}_2 \lim\limits_{m\to\infty} \langle X, e_2\rangle(p_m) = 0. \end{cases} \tag{4.49}$$

If $\bar{\lambda}_1\bar{\lambda}_2 \neq \bar{\lambda}^2$ and $\bar{\lambda} \neq 0$, we get

$$\lim_{m\to\infty}\langle X, e_1\rangle(p_m) = \lim_{m\to\infty}\langle X, e_2\rangle(p_m) = 0.$$

Thus, we know, for $k = 1, 2$,

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0, \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0, \\ \bar{\lambda}_1\bar{h}^{1^*}_{11k} + 3\bar{\lambda}\bar{h}^{1^*}_{12k} + 3\bar{\lambda}_2\bar{h}^{2^*}_{12k} - \bar{\lambda}\bar{h}^{2^*}_{22k} = 0. \end{cases} \tag{4.50}$$

We conclude, for any i, j, k, p,

$$\bar{h}^{p^*}_{ijk} = 0.$$

From (4.47) and (2.25) in Lemma 2.1, we have

$$\begin{aligned} &S \geq \inf H^2 = \bar{H}^2, \\ &S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2) = 0. \end{aligned} \tag{4.51}$$

From Lemma 2.5 and taking limit,

$$\begin{aligned} 0 &\leq \sum_{i,j,k,l,p}(\bar{h}^{p^*}_{ijkl})^2 = \frac{1}{2}\lim_{m\to\infty}\mathcal{L}\sum_{i,j,k,p}(h^{p^*}_{ijk})^2(p_m) \\ &= \bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] - \bar{\lambda}^2\bar{H}^4 \\ &\quad + \bar{H}^2(S + 2 - \frac{3}{2}\bar{H}^2 - \bar{\lambda}_1^2 - \bar{\lambda}_2^2 - 2\bar{\lambda}^2)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2) \\ &< \bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] \\ &\quad + \bar{H}^2(S + 2 - \frac{3}{2}\bar{H}^2 - \bar{\lambda}_1^2 - \bar{\lambda}_2^2 - 2\bar{\lambda}^2)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2). \end{aligned} \tag{4.52}$$

According to (4.51), we have

$$\begin{aligned} &\bar{H}^2\Big[\bar{H}^2 - 2S + \frac{1}{2}\bar{H}^4 - \bar{K}\bar{H}^2 - \bar{K}^2\Big] \\ &\quad + \left(S + 2 - \frac{3}{2}\bar{H}^2 - \frac{1}{\bar{H}^2}\Big(S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4\Big)\right) \\ &\quad \times \left(S(1 - \frac{1}{2}S) - (S - \bar{H}^2)^2 + \frac{1}{2}\bar{H}^4\right) \\ &= \frac{1}{4\bar{H}^2}\left(\bar{H}^8 - 2S\bar{H}^6 - 6S(S-1)\bar{H}^4 + 2S(2-3S)^2\bar{H}^2 - (2-3S)^2S^2\right). \end{aligned}$$

We consider a function $f(t)$ defined by

$$f(t) = t^4 - 2St^3 - 6S(S-1)t^2 + 2S(2-3S)^2t - (2-3S)^2S^2, \tag{4.53}$$

for $0 < t \leq S$. Thus, we get

$$f'(t) = 4t^3 - 6St^2 - 12S(S-1)t + 2S(2-3S)^2, \quad f''(t) = 12(t^2 - St - S(S-1)), \tag{4.54}$$

$f''(t) < 0$ for $t \in (0, S)$. Hence, $f'(t)$ is a decreasing function for $t \in (0, S)$. Since $f'(S) = 4S(S-1)(S-2) = 0$, $f(t)$ is a increasing function for $t \in (0, S)$. According to

$$f(S) = 2(S-1)(S-2)S^2 = 0, \tag{4.55}$$

we conclude $f(t) < 0$ for $t \in (0, S)$. This is a contradiction.
Hence, we have $\bar{\lambda}_1\bar{\lambda}_2 \neq 0$ and $\bar{\lambda} = 0$. In this case, we get for $k = 1, 2$,

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0 \\ \bar{h}^{2^*}_{11k} + \bar{h}^{2^*}_{22k} = 0 \\ (\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{11k} = 0. \end{cases} \tag{4.56}$$

If $\bar{\lambda}_1 = 3\bar{\lambda}_2$, we obtain

$$\inf H^2 = \lim_{m\to\infty} H^2(p_m) = (\bar{\lambda}_1 + \bar{\lambda}_2)^2 = 16\bar{\lambda}_2^2 = \frac{4S}{3},$$

which is impossible from Proposition 4.4. Thus, we get $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$. In this case, we have

$$\bar{h}^{p^*}_{ijk} = 0$$

for any i, j, k, p from (4.56).
From (2.19), we know

$$\bar{H}^{p^*}_{,ij} = \bar{h}^{p^*}_{ij} - \sum_k \bar{h}^{p^*}_{ik}\bar{h}^{1^*}_{jk}\bar{H}$$

