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Profinite groups with restricted centralizers of π-elements

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Abstract
A group G is said to have restricted centralizers if for each g in G the centralizer $C_G(g)$ either is finite or has finite index in G. Shalev showed that a profinite group with restricted centralizers is virtually abelian. Given a set of primes π, we take interest in profinite groups with restricted centralizers of π-elements. It is shown that such a profinite group has an open subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup. This significantly strengthens a result from our earlier paper.

Keywords Profinite groups · Centralizers · π-elements · FC-groups

Mathematics Subject Classification 20E18 · 20F24

1 Introduction

A group G is said to have restricted centralizers if for each g in G the centralizer $C_G(g)$ either is finite or has finite index in G. This notion was introduced by Shalev in [13] where he showed that a profinite group with restricted centralizers is virtually abelian. We say that a profinite group has a property virtually if it has an open subgroup with that property. The article [3] handles profinite groups with restricted centralizers of w-values for a multilinear commutator word w. The theorem proved in [3] says that if w is a multilinear commutator word and G is a profinite group in which the centralizer of any w-value is either finite or open, then the verbal subgroup $w(G)$ is virtually abelian. In [1] we study profinite groups in which p-elements have restricted centralizers, that is, groups in which $C_G(x)$ is either finite or open for any p-element x. The following theorem was proved.

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Theorem 1.1 *Let p be a prime and G a profinite group in which the centralizer of each p-element is either finite or open. Then G has a normal abelian pro-p subgroup N such that G/N is virtually pro-p'.*

The present paper grew out of our desire to determine whether this result can be extended to profinite groups in which the centralizer of each π-element, where π is a fixed set of primes, is either finite or open. As usual, we say that an element x of a profinite group G is a π-element if the order of the image of x in every finite continuous homomorphic image of G is divisible only by primes in π (see [10, Section 2.3] for a formal definition of the order of a profinite group).

It turned out that the techniques used in the proof of Theorem 1.1 were not quite adequate for handling the case of π-elements. The basic difficulty stems from the fact that (pro)finite groups in general do not possess Hall π-subgroups.

In the present paper we develop some new techniques and establish the following theorem about finite groups.

If π is a set of primes and G a finite group, write $O^{\pi'}(G)$ for the unique smallest normal subgroup M of G such that G/M is a π'-group. The conjugacy class containing an element $g \in G$ is denoted by g^G.

Theorem 1.2 *Let n be a positive integer, π be a set of primes, and G a finite group such that $|g^G| \leq n$ for each π-element $g \in G$. Let $H = O^{\pi'}(G)$. Then G has a normal subgroup N such that*

1. *The index $[G : N]$ is n-bounded;*
2. $[H, N] = [H, H]$*;*
3. *The order of $[H, N]$ is n-bounded.*

Throughout the article we use the expression "$(a, b, \ldots)$-bounded" to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \ldots$.

The proof of Theorem 1.2 uses some new results related to Neumann's BFC-theorem [8]. In particular, an important role in the proof is played by a recent probabilistic result from [2]. Theorem 1.2 provides a highly effective tool for handling profinite groups with restricted centralizers of π-elements. Surprisingly, the obtained result is much stronger than Theorem 1.1 even in the case where π consists of a single prime.

Theorem 1.3 *Let π be a set of primes and G a profinite group in which the centralizer of each π-element is either finite or open. Then G has an open subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup.*

Thus, the improvement over Theorem 1.1 is twofold – the result now covers the case of π-elements and provides additional details clarifying the structure of groups in question. Furthermore, it is easy to see that Theorem 1.3 extends Shalev's result [13] which can be recovered by considering the case where $\pi = \pi(G)$ is the set of all prime divisors of the order of G.

We now have several results showing that if the elements of a certain subset X of a profinite group G have restricted centralizers, then the structure of G is very special. This suggests the general line of research whose aim would be to determine which subsets of G have the above property. At present we are not able to provide any insight on the problem. Perhaps one might start with the following question:

Let n be a positive integer. What can be said about a profinite group G such that if $x \in G$ then $C_G(x^n)$ is either finite or open?

Proofs of Theorems 1.2 and 1.3 will be given in Sects. 2 and 3 , respectively.

