



Approximations of the connection Laplacian spectra

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Abstract

We consider a convolution-type operator on vector bundles over metric-measure spaces. This operator extends the analogous convolution Laplacian on functions in our earlier work to vector bundles, and is a natural extension of the graph connection Laplacian. We prove that for Euclidean or Hermitian connections on closed Riemannian manifolds, the spectrum of this operator and that of the graph connection Laplacian both approximate the spectrum of the connection Laplacian.

Keywords Connection Laplacian · Spectral convergence · Discretization

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1 Introduction

This paper is a continuation of our previous works where we approximated, in the spectral sense, the Riemannian Laplace-Beltrami operator with the discrete graph Laplacian [2, 11] and a convolution-type operator [3]. This convolution-type operator, called the ρ -Laplacian (with a small parameter $\rho > 0$), is defined by averaging over metric balls of small radius, and it is a natural extension of the discrete graph Laplacian in a continuous setting. A notable feature of the ρ -Laplacian is that it is not based on differentiation and is readily available on general

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metric-measure spaces. Furthermore, we proved in [3] that the spectrum of the ρ -Laplacian enjoys stability under metric-measure approximations in a large class of metric-measure spaces. Ideally, we define the ρ -Laplacian as a notion of Laplacian on metric-measure spaces (in the spectral sense). Our earlier results in [2, 3, 11] show that the definition makes sense for Riemannian manifolds. We hope that the spectra of the ρ -Laplacians could converge as $\rho \rightarrow 0$, in a large class of metric-measure spaces, with the limit related to known concepts of Laplacian in [5, 6].

The present paper is concerned with the connection Laplacian on vector bundles. In this paper, we introduce an analogous convolution Laplacian acting on vector bundles over metric-measure spaces. This operator can be regarded as a generalization of the ρ -Laplacian (on functions), and its discretization, also known as the *graph connection Laplacian*, is a generalization of the graph Laplacian. We prove that for Euclidean or Hermitian connections on closed Riemannian manifolds, our convolution Laplacian and its discretization both approximate the standard connection Laplacian in the spectral sense. The spectral convergence of the graph connection Laplacians may have applications in numerical computations and manifold learning, in particular analyzing high-dimensional data sets, see e.g. [1, 4, 7, 8, 14–16] and the references therein.

In this introduction, we define our operator for vector bundles over Riemannian manifolds. The general definition for metric-measure spaces can be found in Sect. 2. Let M^n be a compact, connected Riemannian manifold of dimension n without boundary, and let E be a smooth Euclidean (or Hermitian) vector bundle over M equipped with a smooth Euclidean (or Hermitian) connection ∇ . Recall that an Euclidean (resp. Hermitian) connection is a connection that is compatible with the Euclidean (resp. Hermitian) metric on the vector bundle. We denote by $L^2(M, E)$ the space of L^2 -sections of the vector bundle E , and by E_x the fiber over a point $x \in M$. Fix $\rho > 0$ smaller than the injectivity radius $r_{inj}(M)$. Given any pair of points $x, y \in M$ with $d(x, y) \leq \rho$, let $P_{xy} : E_y \rightarrow E_x$ be the parallel transport canonically associated with ∇ from y to x along the unique minimizing geodesic $[yx]$.

For an L^2 -section $u \in L^2(M, E)$, we define the ρ -connection Laplacian operator Δ^ρ by

$$\Delta^\rho u(x) = \frac{2(n+2)}{v_n \rho^{n+2}} \int_{B_\rho(x)} (u(x) - P_{xy}(u(y))) dy, \tag{1.1}$$

where v_n is the volume of the unit ball in \mathbb{R}^n , and $B_\rho(x)$ is the geodesic ball in M of radius ρ centered at $x \in M$.

The operator Δ^ρ is nonnegative and self-adjoint with respect to the standard inner product on $L^2(M, E)$. Furthermore, the lower part of the spectrum of Δ^ρ is discrete. We denote by $\tilde{\lambda}_k$ the k -th eigenvalue of Δ^ρ from the discrete part of the spectrum. Denote by Δ the standard connection Laplacian of the connection ∇ , and by λ_k the k -th eigenvalue of Δ . Our first result states that the spectrum of the ρ -connection Laplacian Δ^ρ approximates the spectrum of the connection Laplacian Δ .

Theorem 1 *There exist constants $C_n > 1$ and $c_n, \sigma_n \in (0, 1]$, depending only on n , such that the following holds. Suppose that the absolute value of the sectional curvatures of M and the norm of the curvature tensor of ∇ are bounded by constants K_M and K_E , respectively. Assume that $\rho > 0$ satisfies*

$$\rho < \min \{r_{inj}(M), c_n K_M^{-1/2}\}. \tag{1.2}$$

Then for every $k \in \mathbb{N}_+$ satisfying $\tilde{\lambda}_k \leq \sigma_n \rho^{-2}$, we have

$$|\tilde{\lambda}_k^{1/2} - \lambda_k^{1/2}| \leq (C_n K_M + \lambda_k) \lambda_k^{1/2} \rho^2 + C_n K_E \rho.$$

Remark 1.1 One can track the dependence of c_n on n and find that it suffices to assume

$$\frac{\sinh^{n-1} c_n}{\sin^{n-1} c_n} < 2.$$

Remark 1.2 In the case of E being the trivial bundle, the connection Laplacian is simply the Laplace-Beltrami operator on functions, and the ρ -connection Laplacian Δ^ρ reduces to an operator on functions

$$\Delta^\rho f(x) = \frac{2(n+2)}{v_n \rho^{n+2}} \int_{B_\rho(x)} (f(x) - f(y)) dy.$$

This operator is the ρ -Laplacian we introduced in [3] up to a normalization adjustment, and its discretization is the graph Laplacian studied in [2]. In this case, Theorem 1 reduces to the convergence of the spectra of ρ -Laplacians (on functions) to the spectrum of the Laplace-Beltrami operator, which is known from Theorem 1 in [2] and Theorem 1.2 in [3].

Now let us turn to the discrete side. We define a discretization of a compact Riemannian manifold M as follows (see [2]).

Definition 1.3 Let $\varepsilon \ll \rho$ and $X_\varepsilon = \{x_i\}_{i=1}^N$ be a finite ε -net in M . The distance function on X_ε is the Riemannian distance d of M restricted onto $X_\varepsilon \times X_\varepsilon$, denoted by $d|_{X_\varepsilon}$. Suppose that X_ε is equipped with a discrete measure $\mu = \sum_{i=1}^N \mu_i \delta_{x_i}$ which approximates the volume on M in the following sense: there exists a partition of M into measurable subsets $\{V_i\}_{i=1}^N$ such that $V_i \subset B_\varepsilon(x_i)$ and $\text{vol}(V_i) = \mu_i$ for every i . Denote this discrete metric-measure space by $\Gamma_\varepsilon = (X_\varepsilon, d|_{X_\varepsilon}, \mu)$, and we write Γ for short.

Let M, E, ∇ be defined as before, and we consider the ρ -connection Laplacian on this discrete metric-measure space Γ , acting on the restriction of the vector bundle E onto X_ε . Namely, let $P = \{P_{x_i x_j} : d(x_i, x_j) < \rho\}$ be the parallel transport between points in X_ε . We call P a ρ -connection on the restriction $E|_{X_\varepsilon}$. For $\bar{u} \in L^2(X_\varepsilon, E|_{X_\varepsilon})$, we define

$$\Delta_\Gamma^\rho \bar{u}(x_i) := \frac{2(n+2)}{v_n \rho^{n+2}} \sum_{d(x_i, x_j) < \rho} \mu_j (\bar{u}(x_i) - P_{x_i x_j} \bar{u}(x_j)). \tag{1.3}$$

This operator is known as the *graph connection Laplacian*. The graph connection Laplacian is a nonnegative self-adjoint operator of dimension $r(E)N$ with respect to the weighted discrete L^2 -inner product, where $r(E)$ is the rank of the vector bundle E . We denote the k -th eigenvalue of Δ_Γ^ρ by $\tilde{\lambda}_k(\Gamma)$.

Our second result can be viewed as a discretized version of Theorem 1.

Theorem 2 *Suppose that the absolute value of the sectional curvatures of M and the norm of the curvature tensor of ∇ are bounded by constants K_M and K_E . Then there exists $\rho_0 = \rho_0(n, K_M, K_E) < r_{inj}(M)/2$, such that for any $\rho < \rho_0, \varepsilon < \rho/4, k \leq r(E)N$ satisfying $\lambda_k < \rho^{-2}/16$, we have*

$$|\tilde{\lambda}_k(\Gamma) - \lambda_k| \leq C_{n, K_M} \left(\rho + \frac{\varepsilon}{\rho} + \lambda_k^{1/2} \rho \right) \lambda_k + C_{n, K_E} \left(\rho + \frac{\varepsilon}{\rho} \right).$$

For compact Riemannian manifolds without boundary, Theorems 1 and 2 imply the closeness between the ρ -connection Laplacian and the graph connection Laplacian in the spectral sense. In the case of trivial bundles, this gives another proof for the closeness between the

ρ -Laplacian (on functions) and the graph Laplacian in the spectral sense (as a special case of Theorem 1.2 in [3]).

This paper is organized as follows. We introduce the general concept of ρ -connection Laplacians for metric-measure spaces in Sect. 2. In Sect. 3, we focus on the case of smooth connections on Riemannian manifolds and prove Theorem 1. We turn to the graph connection Laplacian in Sect. 4 and prove Theorem 2.

2 General metric-measure setup

Let (X, d) be a metric space and E be an (continuous) Euclidean or Hermitian vector bundle over X . That is, we have a fiber bundle $\pi : E \rightarrow X$, and each fiber $E_x := \pi^{-1}(x)$, $x \in X$, is a real or complex vector space equipped with an Euclidean or Hermitian inner product $\langle \cdot, \cdot \rangle_{E_x}$. When it is clear at which point the inner product is taken, we omit the subscript E_x .