because of $H^{1^*} = H$ and $H^{2^*} = 0$. Thus, we get

$$\begin{aligned} &\bar{h}^{1^*}_{1111} + \bar{h}^{1^*}_{2211} = \bar{\lambda}_1 - \bar{\lambda}_1^2\bar{H}, \\ &\bar{h}^{1^*}_{1122} + \bar{h}^{1^*}_{2222} = \bar{\lambda}_2 - \bar{\lambda}_2^2\bar{H}, \\ &\bar{h}^{1^*}_{1112} + \bar{h}^{1^*}_{2212} = \bar{h}^{1^*}_{1121} + \bar{h}^{1^*}_{2221} = 0, \\ &\bar{h}^{2^*}_{1122} + \bar{h}^{2^*}_{2222} = 0, \\ &\bar{h}^{2^*}_{1121} + \bar{h}^{2^*}_{2221} = \bar{\lambda}_2 - \bar{\lambda}_1\bar{\lambda}_2\bar{H}. \end{aligned} \tag{4.57}$$

From Ricci identities (2.8), we obtain

$$\begin{aligned} &\bar{h}^{2^*}_{1122} = \bar{h}^{2^*}_{2211}, \ \bar{h}^{1^*}_{1112} = \bar{h}^{1^*}_{1121}, \\ &\bar{h}^{2^*}_{1112} - \bar{h}^{2^*}_{1121} = (\bar{\lambda}_1 - 2\bar{\lambda}_2)\bar{K}, \\ &\bar{h}^{2^*}_{2212} - \bar{h}^{2^*}_{2221} = 3\bar{\lambda}_2\bar{K}. \end{aligned} \tag{4.58}$$

On the other hand, since S is constant, we know, for $k, l = 1, 2$,

$$0 = -\sum_{i,j,p} \bar{h}^{p^*}_{ijl}\bar{h}^{p^*}_{ijk} = \sum_{i,j,p} \bar{h}^{p^*}_{ij}\bar{h}^{p^*}_{ijkl} = \bar{h}^{1^*}_{11}\bar{h}^{1^*}_{11kl} + 3\bar{h}^{2^*}_{12}\bar{h}^{2^*}_{12kl} = \bar{\lambda}_1\bar{h}^{1^*}_{11kl} + 3\bar{\lambda}_2\bar{h}^{1^*}_{22kl}. \tag{4.59}$$

From (4.57) and (4.59), we have

$$\begin{aligned} &(\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{2211} = \bar{\lambda}_1^2 - \bar{\lambda}_1^3\bar{H}, \\ &(\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{2222} = \bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}_1\bar{\lambda}_2^2\bar{H}, \\ &(\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{1122} = -3\bar{\lambda}_2^2 + 3\bar{\lambda}_2^3\bar{H} \\ &(\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{2212} = 0. \end{aligned} \tag{4.60}$$

Hence, we conclude from (4.57) and (4.60)

$$\bar{h}^{1^*}_{2212} = \bar{h}^{1^*}_{1112} = 0$$

because of $\bar{\lambda}_1 \neq 3\bar{\lambda}_2$. According to (4.60), we obtain

$$(\bar{\lambda}_1 - 3\bar{\lambda}_2)(\bar{h}^{2^*}_{1112} - \bar{h}^{2^*}_{1121}) = -S + (3\bar{\lambda}_2^3 + \bar{\lambda}_1^3)\bar{H} \tag{4.61}$$

because of $S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2$.

For the case $S \equiv 1$, from (4.58), we know

$$\begin{aligned}(\bar{\lambda}_1 - 3\bar{\lambda}_2)&(\bar{h}^{2^*}_{1112} - \bar{h}^{2^*}_{1121})\\ &= (\bar{\lambda}_1 - 3\bar{\lambda}_2)(\bar{\lambda}_1 - 2\bar{\lambda}_2)\bar{K}\\ &= (1 + 3\bar{\lambda}_2^2 - 5\bar{\lambda}_1\bar{\lambda}_2)(\bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}_2^2)\\ &= \bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}_2^2 + 8\bar{\lambda}_1\bar{\lambda}_2^3 - 3\bar{\lambda}_2^4 - 5\bar{\lambda}_1^2\bar{\lambda}_2^2\\ &= \bar{\lambda}_1\bar{\lambda}_2 - 2\bar{\lambda}_2^2 + 8\bar{\lambda}_1\bar{\lambda}_2^3 - 4\bar{\lambda}_1^2\bar{\lambda}_2^2.\end{aligned} \tag{4.62}$$

By a direct calculation and by using $S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 = 1$, we get

$$-1 + (3\bar{\lambda}_2^3 + \bar{\lambda}_1^3)\bar{H} - \left\{\bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}_2^2 + 8\bar{\lambda}_1\bar{\lambda}_2^3 - 3\bar{\lambda}_2^4 - 5\bar{\lambda}_1^2\bar{\lambda}_2^2\right\} = -8\bar{\lambda}_1\bar{\lambda}_2^3 \neq 0.$$

From (4.61) and (4.62), it is impossible.
For the case $S \equiv 2$, we have from (2.25) in Lemma 2.1,

$$\begin{aligned}0 &= S(1 - \frac{3}{2}S) + 2\bar{H}^2 S - \frac{1}{2}\bar{H}^4 - \bar{H}^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)\\ &= -(S - \bar{H}^2)^2 - \frac{1}{2}\bar{H}^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2.\end{aligned} \tag{4.63}$$

We conclude from Proposition 4.4

$$\inf H^2 = \bar{H}^2 = S = \sup H^2.$$

Thus, we know that $H^2 = S$ is constant.

