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Atiyah classes and the essential obstructions in deforming a singular G_2-instanton

Yuanqi Wang[1]

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Abstract

When the rank of the bundle is ≥ 2, in a certain sense, we found an essential obstruction for the gluing construction of G_2-instantons with 1-dimensional singularities. It involves the Atiyah classes generated by contracting a vector in $\mathbb{C}^3$ with the curvature. Intuitively speaking, the gluing does not work if the tangent connection at a component of the 1-dimensional singular locus is not the twisted Fubini-Study connection on a twisted tangent bundle of $\mathbb{P}^2$. Particularly, it fails if the rank of the bundle is ≥ 3.

1 Introduction

Gauge theory plays an important role in the differential topology of 4-manifolds. Corresponding to the groups $SU(3)$, G_2, and $Spin(7)$ in the holonomy list of Berger-Simons [2,20], Donaldson-Thomas [8] and Donaldson-Segal [7] intend to generalize the gauge theory in dimensions $2,\ 3,\ 4$ to $6,\ 7,\ 8$. In dimension 7, the objects of interest are projective G_2-monopoles and instantons. Let ψ be the co-associative 4-form on a 7-manifold with a G_2-structure and a complex Hermitian vector bundle. A G_2-monopole is a Hermitian connection A and a trace-less skew-Hermitian bundle endomorphism σ i.e. section of $ad E$, such that the curvature F_A^0 of the induced $PU(n)$-connection satisfies the following equation

$$\star(F_A^0 \wedge \psi) + d_A\sigma = 0. \quad (1)$$

When the monopole term $\sigma = 0$, we call the connection a projective G_2-instanton.

To understand the boundary of the moduli and to construct examples of singular instantons via gluing, a Fredholm theory is important. For instantons with isolated singularities, the indicial roots are discrete. However, those of 1-dimensional singularities are not. Finite dimensional obstructions can prevent a gluing construction: see the work of Brendle Kapouleas [4] on Einstein metrics. Infinite dimensional obstructions make it even harder: see the work of Chen [5] on twisted connected sum of G_2-structures with conical singularities along circles. An option is to add parameters into the domain Banach space. For ∞-dimensional co-kernel, we need an ∞-dimensional parameter space. On singular G_2-

✉ Yuanqi Wang
yqwang@ku.edu

1 Department of Mathematics, The University of Kansas, Lawrence, KS, USA

instantons and Hermitian Yang-Mills connections, in addition to deforming the connection, we can pull back the G_2-structures by certain diffeomorphisms in which the Frechet partial derivative yields the *Auxiliary operator*. This yields the extended linearized operator. Our main result shows a necessary condition for such a scheme to work for singular G_2-instantons.

Theorem A *In an ideal configuration of G_2-instanton with 1-dimensional singularities (Definition 2.1), the usual linearized operator (3) does not have closed range. The extended linearized operator (7) has closed range only if at each circle γ_i (as in Definition 2.1), the model connection is the twisted Fubini-Study connection on a twisted tangent bundle of $\mathbb{P}^2$. Particularly, the rank of the bundle must be 2.*

The twisted Fubini-Study connection is defined up to a smooth bundle isomorphism on $\mathbb{P}^2$. For bounded linear operators between Banach spaces, the range is not closed implies ∞-dimensional co-kernel. Theorem A implies the non-vanishing of the co-kernel on compact 7-folds, including the twisted connected sums [19]. The co-kernel is called the obstruction but is different from the *essential obstruction* below.

Corollary B *Over a compact 7-manifold M, under the standard weighted Schauder formulation for G_2-instanton with 1-dimensional singularities in Definition 2.1 (1–3), (40), and (41) below, for any $\delta \geq 0$, the usual linearized operator (40) is not surjective. Suppose*

- *the rank of the bundle is 2 but the model connection at some circle is not the twisted Fubini-Study connection on a twisted tangent bundle of $\mathbb{P}^2$, or*
- *the rank of the bundle is ≥ 3.*

Then the extended linearized operator (41) is not surjective.

Gluing construction of G_2-instantons with 1-dimensional singularities on twisted connected sums is mentioned by Jacob-Walpuski in [13]. Corollary B says that the gluing is obstructed if one of the tangent connections is not twisted Fubini-Study. This is different from smooth G_2-instantons on twisted connected sums considered by Sá Earp-Walpuski [19], in which assuming the two Lagrangian subspaces in a sheaf cohomology intersect transversally [19, Theorem 1.2], the co-kernel is trivial [19, Theorem 3.24 Step 2]. This is indeed the case for the concrete examples [16,23].

We do not need the G_2-structure to be globally co-closed though it might well be in the cases of interest. We only need the flexible functorial conditions I–V which can be easily verified for the example in Corollary B. The condition III* holds under surjectivity hypothesis. This is one of the roots of the contradictory argument for Corollary B.

On other geometric objects, there are perturbation theories deforming the singular locus. For example, see Takahashi's deformation [21] of $\mathbb{Z}_2$-harmonic spinors in dimension 3. Very recently, Donaldson [6] developed the deformation for multi-valued harmonic functions. Similarly to [6], here any Green's function must possess a leading term disabling the deformation. In a certain sense it can not be "overcome" by adding the vector fields if the essential obstruction does not vanish (Lemma 6.1 below). For minimal surfaces with non-isolated singularity, please see the work of Mazzeo-Smale [15] that perturbs the singularities away. This generalizes Hardt-Simon perturbation [11] for isolated singularities. Beyond G_2-instantons, for the aforementioned and other geometric objects, we wonder whether there is similar phenomenon of essential obstruction and "rigidity" of tangent cone. This should be related to a certain eigen-space of the link operator of the linearization which we call the indicial eigen-space. In our G_2-instanton setting of Corollary B, the indicial eigen-space is the sheaf cohomology $H^1[\mathbb{P}^2, (End E)(-1)]$. Its complex dimension is always no less than 3, and it contains a distinguished 3-dimensional subspace consisted of Atiyah classes. An Atiyah class in the cohomology is the image under a natural linear injection of the contraction of a (constant) vector in $\mathbb{C}^3$ with the curvature F^0_A. We define the *essential obstruction* as the finite dimensional quotient

$$\frac{H^1[\mathbb{P}^2, (End E)(-1)]}{\{\text{Atiyah classes}\}}. \tag{2}$$

Please see Proposition 4.1 below. On gluing construction of Einstein metrics, Biquard [3] also found an obstruction involving curvature.

Theorem A and Corollary B can be understood as “good news” for the compactification of moduli of smooth instantons, in conjunction with the work of Tian [22] and the co-dimensional 6 conjecture therein. On a compact G_2-manifold with a unitary vector bundle, the ∞-dimensional co-kernel result makes it reasonable to ask “how often” (in a sense that needs to be specified) we can see other model connections than the twisted Fubini-Study as the singularity model of the “limit” of a sequence of smooth ones. We do not know whether the 1-dimensional singularities can “break into” isolated singularities by any sort of gluing construction, as mentioned in [5].

Closer to the language in [6] and [5] and schematically speaking, the reason why the (non-trivial) essential obstruction prevents the deformation is that it “spans” the leading terms of norm $O(\frac{1}{r^2})$ in solutions to the extended linearized equation. On the model space $(\mathbb{C}^3\setminus O) \times \mathbb{S}^1$, r means the distance to the origin in the $\mathbb{C}^3\setminus O$-component, and s means the parameter of the $\mathbb{S}^1$-factor. Please see (32) and (38) below. Pay attention to that the leading term of the modified Bessel function I_0 in (32) is 1 which results in $x_k = O(\frac{1}{r})$ therein. The norm of an eigen-section of the link operator on $\mathbb{S}^5$ is $O(\frac{1}{r})$. These two factors multiply to $O(\frac{1}{r^2})$. However, we need $O(\frac{1}{r})$ deformations due to quadratic non-linearity of instanton equation. These “bad” leading terms are “inevitable” unless the essential obstruction vanishes.

When the tangent connection is indeed twisted Fubini-Study, we do expect a non-standard Fredholm theory incorporating the deformations of singular locus. We plan to address it in the future. Nevertheless, to achieve the goal proposed in [13] i.e. constructing singular G_2-instantons with 1-dimensional singularities, we still need to address the Banach spaces to work with, how to integrate it to a nonlinear theory, the corresponding index theory, and transversality. Except transversality, the other 3 ingredients are no problem for instantons with isolated singularities or smooth instantons.

The vector fields we allow are spanned by all the 7 directions (coordinate vectors) near a component of a circle while the coefficients only depend on r and s. Please see Section 2.2 below. This is the advantage of the Euclidean space $\mathbb{C}^3\setminus O$ as the simplest Calabi-Yau cone: there are constant vector fields on the 7-dimensional product $(\mathbb{C}^3\setminus O) \times \mathbb{S}^1$ deforming the circle $O \times \mathbb{S}^1$ and also generating eigen-sections of the link operator with respect to eigenvalue -1. The case of more general vector fields remains mysterious. We do not know whether there is any analogous structure for general Calabi-Yau cone over a regular Sasakian Einstein 5-manifold.

Sketch of the proof

Briefly speaking, the proof of Theorem A for the extended linearized operator is an assembling of the following 3 facts.

1. The essential obstruction vanishes if and only if the tangent connection is a twisted Fubini-Study connection on $T^{1,0}\mathbb{P}^2(k)$ (Lemma 4.5).

2. The range of the model auxiliary operator is in the "span" of Atiyah classes (Proposition 5.1).
3. Under the surjectivity condition III*, if the essential obstruction is non-trivial, we can construct a singular sequence violating closed range (Lemma 6.1).

On the other hand, heavy but interesting tensor calculations are carried out for the defining Proposition 4.1 of Atiyah classes (self contained in our setting for readers' convenience), and also in a more sophisticated manner for the auxiliary operator formula in Proposition 5.1. Part of the setting was defined in [25], but the actual computations here are different.

- For Proposition 4.1, we need some identities related to the Euler sequence on $\mathbb{P}^2$, contractions between vector fields and transverse quaternion structure (the 3-forms $\frac{d\eta}{2}$, G, H), and the transverse exterior differential d_0 on the standard Sasakian manifold $\mathbb{S}^5$. For example, see the $\overline{\partial}$ and ∂-closedness in Lemma 4.3 and the "partial d_0-closedness" in Lemma 4.4.
- The computation for Proposition 5.1 are routine but with many terms, based on the geometry of $\mathbb{P}^2$, $\mathbb{S}^5$, and the G_2-forms on $\mathbb{R}^7$. It suffices to apply the fine formula (50) for the standard co-associative G_2-form, then exterior differentiate the contraction with a vector field. This yields the 3 Lie derivative terms and 7-terms that has vanishing wedge with the (traceless) curvature (see (55)). Therefore, wedging it with the curvature and taking Hodge dual, only these 3 Lie derivative terms remain. They are handled further in Proposition 5.1. The cancellation of the two "$\frac{X_i}{r} \cdot (e_i \lrcorner H)$" deserves reader's attention.