Let $\rho > 0$ be a small parameter. We denote by $X^2(\rho)$ the set of pairs $(x, y) \in X \times X$ such that $d(x, y) \leq \rho$.

Definition 2.1 A ρ -connection on E is a (Borel measurable) family of uniformly bounded linear maps $P_{xy} : E_y \rightarrow E_x$, $\sup_{(x,y)} \|P_{xy}\| < \infty$, where (x, y) ranges over $X^2(\rho)$. That is, P_{xy} transports vectors from E_y to E_x .

A ρ -connection $P = \{P_{xy}\}$ is said to be *Euclidean* (resp. *Hermitian*) if all maps P_{xy} are Euclidean isometries (resp. unitary operators). A ρ -connection P is said to be *symmetric*, if P_{xy} is invertible and $P_{xy}^{-1} = P_{yx}$ for all $(x, y) \in X^2(\rho)$.

Our primary example of ρ -connection is the one associated with a connection ∇ on an (smooth) Euclidean or Hermitian vector bundle over a Riemannian manifold M . Namely, there is a parallel transport canonically associated with the connection ∇ . Then given two points $x, y \in M$ with $d(x, y) \leq \rho < r_{inj}(M)$, one can transport vectors from E_y to E_x along the unique minimizing geodesic $[yx]$.

Now suppose (X, d) is equipped with a measure μ : we are working in a metric-measure space (X, d, μ) . Assume that X is compact and $\mu(X) < \infty$. We denote by $L^2(X, E)$ the space of L^2 -sections of the vector bundle E .

The following expression $u(x) - P_{xy}(u(y))$, where $u \in L^2(X, E)$ and $x, y \in X$, shows up frequently. We introduce a short notation for it:

$$\Gamma_{xy}(u) := u(x) - P_{xy}(u(y)). \tag{2.1}$$

Note that $\Gamma_{xy}(u)$ is only defined for $(x, y) \in X^2(\rho)$ and it belongs to E_x .

Let $\alpha : X \rightarrow \mathbb{R}_+$ and $\beta : X^2(\rho) \rightarrow \mathbb{R}_+$ be positive L^∞ functions satisfying that α is bounded away from 0 and β is symmetric: $\beta(x, y) = \beta(y, x)$ for all x, y . We define the ρ -connection Laplacian $\Delta_{\alpha,\beta}^\rho$ associated with the ρ -connection P with weights α, β as follows. First we define an L^2 -type inner product $\langle\langle \cdot, \cdot \rangle\rangle_\alpha$ on $L^2(X, E)$ by

$$\langle\langle u, v \rangle\rangle_\alpha := \int_X \alpha(x) \langle u(x), v(x) \rangle_{E_x} d\mu(x),$$

for $u, v \in L^2(X, E)$, and the associated norm $\|\cdot\|_\alpha$ is

$$\|u\|_\alpha^2 := \langle\langle u, u \rangle\rangle_\alpha = \int_X \alpha(x) |u(x)|^2 d\mu(x).$$

Here $|u(x)|^2 = \langle u(x), u(x) \rangle$ is taken with respect to the Euclidean or Hermitian inner product in the fiber E_x . Note that the standard inner product on $L^2(X, E)$ corresponds to the case $\alpha \equiv 1$, in which case the norm is denoted by the usual $\|u\|_{L^2}$.

Next we define a symmetric form D_β^ρ on $L^2(X, E)$ by

$$D_\beta^\rho(u, v) = \frac{1}{2} \iint_{X^2(\rho)} \beta(x, y) \langle \Gamma_{xy}(u), \Gamma_{xy}(v) \rangle_{E_x} d\mu(x) d\mu(y). \tag{2.2}$$

Finally, our Laplacian $\Delta_{\alpha,\beta}^\rho$ is the unique operator from $L^2(X, E)$ to itself satisfying

$$\langle \Delta_{\alpha,\beta}^\rho u, v \rangle_\alpha = D_\beta^\rho(u, v) \tag{2.3}$$

for all $u, v \in L^2(X, E)$. In other words, $\Delta_{\alpha,\beta}^\rho$ is the self-adjoint operator on $L^2(X, \alpha\mu, E)$ associated with the quadratic form D_β^ρ . Note that the boundedness of P_{xy} and $\mu(X)$ imply that both D_β^ρ and $\Delta_{\alpha,\beta}^\rho$ are bounded.

The following proposition gives an explicit formula for $\Delta_{\alpha,\beta}^\rho$.

Proposition 2.2 *Assume that X is compact and $\mu(X) < \infty$. Then $\Delta_{\alpha,\beta}^\rho$ can be written as*

$$\Delta_{\alpha,\beta}^\rho u(x) = \frac{1}{2\alpha(x)} \int_{B_\rho(x)} \beta(x, y) (\Gamma_{xy}(u) - P_{yx}^* \Gamma_{yx}(u)) d\mu(y), \tag{2.4}$$

where $P_{xy}^* : E_x \rightarrow E_y$ is the operator adjoint to P_{xy} with respect to the inner products on the fibers E_x and E_y .

In particular, if the ρ -connection P is Euclidean or Hermitian and is symmetric, then

$$\Delta_{\alpha,\beta}^\rho u(x) = \frac{1}{\alpha(x)} \int_{B_\rho(x)} \beta(x, y) \Gamma_{xy}(u) d\mu(y). \tag{2.5}$$

Proof The proof is a straightforward calculation. For brevity, we write dx and dy instead of $d\mu(x)$ and $d\mu(y)$. By definitions of D_β^ρ and $\Gamma_{xy}(v)$, we have

$$D_\beta^\rho(u, v) = \frac{1}{2} \iint_{X^2(\rho)} \beta(x, y) \langle \Gamma_{xy}(u), v(x) - P_{xy}(v(y)) \rangle dx dy.$$

Expand it and rewrite the term $\langle \Gamma_{xy}(u), P_{xy}(v(y)) \rangle$ as follows:

$$\langle \Gamma_{xy}(u), P_{xy}(v(y)) \rangle_{E_x} = \langle P_{xy}^* \Gamma_{xy}(u), v(y) \rangle_{E_y}.$$

By swapping x, y and using the symmetry of β , one gets

$$\iint_{X^2(\rho)} \beta(x, y) \langle P_{xy}^* \Gamma_{xy}(u), v(y) \rangle dx dy = \iint_{X^2(\rho)} \beta(x, y) \langle P_{yx}^* \Gamma_{yx}(u), v(x) \rangle dx dy.$$

Substituting the last two formulae into the first one yields

$$\begin{aligned} D_\beta^\rho(u, v) &= \frac{1}{2} \iint_{X^2(\rho)} \beta(x, y) \langle \Gamma_{xy}(u) - P_{yx}^* \Gamma_{yx}(u), v(x) \rangle dx dy \\ &= \frac{1}{2} \int_X \alpha(x) \left\langle \frac{1}{\alpha(x)} \int_{B_\rho(x)} \beta(x, y) (\Gamma_{xy}(u) - P_{yx}^* \Gamma_{yx}(u)) dy, v(x) \right\rangle dx. \end{aligned}$$

The right-hand side of the last formula is the $\langle \cdot, \cdot \rangle_\alpha$ -product of the right-hand side of (2.4) and v . This proves (2.4).

To deduce (2.5), observe that $P_{yx}^* = P_{yx}^{-1} = P_{xy}$, since P is Euclidean or Hermitian and is symmetric. Hence by (2.1),

$$P_{yx}^* \Gamma_{yx}(u) = P_{yx}^{-1}(u(y) - P_{yx}(u(x))) = P_{xy}(u(y)) - u(x) = -\Gamma_{xy}(u).$$

This and (2.4) prove (2.5). □

Remark 2.3 We can assume that $\beta(x, y)$ is defined for all pairs $(x, y) \in X^2(\rho)$ and is equal to 0 whenever $d(x, y) > \rho$. This allows us to assume that P_{xy} is defined for all pairs $(x, y) \in X \times X$. It does not matter how P is extended to pairs x, y with $d(x, y) > \rho$, since in this case it will always be multiplied by 0. This allows us to write integration over X rather than over ρ -balls whenever convenient.

Denote by $\sigma(\Delta_{\alpha,\beta}^\rho)$ the spectrum of $\Delta_{\alpha,\beta}^\rho$ and by $\sigma_{ess}(\Delta_{\alpha,\beta}^\rho)$ the essential spectrum.

Corollary 2.4

$$\sigma_{ess}(\Delta_{\alpha,\beta}^\rho) \subset [a(\rho, \alpha, \beta), \infty),$$

where $a(\rho, \alpha, \beta) = \min_{x \in X} \frac{1}{2\alpha(x)} \int_X \beta(x, y) dy$.

In particular, if the ρ -connection P is Euclidean or Hermitian and is symmetric, then

$$\sigma_{ess}(\Delta_{\alpha,\beta}^\rho) \subset [2a(\rho, \alpha, \beta), \infty).$$

Proof Due to (2.1) and (2.4), $\Delta_{\alpha,\beta}^\rho$ is the sum of three operators: the multiplication operator

$$\Delta_1 u(x) = \frac{1}{2\alpha(x)} u(x) \int_X \beta(x, y) dy,$$

the operator

$$\Delta_2 u(x) = \frac{1}{2\alpha(x)} \int_X \beta(x, y) P_{yx}^* P_{yx} u(x) dy,$$

and

$$\Delta_3 u(x) = \frac{1}{2\alpha(x)} \int_X \beta(x, y) (-P_{xy} - P_{yx}^*) u(y) dy.$$

Observe that $\sigma(\Delta_1) \subset [a(\rho, \alpha, \beta), \infty)$; while Δ_2 , as seen from its quadratic form

$$\langle \langle \Delta_2 u, u \rangle \rangle_\alpha = \frac{1}{2} \iint_{X^2(\rho)} \beta(x, y) \langle P_{xy} u, P_{xy} u \rangle dx dy,$$

is non-negative, and Δ_3 is compact as an operator with bounded kernel, which proves the first claim. The second claim follows from the same considerations by using the simpler form (2.5). □

Examples

In this paper, we only need a few choices for α and β . First observe that the Riemannian ρ -connection Laplacian (1.1) is obtained by using $\alpha(x) = 1$ and $\beta(x, y) = \frac{2(n+2)}{v_n \rho^{n+2}}$ if $d(x, y) \leq \rho$. Equivalently, one can use $\alpha(x) = \frac{v_n \rho^{n+2}}{2(n+2)}$ and $\beta(x, y) = 1$ if $d(x, y) \leq \rho$. Another convenient normalization, as seen in [3], is by volumes of ρ -balls: $\beta(x, y) = 1$ if $d(x, y) \leq \rho$

and $\alpha(x) = \rho^2 \mu(B_\rho(x))$ for all $x \in X$. We call the operator $\Delta_{\alpha,\beta}^\rho$ with these α, β the *volume-normalized ρ -connection Laplacian*.