From now on, we consider the case $\bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}^2$. In this case, we have

$$S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 + 4\bar{\lambda}^2 = (\bar{\lambda}_1 + \bar{\lambda}_2)(\bar{\lambda}_1 + 3\bar{\lambda}_2) = \bar{H}(\bar{\lambda}_1 + 3\bar{\lambda}_2).$$

If $S \equiv 1$, from (2.25), we get

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = \frac{1}{2}(\bar{H}^2 - 1)(3\bar{H}^2 - 1) \geq 0.$$

Hence, either $\bar{H}^2 \geq 1$, or $\bar{H}^2 \leq \frac{1}{3}$. If $\bar{H}^2 \geq 1$, then we have $H^2 \equiv 1 = S$ since $\inf H^2 = \bar{H}^2 \leq \sup H^2 = 1$ in view of Proposition 4.4. According to $S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 + 4\bar{\lambda}^2$ and $H^2 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + 2\bar{\lambda}^2$, we know $\bar{\lambda} = 0$ and $\bar{\lambda}_2 = 0$.

If $\bar{H}^2 \leq \frac{1}{3}$, from $S = \bar{H}(\bar{\lambda}_1 + 3\bar{\lambda}_2) = 1$, we obtain $(\bar{\lambda}_1 + 3\bar{\lambda}_2)^2 \geq 3$, which implies $\bar{\lambda}_1 = \bar{\lambda} = 0$ because of $(\bar{\lambda}_1 + 3\bar{\lambda}_2)^2 = \bar{\lambda}_1^2 + 9\bar{\lambda}_2^2 + 6\bar{\lambda}^2 \leq 3\bar{\lambda}_1^2 + 9\bar{\lambda}_2^2 + 12\bar{\lambda}^2 = 3S = 3$. Hence, we have $\inf H^2 = \bar{\lambda}_2^2 = \frac{S}{3} \neq 0$,

$$\bar{H}^{1^*}_{,k} = 0, \quad \bar{H}^{2^*}_{,1} = 0$$

because of $H^{p^*}_{,i} = \sum_k h^{p^*}_{ik}\langle X, e_k\rangle$. Hence, we have, by using the same calculations as in (4.6),

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0 \\ \bar{h}^{2^*}_{111} + \bar{h}^{2^*}_{221} = 0 \\ 3\bar{\lambda}_2\bar{h}^{1^*}_{22k} = 0. \end{cases} \tag{4.64}$$

Hence, we get

$$\bar{h}^{p^*}_{ijk} = 0, \quad \text{except } i = j = k = p^* = 2.$$

If $\bar{h}^{2^*}_{222} \neq 0$, since $\bar{\lambda}_2 \neq 0$, $3\bar{H}^2 = 3\bar{\lambda}_2^2 = S = 1$, we have

$$0 = \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 + S(1 - \frac{3}{2}S) + 2\bar{H}^2 S - \frac{1}{2}\bar{H}^4 - \bar{H}^2\bar{\lambda}_2^2 = \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 > 0.$$

It is impossible. Hence, we know

$$\bar{h}^{p^*}_{ijk} = 0,$$

for any i, j, k, p. From (2.19), we get

$$\bar{H}^{p^*}_{,ij} = \bar{h}^{p^*}_{ij} - \sum_k \bar{h}^{p^*}_{ik}\bar{h}^{1^*}_{jk}\bar{H}.$$

We obtain

$$\bar{h}^{2^*}_{1121} + \bar{h}^{2^*}_{2221} = \bar{\lambda}_2 \neq 0.$$

From (2.34) of Lemma 2.3, we have

$$\frac{1}{2}\mathcal{L}\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = \sum_{i,j,k,l,p} (\bar{h}^{p^*}_{ijkl})^2 > 0. \tag{4.65}$$

From (2.35) of Lemma 2.3, we get

$$\begin{aligned} &\frac{1}{2}\mathcal{L}\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 \\ &= (\bar{H}^2 - 2S)\bar{H}^2 + (3\bar{K} + 2 - \bar{H}^2 + 2S)\sum_{i,j} \bar{H}^2\bar{h}^{1^*}_{ij}\bar{h}^{1^*}_{ij} \\ &\quad - \bar{K}(\bar{H}^4 + \bar{H}^3\bar{h}^{1^*}_{11}) - \sum_{i,j,k,l,p} \bar{H}^2\bar{h}^{1^*}_{jk}\bar{h}^{p^*}_{jk}\bar{h}^{p^*}_{il}\bar{h}^{1^*}_{il} - \sum_{i,j,k} \bar{H}^3\bar{h}^{1^*}_{ik}\bar{h}^{1^*}_{ji}\bar{h}^{1^*}_{jk} \\ &= (\bar{H}^2 - 2S)\bar{H}^2 + (3\bar{K} + 2 - \bar{H}^2 + 2S)\sum_{i,j} \bar{H}^2\bar{\lambda}_2^2 - \bar{K}\bar{H}^4 - \bar{H}^2\bar{\lambda}_2^4 - \bar{H}^3\bar{\lambda}_2^3 \\ &= \bar{H}^6 - 3\bar{H}^4 = -\frac{8}{27}, \end{aligned}$$

which is in contradiction to (4.65). Hence, we get $\inf H^2 = S$, that is, $H^2 = S$ is constant from Proposition 4.4.