Organization of the paper: Almost all definitions related to Theorem A and Corollary B are in Sections 2 and 3 . In Section 4 we define the Atiyah classes in $Eigen_{-1}P$ and use Riemann–Roch to show that the cohomology $H^1[\mathbb{P}^2, (End E)(-1)]$ consists only of Atiyah classes is equivalent to that E is a twisted tangent bundle. In Section 5 we state and prove the formula for the auxiliary operator, leaving routine tensor calculations to the Appendix. In Section 6 we prove the main results using separation of variables, Sasakian geometry of the linearized operator, modified Bessel functions, and functional analysis.

2 Preliminary

In this section we define the configuration required in Theorem A.

Definition 2.1 Throughout, a ball $B(R)$ is always in $\mathbb{C}^3$ and centred at the origin. A tame configuration of G_2-instanton with 1-dimensional singularities consists of:

1. finitely-many disjoint embedded circles (embedded $\mathbb{S}^1$'s) γ_i, $i = 1, ..., l$ with trivial normal bundle in a 7-manifold M, and mutually disjoint tubular neighborhood of γ_i diffeomorphic to $[B(100R_0)\setminus O] \times \mathbb{S}^1$ for some $R_0 > 0$;
2. a smooth unitary connection A on a bundle $E \to M\setminus\gamma$ with rank $n \geq 2$ such that in each tubular ball as above, (A, E) is equal to the pullback of a non-projectively flat Hermitian Yang-Mills connection $(A_i, E_i) \to \mathbb{P}^2$ via the standard fibration map $(\mathbb{C}^3\setminus O)\times\mathbb{S}^1 \to \mathbb{P}^2$;
3. a G_2-structure on M equal to the standard one near each γ_i under the same coordinate;
4. Banach spaces Y, $\mathcal{B}$, and $\chi(M, TM)$ that satisfy condition I, IV, and V below.

A tame configuration is *ideal* if condition II, III, and III* hold.

The reason we can assume R_0 is independent of i is that there are only finitely-many circles. The results in the introduction are independent of R_0 as long as it is > 0. Many discussions below are under the coordinate chart in the first bullet point above. This should be clear from context. For example, see condition II below.

The following terms make it convenient.

Terminology 2.2 The manifold $(\mathbb{C}^3\backslash O)\times\mathbb{S}^1$ is called the *model space*. The open set $B(R)\times\mathbb{S}^1$ and the punched set $[B(R)\backslash O]\times\mathbb{S}^1$ are called the *tubular ball* and *punched tubular ball*, respectively. The punched tubular ball with radius $R=\infty$ is the model space.

Let r denote the distance to the origin in $\mathbb{C}^3$. This is also the Euclidean distance to the circle $O\times\mathbb{S}^1$ in $\mathbb{C}^3\times\mathbb{S}^1$. Sometimes it is denoted by r_x as a function (see (10) below).

2.1 The usual linearization

Let Ω^k_{adE} denote the bundle of adE-valued k-forms. With gauge fixing, the usual linearization of the monopole equation (1) is a first order elliptic operator $\underline{L}$ that maps $C^\infty[M^7\backslash\gamma,\Omega^0_{adE}\oplus\Omega^1_{adE}]$ to itself:

$$\underline{L}\begin{bmatrix}\sigma\\ a\end{bmatrix}=\begin{bmatrix}d_A^\star a\\ d_A\sigma+\star(d_Aa\wedge\psi)\end{bmatrix},\tag{3}$$

where σ is a section and a is a 1-form, both adE-valued. To avoid heavy notation *we henceforth suppress the bundles and even the domain manifold in the notation for the Banach spaces, including the weighted Schauder spaces etc.*

Let the domain of the usual linearized operator be a Banach space Y that is a subspace of $C^1(M\backslash\gamma)$. Likewise, let the target $\mathcal{B}$ be a Banach space that is a subspace of $C^0(M\backslash\gamma)$, such that the following holds.

$$\text{Condition I}:\quad \underline{L}:\ Y\to\mathcal{B}\ \text{ is bounded.}$$

To construct singular sequence, we need two more conditions. The first is the lower bound comparing the norm of Y to the standard weighted C^0-norm whose sections are $O(\frac{1}{r})$ near the circles.

$$\text{Condition II}:\quad \|\xi\|_Y\geq N\|Res|_{\underline{R}_0}\xi\|_{C^0_1[B(\underline{R}_0)\times\mathbb{S}^1]}\ \text{for some } 0<\underline{R}_0<R_0.$$

where $Res|_{\underline{R}_0}$ is the restriction of ξ onto the punched ball of radius $\underline{R}_0$.

The other condition is an upper bound on the $\mathcal{B}$-norms of a particular sequence of compactly supported sections. Namely, let χ be a cutoff function as below (31). We assume there is a unit vector $\zeta\in Eigen_{-1}P$ (which is required to be perpendicular to the Atiyah classes if the essential obstruction is non-trivial) such that

$$\text{Condition III}:\quad ||\frac{y^\delta\chi(ky)K_0(ky)\sin ks}{k}\cdot I\zeta||_{\mathcal{B}}\leq C_{\mathcal{B},k_0}\ \text{for some } \delta\geq 0$$

where $C_{\mathcal{B},k_0}$ is a constant independent of integer $k\geq k_0$ for some $k_0\geq 1$. Moreover, we define

$$\text{Condition III}^\star:\ \ y^\delta\chi(ky)K_0(ky)\sin ks\cdot I\zeta\in Range\underline{L}\ \text{ for any } k \text{ as above.}$$

The range of the extended linearized operator L contains the range of the usual $\underline{L}$. Because of the the exponential decay of the modified Bessel function of second kind $K_0(x)$ for large $x\geq 1$ (see [24] for a comprehensive theory), we expect no difficulty in checking condition III for a specific Banach space $\mathcal{B}$. Please see the proof of Corollary B below.

2.2 The vector fields

Let $(e_i,\ 1 \le i \le 6)$ be the standard basis of $\mathbb{R}^6$ and e^i be the dual basis. Near the circle $O \times \mathbb{S}^1 \subset \mathbb{C}^3 \times \mathbb{S}^1$, we consider vector fields of the following form.

$$X = X_s \frac{\partial}{\partial s} + \Sigma_{i=1}^{6} X_i e_i \text{ where the coefficients } X_s,\ X_i \text{ only depend on } r \text{ and } s. \tag{4}$$

The global vector fields are as follows.

Definition 2.3 Let $\mathfrak{X}(M, TM)$ be a Banach space of vector fields on M which is a subspace in $C^1(M\backslash\gamma)$. We say it satisfies *Condition* IV if the restriction of an arbitrary vector field in $\mathfrak{X}(M, TM)$ onto the tubular ball $B(R_0) \times \mathbb{S}^1$ of each circle defines a bounded linear map from $\mathfrak{X}(M, TM)$ to the space $\mathfrak{X}_{R_0}$ of vector fields of the form (4) (across the circle) with norm

$$||X||_{\mathfrak{X}_{R_0}} = \Sigma_{j=1}^{l} \Sigma_{i=1}^{7} \left\{ |X_i|_{C^{0,1}[B(R_0)\times\mathbb{S}^1]} + \left|\frac{\partial X_i}{\partial r}\right|_{C^0[(B(R_0)\backslash O)\times\mathbb{S}^1]} + \left|\frac{\partial X_i}{\partial s}\right|_{C^0[(B(R_0)\backslash O)\times\mathbb{S}^1]}\right.$$

where $X_7 \triangleq X_s$. We want our vector fields to be Lipschitz even across the circles in line with the existence and uniqueness of flows.

3 The extended linearized operator

3.1 The auxiliary operator

We pullback the G_2-structure via a diffeomorphism χ integrated from a vector field $X \in \mathfrak{X}(M^7, TM^7)$ (at $t = 1$). The monopole equation becomes

$$\star_{\chi^\star\phi}[F_A^0 \wedge (\chi^\star\psi)] + d_A\sigma = 0. \tag{5}$$

By Cartan formula, assuming A is a projective instanton, the linearization in the diffeomorphism at Id_M yields the *Auxiliary operator* :

$$\star_\phi(F_A^0 \wedge d[X\lrcorner\psi]) + \star_\phi[F_A^0 \wedge (X\lrcorner d\psi)]. \tag{6}$$

If A is projectively flat, it vanishes. The second term vanishes in the punched tubular balls near each circle as the G_2-structure therein is standard.

Under Definition 2.3 on the vector fields, we assume the following on the extended linearized operator.

$$\text{Condition V}: \quad L:\ \mathfrak{X}(M^7, TM^7) \oplus Y \to \mathcal{B} \text{ is bounded,}$$

where L is the linearization of (5) with respect to the connection A and the diffeomorphism χ (still with gauge fixing):

$$L \begin{vmatrix} X \\ \sigma \\ a \end{vmatrix} = \begin{vmatrix} d_A^\star a \\ d_A\sigma + \star(d_A a \wedge \psi) + \star[F_A^0 \wedge d(X\lrcorner\psi)] + \star[F_A^0 \wedge (X\lrcorner d\psi)] \end{vmatrix}. \tag{7}$$

This means L is actually the linearization of

$$\begin{cases} \star_{\chi^\star\phi}[F_{A+a}^0 \wedge (\chi^\star\psi)] + d_{A+a}\sigma = 0, \\ d_A^{\star_\phi} a = 0. \end{cases}$$

in χ, σ, a at $\chi = Id$, $\sigma = 0$, and $a = 0$. We assumed $F_A^0 \wedge \psi = 0$ such that the linearization of $\star_{\chi^\star\phi}$ in χ does not contribute. Please compare (7) with formula (3) of the usual linearization. The definitions involved in Theorem A are all established.

Remark 3.1 We only consider the linear operators (7), (9), (3), (8), not the non-linear instanton or monopole equation. Every single result/calculation is about the linearized equations, not the non-linear equations.

3.2 The model problem

We review some standard material in [25]. The model data on $(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1$ is the pullback of a non-projectively flat Hermitian Yang-Mills connection A on a bundle $E \to \mathbb{P}^2$ with rank ≥ 2 and the standard G_2-structure $(\phi_{euc}, \ \psi_{euc})$. Here we abused notation with the bundle "E" on the manifold in Definition 2.1.2. The model usual linearized operator is

$$\underline{L}_0 = I \cdot \left[\frac{\partial}{\partial s} - \underline{T}\left(\frac{\partial}{\partial r} - \frac{P}{r}\right)\right] \tag{8}$$

on the pullback of the bundle

$$Dom = adE^{\oplus(4)} \oplus \Omega^1_{sba}(adE) \to \mathbb{S}^5$$

whose rank is $8 \times rank(adE)$. Moreover,

- $\Omega^1_{sba}(adE)$ is the bundle of semi-basic adE-valued 1-forms i.e. the pullback of $\Omega^1(adE) \to \mathbb{P}^2$,
- and I, $\underline{T}$ are isometries of Dom. They anti-commute and generate a quaternion structure by $I\underline{T} = -K$.

Please see [25, Lemma 5.3]) for more.

Let $\cong$ / $\cong_{\mathbb{C}}$ mean real/complex isomorphisms between two finite dimensional vector spaces. Part of the spectral theory for the link operator P in [25, Theorem A and D] is the following diagram of isomorphisms:

$$\begin{array}{ccc} & \text{“Serre duality”} & \\ Eigen_{-1}P & \text{———————} & Eigen_{-2}P \\ \cong_{\mathbb{C}} \Big| & & \Big| \cong_{\mathbb{C}} \\ H^1[\mathbb{P}^2, EndE(-1)] & \text{————} & H^1[\mathbb{P}^2, EndE(-2)] \\ & \text{Serre duality} & \end{array}$$

The symbol "$Eigen_\mu P$" means the eigen-space of P of the eigen-value μ.