Note that Laplacians on real-or complex-valued functions are a special case of connection Laplacians. Namely, for a Riemannian manifold M , one simply considers the trivial bundle $E = M \times \mathbb{R}$ or $E = M \times \mathbb{C}$ equipped with the trivial connection ∇ . The sections of the trivial bundle are functions on M and the connection Laplacian is simply the Laplace-Beltrami operator (on functions). Similarly, for a metric-measure space X , one can consider the same trivial bundle with the trivial ρ -connection defined by $P_{xy}(y, t) = (x, t)$ for $x, y \in X$ and $t \in \mathbb{R}$ (or $t \in \mathbb{C}$). Then (2.5) boils down to

$$\Delta_{\alpha,\beta}^\rho u(x) = \frac{1}{\alpha(x)} \int_{B_\rho(x)} \beta(x, y)(u(x) - u(y)) d\mu(y),$$

where $u \in L^2(X)$. Such operators are called ρ -Laplacians in [3]. Some analogues of the results of this paper in the case of Laplacians on functions can be found in [2, 3, 10, 11].

Spectra of ρ -connection Laplacians

Since $\Delta_{\alpha,\beta}^\rho$ is self-adjoint with respect to the L^2 -compatible inner product $\langle \cdot, \cdot \rangle_\alpha$ and the corresponding quadratic form D_β^ρ is positive semi-definite, the spectrum of $\Delta_{\alpha,\beta}^\rho$ is contained in $\mathbb{R}_{\geq 0}$.

This spectrum consists of the discrete and essential spectra. It follows from Corollary 2.4 that the essential spectrum has a lower bound in all cases in question. We are only interested in the part of the spectrum below this bound. We enumerate this part of the spectrum as follows (cf. Notation 2.1 in [3]).

Notation 2.5 Denote by $\tilde{\lambda}_\infty = \tilde{\lambda}_\infty(E, P, \rho, \alpha, \beta)$ the infimum of the essential spectrum of $\Delta_{\alpha,\beta}^\rho$. If the essential spectrum is empty (e.g. if X is a discrete space), we set $\tilde{\lambda}_\infty = \infty$. For every $k \in \mathbb{N}_+$, we define $\tilde{\lambda}_k = \tilde{\lambda}_k(E, P, \rho, \alpha, \beta) \in [0, +\infty]$ as follows. Let $0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$ be the eigenvalues of $\Delta_{\alpha,\beta}^\rho$ (counting multiplicities) that are smaller than $\tilde{\lambda}_\infty$. If there are only finitely many of such eigenvalues, we set $\tilde{\lambda}_k = \tilde{\lambda}_\infty$ for all larger values of k .

We abuse the language and refer to $\tilde{\lambda}_k(E, P, \rho, \alpha, \beta)$ as the *k -th eigenvalue* of $\Delta_{\alpha,\beta}^\rho$ even though it may be equal to $\tilde{\lambda}_\infty$.

By the standard min-max formula, for every $k \in \mathbb{N}_+$, we have

$$\tilde{\lambda}_k(E, P, \rho, \alpha, \beta) = \inf_{\dim L=k} \sup_{u \in L \setminus \{0\}} \left(\frac{D_\beta^\rho(u, u)}{\|u\|_\alpha^2} \right), \tag{2.6}$$

where the infimum is taken over all k -dimensional linear subspaces L of $L^2(X, E)$. We emphasize that (2.6) holds in both cases of $\tilde{\lambda}_k < \tilde{\lambda}_\infty$ and $\tilde{\lambda}_k = \tilde{\lambda}_\infty$.

3 Smooth connections on Riemannian manifolds

In this section, suppose $X = M^n$ is a compact Riemannian manifold of dimension n without boundary, and E is a smooth Euclidean (or Hermitian) vector bundle over M equipped with a smooth Euclidean (or Hermitian) connection ∇ . Recall that an Euclidean (resp. Hermitian) connection is a connection that is compatible with the Euclidean (resp. Hermitian) metric

on the vector bundle. For a (sufficiently smooth) section u of E , ∇u is a section of the fiber bundle $\text{Hom}(TM, E)$ over M . Here $\text{Hom}(TM, E)$ is the fiber bundle over M whose fiber over $x \in M$ is the space $\text{Hom}(T_x M, E_x)$ of \mathbb{R} -linear maps from $T_x M$ to E_x . Note that in the case when E is a complex fiber bundle, $\text{Hom}(T_x M, E_x)$ has a natural complex structure. The standard (Euclidean or Hermitian) inner product on $\text{Hom}(T_x M, E_x)$ is defined by

$$\langle \xi, \eta \rangle = \sum_{i=1}^n \langle \xi(e_i), \eta(e_i) \rangle_{E_x}$$

for $\xi, \eta \in \text{Hom}(T_x M, E_x)$, where $\{e_i\}$ is an orthonormal basis of $T_x M$. This defines the standard norm

$$\|\xi\|^2 = \sum_{i=1}^n |\xi(e_i)|^2 \tag{3.1}$$

on fibers and the standard L^2 -norm on sections of $\text{Hom}(TM, E)$.

Let $\Delta = \nabla^* \nabla$ be the connection Laplacian of ∇ (e.g. [12, Chapter 7.3.2] or [9]). It is a nonnegative self-adjoint operator acting on H^2 -sections of E . The corresponding energy functional is given by

$$\langle \Delta u, u \rangle_{L^2} = \|\nabla u\|_{L^2}^2$$

for $u \in H^2(M, E)$.

Since Δ is a nonnegative self-adjoint elliptic operator, it has a discrete spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$, and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. The min-max formula for λ_k takes the form

$$\lambda_k = \inf_{\dim L=k} \sup_{u \in L \setminus \{0\}} \left(\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} \right) \tag{3.2}$$

where the infimum is taken over all k -dimensional linear subspaces L of $H^1(M, E)$. The goal of this section is to prove that λ_k are approximated by eigenvalues of a ρ -connection Laplacian defined below.

Let $\rho > 0$ be smaller than the injectivity radius of M . Let $P = \{P_{xy}\}$ be the ρ -connection associated with the connection ∇ , which is given by the parallel transport from y to x along the unique minimizing geodesic $[yx]$. This particular ρ -connection P is unitary and symmetric since the connection ∇ is Euclidean (or Hermitian). We consider the ρ -connection Laplacian $\Delta^\rho = \Delta_{\alpha, \beta}^\rho$ given by (2.5) with $\alpha(x) = 1$ and $\beta(x, y) = \frac{2(n+2)}{v_n \rho^{n+2}}$ if $d(x, y) \leq \rho$. That is,

$$\Delta^\rho u(x) = \frac{2(n+2)}{v_n \rho^{n+2}} \int_{B_\rho(x)} \Gamma_{xy}(u) dy. \tag{3.3}$$

Recall that $\Gamma_{xy}(u)$ is defined in (2.1). Here and later on in this paper, we denote by dx, dy the integration with respect to the Riemannian volume on M . Denote by $\tilde{\lambda}_k$ the k -th eigenvalue of Δ^ρ (see Notation 2.5).

We introduce a quadratic form D^ρ on $L^2(M, E)$ by

$$D^\rho(u) = \int_M \int_{B_\rho(x)} |\Gamma_{xy}(u)|^2 dx dy. \tag{3.4}$$

Note that for the constant weight $\beta(x, y) = \frac{2(n+2)}{v_n \rho^{n+2}}$ chosen for Δ^ρ , we have

$$D_\beta^\rho(u, u) = \frac{n+2}{v_n \rho^{n+2}} D^\rho(u).$$

Hence (2.6) takes the form

$$\tilde{\lambda}_k = \frac{n+2}{v_n \rho^{n+2}} \inf_{\dim L=k} \sup_{u \in L \setminus \{0\}} \left(\frac{D^\rho(u)}{\|u\|_{L^2}^2} \right). \tag{3.5}$$

The rest of this section is a proof of Theorem 1.

3.1 Preparations and notations

For $x \in M$, denote by $\exp_x : T_x M \rightarrow M$ the Riemannian exponential map. We only need its restriction onto the ρ -ball $B_\rho(0) \subset T_x M$. For $v \in T_x M$, denote by $J_x(v)$ the Jacobian of \exp_x at v . Let $J_{\min}(r)$ and $J_{\max}(r)$ denote the minimum and maximum of $J_x(v)$ over all x, v with $|v| \leq r$. The Rauch Comparison Theorem implies that

$$\left(\frac{\sin r}{r} \right)^{n-1} \leq J_{\min}(r) \leq 1 \leq J_{\max}(r) \leq \left(\frac{\sinh r}{r} \right)^{n-1},$$

for $r < K_M^{1/2} \rho < c_n$, see (1.2). In particular,

$$(1 + C_n K_M \rho^2)^{-1} \leq J_{\min}(r) \leq 1 \leq J_{\max}(r) \leq 1 + C_n K_M \rho^2. \tag{3.6}$$

Moreover, we choose c_n to be sufficiently small such that $J_{\max}(r)/J_{\min}(r) < 2$. Later we will mostly take $r = \rho$ and we denote $J_{\min} := J_{\min}(\rho)$, $J_{\max} := J_{\max}(\rho)$ for short.