For the case $S \equiv 2$, first of all, we will prove $\bar{\lambda} = 0$. If not, we have $S = 2$ and $\bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}^2 \neq 0$. By making use of the same assertion as in the proof of Theorem 4.3, we have

$$\begin{pmatrix} -\bar{\lambda} & 3\bar{\lambda}_2 & 3\bar{\lambda} & \bar{\lambda}_1 & 0 \\ 0 & -\bar{\lambda} & 3\bar{\lambda}_2 & 3\bar{\lambda} & \bar{\lambda}_1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{h}^{2^*}_{222} \\ \bar{h}^{1^*}_{222} \\ \bar{h}^{1^*}_{122} \\ \bar{h}^{1^*}_{112} \\ \bar{h}^{1^*}_{111} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ A \end{pmatrix}, \tag{4.66}$$

where

$$\begin{aligned} A &= \bar{\lambda}_2 \lim_{m\to\infty} \langle X, e_1\rangle(p_m) - \bar{\lambda} \lim_{m\to\infty} \langle X, e_2\rangle(p_m) \\ &= \bar{H} \lim_{m\to\infty} \langle X, e_1\rangle(p_m) \\ &= -\frac{H\bar{\lambda}}{\bar{\lambda}_1} \lim_{m\to\infty} \langle X, e_2\rangle(p_m). \end{aligned} \tag{4.67}$$

Solving this system of linear equations, we have

$$\begin{aligned} \mu \bar{h}^{2^*}_{222} &= \left\{ 12\bar{\lambda}^2 + (\bar{\lambda}_1 - 3\bar{\lambda}_2)^2 \right\} A, \\ \bar{h}^{1^*}_{222} &= -\bar{h}^{1^*}_{211}, \\ \bar{h}^{1^*}_{221} &= -\bar{h}^{1^*}_{111}, \\ \mu \bar{h}^{1^*}_{211} &= \bar{\lambda}(\bar{\lambda}_1 - 3\bar{\lambda}_2) A, \\ \mu \bar{h}^{1^*}_{111} &= -4\bar{\lambda}^2 A \end{aligned} \tag{4.68}$$

with $\mu = 16\bar{\lambda}^2 + (\bar{\lambda}_1 - 3\bar{\lambda}_2)^2$.

$$\lim_{m\to\infty} |\nabla^\perp \vec{H}|^2 = \sum_{i,p} (\bar{H}^{p^*}_{,i})^2 = (\bar{h}^{2^*}_{112} + \bar{h}^{2^*}_{222})^2 = A^2 \tag{4.69}$$

$$\begin{aligned} \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 &= \sum_p (\bar{h}^{p^*}_{111})^2 + \sum_p (\bar{h}^{p^*}_{222})^2 + 3\sum_p (\bar{h}^{p^*}_{221})^2 + 3\sum_p (\bar{h}^{p^*}_{112})^2 \\ &= \sum_p (\bar{h}^{1^*}_{11p})^2 + (\bar{h}^{1^*}_{222})^2 + (\bar{h}^{2^*}_{222})^2 + 3\sum_p (\bar{h}^{1^*}_{22p})^2 + 3(\bar{h}^{1^*}_{112})^2 + 3(\bar{h}^{2^*}_{112})^2 \\ &= \lim_{m\to\infty} |\nabla^\perp \vec{H}|^2. \end{aligned} \tag{4.70}$$

Since $S = 2$ and $\bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}^2 \neq 0$, we obtain

$$S = \bar{\lambda}_1^2 + 3\bar{\lambda}_2^2 + 4\bar{\lambda}_3^2 = \bar{H}(\bar{\lambda}_1 + 3\bar{\lambda}_2) = 2,$$

From (2.25), we have

$$\sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = 4 - 4\bar{H}^2 + \frac{3}{2}\bar{H}^4 = (2 - H^2)^2 + \frac{1}{2}\bar{H}^4 > 0. \tag{4.71}$$

Since $S = 2$ is constant, we get, for $k, l = 1, 2$,

$$-\sum_{i,j,p} \bar{h}^{p^*}_{ijl}\bar{h}^{p^*}_{ijk} = \sum_{i,j,p} \bar{h}^{p^*}_{ij}\bar{h}^{p^*}_{ijkl} = \bar{\lambda}_1\bar{h}^{1^*}_{11kl} + 3\bar{\lambda}\bar{h}^{1^*}_{12kl} + 3\bar{\lambda}_2\bar{h}^{1^*}_{22kl} - \bar{\lambda}\bar{h}^{2^*}_{22kl},$$