The extended linearized operator (7) becomes

$$L_0(\xi, X) = I \cdot \left[\frac{\partial}{\partial s} - \underline{T}\left(\frac{\partial}{\partial r} - \frac{P}{r}\right)\right] + \star[F_A^0 \wedge d(X \lrcorner \psi_{euc})]. \tag{9}$$

3.3 A brief remark about usual weighted Schauder spaces

Following [14] and for Corollary B, we discuss the standard weighted Schauder spaces of bundle sections.

On a punched tubular ball with radius *R* for the bundle *Dom*

Let the Hölder semi-norm be

$$[u]_{C_0^{0,\alpha}} \triangleq \sup_{x,y\in[(B(R)\setminus O)\times\mathbb{S}^1],\ O\times\mathbb{S}^1\cap\overline{xy}=\varnothing} [\min(r_x,r_y)]^{\alpha} \frac{|P_{\overrightarrow{xy}}[u(x)]-u(y)|}{d^{\alpha}(x,y)} \tag{10}$$

where

- r_x is the distance from the $\mathbb{C}^3$-component of x to the origin (see below Terminology 2.2),
- $\overline{xy}$ is the shortest line segment (geodesic) joining $x,\ y$ and realizing the distance $d(x,y)$, and
- $P_{\overrightarrow{xy}}$ is the parallel transport from x to y via the segment and the connection in the tame configuration.

Let the norm $|u|_{C_0^0}$ (and $|u|_{C^0}$ which means the same) be simply $\sup_{x\in[(B(R)\setminus O)\times\mathbb{S}^1]}|u|(x)$. It only depends on the bundle metric thus also applies to a vector field.

The $C_0^{1,\frac{1}{2}}$-norm is defined by

$$|u|_{C_0^{1,\frac{1}{2}}} = |u|_{C_0^{0,\frac{1}{2}}} + \Sigma_{D=\frac{\partial}{\partial r},\frac{\partial}{\partial s},\frac{\nabla_{\mathbb{S}^5,A}}{r}} |Du|_{C_1^{0,\frac{1}{2}}}. \tag{11}$$

When R is finite, according to the principle [10, (6.10)], the above norm treats $O\times\mathbb{S}^1$ as boundary but not the other piece $\partial[B(R)]\times\mathbb{S}^1$.

The weighted Schauder space $C_p^{k,\frac{1}{2}}$ is simply defined by the multiplication with the factor r^p:

$$|\xi|_{C_p^{k,\frac{1}{2}}} \triangleq |r^p\xi|_{C_0^{k,\frac{1}{2}}},\ k=0,\ 1.$$

For example, a section is in $C_1^{k,\frac{1}{2}}$ if and only if the multiplication by r is in $C_0^{k,\frac{1}{2}}$. This implies the norm is $O(\frac{1}{r})$ near the circle $O\times\mathbb{S}^1$.

Over a compact manifold

Under a tame configuration over a compact manifold M, we can finitely cover the whole manifold by

- tubular balls with radius $10R_0$ and
- geodesic convex balls away from the tubular balls of radius $7R_0$ centered at components of γ, such that balls of double radius are still geodesic convex and avoid the same tubular balls.

This can be achieved by taking a small enough ball (regarding R_0 and the Riemannian metric induced by the G_2-structure) at any point not in the tubular balls of radius $10R_0$ (which some of the geodesic convex balls still intersect). Therefore with the tubular balls of radius $10R_0$, an (open) cover is obtained. Then take a finite sub-cover.

The next step is to *simply use partition of unity to patch the local norms to get the global*. On the geodesics balls, the usual Schauder norm is defined as a special case in [14, Definition 4.3]. According to our choice, the cutoff functions ψ_i corresponding to each tubular ball (in the partition of unity) is $\equiv 1$ in the even smaller tubular balls of radius $5R_0$.

4 Atiyah classes

In this section we show that the essential obstruction vanishes if and only if the bundle on $\mathbb{P}^2$ is the twisted tangent bundle $T^{1,0}\mathbb{P}^2(k)$. This is completely different from Theorem A: the "if and only if" here is about vanishing of essential obstruction and the underlying bundle on $\mathbb{P}^2$, but Theorem A is about closed range and vanishing of essential obstruction.

We recall from [25] some Sasakian geometry on the standard round $\mathbb{S}^5$ (of radius 1 in $\mathbb{R}^6$). Let υ and η be the standard Reeb vector field and contact form on $\mathbb{S}^5$. There are three forms $\frac{d\eta}{2}$, H, G on $\wedge^2 D^\star$, where $D^\star \triangleq \eta^\perp$ is the contact co-distribution of rank 4. The metric contraction between a semi-basic ($D^\star$-valued) 1-form with each of the forms is a complex structure on $D^\star$ denoted by $J_0,\ J_H,\ J_G$ respectively. By metric pulling down, these complex structures also act on the contact distribution $D \triangleq \upsilon^\perp$. They form a quaternion structure on both D and $D^\star$. This structure can also be generalized to the bundle Dom for the linearized operator. The action of I on semi-basic 1-forms (including $Eigen_{-1}P$ and $Eigen_{-2}$), is J_0, and the action of $\underline{T}$ on these forms is $-J_G$. The quaternion structure is determined by

$$J_H J_0 = J_G.$$

Let $\star$ denote the Hodge star of the Euclidean metric on the model space $(\mathbb{C}^3\backslash O) \times \mathbb{S}^1$, and $\star_{D^\star}$ the one on the contact co-distribution with respect to the standard metric on $\mathbb{S}^5$.

The pullback of the projective curvature F_A^0 of the Hermitian Yang-Mills connection on $\mathbb{P}^2$ is $\star_{D^\star}$-anti self-dual i.e. invariant under the quaternion structure $J_0,\ J_H,\ J_G$. Let d_0 denote the transverse exterior differential operator $d-\eta\wedge L_\upsilon$. The square d_0^2 does not vanish in general. We call a D-valued vector semi-basic and let $\sharp_{D^\star}$ denote the metric pulling up of a semi-basic vector (field), which is a semi-basic form. Please see [25, Section 3] for a comprehensive discussion.

4.1 The map

Now we define the injection.

Proposition 4.1 *Let $(E, A) \to \mathbb{P}^2$ be a non-projectively flat Hermitian Yang-Mills bundle with rank $n \geq 2$. For any (constant) vector $Y \in \mathbb{R}^6$, the bundle valued 1-form $r(Y \lrcorner F_A^0)$ lies in $Eigen_{-1}P$. The resultant linear map*

$$\boxminus : \mathbb{R}^6 \to Eigen_{-1}P \ \ (\cong_{\mathbb{C}} H^1[\mathbb{P}^2, (End E)(-1)])$$

is a complex injection. It is an isomorphism if and only if $E = (T^{1,0}\mathbb{P}^2)(k)$.

A cohomology class in $Range\boxminus \subset H^1[\mathbb{P}^2, (End E)(-1)]$, in view of [1], is called an Atiyah class. The same term applies to an element in $Range\boxminus \subset Eigen_{-1}P$ via the complex isomorphism in [25, Theorem A and Proposition 8.2].

Notation 4.2 Denote $Range\boxminus$ by

$$\{\text{Atiyah Classes}\}|_{Eigen_{-1}P}.$$

Suppressing the subscript, this is the space on the "denominator" in (2).

We need two facts for Proposition 4.1.

Lemma 4.3 *Let F_A^0 be the curvature of the projective connection induced by a Hermitian connection over a Kähler surface. Suppose F_A^0 is (1, 1).*

- *If X is a holomorphic $(1,0)$-vector field, then $X \lrcorner F_A^0$ is $\overline{\partial}_A$-closed.*
- *If X is an antiholomorphic $(0,1)$-vector field, then $X \lrcorner F_A^0$ is ∂_A-closed.*

Consequently, in either case, $d_A(X \lrcorner F_A^0)$ is $(1,1)$.

Proof It suffices to prove it for holomorphic $(1,0)$ vector fields, it is similar for anti-holomorphic $(0,1)$ vector fields. Under a Kähler geodesic coordinate, we calculate

$$(X^i F^0_{i\bar{1}})_{\bar{2}} - (X^i F^0_{i\bar{2}})_{\bar{1}} = X^i F^0_{i\bar{1},\bar{2}} - X^i F^0_{i\bar{2},\bar{1}} = 0, \tag{12}$$

where the first equal sign holds because X is holomorphic $(1,0)$, the second is by Bianchi identity for F_A^0 and that the curvature is $(1,1)$. □

We henceforth suppress the connection in the derivatives. The other fact is the following.

Lemma 4.4 *For any constant vector $Y \in \mathbb{R}^6$, $d_0(rY \lrcorner F_A^0)$ is $(1,1)$. Consequently,*

$$[d_0(rY \lrcorner F_A^0)] \lrcorner G = [d_0(rY \lrcorner F_A^0)] \lrcorner H = 0.$$

Proof Let $Z_0,\ Z_1,\ Z_2$ be the complex coordinates of $\mathbb{C}^3$. It suffices to prove it for the complexified version for the constant holomorphic vectors

$$\frac{\partial}{\partial Z_0},\ \frac{\partial}{\partial Z_1},\ \frac{\partial}{\partial Z_2} \tag{13}$$

and anti-holomorphic vectors

$$\frac{\partial}{\partial \overline{Z}_0},\ \frac{\partial}{\partial \overline{Z}_1},\ \frac{\partial}{\partial \overline{Z}_2}. \tag{14}$$

We only do it for $\frac{\partial}{\partial Z_0}$ on the dense open set

$$V_{\mathbb{C}^3 \setminus O} = \{Z_0 \neq 0,\ Z_1 \neq 0,\ Z_2 \neq 0\} \subset \mathbb{C}^3 \setminus O. \tag{15}$$

Then it follows by continuity. The proof for the other five vectors are similar.

We calculate

$$\begin{aligned} &d_0\left(r\frac{\partial}{\partial Z_0} \lrcorner F_A^0\right) = d_0\left(\frac{r}{Z_0} Z_0 \frac{\partial}{\partial Z_0} \lrcorner F_A^0\right) \\ &\left\{d_0\left(\frac{r}{Z_0}\right) \wedge \left[\pi_\star\left(Z_0\frac{\partial}{\partial Z_0}\right)\right] \lrcorner F_A^0\right\} + \frac{r}{Z_0} \cdot d_{\mathbb{P}^2}\left(\left[\pi_\star\left(Z_0\frac{\partial}{\partial Z_0}\right)\right] \lrcorner F_A^0\right) \end{aligned} \tag{16}$$

The radius r equals 1 on $\mathbb{S}^5$. According to an identity in [25, Proof of Lemma 8.7], $d_0(\frac{r}{Z_0}) = d_0(\frac{1}{Z_0})$ is $(1,0)$. Since $[\pi_\star(Z_0\frac{\partial}{\partial Z_0})] \lrcorner F_A^0$ is $(0,1)$, the first term in (16) is $(1,1)$. So is the second term by Lemma 4.3. □

4.2 Riemann–Roch formula

The map $\boxminus$ being surjective implies rigidity of the connection.