As a consequence of (3.6), Corollary 2.4 implies that

$$\tilde{\lambda}_\infty \geq \frac{2(n+2)}{v_n \rho^{n+2}} \min_{x \in M} \mu(B_\rho(x)) \geq \frac{2(n+2)}{\rho^2} \frac{1}{1 + C_n K_M \rho^2} \geq \sigma_n \rho^{-2}, \tag{3.7}$$

for some constant σ_n depending only on n due to our choice of ρ in (1.2). This shows that $\tilde{\lambda}_\infty$ is of order ρ^{-2} in the present case.

Later we use the following well-known inequality. We did not find a precise reference for it, so we give a short proof here.

Lemma 3.1 *Let $\gamma_s : [0, 1] \rightarrow M$ be a smooth family of paths from a fixed point $y \in M$ to $x(s)$, $s \in [-\varepsilon, \varepsilon]$. Let $P_{\gamma_s}(v)$ be the ∇ -parallel transport along γ_s of $v \in E_y$ to $E_{x(s)}$. Then*

$$|\nabla_s P_{\gamma_s}(v)| \leq |v| \cdot K_E \cdot \text{length}(\gamma_s) \cdot \sup_{t \in [0,1]} \left| \frac{d\gamma_s(t)}{ds} \right|. \tag{3.8}$$

Proof Let $v(t, s) = P_{\gamma_s[0,t]}(v)$, where $P_{\gamma_s[t_1,t_2]}$ is the ∇ -parallel transport along γ_s from $\gamma_s(t_1)$ to $\gamma_s(t_2)$. Note that $P_{\gamma_s[t_1,t_2]}$ is a unitary operator.

Observe that $\nabla_t v(t, s) = 0$ and thus $\nabla_s \nabla_t v(t, s) = 0$. Hence, using the definition of the curvature operator R_E , we see that

$$\nabla_t \nabla_s v(t, s) = R_E \left(\frac{d\gamma_s(t)}{dt}, \frac{d\gamma_s(t)}{ds} \right) v(t, s) \in E_{\gamma_s(t)}. \tag{3.9}$$

Estimating the right-hand side yields

$$|\nabla_t \nabla_s v(t, s)| \leq |v| \cdot K_E \cdot \left| \frac{d\gamma_s(t)}{dt} \right| \cdot \sup_{t \in [0,1]} \left| \frac{d\gamma_s(t)}{ds} \right|, \tag{3.10}$$

where we have used the fact that $|v(t, s)| = |v|$ since the parallel transport P is unitary. The plan is to integrate (3.9) with respect to t . However, the fibers $E_{\gamma_s(t)}$ vary with t . Thus, we

use $P_{\gamma_s[t,1]}$ to identify them with $E_{x(s)}$. Recall that by the definition of parallel translations, for any vector field X along γ_s , one has

$$\frac{d}{dt}(P_{\gamma_s[t,1]}X(t)) = P_{\gamma_s[t,1]}(\nabla_t X(t)).$$

Note that the vectors under $\frac{d}{dt}$ in this formula lie in the same vector space $E_{x(s)}$ for all t . We apply this to $X(t) = \nabla_s v(t, s)$ and obtain

$$\frac{d}{dt}(P_{\gamma_s[t,1]}\nabla_s v(t, s)) = P_{\gamma_s[t,1]}(\nabla_t \nabla_s v(t, s)).$$

Integrating with respect to t and taking into account that $\nabla_s v(0, s) = 0$ yield

$$\nabla_s P_{\gamma_s}(v) = \nabla_s v(1, s) = \int_0^1 P_{\gamma_s[t,1]}(\nabla_t \nabla_s v(t, s)) dt.$$

Therefore, using the fact that $P_{\gamma_s[t,1]}$ is unitary, we have

$$|\nabla_s P_{\gamma_s}(v)| \leq \int_0^1 |\nabla_t \nabla_s v(t, s)| dt \leq |v| \cdot K_E \cdot \sup_{t \in [0,1]} \left| \frac{d\gamma_s(t)}{ds} \right| \cdot \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right| dt,$$

where the second inequality follows from (3.10). This formula is exactly (3.8). □

We need the following elementary fact from the linear algebra (e.g. [2, §2.3]): If S is a quadratic form on \mathbb{R}^n , then

$$\int_{B_\rho(0)} S(x) dx = \frac{v_n \rho^{n+2}}{n+2} \text{trace}(S). \tag{3.11}$$

In the following two subsections, we control the upper and lower bounds for $\tilde{\lambda}_k$ by following the method we established in [2].

3.2 Upper bound for $\tilde{\lambda}_k$

Lemma 3.2 *For any $u \in H^1(M, E)$, we have*

$$D^\rho(u) \leq J_{\max} \frac{v_n}{n+2} \rho^{n+2} \|\nabla u\|_{L^2}^2.$$

Proof This lemma is similar to Lemma 3.3 in [2] and the proof is essentially the same. We may assume that u is smooth. By substituting $y = \exp_x(v)$, we have

$$\begin{aligned} \int_{B_\rho(x)} |\Gamma_{xy}(u)|^2 dy &= \int_{B_\rho(0) \subset T_x M} |\Gamma_{x, \exp_x(v)}(u)|^2 J_x(v) dv \\ &\leq J_{\max} \int_{B_\rho(0)} |\Gamma_{x, \exp_x(v)}(u)|^2 dv. \end{aligned}$$

Hence

$$D^\rho(u) \leq J_{\max} A, \tag{3.12}$$

where

$$A = \int_M \int_{B_\rho(0) \subset T_x M} |\Gamma_{x, \exp_x(v)}(u)|^2 dv dx.$$

Note that the right-hand side is an integral with respect to the Liouville measure on TM . Let us estimate A .

For every constant-speed minimizing geodesic $\gamma : [0, 1] \rightarrow M$, we have

$$\Gamma_{\gamma(0)\gamma(1)} = u(\gamma(0)) - P_{\gamma(0)\gamma(1)}(u(\gamma(1))) = - \int_0^1 \frac{d}{dt} P_{\gamma(0)\gamma(t)}(u(\gamma(t))) dt,$$

and

$$\frac{d}{dt} P_{\gamma(0)\gamma(t)}(u(\gamma(t))) = P_{\gamma(0)\gamma(t)}(\nabla_{\dot{\gamma}(t)}u)$$

by the definition of the parallel transport P . Therefore,

$$|\Gamma_{\gamma(0)\gamma(1)}| \leq \int_0^1 |P_{\gamma(0)\gamma(t)}(\nabla_{\dot{\gamma}(t)}u)| = \int_0^1 |\nabla_{\dot{\gamma}(t)}u| dt,$$

where the last equality is due to P being unitary. Then,

$$|\Gamma_{\gamma(0)\gamma(1)}|^2 \leq \int_0^1 |\nabla_{\dot{\gamma}(t)}u|^2 dt. \tag{3.13}$$

For $x \in M$ and $v \in \mathcal{B}_\rho(0) \subset T_xM$, denote by $\gamma_{x,v}$ the constant-speed geodesic with the initial data $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. Equivalently, $\gamma_{x,v}(t) = \exp_x(tv)$. Applying (3.13) to $\gamma_{x,v}$ yields

$$|\Gamma_{x,\exp_x(v)}(u)|^2 \leq \int_0^1 |\nabla_{\dot{\gamma}_{x,v}(t)}u|^2 dt.$$

This and the definition of A imply that

$$A \leq \int_0^1 f(t) dt,$$

where

$$f(t) = \int_M \int_{\mathcal{B}_\rho(0) \subset T_xM} |\nabla_{\dot{\gamma}_{x,v}(t)}u|^2 dv dx.$$

Note that $\dot{\gamma}_{x,v}(t)$ is the image of v under the time t map of the geodesic flow. Since the geodesic flow preserves the Liouville measure and the subset $\{(x, v) \in TM : v \in \mathcal{B}_\rho(0) \subset T_xM\}$, $f(t)$ does not depend on t . Hence,

$$A \leq f(0) = \int_M \int_{\mathcal{B}_\rho(0) \subset T_xM} |\nabla_v u|^2 dv dx = \int_M \frac{v_n \rho^{n+2}}{n+2} \|\nabla u(x)\|^2 dx = \frac{v_n \rho^{n+2}}{n+2} \|\nabla u\|_{L^2}^2,$$

where the second equality follows from (3.11). This and (3.12) yield the lemma. □

The lemma above gives an upper bound for $\tilde{\lambda}_k$.

Proposition 3.3 *For every $k \in \mathbb{N}_+$, we have*

$$\tilde{\lambda}_k \leq J_{\max} \lambda_k \leq (1 + C_n K_M \rho^2) \lambda_k.$$

Proof The second inequality follows from (3.6). The first one follows immediately from combining the min-max formulae (3.2) and (3.5) and Lemma 3.2. □

3.3 Lower bound for $\tilde{\lambda}_k$

As in [2, Section 5], define $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by

$$\psi(t) = \begin{cases} \frac{n+2}{2v_n}(1-t^2), & 0 \leq t \leq 1, \\ 0, & t \geq 1. \end{cases}$$

The normalization constant $\frac{n+2}{2v_n}$ is chosen so that $\int_{\mathbb{R}^n} \psi(|x|) dx = 1$.

We define $k_\rho : M \times M \rightarrow \mathbb{R}_{\geq 0}$ by

$$k_\rho(x, y) = \rho^{-n} \psi\left(\frac{d(x, y)}{\rho}\right), \tag{3.14}$$

and $\theta : M \rightarrow \mathbb{R}_{\geq 0}$ by

$$\theta(x) = \int_M k_\rho(x, y) dy = \int_{B_\rho(x)} k_\rho(x, y) dy. \tag{3.15}$$

(The second identity follows from the fact that $k_\rho(x, y) = 0$ if $d(x, y) \geq \rho$.)