namely,

$$\begin{aligned}
&\bar{\lambda}_1\bar{h}^{1^*}_{1122}+3\bar{\lambda}\bar{h}^{1^*}_{1222}+3\bar{\lambda}_2\bar{h}^{1^*}_{2222}-\bar{\lambda}\bar{h}^{2^*}_{2222}\\
&\quad=-(\bar{h}^{1^*}_{112})^2-3(\bar{h}^{2^*}_{112})^2-3(\bar{h}^{2^*}_{212})^2-(\bar{h}^{2^*}_{222})^2\\
&\quad=-\frac{12\bar{\lambda}^2+(\bar{\lambda}_1-3\bar{\lambda}_2)^2}{\mu}A^2,\\
&\bar{\lambda}_1\bar{h}^{1^*}_{1112}+3\bar{\lambda}\bar{h}^{1^*}_{1212}+3\bar{\lambda}_2\bar{h}^{1^*}_{2212}-\bar{\lambda}\bar{h}^{2^*}_{2212}\\
&\quad=-\bar{h}^{1^*}_{111}\bar{h}^{1^*}_{112}-3\bar{h}^{2^*}_{111}\bar{h}^{2^*}_{112}-3\bar{h}^{2^*}_{121}\bar{h}^{2^*}_{122}-\bar{h}^{2^*}_{221}\bar{h}^{2^*}_{222}\\
&\quad=\frac{\bar{\lambda}(\bar{\lambda}_1-3\bar{\lambda}_2)}{\mu}A^2.
\end{aligned}\tag{4.72}$$

From (2.19) and taking limit, we know, for $i, j, p = 1, 2$

$$\bar{h}^{p^*}_{11ij}+\bar{h}^{p^*}_{22ij}=\sum_k\bar{h}^{p^*}_{ikj}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)+\bar{h}^{p^*}_{ij}-\sum_k\bar{h}^{p^*}_{ik}\bar{h}^{1^*}_{jk}\bar{H},$$

it means that,

$$\begin{aligned}
\bar{h}^{1^*}_{1122}+\bar{h}^{1^*}_{2222}&=\sum_k\bar{h}^{1^*}_{2k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)+\bar{\lambda}_2-(\bar{\lambda}_2^2+\bar{\lambda}^2)\bar{H},\\
\bar{h}^{1^*}_{1112}+\bar{h}^{1^*}_{2212}&=\sum_k\bar{h}^{1^*}_{1k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)+\bar{\lambda}-\bar{\lambda}\bar{H}^2,\\
\bar{h}^{2^*}_{1122}+\bar{h}^{2^*}_{2222}&=\sum_k\bar{h}^{2^*}_{2k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)-\bar{\lambda},
\end{aligned}$$

Since, from (4.49) and (4.67) and $S=\bar{H}(\bar{\lambda}_1+3\bar{\lambda}_2)=2$,

$$\begin{aligned}
\sum_k\bar{h}^{1^*}_{2k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)&=\frac{\bar{\lambda}_1A^2}{\mu},\\
\sum_k\bar{h}^{2^*}_{2k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)&=\frac{3\bar{\lambda}}{\mu}A^2-\frac{\bar{\lambda}_1^2+3\bar{\lambda}^2}{2\bar{\lambda}}A^2,\\
\sum_k\bar{h}^{1^*}_{1k2}\lim_{m\to\infty}\langle X,e_k\rangle(p_m)&=-\frac{3\bar{\lambda}}{\mu}A^2,
\end{aligned}$$

we obtain

$$\begin{cases}
\bar{\lambda}_1\bar{h}^{1^*}_{1122}+3\bar{\lambda}\bar{h}^{1^*}_{1222}+3\bar{\lambda}_2\bar{h}^{1^*}_{2222}-\bar{\lambda}\bar{h}^{2^*}_{2222}=-\dfrac{12\bar{\lambda}^2+(\bar{\lambda}_1-3\bar{\lambda}_2)^2}{\mu}A^2,\\
\bar{\lambda}_1\bar{h}^{1^*}_{1112}+3\bar{\lambda}\bar{h}^{1^*}_{1212}+3\bar{\lambda}_2\bar{h}^{1^*}_{2212}-\bar{\lambda}\bar{h}^{2^*}_{2212}=\dfrac{\bar{\lambda}(\bar{\lambda}_1-3\bar{\lambda}_2)}{\mu}A^2.\\
\bar{h}^{1^*}_{1122}+\bar{h}^{1^*}_{2222}=\dfrac{\bar{\lambda}_1A^2}{\mu}+\bar{\lambda}_2-(\bar{\lambda}_2^2+\bar{\lambda}^2)\bar{H},\\
\bar{h}^{1^*}_{1112}+\bar{h}^{1^*}_{2212}=-\dfrac{3\bar{\lambda}}{\mu}A^2+\bar{\lambda}-\bar{\lambda}\bar{H}^2,\\
\bar{h}^{2^*}_{1122}+\bar{h}^{2^*}_{2222}=\dfrac{3\bar{\lambda}}{\mu}A^2-\dfrac{\bar{\lambda}_1^2+3\bar{\lambda}^2}{2\bar{\lambda}}A^2-\bar{\lambda}.
\end{cases}\tag{4.73}$$