Lemma 4.5 *Let $(E, A) \to \mathbb{P}^2$ be a non-projectively flat Hermitian Yang-Mills bundle with rank $n \geq 2$. Suppose $c_2(End E) \leq 3$. Then $n = 2$. Moreover, as a holomorphic bundle, $(E, \overline{\partial}_A)$ is isomorphic to the twisted Fubini-Study connection on $(T^{1,0}\mathbb{P}^2)(k)$ for some integer k. In particular, the equality $c_2(End E) = 3$ holds.*

Consequently, the essential obstruction

$$\frac{H^1[\mathbb{P}^2, (EndE)(-1)]}{\{Atiyah\ classes\}}$$

of a non-projectively flat Hermitian Yang-Mills bundle with rank ≥ 2 *on* $\mathbb{P}^2$ *vanishes if and only if it is isomorphic to the twisted Fubini-Study connection on a twisted tangent bundle of* $\mathbb{P}^2$.

In the above case, we recall that $c_2(EndE) = 2nc_2(E) - (n-1)c_1^2(E)$.

Because the subspace $\{Atiyah\ classes\}$ has (complex) dimension 3, the vanishing of the essential obstruction and Riemann–Roch formula (see [25, Lemma 17.10]) implies the dimension condition $h^1[\mathbb{P}^2, (EndE)(-1)] = c_2(EndE) \leq 3$. We note that the injection in Proposition 4.1 already says $h^1[\mathbb{P}^2, (EndE)(-1)] \geq 3$. It is consistent with the upshot $c_2(EndE) = 3$.

Proof of Lemma 4.5 The Hermitian Yang-Mills condition implies poly-stability i.e.

$$E = E_1 \oplus ... \oplus E_m$$

where the m components are stable bundles of the same slope. Any (holomorphic) endomorphism of E is determined by the induced homomorphism

$$E_i \to E_j \text{ for any } i,\ j = 1, ..., m.$$

Then Lemma 7.5 below yields a natural complex injection

$$H^0[\mathbb{P}^2, EndE] \to gl(m, \mathbb{C}).$$

This implies $h^0(\mathbb{P}^2, EndE) \leq m^2$.

Step 1: We show that E must be stable and rank$E = 2$ i.e. $m = 1$. Because

$$h^0[\mathbb{P}^2, (EndE)(-3)] = 0,$$

the cohomology formula (for example, see [25, Lemma 18.10]) implies

$$0 \leq h^1[\mathbb{P}^2, EndE] = 2nc_2(E) - (n-1)c_1^2(E) + (m^2 - n^2) \leq m^2 + 3 - n^2.$$

Hence

$$n^2 \leq 3 + m^2. \tag{17}$$

But

$$n = n_1 + ... + n_m$$

is the sum of the ranks of the sub-bundles. Then either

- $n_1 = ... = n_m = 1$,
- or $m = 1$.

This is because if $m \geq 2$ and there is at least one summand with rank ≥ 2, the inequality

$$n^2 \geq (m+1)^2 = m^2 + 2m + 1 > 3 + m^2$$

contradicts (17). The first bullet point condition implies E is projectively flat, which contradicts our assumption. The second says E is stable therefore rank$E = 2$ by (17).

Step 2: It suffices to show E must be a twisted tangent bundle using (the other conditions and)

$$0 \leq 4c_2(E) - c_1^2(E) \leq 3.$$

Because the Chern numbers $c_1(E)$ and $c_2(E)$ are both integers, $4c_2(E) - c_1^2(E)$ can not be 1 or 2 mod 4. This is because in mod 4 congruence classes, $4c_2(E) = 0$ and the square of an integer (including $c_1^2(E)$) does not equal 2 or 3. Hence

$$4c_2(E) - c_1^2(E) = \text{ 3 or 0}.$$

Case 1. Suppose $4c_2(E) - c_1^2(E) = 3$. Then $c_1(E)$ must be odd. Let $E(k_E)$ be the normalization of E such that $c_1[E(k_E)] = -1$, we have $c_2[E(k_E)] = 1$. Then $E(k_E)$ must be topologically isomorphic to $(T^{1,0}\mathbb{P}^2)(-2)$. Mukai [17] shows that they must also be isomorphic as holomorphic bundles.

Case 2. Suppose $4c_2(E) - c_1^2(E) = 0$. The equality in Bogomolov inequality is attained. It must be projectively flat, but E is stable with rank ≥ 2. This is a contradiction.

The above says E must be isomorphic to $(T^{1,0}\mathbb{P}^2)(k)$ as holomorphic vector bundles. □

Postscript: The reason $c_1^2(E)$ is a squared integer is that the Picard group of $\mathbb{P}^2$ is generated by $O(1)$ and the Chern number $c_1^2[O(1)]$ is equal to 1 i.e.

$$\int_{\mathbb{P}^2} c_1[O(1)] \wedge c_1[O(1)] = 1 \text{ as Chern number.}$$

This implies $c_1^2(E) = (degE)^2$ where $detE = O(degE)$.

4.3 Proof of Proposition 4.1

In conjunction with the review of the standard material in [25] that we need here (see the beginning of Section 4 and 3.2), let $\star_{\mathbb{C}^3}$ denote the hodge star on $\mathbb{C}^3$ (under standard Euclidean metric). Let L denote the Lie derivative with respect a vector field.

Step 1: $r(Y \lrcorner F_A^0)$ is d_0-co-closed.

We first show it is $d_{\mathbb{C}^3}$ co-closed. Similarly to Lemma 4.4, we show the complexified version for the holomorphic and anti-holomorphic vector fields (13), (14). This is straightforward because the pullback F_A^0 is (1, 1) on $\mathbb{C}^3 \setminus O$, and the projective Hermitian Yang-Mills condition $F_A^0 \lrcorner \frac{d\eta}{2} = 0$ on $\mathbb{P}^2$ implies $F_A^0 \lrcorner \omega_{\mathbb{C}^3} = 0$ as pullback. By Bianchi identity, for any $i = 0, 1, 2$, $F^0_{A,i\bar{j},j} = 0$ on $\mathbb{C}^3$. Hence there is a constant c_0 such that

$$d_{\mathbb{C}^3}^{\star_{\mathbb{C}^3}}\left(r\frac{\partial}{\partial Z_i} \lrcorner F_A^0\right) = c_0\left(rF^0_{A,i\bar{j}}\right)_j = c_0 r_j F^0_{A,i\bar{j}} + c_0\left(rF^0_{A,i\bar{j},j}\right) = c_0 F_A^0\left(\frac{\partial}{\partial Z_0}, \frac{\partial}{\partial r}\right) = 0. \tag{18}$$

Because $d_{\mathbb{C}^3}^{\star_{\mathbb{C}^3}} = \frac{d_0^{\star_D \star}}{r^2}$ on semi-basic 1-forms pullback from $\mathbb{S}^5$, we find

$$d_0^{\star_D \star}\left(r\frac{\partial}{\partial Z_i} \lrcorner F_A^0\right) = 0.$$

Similarly proof yields the following for any $i = 0,\ 1,\ 2$.

$$d_0^{\star_D \star}\left(r\frac{\partial}{\partial \overline{Z}_i} \lrcorner F_A^0\right) = 0.$$

Hence for any constant vector $Y \in \mathbb{R}^6$, we have

$$d_0^{\star D\star}(rY \lrcorner F_A^0) = 0.$$

Step 2: $r(Y \lrcorner F_A^0)$ is an eigen-section of P of eigenvalue -1.

It is semi-basic. Because $F_{A_0}^0$ is $(1,1)$ on $\mathbb{C}^3 \backslash O$ and $\mathbb{R}^6$, the $J_{\mathbb{C}^3}$ and J_0 invariance of $F_{A_0}^0$ tells us that

$$[J_{\mathbb{C}^3}(Y)] \lrcorner F_{A_0}^0 = J_{\mathbb{C}^3}(Y \lrcorner F_{A_0}^0) = J_0(Y \lrcorner F_{A_0}^0) \text{ for any } Y \in \mathbb{C}^3. \tag{19}$$

We used that on semi-basic vectors and forms, $J_{\mathbb{C}^3}$ coincides with J_0 (see [25, Appendix]). Consequently,

$$d_0^{\star D\star} J_0(rY \lrcorner F_A^0) = d_0^{\star D\star}[r J_{\mathbb{C}^3}(Y) \lrcorner F_A^0] = 0$$

for any $Y \in \mathbb{R}^6$ as well. The Lie derivative in the Reeb vector field is

$$L_v(rY \lrcorner F_A^0) = rL_v(Y \lrcorner F_A^0) = r(J_{\mathbb{C}^3}Y) \lrcorner F_A^0 = rJ_{\mathbb{C}^3}(Y \lrcorner F_A^0) = J_0(rY \lrcorner F_A^0). \tag{20}$$

Apply J_0 to both hand sides and using that $L_v J_0 = J_0 L_v$, we find

$$L_v[J_0(rY \lrcorner F_A^0)] = -rY \lrcorner F_A^0. \tag{21}$$

Via the formula for P in [25, Lemma 5.3], the above means $rY \lrcorner F_A^0$ is an eigen-section of P of eigenvalue -1. It defines the map

$$\boxminus : \mathbb{R}^6 \to Eigen_{-1}P \quad \text{via} \quad \boxminus(Y) \triangleq rY \lrcorner F_A^0.$$

Step 3. $\boxminus$ is injective.

Let p be a point on $\mathbb{S}^5$ at which $v = (rY)^{\|_D}$ is nonzero (see Fact 7.1 below). Then we normalize it via $e_1 = \frac{v}{|v|}$, and complete it into an orthonormal frame (e_1, e_2, e_3, e_4) for the contact distribution D at p. That F_A^0 is anti self-dual means

$$F_A^0 = F_{A,I}^0(e^{12} - e^{34}) + F_{A,II}^0(e^{13} - e^{42}) + F_{A,III}^0(e^{14} - e^{23}). \tag{22}$$

The condition $e_1 \lrcorner F_A^0 = 0$ at p implies that $0 = F_{A,I}^0 e^2 + F_{A,II}^0 e^3 + F_{A,III}^0 e^4$. This in turn implies $F_{A,I}^0 = F_{A,II}^0 = F_{A,III}^0 = 0$ i.e. $F_A^0 = 0$ at p. Because v is non-zero on a dense open set on $\mathbb{S}^5$, $F_A^0 = 0$ on the same set. By continuity of F_A^0, it vanishes everywhere on $\mathbb{S}^5$.

When E is a twisted tangent bundle of $\mathbb{P}^2$, by Lemma 4.5, the injection $\boxminus$ is an isomorphism since the dimension of the domain equals the dimension of the range. The proof is complete.

5 Formula of the auxiliary operator

The Atiyah classes originally defined in $Eigen_{-1}P$ can also be defined in $Eigen_{-2}P$ via the isometry $\underline{T}$ i.e.

$$\{\text{Atiyah classes}\}|_{Eigen_{-2}P} \triangleq \underline{T}[\{\text{Atiyah classes}\}|_{Eigen_{-1}P}].$$

The desired formula involves both.