We need the following estimates on θ :

$$J_{\min} \leq \theta(x) \leq J_{\max} \tag{3.16}$$

and

$$|d_x \theta| \leq C_n K_M \rho \tag{3.17}$$

for all $x \in M$. See [2, Lemma 5.1] for a proof.

Define a convolution operator $I : L^2(M, E) \rightarrow C^{0,1}(M, E)$ by

$$Iu(x) = \frac{1}{\theta(x)} \int_M k_\rho(x, y) P_{xy}(u(y)) dy = \frac{1}{\theta(x)} \int_{B_\rho(x)} k_\rho(x, y) P_{xy}(u(y)) dy,$$

for $x \in M$ (compare with [2, Definition 5.2]). We estimate the energy and the L^2 -norm of Iu in the following two lemmas.

Lemma 3.4 *For any $u \in L^2(M, E)$, we have*

$$\|Iu\|_{L^2}^2 \geq \frac{J_{\min}}{J_{\max}} \|u\|_{L^2}^2 - \frac{n+2}{2v_n \rho^n} D^\rho(u).$$

Proof Consider the weighted ρ -connection Laplacian $\Delta_{\theta, k_\rho}^\rho$ defined in Section 2. By (2.5),

$$\Delta_{\theta, k_\rho}^\rho u(x) = \frac{1}{\theta(x)} \int_{B_\rho(x)} k_\rho(x, y) (u(x) - P_{xy}(u(y))) dy = u(x) - Iu(x).$$

where the second equality follows from the definition (3.15). Equivalently, $Iu = u - \Delta_{\theta, k_\rho}^\rho u$. Therefore by (2.3),

$$\|Iu\|_\theta^2 = \|u\|_\theta^2 - 2\langle \Delta_{\theta, k_\rho}^\rho u, u \rangle_\theta + \|\Delta_{\theta, k_\rho}^\rho u\|_\theta^2 \geq \|u\|_\theta^2 - 2\langle \Delta_{\theta, k_\rho}^\rho u, u \rangle_\theta = \|u\|_\theta^2 - 2D_{k_\rho}^\rho(u, u).$$

Observe that

$$D_{k_\rho}^\rho(u, u) \leq \frac{1}{2} \max_{x,y} k_\rho(x, y) \cdot D^\rho(u) \leq \frac{n+2}{4v_n \rho^n} D^\rho(u).$$

Thus

$$\|Iu\|_{\theta}^2 \geq \|u\|_{\theta}^2 - \frac{n+2}{2v_n\rho^n} D^{\rho}(u).$$

Then the lemma follows from the inequality above, $J_{\max} \geq 1$, and the following trivial estimates:

$$\|Iu\|_{\theta}^2 \leq \max_x \theta(x) \cdot \|Iu\|_{L^2}^2 \leq J_{\max} \|Iu\|_{L^2}^2,$$

and

$$\|u\|_{\theta}^2 \geq \min_x \theta(x) \cdot \|u\|_{L^2}^2 \geq J_{\min} \|u\|_{L^2}^2,$$

due to (3.16). □

Lemma 3.5 *For any $u \in L^2(M, E)$, we have*

$$\|\nabla Iu\|_{L^2} \leq (1 + C_n K_M \rho^2) \sqrt{\frac{n+2}{v_n \rho^{n+2}}} D^{\rho}(u) + C_n K_E \rho \|u\|_{L^2}.$$

Proof We rewrite the definition of I as

$$Iu(x) = \int_M \tilde{k}_{\rho}(x, y) P_{xy}(u(y)) dy, \tag{3.18}$$

where

$$\tilde{k}_{\rho}(x, y) = \theta(x)^{-1} k_{\rho}(x, y).$$

Note that for any $x \in M$,

$$\int_M \tilde{k}_{\rho}(x, y) dy = 1.$$

Differentiating this identity, we get

$$\int_M d_x \tilde{k}_{\rho}(x, y) dy = 0, \tag{3.19}$$

where d_x denotes the differential with respect to x . Differentiating (3.18) yields

$$\nabla Iu(x) = A_0(x) + A_3(x),$$

where

$$A_0(x) = \int_M d_x \tilde{k}_{\rho}(x, y) \otimes P_{xy}(u(y)) dy = \int_{B_{\rho}(x)} d_x \tilde{k}_{\rho}(x, y) \otimes P_{xy}(u(y)) dy,$$

and

$$A_3(x) = \int_M \tilde{k}_{\rho}(x, y) \nabla V_{y,u}(x) dy = \int_{B_{\rho}(x)} \tilde{k}_{\rho}(x, y) \nabla V_{y,u}(x) dy.$$

In the above, $V_{y,u}(\cdot)$ is the section of $E|_{B_{\rho}(y)}$ defined by

$$V_{y,u}(z) = P_{zy}(u(y)). \tag{3.20}$$

For better understanding, observe that $\nabla V_{y,u} = 0$ if ∇ is a flat connection. In general, we have an estimate

$$\|\nabla V_{y,u}(x)\| \leq C_n K_E d(x, y) |u(y)| \leq C_n K_E \rho |u(y)|,$$

where the norm at the left-hand side is defined by (3.1). Indeed, for any unit vector $w \in T_x M$, we have

$$|\nabla_w V_{y,u}(x)| \leq K_E d(x, y) |u(y)| |w| = K_E d(x, y) |u(y)|. \tag{3.21}$$

This follows from Lemma 3.1 applied to $u(y)$ in place of v , and applied to a family of minimizing geodesics from y to points $x(s)$ where $x(0) = x$ and $\dot{x}(0) = w$. Note that due to the curvature bound $\text{Sec}_M \leq K_M$, the Rauch comparison theorem implies that $\sup_t \left| \frac{dy_s(t)}{ds} \right|$ in (3.8) is attained at $t = 1$ when ρ satisfies (1.2), see e.g. [13, Chapter 4, Corollary 2.8(1)].

Hence by the Cauchy–Schwarz inequality,

$$\|A_3(x)\| \leq C_n K_E \rho \int_{B_\rho(x)} \tilde{k}_\rho(x, y) |u(y)| dy \leq C_n K_E \rho^{1-\frac{n}{2}} \left(\int_{B_\rho(x)} |u(y)|^2 dy \right)^{\frac{1}{2}},$$

where we used the estimates $\tilde{k}_\rho(x, y) \leq C_n \rho^{-n}$ and $\text{vol}(B_\rho(x)) \leq C_n \rho^n$ due to (3.6). Thus

$$\|A_3\|_{L^2} = \left(\int_M \|A_3(x)\|^2 dx \right)^{\frac{1}{2}} \leq C_n K_E \rho \|u\|_{L^2}. \tag{3.22}$$

Next we turn to A_0 . Using (3.19),

$$A_0(x) = \int_{B_\rho(x)} d_x \tilde{k}_\rho(x, y) \otimes (P_{xy}(u(y)) - u(x)) dy = - \int_{B_\rho(x)} d_x \tilde{k}_\rho(x, y) \otimes \Gamma_{xy}(u) dy.$$

Since $d_x \tilde{k}_\rho(x, y) = \theta(x)^{-1} d_x k_\rho(x, y) - \theta(x)^{-2} d_x \theta(x) \cdot k_\rho(x, y)$, we split A_0 into two terms:

$$A_0(x) = A_1(x) + A_2(x),$$

where

$$A_1(x) = -\theta(x)^{-1} \int_{B_\rho(x)} d_x k_\rho(x, y) \otimes \Gamma_{xy}(u) dy,$$

and

$$A_2(x) = \theta(x)^{-2} d_x \theta(x) \otimes \int_{B_\rho(x)} k_\rho(x, y) \Gamma_{xy}(u) dy.$$

In view of (3.6), (3.16) and (3.17),

$$\|A_2(x)\| \leq C_n K_M \rho^{1-\frac{n}{2}} \left(\int_{B_\rho(x)} |\Gamma_{xy}(u)|^2 dy \right)^{\frac{1}{2}}.$$

This inequality and the definition (3.4) of $D^\rho(u)$ yield that

$$\|A_2\|_{L^2} \leq C_n K_M \rho^{1-\frac{n}{2}} \sqrt{D^\rho(u)} = C_n K_M \rho^2 \sqrt{\frac{1}{\rho^{n+2}} D^\rho(u)}. \tag{3.23}$$

Now we estimate A_1 . From the definition of k_ρ , for $w \in T_x M$, we have

$$d_x k_\rho(x, y) \cdot w = \frac{n+2}{v_n \rho^{n+2}} \langle \exp_x^{-1}(y), w \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian inner product in $T_x M$. Substituting this into the formula for A_1 yields

$$A_1(x) \cdot w = -\theta(x)^{-1} \frac{n+2}{v_n \rho^{n+2}} \int_{B_\rho(x)} \langle \exp_x^{-1}(y), w \rangle \Gamma_{xy}(u) dy.$$

Using the substitution $y = \exp_x(v)$, we get

$$A_1(x) \cdot w = -\theta(x)^{-1} \frac{n+2}{v_n \rho^{n+2}} \int_{B_\rho(0) \subset T_x M} \langle v, w \rangle \varphi(v) J_x(v) dv, \tag{3.24}$$

where

$$\varphi(v) = \Gamma_{x, \exp_x(v)}(u) \in E_x.$$

To proceed we need the following sublemma.