Taking covariant differentiation of (2.25) and using (4.47) and (4.48), we obtain

$$0 = \bar{h}^{1^*}_{111}\bar{h}^{1^*}_{1112} + 4\bar{h}^{2^*}_{111}\bar{h}^{1^*}_{1122} + 6\bar{h}^{1^*}_{122}\bar{h}^{1^*}_{1222} + 4\bar{h}^{1^*}_{222}\bar{h}^{1^*}_{2222} + \bar{h}^{2^*}_{222}\bar{h}^{2^*}_{2222} - \bar{H}^2\Big\{\bar{\lambda}(\bar{h}^{2^*}_{112} + \bar{h}^{2^*}_{222}) + \bar{\lambda}_1\bar{h}^{2^*}_{111} + 2\bar{\lambda}\bar{h}^{1^*}_{122} + \bar{\lambda}_2\bar{h}^{1^*}_{222}\Big\}$$

Since

$$\bar{H}^2\Big\{\bar{\lambda}(\bar{h}^{2^*}_{112} + \bar{h}^{2^*}_{222}) + \bar{\lambda}_1\bar{h}^{2^*}_{111} + 2\bar{\lambda}\bar{h}^{1^*}_{122} + \bar{\lambda}_2\bar{h}^{1^*}_{222}\Big\} = \bar{H}^2\frac{\bar{\lambda}A}{\mu}(2+\mu),$$
$$\bar{h}^{1^*}_{111}\bar{h}^{1^*}_{1112} + 4\bar{h}^{2^*}_{111}\bar{h}^{1^*}_{1122} + 6\bar{h}^{1^*}_{122}\bar{h}^{1^*}_{1222} + 4\bar{h}^{1^*}_{222}\bar{h}^{1^*}_{2222} + \bar{h}^{2^*}_{222}\bar{h}^{2^*}_{2222}$$
$$= \frac{A}{\mu}\bigg(\Big\{-16\bar{\lambda}^2 + (\bar{\lambda}_1 - 3\bar{\lambda}_2)^2\Big\}\bar{h}^{2^*}_{2222} - 8\bar{\lambda}(\bar{\lambda}_1 - 3\bar{\lambda}_2)\bar{h}^{1^*}_{2222}\bigg) + \frac{\bar{\lambda}A^3}{\mu}\Big\{\frac{84\bar{\lambda}^2 + 4\lambda_1^2}{\mu} - 14(\bar{\lambda}_1^2 + 3\bar{\lambda}^2)\Big\} + \frac{4\bar{\lambda}A}{\mu}(-7\bar{\lambda}^2 - 3\lambda_2^2 + 3\bar{\lambda}_2^2\bar{H}^2),$$

we have

$$\bar{H}^2\frac{\bar{\lambda}A}{\mu}(2+\mu)$$
$$= \frac{A}{\mu}\bigg(\Big\{-4\bar{\lambda}\frac{21\bar{\lambda}^2 + \bar{\lambda}_1^2}{\mu} + \frac{(\bar{\lambda}_1^2 + 3\bar{\lambda}^2)(18\bar{\lambda}^2 - \bar{\lambda}_1^2 - 9\bar{\lambda}_2^2)}{2\bar{\lambda}}\Big\}A^2 + 10\bar{\lambda}^3 - 6\bar{\lambda}\bar{\lambda}_2^2 + 4\bar{\lambda}^3\bar{H}^2\bigg)$$
$$+ \frac{\bar{\lambda}A^3}{\mu}\Big\{\frac{84\bar{\lambda}^2 + 4\lambda_1^2}{\mu} - 14(\bar{\lambda}_1^2 + 3\bar{\lambda}^2)\Big\} + \frac{4\bar{\lambda}A}{\mu}(-7\bar{\lambda}^2 - 3\lambda_2^2 + 3\bar{\lambda}_2^2\bar{H}^2)$$
$$= \frac{A^3}{\mu}\frac{(\bar{\lambda}_1^2 + 3\bar{\lambda}^2)(-10\bar{\lambda}^2 - \bar{\lambda}_1^2 - 9\bar{\lambda}_2^2)}{2\bar{\lambda}} + \frac{\bar{\lambda}A}{\mu}(-18\bar{\lambda}^2 - 18\lambda_2^2 + 4(\bar{\lambda}^2 + 3\bar{\lambda}_2^2)\bar{H}^2),$$

which is impossible because of $2+\mu = 2\bar{\lambda}_1^2 + 12\bar{\lambda}_2^2 + 14\bar{\lambda}^2$. Hence, we have $\bar{\lambda} = 0$, that is, $\bar{\lambda}_1\bar{\lambda}_2 = 0$.

If $\bar{\lambda}_2 = 0$, we get $\inf H^2 = S = \sup H^2$ from Proposition 4.4. Namely, $H^2 = S$ is constant.