2 Proof of Theorem 1.2

The following lemma is taken from [1]. If $X \subseteq G$ is a subset of a group G, we write $\langle X \rangle$ for the subgroup generated by X and $\langle X^G \rangle$ for the minimal normal subgroup of G containing X.

Lemma 2.1 *Let i, j be positive integers and G a group having a subgroup K such that $|x^G| \leq i$ for each $x \in K$. Suppose that $|K| \leq j$. Then $\langle K^G \rangle$ has finite (i, j)-bounded order.*

If K is a subgroup of a finite group G, we denote by

$$Pr(K, G) = \frac{|\{(x, y) \in K \times G : [x, y] = 1\}|}{|K||G|}$$

the relative commutativity degree of K in G, that is, the probability that a random element of G commutes with a random element of K. Note that

$$Pr(K, G) = \frac{\sum_{x \in K} |C_G(x)|}{|K||G|}.$$

It follows that if $|x^G| \leq n$ for each $x \in K$, then $Pr(K, G) \geq \frac{1}{n}$.

The next result was obtained in [2, Proposition 1.2]. In the case where $K = G$ this is a well known theorem, due to P. M. Neumann [9].

Proposition 2.2 *Let $\epsilon > 0$, and let G be a finite group having a subgroup K such that $Pr(K, G) \geq \epsilon$. Then there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indexes $[G : T]$ and $[K : B]$, and the order of the commutator subgroup $[T, B]$ are ϵ-bounded.*

We will now embark on the proof of Theorem 1.2.

Assume the hypothesis of Theorem 1.2. Let X be the set of all π-elements of G. Clearly, $H = \langle X \rangle$. Given an element $g \in H$, we write $l(g)$ for the minimal number l with the property that g can be written as a product of l elements of X. The following result is straightforward from [4, Lemma 2.1].

Lemma 2.3 *Let $K \leq H$ be a subgroup of index m in H, and let $b \in H$. Then the coset Kb contains an element g such that $l(g) \leq m - 1$.*

Let m be the maximum of indices of $C_H(x)$ in H where $x \in X$. Obviously, we have $m \leq n$.

Lemma 2.4 *For any $x \in X$ the subgroup $[H, x]$ has m-bounded order.*

Proof Take $x \in X$. Since the index of $C_H(x)$ in H is at most m, by Lemma 2.3, we can choose elements $y_1, \ldots, y_m$ in H such that $l(y_i) \leq m - 1$ and the subgroup $[H, x]$ is generated by the commutators $[y_i, x]$, for $i = 1, \ldots, m$. For any such i write $y_i = y_{i1} \ldots y_{i(m-1)}$, with $y_{ij} \in X$. Using standard commutator identities we can rewrite $[y_i, x]$ as a product of conjugates in H of the commutators $[y_{ij}, x]$. Let $\{h_1, \ldots, h_s\}$ be the conjugates in H of all elements from the set $\{x, y_{ij} \mid 1 \leq i, j \leq m\}$. Note that the number s here is m-bounded. This follows form the fact that $C_H(x)$ has index at most m in H for each $x \in X$. Put $T = \langle h_1, \ldots, h_s \rangle$. Since $[H, x]$ is contained in the commutator subgroup T', it is sufficient to show that T' has m-bounded order. Observe that the centre $Z(T)$ has index at most m^s in T, since the index of $C_H(h_i)$ is at most m in H for any $i = 1, \ldots, s$. Thus, by Schur's theorem [11, 10.1.4], we conclude that the order of T' is m-bounded, as desired. □

Select $a \in X$ such that $|a^H| = m$. Choose $b_1, \ldots, b_m$ in H such that $l(b_i) \leq m-1$ and $a^H = \{a^{b_i}; i = 1, \ldots, m\}$. The existence of the elements b_i is guaranteed by Lemma 2.3. Set $U = C_G(\langle b_1, \ldots, b_m \rangle)$. Note that the index of U in G is n-bounded. Indeed, since $l(b_i) \leq m-1$ we can write $b_i = b_{i1} \ldots b_{i(m-1)}$, where $b_{ij} \in X$ and $i = 1, \ldots, m$. By the hypothesis the index of $C_G(b_{ij})$ in G is at most n for any such element b_{ij}. Thus, $[G : U] \leq n^{(m-1)m}$.