Sublemma *For any L^2 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, one has*

$$\left| \int_{B_\rho(0) \subset \mathbb{R}^n} f(v) v dv \right|^2 \leq \frac{v_n \rho^{n+2}}{n+2} \int_{B_\rho(0)} f(v)^2 dv.$$

Proof Denote

$$F = \int_{B_\rho(0) \subset \mathbb{R}^n} f(v) v dv \in \mathbb{R}^n.$$

Let $v_0 \in \mathbb{R}^n$ be the unit vector in the direction of F . Then

$$|F| = \langle F, v_0 \rangle = \int_{B_\rho(0)} f(v) \langle v, v_0 \rangle dv.$$

Therefore

$$|F|^2 \leq \left(\int_{B_\rho(0)} f(v)^2 dv \right) \left(\int_{B_\rho(0)} \langle v, v_0 \rangle^2 dv \right).$$

Using (3.11), the second integral equals to $\frac{v_n \rho^{n+2}}{n+2}$. The sublemma follows. □

We use the sublemma and (3.24) to estimate A_1 . Fix $x \in M$ and an orthonormal basis $\zeta_1, \dots, \zeta_m \in E_x$, where $m = \dim E_x$. Then $\varphi(v) = \sum_j (f_j(v) + i g_j(v)) \zeta_j$ for some functions $f_j, g_j : B_\rho(0) \rightarrow \mathbb{R}$, $j = 1, \dots, m$. Then the formula (3.24) takes the form

$$A_1(x) \cdot w = -\theta(x)^{-1} \frac{n+2}{v_n \rho^{n+2}} \sum_{j=1}^m (\langle a_j, w \rangle + i \langle b_j, w \rangle) \zeta_j,$$

for any $w \in T_x M$, where $a_j, b_j \in T_x M$ are given by

$$a_j = \int_{B_\rho(0)} f_j(v) v J_x(v) dv, \quad b_j = \int_{B_\rho(0)} g_j(v) v J_x(v) dv.$$

Hence by using (3.1),

$$\|A_1(x)\|^2 = \left(\theta(x)^{-1} \frac{n+2}{v_n \rho^{n+2}} \right)^2 \sum_{j=1}^m (a_j^2 + b_j^2).$$

Applying the Sublemma to $f_j J_x$ and $g_j J_x$ in place of f , we obtain

$$a_j^2 + b_j^2 \leq \frac{v_n \rho^{n+2}}{n+2} \int_{B_\rho(0)} (f_j(v)^2 + g_j(v)^2) J_x(v)^2 dv.$$

Thus,

$$\begin{aligned} \|A_1(x)\|^2 &\leq \theta(x)^{-2} \frac{n+2}{v_n \rho^{n+2}} \int_{B_\rho(0)} |\varphi(v)|^2 J_x(v)^2 dv \\ &\leq \theta(x)^{-2} \frac{n+2}{v_n \rho^{n+2}} J_{\max} \int_{B_\rho(x)} |\Gamma_{xy}(u)|^2 dy, \end{aligned}$$

where we used again the substitution $y = \exp_x(v)$ and the definition of φ in the last inequality.

By (3.16) and (3.6), we have $\theta(x)^{-2} J_{\max} \leq 1 + C_n K_M \rho^2$. Hence,

$$\|A_1(x)\|^2 \leq (1 + C_n K_M \rho^2) \frac{n+2}{v_n \rho^{n+2}} \int_{B_\rho(x)} |\Gamma_{xy}(u)|^2 dy.$$

Integrating over M yields that

$$\|A_1\|_{L^2} \leq (1 + C_n K_M \rho^2) \sqrt{\frac{n+2}{v_n \rho^{n+2}}} D^\rho(u). \tag{3.25}$$

From $\nabla Iu = A_1 + A_2 + A_3$, the lemma follows from (3.25), (3.23), and (3.22). □

The lower bound for $\tilde{\lambda}_k$ is a consequence of Lemma 3.4 and Lemma 3.5.

Proposition 3.6 *For every $k \in \mathbb{N}_+$ satisfying $\tilde{\lambda}_k \leq \sigma_n \rho^{-2}$, we have*

$$\lambda_k^{1/2} \leq (1 + C_n K_M \rho^2 + \tilde{\lambda}_k \rho^2) \tilde{\lambda}_k^{1/2} + C_n K_E \rho.$$

Proof Let $u_1, \dots, u_k \in L^2(M, E)$ be the first k eigen-sections of Δ^ρ corresponding to $\tilde{\lambda}_1, \dots, \tilde{\lambda}_k$. Let \tilde{L} be the linear span of u_1, \dots, u_k . Then \tilde{L} realizes the infimum in the min-max formula (3.5). Hence for all $u \in \tilde{L}$,

$$D^\rho(u) \leq \frac{v_n \rho^{n+2}}{n+2} \tilde{\lambda}_k \|u\|_{L^2}^2. \tag{3.26}$$

This and Lemma 3.4 imply that

$$\|Iu\|_{L^2}^2 \geq \frac{J_{\min}}{J_{\max}} \|u\|_{L^2}^2 - \frac{n+2}{2v_n \rho^n} D^\rho(u) \geq \left(\frac{J_{\min}}{J_{\max}} - \frac{1}{2} \rho^2 \tilde{\lambda}_k \right) \|u\|_{L^2}^2. \tag{3.27}$$

For $\rho^2 \tilde{\lambda}_k \leq 1$ and $\frac{J_{\min}}{J_{\max}} > \frac{1}{2}$, the left-hand side of the inequality above is positive for all $u \in \tilde{L} \setminus \{0\}$. In particular, I is injective on \tilde{L} .

Define $L = I(\tilde{L}) \subset C^{0,1}(M, E)$. Since $I|_{\tilde{L}}$ is injective, we have $\dim L = \dim \tilde{L} = k$. Now the min-max formula (3.2), Lemma 3.5 and (3.27) imply that

$$\begin{aligned} \lambda_k^{1/2} &\leq \sup_{u \in L \setminus \{0\}} \frac{\|\nabla u\|_{L^2}}{\|u\|_{L^2}} = \sup_{u \in \tilde{L} \setminus \{0\}} \frac{\|\nabla Iu\|_{L^2}}{\|Iu\|_{L^2}} \\ &\leq \frac{(1 + C_n K_M \rho^2) \sqrt{\frac{n+2}{v_n \rho^{n+2}}} D^\rho(u) + C_n K_E \rho \|u\|_{L^2}}{\left(\frac{J_{\min}}{J_{\max}} - \frac{1}{2} \rho^2 \tilde{\lambda}_k \right)^{\frac{1}{2}} \|u\|_{L^2}}. \end{aligned}$$

This and (3.26) imply that

$$\lambda_k^{1/2} \leq \frac{(1 + C_n K_M \rho^2) \tilde{\lambda}_k^{1/2} + C_n K_E \rho}{\left(\frac{J_{\min}}{J_{\max}} - \frac{1}{2} \rho^2 \tilde{\lambda}_k\right)^{\frac{1}{2}}}. \tag{3.28}$$

Using the Jacobian estimate (3.6), one sees that $\frac{J_{\min}}{J_{\max}} \geq 1 - C_n K_M \rho^2$, which implies that

$$\left(\frac{J_{\min}}{J_{\max}} - \frac{1}{2} \rho^2 \tilde{\lambda}_k\right)^{-\frac{1}{2}} \leq 1 + C_n K_M \rho^2 + \tilde{\lambda}_k \rho^2.$$

Then the proposition follows. □

Proof of Theorem 1 The estimate directly follows from Proposition 3.3 and Proposition 3.6, after converting all error terms involving $\tilde{\lambda}_k$ to λ_k by using Proposition 3.3. □

4 Discretization of the connection Laplacian

In this section we prove Theorem 2. Let M^n be a compact, connected Riemannian manifold of dimension n without boundary, and let E be a smooth Euclidean (or Hermitian) vector bundle over M equipped with a smooth Euclidean (or Hermitian) connection ∇ . Suppose $P = \{P_{xy}\}$ is the ρ -connection given by the parallel transport canonically associated with the connection ∇ . Recall that P is unitary and symmetric. Let $\Gamma = (X_\varepsilon, d|_{X_\varepsilon}, \mu)$ (short for Γ_ε) be the discrete metric-measure space defined in Definition 1.3, where $X_\varepsilon = \{x_i\}_{i=1}^N$ is a finite ε -net in M for $\varepsilon \ll \rho$. We consider the ρ -connection Laplacian (2.5) on this discrete metric-measure space Γ , acting on the restriction of the vector bundle E onto X_ε .

The vector bundle E restricted onto X_ε is equipped with the norm

$$\|\bar{u}\|_\Gamma^2 = \sum_{i=1}^N \mu_i |\bar{u}(x_i)|^2, \tag{4.1}$$

for $\bar{u} \in L^2(X_\varepsilon, E|_{X_\varepsilon})$. Choosing the weights $\alpha(x) = 1$ and $\beta(x, y) = \frac{2(n+2)}{v_n \rho^{n+2}}$ in (2.5) the same as in the case of smooth connections, the graph connection Laplacian Δ_Γ^ρ is given by

$$\Delta_\Gamma^\rho \bar{u}(x_i) = \frac{2(n+2)}{v_n \rho^{n+2}} \sum_{d(x_i, x_j) < \rho} \mu_j (\bar{u}(x_i) - P_{x_i x_j} \bar{u}(x_j)), \tag{4.2}$$

and its energy (2.2) is given by

$$\|\delta \bar{u}\|^2 := \frac{n+2}{v_n \rho^{n+2}} \sum_i \sum_{j: d(x_i, x_j) < \rho} \mu_i \mu_j |\bar{u}(x_i) - P_{x_i x_j} \bar{u}(x_j)|^2. \tag{4.3}$$

Denote by $\tilde{\lambda}_k(\Gamma)$ the k -th eigenvalue of Δ_Γ^ρ . Our goal is to prove that $\tilde{\lambda}_k(\Gamma)$ approximates the eigenvalue λ_k of the connection Laplacian Δ for every k , as $\rho + \frac{\varepsilon}{\rho} \rightarrow 0$. In the light of what we have already done in Sect. 3, we only need to obtain a few more estimates.