If $\bar{\lambda}_1 = 0$, we have

$$\bar{\lambda}_2 \neq 0, \quad S = 3\inf H^2, \quad \bar{H}^{1^*}_{,k} = 0, \ k = 1, 2.$$

Hence, we have, by using the same calculations as in (4.6),

$$\begin{cases} \bar{h}^{1^*}_{11k} + \bar{h}^{1^*}_{22k} = 0 \\ \bar{h}^{2^*}_{111} + \bar{h}^{2^*}_{221} = 0 \\ 3\bar{\lambda}_2\bar{h}^{1^*}_{22k} = 0. \end{cases} \tag{4.74}$$

Hence, we have

$$\bar{h}^{p^*}_{ijk} = 0, \quad \text{except } i = j = k = p^* = 2.$$

If $\bar{h}^{2^*}_{222} = 0$, we get

$$\bar{h}^{p^*}_{ijk} = 0,$$

for any i, j, k, p. According to Lemma 2.1, we have

$$0 = \sum_{i,j,k,p} (h^{p^*}_{ijk})^2 + S(1 - \frac{3}{2}S) + 2H^2 S - \frac{1}{2}H^4 - \sum_{j,k,p,q} H^{p^*} h^{p^*}_{jk} H^{q^*} h^{q^*}_{jk} = -2.$$

This is impossible.
If $\bar{h}^{2^*}_{222} \neq 0$, from Lemma 2.1, we obtain

$$|\nabla^{\perp} \vec{H}|^2 = \sum_{i,j,k,p} (\bar{h}^{p^*}_{ijk})^2 = (\bar{h}^{2^*}_{222})^2 = 2.$$

Since $S = \sum_{i,j,p} (h^{p^*}_{ij})^2$ is constant, we have

$$\sum_{i,j,p} h^{p^*}_{ij} h^{p^*}_{ijk} = 0, \ k = 1, 2$$

and

$$\sum_{i,j,p} h^{p^*}_{ijl} h^{p^*}_{ijk} + \sum_{i,j,p} h^{p^*}_{ij} h^{p^*}_{ijkl} = 0, \ k, l = 1, 2.$$

Then, for $k, l = 1, 2$, we get

$$\sum_{i,j,p} \bar{h}^{p^*}_{ijl} \bar{h}^{p^*}_{ijk} = - \sum_{i,j,p} \bar{h}^{p^*}_{ij} \bar{h}^{p^*}_{ijkl} = -\bar{h}^{1^*}_{22} \bar{h}^{1^*}_{22kl} - 2\bar{h}^{2^*}_{12} \bar{h}^{2^*}_{12kl} = -3\bar{\lambda}_2 \bar{h}^{1^*}_{22kl}.$$

If $k = l = 1$, we have

$$\bar{h}^{1^*}_{2211} = 0. \tag{4.75}$$

From (2.19), we know

$$\bar{H}^{p^*}_{,ij} = \bar{h}^{p^*}_{ij} - \sum_k \bar{h}^{p^*}_{ik} \bar{h}^{1^*}_{jk} \bar{H}.$$

Let $p = i = 2$, $j = 1$, we get

$$\bar{h}^{2^*}_{1121} + \bar{h}^{2^*}_{2221} = \bar{\lambda}_2.$$

From (4.75), we obtain

$$\bar{h}^{2^*}_{2221} = \bar{\lambda}_2 \neq 0.$$

On the other hand, from Lemma 2.1, we have

$$2 \sum_{i,j,k,p} \bar{h}^{p^*}_{ijk} \bar{h}^{p^*}_{ijk1} = (\sum_{j,k,p,q} \bar{H}^{p^*} \bar{h}^{p^*}_{jk} \bar{H}^{q^*} \bar{h}^{q^*}_{jk})_{,1} = 0$$

because $\bar{H}^{1^*}_{,i} = 0$, $h^{q^*}_{ij1} = 0$. Since

$$2 \sum_{i,j,k,p} \bar{h}^{p^*}_{ijk} \bar{h}^{p^*}_{ijk1} = 2\bar{h}^{2^*}_{222} \bar{h}^{2^*}_{2221} = 2\bar{\lambda}_2 \bar{h}^{2^*}_{222} \neq 0,$$

it is a contradiction. Thus, we know that $H^2 = S$ is constant. □

Proof of Theorem 1.1 From Theorem 4.1 and Theorem 4.2, we know $S = 0$, $S = 1$ or $S = 2$. According to the result of Cheng and Peng [8], we only consider the case $S \equiv 2$ and $S \equiv 1$. Therefore, the mean curvature $H^2 = S$ is constant from Theorem 4.4.

If $H^2 = S = 2$, from (2.25) in Lemma 2.1, we have

$$\sum_{i,j,k,p} (h^{p^*}_{ijk})^2 \equiv 0, \quad \lambda_1 = \lambda_2 \neq 0.$$

According to

$$H^{p^*}_{,i} = \sum_k h^{p^*}_{ik} \langle X, e_k \rangle, \quad \text{for } i, p = 1, 2,$$

we know, at any point,

$$0 = h^{1^*}_{11i} + h^{1^*}_{22i} = H^{1^*}_{,i} = \lambda_i \langle X, e_i \rangle.$$

Hence, we get $\langle X, e_i \rangle = 0$ for $i = 1, 2$ at any point. Thus, $|X|^2$ is constant. According to

$$\frac{1}{2}\mathcal{L}|X|^2 = 2 - |X|^2$$

we obtain

$$|X|^2 \equiv 2,$$

it means that, $X : M^2 \to \mathbb{R}^4$ becomes a complete surface in the sphere $S^3(\sqrt{2})$. Because $S = 2$, it is easy to prove that $X : M^2 \to S^3(\sqrt{2})$ is minimal and its Gaussian curvature is zero. Thus, we conclude that $X : M^2 \to \mathbb{R}^4$ is the Clifford torus $S^1(1) \times S^1(1)$.