The next result is somewhat analogous to [14, Lemma 4.5].

Lemma 2.5 *If $u \in U$ and $ua \in X$, then $[H, u] \leq [H, a]$.*

Proof Assume that $u \in U$ and $ua \in X$. For each $i = 1, \ldots, m$ we have $(ua)^{b_i} = ua^{b_i}$, since u belongs to U. We know that $ua \in X$ so taking into account the hypothesis on the cardinality of the conjugacy class of ua in H, we deduce that $(ua)^H$ consists exactly of the elements ua^{b_i}, for $i = 1, \ldots, m$. Thus, given an arbitrary element $h \in H$, there exists $b \in \{b_1, \ldots, b_m\}$ such that $(ua)^h = ua^b$ and so $u^h a^h = ua^b$. It follows that $[u, h] = a^b a^{-h} \in [H, a]$, and the result holds. □

Lemma 2.6 *The order of the commutator subgroup of H is n-bounded.*

Proof Let U_0 be the maximal normal subgroup of G contained in U. Recall that, by the remark made before Lemma 2.5, U has n-bounded index in G. It follows that the index $[G : U_0]$ is n-bounded as well.

By the hypothesis a has at most n conjugates in G, say $\{a^{g_1}, \ldots, a^{g_n}\}$. Let T be the normal closure in G of the subgroup $[H, a]$. Note that the subgroups $[H, a^{g_i}]$ are normal in H, therefore $T = [H, a^{g_1}] \ldots [H, a^{g_n}]$. By Lemma 2.4 each of the subgroups $[H, a^{g_i}]$ has n-bounded order. We conclude that the order of T is n-bounded.

Let $Y = Xa^{-1} \cap U$. Note that for any $y \in Y$ the product ya belongs to X. Therefore, by Lemma 2.5, for any $y \in Y$, the subgroup $[H, y]$ is contained in $[H, a]$. Thus,

$$[H, Y] \leq T. \tag{1}$$

Observe that for any $u \in U_0$ the commutator $[u, a^{-1}] = a^u a^{-1}$ lies in Y and so

$$\left[H, \left[U_0, a^{-1}\right]\right] \leq [H, Y]. \tag{2}$$

Since $[U_0, a^{-1}] = [U_0, a]$, we deduce from (1) and (2) that

$$[H, [U_0, a]] \leq T. \tag{3}$$

Since T has n-bounded order, it is sufficient to show that the derived group of the quotient H/T has finite n-bounded order. We pass now to the quotient G/T and for the sake of simplicity the images of G, H, U, U_0, X and Y will be denoted by the same symbols. Note that by (1) the set Y becomes central in H modulo T. Containment (3) shows that $[U_0, a] \leq Z(H)$. This implies that if $b \in U_0$ is a π-element, then $[b, a] \in Z(H)$ and the subgroup $\langle a, b \rangle$ is nilpotent. Thus the product ba is a π-element too and so $b \in Y$. Hence, all π-elements of U_0 are contained in Y and, in view of (1), we deduce that they are contained in $Z(H)$.

Next we consider the quotient $G/Z(H)$. Since the image of U_0 in $G/Z(H)$ is a π'-group and has n-bounded index in G, we deduce that the order of any π-subgroup in $G/Z(H)$ is n-bounded. In particular, there is an n-bounded constant C such that for every $p \in \pi$ the order of the Sylow p-subgroup of $G/Z(H)$ is at most C. Because of Lemma 2.1 for any $p \in \pi$ each Sylow p-subgroup of $G/Z(H)$ is contained in a normal subgroup of n-bounded order. We deduce that all such Sylow subgroups of $G/Z(H)$ are contained in a normal subgroup of n-bounded order. Since H is generated by π-elements, it follows that the order of $H/Z(H)$ is n-bounded. Thus, in view of Schur's theorem [11, 10.1.4], we conclude that $|H'|$ is n-bounded, as desired. □

We will now complete the proof of Theorem 1.2.