For the upper bound for $\tilde{\lambda}_k(\Gamma)$, we follow Section 4 in [2] and define the following discretization operator $Q : L^2(M, E) \rightarrow L^2(X_\varepsilon, E|_{X_\varepsilon})$ by

$$Qu(x_i) = \frac{1}{\mu_i} \int_{V_i} P_{x_i y} u(y) dy. \tag{4.4}$$

Define an extension operator $Q^* : L^2(X_\varepsilon, E|_{X_\varepsilon}) \rightarrow L^2(M, E)$ by

$$Q^* \bar{u}(y) = \sum_{i=1}^N P_{yx_i} \bar{u}(x_i) 1_{V_i}(y), \tag{4.5}$$

where 1_{V_i} denotes the characteristic function of the set V_i . Note that $Q \circ Q^* = Id_{L^2(X_\varepsilon, E|_{X_\varepsilon})}$. The energy of Qu for $u \in L^2(M, E)$ is given by

$$\|\delta(Qu)\|^2 = \frac{n+2}{v_n \rho^{n+2}} \sum_i \sum_{j: d(x_i, x_j) < \rho} \mu_i \mu_j |Qu(x_i) - P_{x_i x_j} Qu(x_j)|^2. \tag{4.6}$$

To control the upper bound for $\tilde{\lambda}_k(\Gamma)$, we need to estimate $\|Qu\|_\Gamma$ and $\|\delta(Qu)\|$. We start with the following lemma as an application of Lemma 3.1.

Lemma 4.1 *Let $\rho < r_{inj}(M)/2$, $\varepsilon < \rho/4$, and $x_i, x_j, y, z \in M$ be given satisfying $d(x_i, x_j) < \rho$, $d(y, x_i) < \varepsilon$, $d(z, x_j) < \varepsilon$. Then for any $v \in E_z$, we have*

$$|P_{yx_i} P_{x_i z} v - P_{yz} v| \leq K_E(\rho + 2\varepsilon)\varepsilon|v|,$$

and

$$|P_{yx_i} P_{x_i x_j} P_{x_j z} v - P_{yz} v| \leq 2K_E(\rho + 2\varepsilon)\varepsilon|v|.$$

Proof Let $V_{z,v}(x) = P_{xz}(v)$, and let $\gamma_{y,x_i} : [0, d(y, x_i)] \rightarrow M$ be the unique minimizing geodesic from y to x_i with arclength parametrization. By definition,

$$\nabla_s V_{z,v}(\gamma_{y,x_i}(s)) = \lim_{s' \rightarrow 0} \frac{P_{s,s+s'} V_{z,v}(\gamma_{y,x_i}(s + s')) - V_{z,v}(\gamma_{y,x_i}(s))}{s'},$$

where $P_{s,s+s'}$ denotes the parallel transport from $\gamma_{y,x_i}(s + s')$ to $\gamma_{y,x_i}(s)$ along the geodesic γ_{y,x_i} . Apply $P_{0,s}$ to both sides:

$$P_{0,s}(\nabla_s V_{z,v}(\gamma_{y,x_i}(s))) = \lim_{s' \rightarrow 0} \frac{P_{0,s+s'} V_{z,v}(\gamma_{y,x_i}(s + s')) - P_{0,s} V_{z,v}(\gamma_{y,x_i}(s))}{s'}.$$

Observe that $P_{0,\cdot} V_{z,v}(\gamma_{y,x_i}(\cdot))$ is a curve in E_y . Since P is unitary, the formula above shows that the tangent vectors of this E_y -curve have lengths bounded by $K_E(\rho + 2\varepsilon)|v|$ due to (3.21). Thus,

$$|P_{yx_i} V_{z,v}(x_i) - V_{z,v}(y)| = \left| \int_0^{d(x_i,y)} P_{0,s}(\nabla_s V_{z,v}(\gamma_{y,x_i}(s))) ds \right| \leq K_E(\rho + 2\varepsilon)\varepsilon|v|.$$

Then the first conclusion directly follows from the definition $V_{z,v}(x) = P_{xz}(v)$.

The second conclusion can be derived using the first conclusion. Namely, we apply the first conclusion with x_j, z, y in place of y, x_i, z , and $P_{yz}v$ in place of v :

$$|P_{x_j z} P_{z y}(P_{yz}v) - P_{x_j y}(P_{yz}v)| \leq K_E(\rho + 2\varepsilon)\varepsilon|P_{yz}v| = K_E(\rho + 2\varepsilon)\varepsilon|v|.$$

Since P is symmetric and unitary, the inequality above is equivalent to

$$|P_{yx_j} P_{x_j z} v - P_{yz} v| \leq K_E(\rho + 2\varepsilon)\varepsilon|v|. \tag{4.7}$$

Applying the first conclusion with y, x_i, x_j in place of y, x_i, z , and $P_{x_j z}v$ in place of v gives

$$|P_{yx_i} P_{x_i x_j}(P_{x_j z}v) - P_{yx_j}(P_{x_j z}v)| \leq K_E(\rho + 2\varepsilon)\varepsilon|P_{x_j z}v| = K_E(\rho + 2\varepsilon)\varepsilon|v|. \tag{4.8}$$

Thus the second conclusion follows from (4.7), (4.8) and the triangle inequality. \square

The following two lemmas enable us to obtain an upper bound for $\tilde{\lambda}_k(\Gamma)$.

Lemma 4.2 *For any $u \in L^2(M, E)$, we have*

$$\|u\|_{L^2} - \|Qu\|_{\Gamma} \leq \frac{C}{v_n(\rho - \varepsilon)^n} D^\rho(u) + CK_E^2 \rho^4 \|u\|_{L^2}^2.$$

Proof Observe that Q^* preserves the norms. Hence,

$$\|u\|_{L^2} - \|Qu\|_{\Gamma} = \|u\|_{L^2} - \|Q^*Qu\|_{L^2} \leq \|u - Q^*Qu\|_{L^2}.$$

By the definitions of Q and Q^* ,

$$\begin{aligned} \|u - Q^*Qu\|_{L^2}^2 &= \sum_{i=1}^N \int_{V_i} |u(x) - P_{xx_i}Qu(x_i)|^2 dx \\ &= \sum_{i=1}^N \int_{V_i} \left| u(x) - \frac{1}{\mu_i} \int_{V_i} P_{xx_i}P_{x_iy}u(y) dy \right|^2 dx. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\|u - Q^*Qu\|_{L^2}^2 \leq \sum_{i=1}^N \frac{1}{\mu_i} \int_{V_i} \int_{V_i} |u(x) - P_{xx_i}P_{x_iy}u(y)|^2 dx dy.$$

Since $V_i \subset B_\varepsilon(x_i)$, then Lemma 4.1 yields that

$$\|u - Q^*Qu\|_{L^2}^2 \leq \sum_{i=1}^N \frac{1}{\mu_i} \int_{V_i} \int_{V_i} (|u(x) - P_{xy}u(y)| + 2K_E \varepsilon^2 |u(y)|)^2 dx dy.$$

To deal with the first term, we follow the proof of Lemma 3.4 in [2]. We fix $x, y \in V_i$ and consider the set $U = B_\rho(x) \cap B_\rho(y)$. Observe that U contains the ball of radius $\rho - |xy|/2 \geq \rho - \varepsilon$ centered at the midpoint between x and y . Hence we have $\text{vol}(U) \geq C v_n(\rho - \varepsilon)^n$ by (3.6). Recall that P is unitary and symmetric. Then for every $z \in U$, we have

$$\begin{aligned} |u(x) - P_{xy}u(y)| &\leq |u(x) - P_{xz}u(z)| + |P_{xz}u(z) - P_{xy}u(y)| \\ &= |u(x) - P_{xz}u(z)| + |u(z) - P_{zx}P_{xy}u(y)| \\ &\leq |u(x) - P_{xz}u(z)| + |u(z) - P_{zy}u(y)| + K_E \rho^2 |u(y)|, \end{aligned}$$

where we applied Lemma 4.1 in the last inequality. Then

$$\begin{aligned} |u(x) - P_{xy}u(y)|^2 &\leq \frac{C}{\text{vol}(U)} \int_U (|u(x) - P_{xz}u(z)|^2 + |u(y) - P_{yz}u(z)|^2 + K_E^2 \rho^4 |u(y)|^2) dz \\ &\leq \frac{C}{\text{vol}(U)} (F(x) + F(y)) + CK_E^2 \rho^4 |u(y)|^2, \end{aligned}$$

where $F(x) = \int_{B_\rho(x)} |u(x) - P_{xz}u(z)|^2 dz$. Hence by definition (3.4), we obtain

$$\begin{aligned} \|u - Q^*Qu\|_{L^2}^2 &\leq \sum_{i=1}^N \frac{C}{\mu_i} \int_{V_i} \int_{V_i} \frac{1}{\text{vol}(U)} (F(x) + F(y)) dx dy + CK_E^2 \rho^4 \|u\|_{L^2}^2 \\ &= \frac{C}{\text{vol}(U)} D^\rho(u) + CK_E^2 \rho^4 \|u\|_{L^2}^2 \leq \frac{C}{v_n(\rho - \varepsilon)^n} D^\rho(u) + CK_E^2 \rho^4 \|u\|_{L^2}^2. \end{aligned}$$

□

Lemma 4.3 For any $u \in L^2(M, E)$, we have

$$\|\delta(Qu)\|^2 \leq \frac{n+2}{v_n \rho^{n+2}} (1 + 4\rho^2) D^{\rho+2\varepsilon}(u) + C_n K_E^2 \left(\frac{\varepsilon}{\rho}\right)^2 \|u\|_{L^2}^2.$$

Proof The definition of Q yields that

$$Qu(x_i) - P_{x_i x_j} Qu(x_j) = \frac{1}{\mu_i \mu_j} \int_{V_i} \int_{V_j} (P_{x_i y} u(y) - P_{x_i x_j} P_{x_j z} u(z)) dy dz.$$