Proposition 5.1 *Let $X \in C^1[(B(R)\backslash O) \times \mathbb{S}^1]$ be a vector field of the form (4), $0 < R \leq \infty$. The following holds therein.*

$$Aux(X) = -\Sigma_{i=1}^6 \left\{ \frac{\partial X_i}{\partial s} \cdot [(J_{\mathbb{C}^3} e_i) \lrcorner F_A^0] + \frac{\partial X_i}{\partial r} \cdot J_H(e_i \lrcorner F_A^0) \right\}. \tag{23}$$

Consequently, $RangeAux$ lies in the span (by continuous functions only of r, s on the same tubular ball) of Atiyah classes in $Eigen_{-1}P$ and $Eigen_{-2}P$.

Proof of Lemma 5.1: It suffices to apply Lemma 7.3 and calculate the Lie derivatives therein. The condition that the co-efficients of X only depend on r, s is also used in Step 3 of the proof of the preliminary formula (49) below.

Because the $\frac{\partial}{\partial s}$-component does not contribute to the operator at all (see Fact 7.4 below), we can assume X is perpendicular to $\frac{\partial}{\partial s}$. The Lie derivatives in the Reeb-vector field υ, radial vector field $\frac{\partial}{\partial r}$, and $\frac{\partial}{\partial s}$ of the symmetric bi-linear forms ds^2, dr^2, and $g_{\mathbb{S}^5}$ all vanish. We note that $\frac{\partial}{\partial r}$ is not Killing for the Euclidean metric $ds^2 + dr^2 + r^2 g_{\mathbb{S}^5}$. We compute via the Lie derivative formulas in Lemma 7.2 and elementary Riemannian geometry that

$$L_{\frac{\partial}{\partial r}} X = \Sigma_{i=1}^{6}\left[\frac{\partial X_i}{\partial r} e_i - X_i \frac{e_i^{\perp \frac{\partial}{\partial r}}}{r}\right],\ L_{\upsilon} X = -\Sigma_{i=1}^{6} X_i J_{\mathbb{C}^3}(e_i),\ L_{\frac{\partial}{\partial s}} X = \Sigma_{i=1}^{6} \frac{\partial X_i}{\partial s} e_i. \tag{24}$$

Then the 3 Lie-derivative contractions in (49) can be calculated as follows.

$$(L_{\frac{\partial}{\partial s}} X) \lrcorner \frac{d\eta}{2} = \Sigma_{i=1}^{6} \frac{\partial X_i}{\partial s}\left(e_i \lrcorner \frac{d\eta}{2}\right),$$
$$(L_{\frac{\partial}{\partial r}} X) \lrcorner H = \Sigma_{i=1}^{6}\left(\frac{\partial X_i}{\partial r} - \frac{X_i}{r}\right)(e_i \lrcorner H),$$
$$\frac{1}{r}(L_{\upsilon} X) \lrcorner G = -\Sigma_{i=1}^{6} \frac{X_i}{r}\left[J_{\mathbb{C}^3}(e_i) \lrcorner G\right] = \Sigma_{i=1}^{6} \frac{X_i}{r}(e_i \lrcorner H).$$

Summing the above 3 equalities and combining coefficients of similar terms, the two terms containing $e_i \lrcorner H$ cancel out and we find

$$\left(\frac{\partial X}{\partial s}\right) \lrcorner \frac{d\eta}{2} + (L_{\frac{\partial}{\partial r}} X) \lrcorner H + \frac{(L_{\upsilon} X) \lrcorner G}{r} = \Sigma_{i=1}^{6}\left\{\frac{\partial X_i}{\partial s} \cdot \left(e_i \lrcorner \frac{d\eta}{2}\right) + \frac{\partial X_i}{\partial r} \cdot (e_i \lrcorner H)\right\}.$$

Here we applied again the remark below (19) about the relation between $J_{\mathbb{C}^3}$ and J_0. Using (49) and contracting the above with $-F_A^0$, the proof of the desired formula (23) is complete. □

6 The Dirac system and proof of Theorem A and Corollary B

In this section we assemble the established tools to prove the main results. Via separation of variables, the singular sequence is constructed via a linear system of two partial differential equations in r and s.

6.1 The Dirac system

Let (ξ, X) be the independent variable of the model extended linearized operator L_0, where X is the vector field and ξ is the section of the domain bundle $adE \oplus \Omega^1_{adE}$. Because $RangeAux$ *is spanned by functions in r and s of Atiyah classes* in both $Eigen_{-1}P$ and $Eigen_{-2}P$, in the perpendicular direction, the extended linearized operator coincides with the usual linearized operator in the following sense. In the Hilbert space $L^2[\mathbb{S}^5, Dom]$, let $\|_{Atiyah}$ denote the projection to the 12 dimensional subspace

$$\{\text{Atiyah classes}\}|_{Eigen_{-1}P} \oplus \{\text{Atiyah classes}\}|_{Eigen_{-2}P},$$

and $\perp_{Atiyah}$ the projection to the orthogonal complement. We have

$$[Aux(X)]^{\|_{Atiyah}} = Aux(X)$$

for any differentiable vector field X in the punched tubular ball.

For any $\xi \in C^1[(B(R_0)\backslash O) \times \mathbb{S}^1]$,

$$[L_0(\xi, X)]^{\perp_{Atiyah}} = (\underline{L}_0\xi)^{\perp_{Atiyah}} = \underline{L}_0(\xi^{\perp_{Atiyah}}). \tag{25}$$

But Aux does appear in the Atiyah class component:

$$[L_0(\xi, X)]^{\|_{Atiyah}} = (\underline{L}_0\xi)^{\|_{Atiyah}} + Aux(X) = \underline{L}_0(\xi)^{\|_{Atiyah}} + Aux(X). \tag{26}$$

Gram-Schmit process for each eigen-space of P yields a complete orthonormal P-eigen-basis $(\phi_\beta,\ \beta \in Spec^{mul} P)$ for $L^2[\mathbb{S}^5, Dom]$ such that

- the eigen-section ζ perpendicular to the Atiyah classes in condition III appears as an element in the basis if essential obstruction is non-trivial,
- 6 elements of the eigen-basis form an I-invariant orthonormal basis for $\{\text{Atiyah classes}\}|_{Eigen_{-1}P}$, and applying $\underline{T}$ yields that of $\{\text{Atiyah classes}\}|_{Eigen_{-2}P}$.

Via separation of variables, we need to solve equations for the Fourier-coefficient of an arbitrary section ϕ_β in the eigen-basis. However, because of the endomorphism $\underline{T}$ in the Dirac operator (8) (see [25, Lemma 5.3]), we need to consider ϕ_β and $\underline{T}\phi_\beta$ simultaneously. Particularly, in line with (25) and that ζ is perpendicular to the Atiyah classes, the operator $-I \cdot \underline{L}_0$ also preserves the span by functions in $r,\ s$ of ζ and $\underline{T}\zeta$:

$$(-I \cdot \underline{L}_0\xi)^{\|_{span\{\zeta,\ \underline{T}\zeta\}}} = (-I \cdot \underline{L}_0)(\xi)^{\|_{span\{\zeta,\ \underline{T}\zeta\}}} \text{ for any } \xi \in C^1[(B(R_0)\backslash O) \times \mathbb{S}^1]. \tag{27}$$

The equation in $span\{\zeta,\ \underline{T}\zeta\}$ of two unknowns x and y reads

$$-I \cdot L(x\zeta + y\underline{T}\zeta) = f\zeta + g\underline{T}\zeta.$$

According to formula (8) for the usual linearized operator, this is equivalent to the *Dirac system* of two variables:

$$\begin{cases} \dfrac{\partial x}{\partial s} + \dfrac{\partial y}{\partial r} + \dfrac{2y}{r} = f; \\ \dfrac{\partial y}{\partial s} - \dfrac{\partial x}{\partial r} - \dfrac{x}{r} = g. \end{cases} \tag{28}$$

Plugging

$$\frac{\partial y}{\partial s} = \frac{\partial x}{\partial r} + \frac{x}{r} + g. \tag{29}$$

into $\frac{\partial}{\partial s}$ of the first equation, we find a second order equation only in x.

$$\frac{\partial^2 x}{\partial r^2} + \frac{\partial^2 x}{\partial s^2} + \frac{3}{r}\frac{\partial x}{\partial r} + \frac{x}{r^2} = \frac{\partial f}{\partial s} - \frac{\partial g}{\partial r} - \frac{2g}{r} \triangleq h.$$

The equation of the Fourier co-efficient of $\cos ks$ and $\sin ks$ reads

$$x_k'' + \frac{3x_k'}{r} + \frac{x_k}{r^2} - k^2 x_k = h_k. \tag{30}$$

This ordinary differential equation can be solved elementarily.

6.2 The singular sequence

Now we construct a sequence that violates the closeness of the range. We only consider positive independent variable for the special functions. Let k be a positive integer and

$$h_k(y) \triangleq y^{\delta}\chi(ky)K_0(ky), \tag{31}$$

where $\chi(r)$ is a cut-off function that is $\equiv 0$ when $r \leq 1$ or $r \geq 4$, but $\equiv 1$ when $r \in [2, 3]$.

In the following, h_k and x_k are specific as (31) and (32), but in the previous section they are general.

Lemma 6.1 *For any non-negative number δ there is a positive constant C_δ with the following property. Let h_k be as (31). The only solution to (30) that $= O(1)$ as $r \to 0$ is*

$$x_k = -\frac{K_0(kr)}{r}\int_0^r I_0(ky)y^2h_k(y)dy + \frac{I_0(kr)}{r}\int_0^r K_0(ky)y^2h_k(y)dy. \tag{32}$$

The following holds for any positive integer k and real number r such that $kr \geq 10$.

$$|x_k| \geq \frac{C_\delta \cdot e^{kr}}{k^{\frac{7}{2}+\delta}r^{\frac{3}{2}}}. \tag{33}$$

Consequently, $\lim_{k\to\infty}|x_k| = +\infty$ *uniformly on any compact subset of* $(0, \infty)$.

Remark 6.2 The solution x_k is supported away from 0 since h_k is. The constant C_δ is given by integral and point-wise bounds on the special functions.

Proof The trick is to consider kr instead of r alone. The general solution to the ODE is

$$-\frac{K_0(kr)}{r}\int_0^r I_0(ky)y^2h_k(y)dy + \frac{I_0(kr)}{r}\int_0^r K_0(ky)y^2h_k(y)dy + \frac{aK_0(kr)}{r} + \frac{bI_0(ky)}{r}. \tag{34}$$

The main part x_k is compactly supported away from 0, but the homogeneous solutions $\frac{K_0(kr)}{r}$ and $\frac{I_0(kr)}{r}$ have leading terms $\frac{C\log r}{r}$ and $\frac{C}{r}$ for nonzero constant $C's$, respectively. Since we require x to be $O(1)$, these two homogeneous solutions can not appear i.e. $a,\ b$ must be 0.