Proof Assume first that H is abelian. In this case the set X of π-elements is a subgroup, that is, $X = H$. By the hypothesis we have $|x^G| \leq n$ for any element $x \in H$ and so the relative commutativity degree $Pr(H, G)$ of H in G is at least $\frac{1}{n}$. Thus, by virtue of Proposition 2.2, there is a normal subgroup $T \leq G$ and a subgroup $B \leq H$ such that the indexes $[G : T]$ and $[H : B]$, and the order of the commutator subgroup $[T, B]$ are n-bounded.

Since H is a normal π-subgroup and $[G : H]$ is a π'-number, by the Schur–Zassenhaus Theorem [5, Theorem 6.2.1] the subgroup H admits a complement L in G such that $G = HL$ and L is a π'-subgroup. Set $T_0 = T \cap L$. Observe that the index $[L : T_0]$ is n-bounded since it is at most the index of T in G. Thus we deduce that the index of HT_0 is n-bounded in G, as well.

We claim that the order of $[H, T_0]$ is n-bounded. Indeed, the π'-subgroup T_0 acts coprimely on the the abelian π-subgroup $B_1 = B[B, T_0]$, and so we have $B_1 = C_{B_1}(T_0) \times [B_1, T_0]$ ([7, Corollary 1.6.5]). Note that $[B_1, T_0] = [B, T_0]$. Since the oder of $[B, T_0]$ is n-bounded (being at most the order of $[T, B]$), we deduce that the index $[B_1 : C_{B_1}(T_0)]$ is n-bounded. In combination with the fact that $[H : B]$ is n-bounded, we obtain that the index $[H : C_{B_1}(T_0)]$ is n-bounded and so in particular $[H : C_H(T_0)]$ is n-bounded. Since T_0 acts coprimely on the abelian normal π-subgroup H, we have $H = C_H(T_0) \times [H, T_0]$. Thus we obtain that the order of the commutator subgroup $[H, T_0]$ is n-bounded, as claimed. Let $T_1 = C_{T_0}([H, T_0])$ and remark that the index $[T_0 : T_1]$ of T_1 in T_0 is n-bounded too. Set $N = HT_1$. From the fact that the indexes $[T_0 : T_1]$ and $[G : HT_0]$ are both n-bounded, we deduce that the index of N in G is n-bounded, as well.

Note that N is normal in G since the image of N in $G/H \cong L$ is isomorphic to T_1 which is normal in L. Furthermore, we have $[H, T_1, T_1] = 1$, since $T_1 = C_{T_0}([H, T_0])$. Hence by the standard properties of coprime actions we have $[H, T_1] = 1$ ([7, Corollary 1.6.4]). Therefore $[H, N] = 1$. This proves the theorem in the particular case where H is abelian.

In the general case, in view of Lemma 2.6, the commutator subgroup $[H, H]$ is of n-bounded order. We pass to the quotient $\overline{G} = G/[H, H]$. The above argument shows that $\overline{G}$ has a normal subgroup $\overline{N}$ of n-bounded index such that $\overline{H} \leq Z(\overline{N})$. Here $Z(\overline{N})$ stands for the centre of $\overline{N}$. Let N be the inverse image of $\overline{N}$. We have $[H, N] = [H, H]$ and so N has the required properties. The proof is now complete. □

3 Proof of Theorem 1.3

We will require the following result taken from [1, Lemma 4.1].

Lemma 3.1 *Let G be a locally nilpotent group containing an element with finite centralizer. Suppose that G is residually finite. Then G is finite.*

Profinite groups have Sylow p-subgroups and satisfy analogues of the Sylow theorems. Prosoluble groups satisfy analogues of the theorems on Hall π-subgroups. We refer the reader to the corresponding chapters in [10, Ch. 2] and [15, Ch. 2].

Recall that an automorphism ϕ of a group G is called fixed-point-free if $C_G(\phi) = 1$, that is, the fixed-point subgroup is trivial. It is a well-known corollary of the classification of finite simple groups that if G is a finite group admitting a fixed-point-free automorphism, then G is soluble (see for example [12] for a short proof). A continuous automorphism ϕ of a profinite group G is coprime if for any open ϕ-invariant normal subgroup N of G the order of the automorphism of G/N induced by ϕ is coprime to the order of G/N. It follows that if a profinite group G admits a coprime fixed-point-free automorphism, then G is prosoluble. This will be used in the proof of Theorem 1.3.