If $H^2 = S = 1$, from (2.25) in Lemma 2.1, we have

$$\sum_{i,j,k,p} (h^{p^*}_{ijk})^2 \equiv 0, \quad \lambda = 0, \quad \lambda_2 = 0.$$

From the results of Yau in [29], we know that $X : M^2 \to \mathbb{R}^4$ is $S^1(1) \times \mathbb{R}^1$. It completes the proof of Theorem 1.1. □

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References

1. Abresch, U., Langer, J.: The normalized curve shortening flow and homothetic solutions. J. Differ. Geom. **23**, 175–196 (1986)
2. Anciaux, H.: Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$. Geom. Dedic. **120**, 37–48 (2006)
3. Brendle, S.: Embedded self-similar shrinkers of genus 0. Ann. Math. **183**, 715–728 (2016)
4. Cao, H.-D., Li, H.: A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension. Calc. Var. Part. Differ. Equ. **46**, 879–889 (2013)
5. Castro, I., Lerma, A.M.: Hamiltonian stationary self-similar solutions for Lagrangian mean curvature flow in the complex Euclidean plane. Proc. Am. Math. Soc. **138**, 1821–1832 (2010)
6. Castro, I., Lerma, A.M.: The Clifford torus as a self-shrinker for the Lagrangian mean curvature flow. Int. Math. Res. Not. **6**, 1515–1527 (2014)
7. Cheng, Q.-M., Ogata, S.: 2-Dimensional complete self-shrinkers in $\mathbb{R}^3$. Math. Z. **284**, 537–542 (2016)
8. Cheng, Q.-M., Peng, Y.: Complete self-shrinkers of the mean curvature flow. Calc. Var. Part. Differ. Equ. **52**, 497–506 (2015). https://doi.org/10.1007/s00526-014-0720-2

9. Cheng, Q.-M., Wei, G.: A gap theorem for self-shrinkers. Trans. Am. Math. Soc. **367**, 4895–4915 (2015)
10. Cheng, X., Zhou, D.: Volume estimate about shrinkers. Proc. Am. Math. Soc. **141**, 687–696 (2013)
11. Colding, T.H., Minicozzi, W.P., II.: Generic mean curvature flow I; Generic singularities. Ann. Math. **175**, 755–833 (2012)
12. Ding, Q., Xin, Y.L.: Volume growth, eigenvalue and compactness for self-shrinkers. Asia J. Math. **17**, 443–456 (2013)
13. Ding, Q., Xin, Y.L.: The rigidity theorems of self shrinkers. Trans. Am. Math. Soc. **366**, 5067–5085 (2014)
14. Halldorsson, H.: Self-similar solutions to the curve shortening flow. Trans. Am. Math. Soc. **364**, 5285–5309 (2012)
15. Huisken, G.: Local and global behaviour of hypersurfaces moving by mean curvature, Differential geometry: partial differential equations on manifolds (Los Angeles, CA: Proc. Sympos. Pure Math., 54, Part 1, Am. Math. Soc. Providence, R I **1993**, 175–191 (1990)
16. Huisken, G.: Flow by mean curvature convex surfaces into spheres. J. Differ. Geom. **20**, 237–266 (1984)
17. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**, 285–299 (1990)
18. Kleene, S., Møller, N.M.: Self-shrinkers with a rotation symmetry. Trans. Am. Math. Soc. **366**, 3943–3963 (2014)
19. Lawson, H.B.: Local rigidity theorems for minimal hypersurfaces. Ann. Math. **89**, 187–197 (1969)
20. Le, N.Q., Sesum, N.: Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers. Commun. Anal. Geom. **19**, 633–659 (2011)
21. Lee, Y.-I., Wang, M.-T.: Hamiltonian stationary cones and self-similar solutions in higher dimension. Trans. Am. Math. Soc. **362**, 1491–1503 (2010)
22. Li, H., Wang, X.F.: New characterizations of the Clifford torus as a Lagrangian self-shrinkers. J. Geom. Anal. **27**, 1393–1412 (2017)
23. Li, H., Wei, Y.: Classification and rigidity of self-shrinkers in the mean curvature flow. J. Math. Soc. Jpn. **66**, 709–734 (2014)
24. Neves, A.: Singularities of Lagrangian mean curvature flow: monotone case. Math. Res. Lett. **17**, 109–126 (2010)
25. Neves, A.: Recent progress on singularities of Lagrangian mean curvature flow. In: Surveys in geometric analysis and relativity, pp.413–438, Advanced Lectures in Mathematics (ALM), **20**(2011), International Press, Somerville
26. Smoczyk, K.: The Lagrangian mean curvature flow, vol. **102** S. Univ. Leipzig (Habil), Leipzig (2000)
27. Smoczyk, K.: Self-shrinkers of the mean curvature flow in arbitrary co-dimension. Int. Math. Res. Not. **48**, 2983–3004 (2005)
28. Strominger, A., Yau, S.T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. B **479**, 243–259 (1996)
29. Yau, S.T.: Submanifolds with constant mean curvature I. Am. J. Math. **96**, 346–366 (1974)