Then by the Cauchy-Schwarz inequality and the fact that P is unitary and symmetric,

$$\begin{aligned} |Qu(x_i) - P_{x_i x_j} Qu(x_j)|^2 &\leq \frac{1}{\mu_i \mu_j} \int_{V_i} \int_{V_j} |P_{x_i y} u(y) - P_{x_i x_j} P_{x_j z} u(z)|^2 dy dz \\ &= \frac{1}{\mu_i \mu_j} \int_{V_i} \int_{V_j} |u(y) - P_{y x_i} P_{x_i x_j} P_{x_j z} u(z)|^2 dy dz. \end{aligned}$$

The parallel transport appeared in the quantity above goes through the path $[z x_j x_i y]$, while what we need is to go through the minimizing geodesic $[zy]$. Thus (4.6) and Lemma 4.1 imply that

$$\begin{aligned} \|\delta(Qu)\|^2 &\leq \frac{n+2}{v_n \rho^{n+2}} \sum_i \sum_{j: d(x_i, x_j) < \rho} \int_{V_i} \int_{V_j} |u(y) - P_{y x_i} P_{x_i x_j} P_{x_j z} u(z)|^2 dy dz \\ &\leq \frac{n+2}{v_n \rho^{n+2}} \sum_i \sum_{j: d(x_i, x_j) < \rho} \int_{V_i} \int_{V_j} (|u(y) - P_{y z} u(z)| + 4K_E \rho \varepsilon |u(z)|)^2 dy dz \\ &\leq \frac{n+2}{v_n \rho^{n+2}} \sum_i \sum_{j: d(x_i, x_j) < \rho} \int_{V_i} \int_{V_j} \left((1 + 4\rho^2) |u(y) - P_{y z} u(z)|^2 + CK_E^2 \varepsilon^2 |u(z)|^2 \right) dy dz. \end{aligned}$$

Here the last inequality above used the inequality that

$$2K_E \rho \varepsilon |u(z)| \cdot |u(y) - P_{y z} u(z)| \leq \rho^2 |u(y) - P_{y z} u(z)|^2 + K_E^2 \varepsilon^2 |u(z)|^2.$$

Since $\bigcup_{j: d(x_i, x_j) < \rho} V_j \subset B_{\rho+2\varepsilon}(y)$ for $y \in V_i$, we have

$$\begin{aligned} \|\delta(Qu)\|^2 &\leq \frac{n+2}{v_n \rho^{n+2}} (1 + 4\rho^2) \int_M \int_{B_{\rho+2\varepsilon}(y)} |u(y) - P_{y z} u(z)|^2 dy dz + C_n K_E^2 \frac{\varepsilon^2 (\rho + 2\varepsilon)^n}{\rho^{n+2}} \|u\|_{L^2}^2 \\ &\leq \frac{n+2}{v_n \rho^{n+2}} (1 + 4\rho^2) D^{\rho+2\varepsilon}(u) + C_n K_E^2 \left(\frac{\varepsilon}{\rho}\right)^2 \|u\|_{L^2}^2. \end{aligned}$$

□

The lower bound for $\tilde{\lambda}_k(\Gamma)$ almost immediately follows from Lemmas 3.4 and 3.5, since these two lemmas hold for any L^2 section. For any $\bar{u} \in L^2(X_\varepsilon, E|_{X_\varepsilon})$, we consider $Q^* \bar{u} \in L^2(M, E)$ and apply those two lemmas to $Q^* \bar{u}$. Recall that $\|Q^* \bar{u}\|_{L^2}^2 = \|\bar{u}\|_\Gamma^2$. The only part left is to estimate $D^\rho(Q^* \bar{u})$ in terms of $\|\delta \bar{u}\|^2$.

Lemma 4.4 For any $\bar{u} \in L^2(X_\varepsilon, E|_{X_\varepsilon})$, we have

$$D^{\rho-2\varepsilon}(Q^* \bar{u}) \leq \frac{v_n \rho^{n+2}}{n+2} (1 + 4\rho^2) \|\delta \bar{u}\|^2 + C_n K_E^2 \varepsilon^2 (\rho + \varepsilon)^n \|\bar{u}\|_\Gamma^2.$$

Proof Since $B_{\rho-2\varepsilon}(y) \subset \bigcup_{j:d(x_i,x_j)<\rho} V_j$ for $y \in V_i$, we have

$$\begin{aligned} D^{\rho-2\varepsilon}(Q^*\bar{u}) &= \int_M \int_{B_{\rho-2\varepsilon}(y)} |Q^*\bar{u}(y) - P_{yz}Q^*\bar{u}(z)|^2 dz dy \\ &\leq \sum_i \sum_{j:d(x_i,x_j)<\rho} \int_{V_i} \int_{V_j} |Q^*\bar{u}(y) - P_{yz}Q^*\bar{u}(z)|^2 dz dy. \end{aligned}$$

By the definition of Q^* , for any $y \in V_i, z \in V_j$,

$$Q^*\bar{u}(y) - P_{yz}Q^*\bar{u}(z) = P_{yx_i}\bar{u}(x_i) - P_{yz}P_{zx_j}\bar{u}(x_j).$$

Since P is unitary and symmetric, Lemma 4.1 implies that

$$\begin{aligned} |Q^*\bar{u}(y) - P_{yz}Q^*\bar{u}(z)|^2 &= |\bar{u}(x_i) - P_{x_iy}P_{yz}P_{zx_j}\bar{u}(x_j)|^2 \\ &\leq (|\bar{u}(x_i) - P_{x_ix_j}\bar{u}(x_j)| + 4K_E\rho\varepsilon|\bar{u}(x_j)|)^2 \\ &\leq (1 + 4\rho^2)|\bar{u}(x_i) - P_{x_ix_j}\bar{u}(x_j)|^2 + CK_E^2\varepsilon^2|\bar{u}(x_j)|^2. \end{aligned}$$

Integrating the last inequality over V_i, V_j , by definition (4.3), we obtain

$$\begin{aligned} D^{\rho-2\varepsilon}(Q^*\bar{u}) &\leq (1 + 4\rho^2) \sum_{i,j} \mu_i\mu_j |\bar{u}(x_i) - P_{x_ix_j}\bar{u}(x_j)|^2 + CK_E^2\varepsilon^2 \sum_{i,j} \mu_i\mu_j |\bar{u}(x_j)|^2 \\ &\leq \frac{\nu_n\rho^{n+2}}{n+2} (1 + 4\rho^2) \|\delta\bar{u}\|^2 + C_n K_E^2 \varepsilon^2 (\rho + \varepsilon)^n \|\bar{u}\|_\Gamma^2, \end{aligned}$$

where the last inequality used the fact that $\sum_{i:d(x_i,x_j)<\rho} \mu_i \leq \text{vol}(B_{\rho+\varepsilon}(x_j))$. □

Proof of Theorem 2 The upper bound for $\tilde{\lambda}_k(\Gamma)$ follows from Lemmas 4.2, 4.3 and 3.2. The lower bound follows from Lemmas 4.4, 3.4 and 3.5. The calculations are straightforward, similar to Proposition 3.6. □

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References

1. Belkin, M., Niyogi, P.: Convergence of Laplacian eigenmaps. Adv. NIPS (2007)
2. Burago, D., Ivanov, S., Kurylev, Y.: A graph discretization of the Laplace–Beltrami operator. J. Spectr. Theory 4(4), 675–714 (2014). [arXiv:1301.2222](https://arxiv.org/abs/1301.2222)
3. Burago, D., Ivanov, S., Kurylev, Y.: Spectral stability of metric-measure Laplacians. Israel J. Math. 232(1), 125–158 (2019). [arXiv:1506.06781](https://arxiv.org/abs/1506.06781)

4. García Trillos, N., Gerlach, M., Hein, M., Slepčev, D.: Error estimates for spectral convergence of the graph Laplacian on random geometric graphs towards the Laplace–Beltrami operator. *Found. Comput. Math.* **20**, 827–887 (2020)
5. Gigli, N.: On the differential structure of metric-measure spaces and applications. *Mem. Amer. Math. Soc.* **236**(1113) (2015). [arXiv:1205.6622](https://arxiv.org/abs/1205.6622)
6. Gigli, N.: Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. *Mem. Amer. Math. Soc.* (2018). [arXiv:1407.0809](https://arxiv.org/abs/1407.0809)
7. Jiang, SW., Harlim, J.: Ghost point diffusion maps for solving elliptic PDEs on manifolds with classical boundary conditions, to appear in *Comm. Pure Appl. Math.* [arXiv:2006.04002](https://arxiv.org/abs/2006.04002)
8. Karoui, N., Wu, H.: Graph connection Laplacian methods can be made robust to noise. *Ann. Stat.* **44**(1), 346–372 (2016)
9. Kurylev, Y., Oksanen, L., Paternain, G.: Inverse problems for the connection Laplacian. *J. Differ. Geom.* **110**, 457–494 (2018)
10. Lebeau, G., Michel, L.: Semi-classical analysis of a random walk on a manifold. *Ann. Probab.* **38**(1), 277–315 (2010)
11. Lu, J.: Graph approximations to the Laplacian spectra, to appear in *Journal of Topology and Analysis*. [arXiv:1910.09224](https://arxiv.org/abs/1910.09224)
12. Petersen, P.: *Riemannian Geometry*, 2nd edn. Springer, Berlin (2006)
13. Sakai, T.: *Riemannian Geometry*. AMS, Providence (1996)
14. Singer, A., Wu, H.: Vector diffusion maps and the connection Laplacian. *Commun. Pure Appl. Math.* **65**, 1067–1144 (2012)
15. Singer, A., Wu, H.: Spectral convergence of the connection Laplacian from random samples. *Inf. Infer.* **6**, 58–123 (2017)
16. Wormell, C., Reich, S.: Spectral convergence of diffusion maps: improved error bounds and an alternative normalization. *SIAM J. Numer. Anal.* **59**, 1687–1734 (2021)

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