Proof of Theorem 1.3 Recall that π is a set of primes and G is a profinite group in which the centralizer of every π-element is either finite or open. We wish to show that G has an open subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup.

Let X be the set of π-elements in G. Consider first the case where the conjugacy class x^G is finite for any $x \in X$. For each integer $i \geq 1$ set

$$S_i = \{x \in X;\ |x^G| \leq i\}.$$

The sets S_i are closed. Thus, we have countably many sets which cover the closed set X. By the Baire Category Theorem [6, Theorem 34] at least one of these sets has non-empty interior. It follows that there is a positive integer k, an open normal subgroup M, and an element $a \in X$ such that all elements in $X \cap aM$ are contained in S_k.

Note that $\langle a^G \rangle$ has finite commutator subgroup, which we will denote by T. Indeed, the subgroup $\langle a^G \rangle$ is generated by finitely many elements whose centralizer is open. This implies that the centre of $\langle a^G \rangle$ has finite index in $\langle a^G \rangle$, and by Schur's theorem [11, 10.1.4], we conclude that T is finite, as claimed.

Let $x \in X \cap M$. Note that the product ax is not necessarily in X. On the other hand, ax is a π-element modulo T. This is because $\langle a^G \rangle$ becomes an abelian normal π-subgroup modulo T and the image of ax in the quotient $G/\langle a^G \rangle$ is a π-element. In other words, there are $y \in X \cap aM$ and $t \in T$ such that $ax = ty$. Observe that t has an open centralizer in G since $t \in T$. In fact $[G : C_G(t)] \leq |T|$. From the equality $ax = ty$ deduce that $|x^G| \leq k^2|T|$. This happens for any $x \in X \cap M$. Using a routine inverse limit argument in combination with Theorem 1.2 we obtain that M has an open normal subgroup N such that the index $[M : N]$ and the order of $[H, N]$ are finite. Here H stands for the subgroup generated by all π-elements of M. Choose an open normal subgroup U in G such that $U \cap [H, N] = 1$. Then $U \cap M$ is an open normal subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup. This proves the theorem in the case where all π-elements of G have open centralizers.

Assume now that G has a π-element, say b, of infinite order. Since the procyclic subgroup $\langle b \rangle$ is contained in the centralizer $C_G(b)$, it follows that $C_G(b)$ is open in G. This implies that all elements of $X \cap C_G(b)$ have open centralizers (because they centralize the procyclic subgroup $\langle b \rangle$). In view of the above $C_G(b)$ has an open subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup and we are done.

We will therefore assume that G is infinite while all π-elements of G have finite orders and there is at least one π-element, say d, such that $C_G(d)$ is finite. The element d is a product of finitely many π-elements of prime power order. At least one of these elements must have finite centralizer. So without loss of generality we can assume that d is a p-element for a prime $p \in \pi$.

Let P_0 be a Sylow p-subgroup of G containing d. Since P_0 is torsion, we deduce from Zelmanov's theorem [16] that P_0 is locally nilpotent. The centralizer $C_G(d)$ is finite and so in view of Lemma 3.1 the subgroup P_0 is finite. Choose an open normal pro-p' subgroup L such that $L \cap C_G(d) = 1$. Note that any finite homomorphic image of L admits a coprime fixed-point-free automorphism (induced by the coprime action of d on L). Hence

L is prosoluble. Let K be a Hall π-subgroup of L. Since any element in K has restricted centralizer, Shalev's result [13] shows that K is virtually abelian. We therefore can choose an open normal subgroup J in L such that $J \cap K$ is abelian. If $J \cap K$ is finite then G is virtually pro-π' and we are done. If $J \cap K$ is infinite, then all π-elements of J have infinite centralizers. This yields that all π-elements of J have open centralizers in J and in view of the first part of the proof, J has an open normal subgroup of the form $P \times Q$, where P is an abelian pro-π subgroup and Q is a pro-π' subgroup. This establishes the theorem. □

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