In order to bound the first term in (32), we estimate the integral for any r:

$$|\int_0^r I_0(ky)y^2h_k(y)dy| \leq \frac{1}{k^{3+\delta}}\int_0^\infty \chi(w)I_0(w)|K_0(w)|w^{2+\delta}dw \leq \frac{\underline{C}_{2,\delta}}{k^{3+\delta}}, \tag{35}$$

where $\underline{C}_{2,\delta}$ is the value of the integral

$$\int_1^4 I_0(w)|K_0(w)|w^{2+\delta}dw.$$

Please notice that χ is supported in the interval. Then if $kr \geq 1$, using the bound on $\frac{K_0(x)}{x}$ when $x \geq 1$, we find

$$|-\frac{K_0(kr)}{r}\int_0^r I_0(ky)h_k(y)y^2dy| \leq \frac{\underline{C}_{2,\delta}}{k^{2+\delta}} \cdot \frac{|K_0(kr)|}{kr} \leq \frac{\underline{C}_{3,\delta}}{k^{2+\delta}}. \tag{36}$$

To bound the second term in (32) from below, we compute

$$\frac{I_0(kr)}{r}\int_0^r K_0(ky)y^2h_k(y)dy = \frac{I_0(kr)}{k^{3+\delta}r}\int_0^{kr}\chi(w)K_0^2(w)w^{2+\delta}dw \geq \frac{\underline{C}_{4,\delta}\cdot I_0(kr)}{k^{3+\delta}r}$$
$$\geq \frac{\underline{C}_{4,\delta}\cdot \underline{C}_5 e^{kr}}{k^{3+\delta}r\cdot\sqrt{kr}}.$$

where the constant $\underline{C}_{4,\delta}$ equals $\int_1^4 K_0^2(w)w^{2+\delta}dw$, and $\underline{C}_5$ equals the positive lower bound on $e^{-w}\sqrt{w}I_0(w)$ for $w \geq 1$. Let C_δ be large enough regarding these two constants and $\underline{C}_{3,\delta}$, the proof of (33) is complete. □

6.3 Proof of Theorem A and Corollary B

In functional analysis, closed range is equivalent to existence of "a priori estimate" in the following sense.

Fact 6.3 Suppose $L : X \to Y$ is a bounded linear map between Banach spaces. Then $RangeL$ is closed if and only if there is a non-negative constant N such that for any $y \in RangeL$, there exists a solution x to the equation $Lx = y$ with the bound

$$||x||_X \leq N||y||_Y. \tag{37}$$

Proof of Theorem A: *The idea is to construct a singular sequence violating closed range whenever the essential obstruction does not vanish.* By Lemma 4.5, this happens if the connection is not isomorphic to the twisted Fubini-Study on a twisted tangent bundle of $\mathbb{P}^2$. We only show it for the extended linearized operator using conditions II–V and III*. Similar argument applies to the usual $\underline{L}$ under conditions I–III and III*.

Definition 2.1 of the configuration says that we are in the model setting in the tubular ball. Given a large enough positive integer k, we specify the single variable function h_k in y (the radius) as in (31) and let $f_k = \frac{h_k}{k}$. Again, let ζ be the eigen-section in condition III.

Because the auxiliary operator does not cover ζ which is perpendicular to all the Atiyah classes in $Eigen_{-1}P$, and that $-I\cdot L_0$ commutes with the projection to $span\{\zeta,\ \underline{T}\zeta\}$ ($-I\cdot L_0 = -I\cdot \underline{L}_0$ thereon, see (25) and (27)), the $\zeta\cos ks$-component of any solution $\xi_k = O(\frac{1}{r})$ to

$$L_0\xi_k = (f_k \sin ks)I\zeta \tag{38}$$

must be $O(1)$ thus equals the x_k in (32). To see this, in view of the argument from (27) to (30) on Dirac system, we simply project both sides of (38) onto $span\{\zeta,\ \underline{T}\zeta\}$ according to (27), then take the Fourier co-efficient of $\cos ks$ and apply Lemma 6.1. Therefore the L^2-norm of ξ_k on the stripe defined by $\underline{R}_0/10 < r < \underline{R}_0/5$ tends to ∞ as $k\to\infty$. As the C_1^0-norm on the punched ball of radius $\underline{R}_0$ is stronger than this L^2-norm, condition II implies

$$|\xi_k|_Y \to \infty \text{ as } k\to\infty.$$

Condition III* says that $(f_k \sin ks)I\zeta$ is in $RangeL$ and III says their $\mathcal{B}$-norm are uniformly bounded. According to the characterization of closed range in Fact 6.3, $RangeL$ is not closed. □

Under a tame configuration over a compact 7-fold, let

$$C^2_{r,s}[M, TM] \subset C^2[M, TM] \tag{39}$$

be the subspace of C^2 vector fields that restrict to the form (4) in the punched tubular ball i.e. only depending on r and s in $B(R_0) \times \mathbb{S}^1$ near each circle. Between weighted Schauder spaces as in Section 3.3, consider the usual linearized operator

$$\underline{L}: C^{1,\frac{1}{2}}_{1-\delta}[M^7\backslash\gamma, \Omega^0_{adE} \oplus \Omega^1_{adE}] \to C^{0,\frac{1}{2}}_{2-\delta}[M^7\backslash\gamma, \Omega^0_{adE} \oplus \Omega^1_{adE}] \quad (40)$$

and the extended linearized operator

$$L: C^{1,\frac{1}{2}}_{1}[M\backslash\gamma, \Omega^0_{adE} \oplus \Omega^1_{adE}] \oplus C^2_{r,s}[M, TM] \to C^{0,\frac{1}{2}}_{2}[M^7\backslash\gamma, \Omega^0_{adE} \oplus \Omega^1_{adE}]. \quad (41)$$

Both are bounded. The reason we let $\delta = 0$ for the extended linearization is that we do not know whether Aux has a better bound than $C^{0,\frac{1}{2}}_2$, due to the quadratic growth of the norm of the curvature near the circles.

Proof of Corollary B: It is straight-forward to verify conditions I–V. Was L surjective, condition III* holds as well i.e. the configuration is ideal. Then Theorem A says $RangeL$ is not closed, which is a contradiction. Similar argument applies to the usual $\underline{L}$.

For the reader's convenience, we still provide the detail in checking the conditions.

- Condition I (saying $\underline{L}$ is bounded) holds by formula (3), our choice (41), and definition of the weighted Schauder spaces. The weight for the first derivatives has 1 more power than that for the section itself.
- Condition II (coerciveness) holds because restricted to the tubular ball, the norm C^0_1 is weaker than $C^{1,\frac{1}{2}}_1$ ($C^{1,\frac{1}{2}}_{1-\delta}$).
- Condition III (bound on the particular sequence) follows simply from the decay of the modified Bessel function K_0 and that χ is non-negative, supported in $(1, 4)$, and bounded by 1. Namely, the following holds for large positive integer k.

$$\sup_{r\in(0,\infty)} r^{2-\delta}\left|\frac{r^\delta\chi(kr)K_0(kr)I\zeta\cdot\sin ks}{k}\right| = \sup_{r\in(0,\infty)}\left|\frac{(kr)\chi(kr)K_0(kr)}{k^2}\right|\cdot|rI\zeta| \le C.$$

The $r^{3-\delta}$-weighted bounds on the $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial s}$, and $\frac{\nabla_{\mathbb{S}^5}}{r}$ of $\frac{r^\delta\chi(kr)K_0(kr)I\zeta\cdot\sin ks}{k}$ follow similarly. Then

$$\left|\frac{r^\delta\chi(kr)K_0(kr)I\zeta\cdot\sin ks}{k}\right|_{C^1_{2-\delta}} \le C. \quad (42)$$

This implies the $C^{0,\frac{1}{2}}_{2-\delta}$-bound of the same thing by interpolation of weighted Schauder norms.
- Condition III* is simply the contradiction hypothesis that the linearization is surjective. Condition IV holds automatically because of our vector fields (39).
- Condition V (saying L is bounded) holds by formula (7), our choice (41), and the simple weighted Hölder bound on the auxiliary operator:

$$|\star(F^0_A \wedge d[X\lrcorner\psi]) + \star(F^0_A \wedge [X\lrcorner d\psi])|_{C^{0,\frac{1}{2}}_2} \le |X|_{C^2}.$$

Because it involves first partial derivatives of X, we need X to be C^2.

□

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7 Appendix

7.1 Non-vanishing of a certain projection of coordinate vector fields on $\mathbb{R}^6$

It is routine to check the following "non-vanishing" that applies to the proof of Proposition 4.1. Let (Z_0, Z_1, Z_2) be the coordinates for $\mathbb{C}^3$ and υ be the standard Reeb vector field on $\mathbb{S}^5$.

Fact 7.1 Let $Y \in \mathbb{R}^6 \backslash O$ be a (constant) nonzero vector. There exists a dense open set on $\mathbb{S}^5$ on which $(rY)^{\|_D}$ is non-zero everywhere.

Proof of Fact 7.1: We write $Y = Y^{1,0} + Y^{0,1}$, where

$$Y^{1,0} = a_Y \frac{\partial}{\partial Z_0} + b_Y \frac{\partial}{\partial Z_1} + c_Y \frac{\partial}{\partial Z_2}, \text{ and } Y^{0,1} = \overline{Y^{1,0}}$$

for some complex constants $a_Y,\ b_Y,\ c_Y$. Under the Sasakian coordinate in $U_{0,\mathbb{S}^5} \subset \mathbb{S}^5$ defined by $Z_0 \neq 0$ [25, (15)], using formula (46) and

$$Z_0 \frac{\partial}{\partial Z_1} = \frac{Z_0 \bar{Z}_1}{2r} \frac{\partial}{\partial r} + \frac{\partial}{\partial u_1};\ Z_0 \frac{\partial}{\partial Z_2} = \frac{Z_0 \bar{Z}_2}{2r} \frac{\partial}{\partial r} + \frac{\partial}{\partial u_2}$$

for $(1, 0)$ coordinate vectors in $\mathbb{C}^3$, we calculate the projection onto the contact distribution over $\mathbb{S}^5$:

$$\begin{aligned}(rY^{1,0})^{\|_D} &= \frac{r}{Z_0}\left[(b_Y - a_Y u_1)\frac{\partial}{\partial u_1} + (c_Y - a_Y u_2)\frac{\partial}{\partial u_2}\right]^{\|_D} \\ &= \frac{r}{Z_0}\left[(b_Y - a_Y u_1)\Big(\frac{\partial}{\partial u_1} - \eta\Big(\frac{\partial}{\partial u_1}\Big)\upsilon\Big) + (c_Y - a_Y u_2)\left(\frac{\partial}{\partial u_2} - \eta\left(\frac{\partial}{\partial u_2}\right)\upsilon\right)\right].\end{aligned}$$

When $(b_Y - a_Y u_1) \neq 0$ or $(c_Y - a_Y u_2) \neq 0$, we have $(rY^{1,0})^{\|_D} \neq 0$. These two non-vanishing conditions together with $Z_0 \neq 0$ define a dense open set on $\mathbb{S}^5$. □

7.2 The Lie derivatives of the vector fields on $\mathbb{C}^3\backslash O$

Proposition 5.1 applies the following formulas of the Lie derivatives.

Lemma 7.2 *1. Let υ be the standard Reeb vector field on $\mathbb{S}^5$. Then*

$$L_\upsilon\left(Z_i \frac{\partial}{\partial Z_i}\right) = 0,\ \ L_\upsilon\left(\bar{Z}_i \frac{\partial}{\partial \bar{Z}_i}\right) = 0. \tag{43}$$

Consequently, on $\mathbb{R}^6$ and its complexfication, $L_\upsilon = -J_{\mathbb{C}^3}$ is equal to the negative of the standard complex structure i.e.

$$L_\upsilon \frac{\partial}{\partial Z_i} = -\sqrt{-1}\frac{\partial}{\partial Z_i},\ L_\upsilon \frac{\partial}{\partial \bar{Z}_i} = \sqrt{-1}\frac{\partial}{\partial \bar{Z}_i}. \tag{44}$$

Particularly, for any vector $Y \in \mathbb{R}^6$, $L_\upsilon = -J_{\mathbb{C}^3} Y$.

2. $L_{\frac{\partial}{\partial r}} \frac{\partial}{\partial Z_i} = -\frac{(\frac{\partial}{\partial Z_i})^{\perp \frac{\partial}{\partial r}}}{r}$. The complex conjugate $L_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \bar{Z}_i} = -\frac{(\frac{\partial}{\partial \bar{Z}_i})^{\perp \frac{\partial}{\partial r}}}{r}$ also holds. This means $L_{\frac{\partial}{\partial r}}$ is $-\frac{1}{r}$ times the projection to the orthogonal complement of $\frac{\partial}{\partial r}$. Particularly, for any vector $Y \in \mathbb{R}^6$, $L_{\frac{\partial}{\partial r}} Y = -\frac{1}{r} \cdot Y^{\perp \frac{\partial}{\partial r}}$.

Proof We prove them in the Zariski open set $V_{\mathbb{C}^3\setminus O}$ (15). Then the global equations follow by continuity.

On item 1, recall again [25, (15)] about the Sasakian coordinate system and the local Kähler potentials ϕ_i of the Fubini-Study metric $\frac{d\eta}{2}$ such that

$$Z_i = \frac{r}{\sqrt{\phi_i}} e^{i\theta_i}. \tag{45}$$

In the $i-th$ Sasakian coordinate chart, the Reeb vector field υ equals $\frac{\partial}{\partial\theta_i}$ ([25, Fact 3.4]). On the Euler sequence, $Z_i\frac{\partial}{\partial Z_i}$ are the scaling invariant holomorphic vector fields on $\mathbb{C}^3\setminus O$ whose projections to $\mathbb{P}^2$ span the holomorphic tangent bundle point-wisely. We directly verify the following identities.

$$Z_0\frac{\partial}{\partial Z_0} = \frac{r}{2\phi_0}\frac{\partial}{\partial r} - \frac{\sqrt{-1}}{2}\frac{\partial}{\partial\theta_0} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} \text{ in } U_{0,\mathbb{C}^3}; \tag{46}$$

$$Z_1\frac{\partial}{\partial Z_1} = \frac{r}{2\phi_1}\frac{\partial}{\partial r} - \frac{\sqrt{-1}}{2}\frac{\partial}{\partial\theta_1} - v_0\frac{\partial}{\partial v_0} - v_2\frac{\partial}{\partial v_2} \text{ in } U_{1,\mathbb{C}^3}; \tag{47}$$

$$Z_2\frac{\partial}{\partial Z_2} = \frac{r}{2\phi_2}\frac{\partial}{\partial r} - \frac{\sqrt{-1}}{2}\frac{\partial}{\partial\theta_2} - w_0\frac{\partial}{\partial w_0} - w_1\frac{\partial}{\partial w_1} \text{ in } U_{2,\mathbb{C}^3}. \tag{48}$$

Using the above identities, the desired (43) follows because each term in (46)–(48) has vanishing Lie bracket with the Reeb vector field. By (45) and the characterization of υ above, we simply obtain

$$L_\upsilon Z_i = \sqrt{-1}Z_i$$

from which (44) follows.

We now prove item 2.

$$\begin{aligned} L_{\frac{\partial}{\partial r}}\frac{\partial}{\partial Z_i} &= \left[\frac{\partial}{\partial r}, \frac{1}{Z_i}\left(Z_i\frac{\partial}{\partial Z_i}\right)\right] = \left[\frac{\partial}{\partial r}\left(\frac{1}{Z_i}\right)\right]\cdot\left(Z_i\frac{\partial}{\partial Z_i}\right) + \frac{1}{Z_i}\left[\frac{\partial}{\partial r}, Z_i\frac{\partial}{\partial Z_i}\right] \\ &= -\frac{1}{rZ_i}\left(Z_i\frac{\partial}{\partial Z_i}\right) + \frac{1}{2\phi_i Z_i}\frac{\partial}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial Z_i} + \frac{1}{2\phi_i Z_i}\frac{\partial}{\partial r} \\ &= -\frac{\left(\frac{\partial}{\partial Z_i}\right)^{\perp\frac{\partial}{\partial r}}}{r}, \end{aligned}$$

where we used that the orthogonal projection of $\frac{\partial}{\partial Z_i}$ to $\frac{\partial}{\partial r}$ is $\frac{r}{2\phi_i Z_i}\frac{\partial}{\partial r}$. □

7.3 On the Auxiliary operator

We provide the routine tensor calculation for Proposition 5.1.

Lemma 7.3 *Under the conditions in Proposition 5.1 and the splitting of tangent bundle*

$$T[(\mathbb{C}^3\setminus O)\times\mathbb{S}^1] = Span\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}, \upsilon\right)\oplus D$$

where D is the contact distribution on $\mathbb{S}^5$ and υ is the Reeb vector field, we write the vector field (whose co-efficients under the standard Euclidean basis only depend on r, s, see our *assumption* (4) *as*

$$X = X_s \frac{\partial}{\partial s} + X_r \frac{\partial}{\partial r} + X_\upsilon \upsilon + X_0$$

such that X_0 *is* D*-valued. Then the auxiliary operator is*

$$\begin{aligned}\star(F_A^0 \wedge d[X \lrcorner \psi]) &= \star_{D^\star}\left\{\left[(L_{\frac{\partial}{\partial s}} X_0) \lrcorner \frac{d\eta}{2} + (L_{\frac{\partial}{\partial r}} X_0) \lrcorner H + \frac{(L_\upsilon X_0) \lrcorner G}{r}\right] \wedge F_A^0\right\} \\ &= \star_{D^\star}\left\{\left[(L_{\frac{\partial}{\partial s}} X) \lrcorner \frac{d\eta}{2} + (L_{\frac{\partial}{\partial r}} X) \lrcorner H + \frac{(L_\upsilon X) \lrcorner G}{r}\right] \wedge F_A^0\right\} \\ &= -\left[(L_{\frac{\partial}{\partial s}} X) \lrcorner \frac{d\eta}{2} + (L_{\frac{\partial}{\partial r}} X) \lrcorner H + \frac{(L_\upsilon X) \lrcorner G}{r}\right] \lrcorner F_A^0. \end{aligned} \tag{49}$$

Strategy: it is completely routine. We simply calculate

1. the exterior derivative of $X \lrcorner \psi_{euc}$, then
2. wedge it with the curvature then apply $\star$.

The idea for the first step is to separate $d(X_0 \lrcorner \psi)$ into two parts, such that the first part contains $ds \wedge dr \wedge \eta$ as a byte, but the other does not. Then carrying out the second step, the first part yields the first line on the right side of (49), the other part yields the supplementary term $Q(X_0)$ (see (55) below) which has vanishing wedge with the curvature.

Proof of Lemma 7.3: The standard co-associative form on $\mathbb{C}^3 \times \mathbb{S}^1$ is

$$\begin{aligned}\psi_{euc} &= \frac{\omega_{euc}^2}{2} + Im\Omega_{euc} \wedge ds \\ &= r^3 dr \wedge \eta \wedge \frac{d\eta}{2} + \frac{r^4}{2}\left(\frac{d\eta}{2}\right)^2 - r^3 ds \wedge \eta \wedge H + r^2 ds \wedge dr \wedge G. \end{aligned} \tag{50}$$

□

Step 1: The semi-basic component of the vector field

Let X_0 be a semi-basic vector field (contact distribution D-valued) on $\mathbb{C}^3 \setminus O$, we compute

$$\begin{aligned}X_0 \lrcorner \psi_{euc} = r^3 dr \wedge \eta \wedge (X_0 \lrcorner \frac{d\eta}{2}) - r^3 ds \wedge \eta \wedge (X_0 \lrcorner H) \\ + r^2 ds \wedge dr \wedge (X_0 \lrcorner G) + \frac{r^4}{2} X_0 \lrcorner (\frac{d\eta}{2})^2. \end{aligned} \tag{51}$$

We successively calculate the exterior derivative of each term in (51) using the Reeb Lie derivatives in [25, Section 3.4]:

$$\begin{aligned} d[r^3 dr \wedge \eta \wedge (X_0 \lrcorner \frac{d\eta}{2})] = r^3 ds \wedge dr \wedge \eta \wedge [\frac{\partial X_0}{\partial s} \lrcorner \frac{d\eta}{2}] - 2r^3 dr \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner \frac{d\eta}{2}) \\ + r^3 dr \wedge \eta \wedge d_0 (X_0 \lrcorner \frac{d\eta}{2}), \end{aligned} \tag{52}$$

$$\begin{aligned} d[r^3 ds \wedge \eta \wedge (X_0 \lrcorner H)] = -3r^2 ds \wedge dr \wedge \eta \wedge (X_0 \lrcorner H) - r^3 ds \wedge dr \wedge \eta \wedge \frac{\partial}{\partial r}(X_0 \lrcorner H) \\ -2r^3 ds \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner H) + r^3 ds \wedge \eta \wedge d_0 (X_0 \lrcorner H), \end{aligned} \tag{53}$$

$$\begin{aligned} d[r^2 ds \wedge dr \wedge (X_0 \lrcorner G)] = r^2 ds \wedge dr \wedge \eta \wedge [(L_\upsilon X_0) \lrcorner G] - 3r^2 ds \wedge dr \wedge \eta \wedge (X_0 \lrcorner H) \\ + r^2 ds \wedge dr \wedge d_0 (X_0 \lrcorner G). \end{aligned} \tag{54}$$

Using the above 3 identities and (51), we find

$$d(X_0 \lrcorner \psi_{euc}) = r^3 ds \wedge dr \wedge \eta \wedge [\frac{\partial X_0}{\partial s} \lrcorner \frac{d\eta}{2} + \frac{\partial X_0}{\partial r} \lrcorner H + \frac{(L_\upsilon X_0) \lrcorner G}{r}] + Q(X_0). \quad (55)$$

where

$$Q(X_0) = -2r^3 dr \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner \frac{d\eta}{2}) + \frac{r^4}{2} d[X_0 \lrcorner (\frac{d\eta}{2})^2] + 2r^3 ds \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner H)$$
$$-r^3 ds \wedge \eta \wedge d_0(X_0 \lrcorner H) + r^2 ds \wedge dr \wedge d_0(X_0 \lrcorner G) + r^3 dr \wedge \eta \wedge d_0(X_0 \lrcorner \frac{d\eta}{2})$$
$$+2r^3 dr \wedge [X_0 \lrcorner (\frac{d\eta}{2})^2]. \quad (56)$$

The term $-3r^2 ds \wedge dr \wedge \eta \wedge (X_0 \lrcorner H)$ in (53) and (54) cancels out.

Because the Hodge star of $ds \wedge dr \wedge \eta$ is semi basic, wedging (55) by F_A^0, it is to routine to verify that

$$Aux(X_0) = \star\{F_A^0 \wedge (r^3 ds \wedge dr \wedge \eta) \wedge [\frac{\partial X_0}{\partial s} \lrcorner \frac{d\eta}{2} + \frac{\partial X_0}{\partial r} \lrcorner H + \frac{(L_\upsilon X_0) \lrcorner G}{r}]\}$$
$$+\star[F_A^0 \wedge Q(X_0)]$$
$$= \star_{D^\star}\{[(L_{\frac{\partial}{\partial s}} X_0) \lrcorner \frac{d\eta}{2} + (L_{\frac{\partial}{\partial r}} X_0) \lrcorner H + \frac{(L_\upsilon X_0) \lrcorner G}{r}] \wedge F_A^0\}$$
$$+\star[F_A^0 \wedge Q(X_0)]. \quad (57)$$

Step 2: The component of X perpendicular to the contact distribution has no contribution to the auxiliary operator.

Fact 7.4 For any C^1-functions $X_s,\ X_\upsilon,\ X_r$ defined on a punched tubular ball in the model space,

$$Aux(X_s \frac{\partial}{\partial s} + X_\upsilon \upsilon + X_r \frac{\partial}{\partial r}) = 0. \quad (58)$$

The proof is completely routine. The distribution $span(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}, \upsilon)$ is integrable (involutive) of which $X - X_0$ is a section. The observation is that the exterior differential of each term in

$$(X_s \frac{\partial}{\partial s} + X_\upsilon \upsilon + X_r \frac{\partial}{\partial r}) \lrcorner \psi_{euc} \quad (59)$$

contains at least one among the 3-forms $\frac{d\eta}{2},\ G,\ H$ as a byte. This is because every term in ψ_{euc} itself contains one of these as byte, and applies the identities

$$dH = 3\eta \wedge G,\ dG = -3\eta \wedge H.$$

Therefore the wedge of (59) and the anti self-dual curvature F_A^0 (as an $End E$-valued section of $\wedge^2 D^\star$) vanishes.

To complete the proof of the Lemma, it suffices to show the following which indeed requires that the co-efficients of the vector field X only depend on $r,\ s$. This condition is not applied so far.

Step 3: the wedge between each term in $Q(X_0)$ and F_A^0 is 0.

We first show it for the 3 terms in line 2 of (56). The observation is that the multiplication by a differentiable function of only $r,\ s$ commutes with the transverse Hodge dual operator

$d_0^{\star_{D^\star}}$. Namely, on the first term among the 3, it suffices to show

$$d_0(X_0 \lrcorner H) \wedge F_A^0 = 0.$$

Taking $\star_{D^\star}$, the above is equivalent to

$$-d_0^{\star_{D^\star}}[(J_H X_0) \lrcorner F_A^0] = 0.$$

Using J_H invariance of the curvature, it suffices to observe

$$d_0^{\star_{D^\star}}[(J_H X_0) \lrcorner F_A^0] = \Sigma_{i=1}^6 \frac{X_i}{r} \cdot d_0^{\star_{D^\star}}[J_H(re_i \lrcorner F_A^0)] = 0$$

where we used

$$d_0 X_i = 0 \text{ for all } i = 1, .., 6$$

because these co-efficients only depend on r and s. The other two terms are similar.

To show the four terms in line 1 and 3 of (56) have vanishing wedge with the curvature, using the identity $X_0 \lrcorner [\frac{1}{2}(\frac{d\eta}{2})^2] = \star_{D^\star}(X_0^{\sharp_{D^\star}})$, we calculate the the second term in line 1 of (56):

$$d[X_0 \lrcorner \frac{(d\eta)^2}{4}] = ds \wedge [\frac{\partial X_0}{\partial s} \lrcorner \frac{(d\eta)^2}{4}] + dr \wedge [\frac{\partial X_0}{\partial r} \lrcorner \frac{(d\eta)^2}{4}] + \eta \wedge [L_\upsilon(X_0) \lrcorner \frac{(d\eta)^2}{4}]$$
$$+ d_0[X_0 \lrcorner \frac{(d\eta)^2}{4}]. \tag{60}$$

Any form with a byte in $\wedge^5 D^\star$ must vanish because the $(\mathbb{R})$ rank of the contact distribution $D^\star$ is 4. Because the curvature F_A^0 is an endomorphism-valued semi basic 2-form (pullback from $\mathbb{P}^2$), any form with a byte in $\wedge^3 D^\star$ has vanishing wedge with the curvature. This is precisely the case for every term in (60). The reason why the last term is semi-basic is simply that the transverse exterior differential d_0 of a semi basic form remains semi basic. In summary, we find

$$\frac{r^4}{2} d[X_0 \lrcorner (\frac{d\eta}{2})^2] \wedge F_A^0 = 0.$$

Similarly, the other 3 terms in line 1 and line 3 of (56) also has byte of semi basic 3-form. Then their wedge with the curvature also vanish

$$2r^3 dr \wedge [X_0 \lrcorner (\frac{d\eta}{2})^2] \wedge F_A^0 = -2r^3 dr \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner \frac{d\eta}{2}) \wedge F_A^0$$
$$= 2r^3 ds \wedge \frac{d\eta}{2} \wedge (X_0 \lrcorner H) \wedge F_A^0 = 0.$$

This means $Q(X_0)$ has no contribution to the auxiliary operator i.e.

$$Q(X_0) \wedge F_A^0 = 0.$$

The first two equal signs in (49) is proved by (57) and (58). The curvature F_A^0 is $\star_{D^\star}$ anti self dual. Then

$$\star_{D^\star}(\theta \wedge F_A^0) = -\theta \lrcorner F_A^0$$

for any semi basic 1-form θ. The last line in (49) is proved. □

7.4 Homomorphism between stable bundles on $\mathbb{P}^2$

In proving Lemma 4.5, under the Chern number condition and others therein, the following is crucial to bound $h^0[\mathbb{P}^2, End E]$ and to show that the poly-stable bundle E is stable.

Lemma 7.5 *On $\mathbb{P}^n$, any nontrivial sheaf homomorphism between two stable locally free sheaves of the same slope is an isomorphism.*

Consequently, the space of such homomorphisms is either (complex) 0 *or* 1*-dimensional.*

On projective curves, the similar result is well recorded in literature. But this particular version we need does not seem very easy to find. Following [18] verbatim, we still give the detail for the reader's convenience.

Proof Let $\phi : E_1 \to E_2$ denote the nontrivial homomorphism and stable bundles. [18, Lemma 1.2.8] says ϕ must be injective or generically surjective i.e. surjective on stalks at an arbitrary point away from the singular locus of $Coker\phi$. By [18, Corollary page 171], it suffices to show $rank E_1 = rank E_2$ by ruling out the following two cases.

Case A: suppose $rank E_1 < rank E_2$. Then ϕ must be injective and $Image\phi$ is a sub-sheaf of E_2 of the same slope but lower rank. This contradicts the stability of E_2.

Case B: suppose $rank E_1 > rank E_2$, then it must be generically surjective. Using that $Image\phi$ is a torsion free coherent quotient of E_1 [18, Proof in page 170], we find $rank Image\phi = rank E_2$. Moreover, we have $c_1(Image\phi) \leq c_1(E_2)$ [18, Proof 1 in page 161]. Thus the torsion free quotient has less or equal slope:

$$\mu(Image\phi) \leq \mu(E_2) = \mu(E_1).$$

This contradicts the stability of E_1.

The consequence holds by simpleness of stable bundles. □

References

1. Atiyah, M.: Complex Analytic Connections in Fibre Bundles. Trans. Am. Math. Soc. **85**, 181–207 (1957)
2. Berger, M.: Sur les groupes d'holonomie homogènes des variétés a connexion affines et des variétés riemanniennes. Bull. Soc. Math. France **83**, 279–330 (1953)
3. Biquard, O.: Désingularisation de métriques dÉinstein. I. Invent. Math. **192**(1), 197–252 (2013)
4. Brendle, S., Kapouleas, N.: Gluing Eguchi–Hanson metrics and a question of Page. Commun. Pure Appl. Math. **70**(7), 1366–1401 (2017)
5. Chen, G.: G_2 manifolds with nodal singularities along circles. J. Geomet. Anal. https://doi.org/10.1007/s12220-019-00283-3
6. Donaldson, S.: Deformations of multivalued harmonic functions. arXiv:1912.08274
7. Donaldson, S., Segal, E.: Gauge theory in higher dimensions, II. Geometry of Special Holonomy and Related Topics 141. Surveys in Differential Geometry, vol. 16. International Press, Somerville, MA (2011)
8. Donaldson, S., Thomas, R.: Gauge theory in higher dimensions. In: From:"The Geometric Universe", pp. 31–47. Oxford University Press (1998)
9. Freed, D., Uhlenbeck, K.: Instantons and 4-manifolds. Springer, Berlin (1984)
10. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
11. Hardt, R., Simon, L.: Area minimizing hypersurfaces with isolated singularities. J. Reine Angew. Math. **362**, 102–129 (1985)
12. Jacob, A., Sá Earp, H., Walpuski, T.: Tangent cones of Hermitian Yang–Mills connections with isolated singularities. Math. Res. Lett. **25**(5), 1429–1445 (2018)
13. Jacob, A., Walpuski, T.: Hermitian Yang–Mills metrics on reflexive sheaves over asymptotically cylindrical Kähler manifolds. Commun. Partial. Differ. Equ. **43**(11), 1566–1598 (2018)
14. Lotay, J.: Coassociative 4−folds with conical singularities. Commun. Anal. Geom. **15**(5), 891–946 (2007)
15. Mazzeo, R., Smale, N.: Perturbing away higher dimensional singularities from area minimizing hypersurfaces. Commun. Anal. Geom. **2**(2), 313–336 (1994)

16. Menet, G., Nordström, J., Sá Earp, H.: Construction of G_2-instantons via twisted connected sums. arXiv:1510.03836
17. Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface. Invent. Math. **77**, 101–116 (1984)
18. Okonek, C., Schneider, M., Spindler, H.: Vector bundles on complex projective spaces. Prog. Math. (1952)
19. Sá Earp, H., Walpuski, T.: G_2-instantons over twisted connected sums. Geom. Topol. **19**, 1263–1285 (2015)
20. Simons, J.: On the transitivity of holonomy systems. Ann. Math. **76**(2), 213–234 (1962)
21. Takahashi, R.: The moduli space of $\mathbb{S}^1$-type zero loci for $\frac{Z}{2}$-harmonic spinors in dimension 3. arXiv:1503.00767
22. Tian, G.: Gauge theory and calibrated geometry. Ann. Math. **151**, 193–268 (2000)
23. Walpuski, T.: G_2-instantons over twisted connected sums: an example. Math. Res. Lett. **23**(2), 529–544 (2016)
24. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge Mathematical Library, Cambridge (1922)
25. Wang, Y.Q.: The spectrum of an operator associated with G_2-instantons with 1-dimensional singularities and Hermitian Yang–Mills connections with isolated singularities. arXiv:1911.11979