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Exponential sums over cubes of primes in short intervals and its applications

Taiyu Li[1] · Yanjun Yao[2]

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Abstract
In this paper, we treat the estimate on exponential sums over cubes of primes in short intervals, and improve a previous bound of Kumchev (in: Number Theory: Arithmetic in Shangri-La (Proc. China-Japan Seminar Number Theory), pp. 116–131, World Scientific, Singapore, 2013). Moreover, we present some applications to the cubic Waring–Goldbach problem.

Keywords Exponential sums over primes · short intervals · Waring–Goldbach problem

Mathematics Subject Classification 11L15 · 11L26 · 11P32

1 Introduction

Let $\Lambda(n)$ be the von Mangoldt function, $2 \le y \le x$, and $e(z) = \exp(2\pi iz)$. In this note, we pursue bounds for exponential sums over cubes of primes of the form

$$f(\alpha; x, y) = \sum_{x<n\le x+y} \Lambda(n)e(n^3\alpha).$$

When $y = x^\theta$ with $\theta < 1$, such exponential sums play a key role in the study of additive problems with almost equal cubes of primes (see [7,8,11,13]).

By Dirichlet's lemma on Diophantine approximations, every real number $\alpha \in [1/Q, 1 + 1/Q]$ has a rational approximation a/q, where a and q are integers subject to

$$1 \le q \le Q, \quad (a, q) = 1, \quad |\alpha - a/q| \le (qQ)^{-1}. \tag{1.1}$$

✉ Yanjun Yao
yanjun.yao@sxu.edu.cn

Taiyu Li
tyli@sdu.edu.cn

1 School of Mathematics and Statistics, Shandong University, 180 Wenhua Xilu, Weihai 264209, Shandong, China

2 School of Mathematical Sciences, Shanxi University, 92 Wucheng Lu, Taiyuan 030006, Shanxi, China

For a given positive parameter P with $1 < P < Q/2$, define

$$\mathfrak{M}(P) = \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \left\{\alpha : \alpha = a/q + \lambda,\ |\lambda| \le (qQ)^{-1}\right\},$$

and denote by $\mathfrak{m}(P) = [1/Q, 1 + 1/Q]\backslash\mathfrak{M}(P)$. In the terminology of the circle method, $\mathfrak{M}(P)$ is a set of *major arcs* and $\mathfrak{m}(P)$ is the respective set of *minor arcs*. The first goal of this paper is to establish the following bound of $f(\alpha; x, y)$ on sets of minor arcs.

Theorem 1 *Let* $\frac{8}{9} < \theta \le 1$ *and* $0 < \rho \le \min\left(\frac{3\theta-2}{12}, \frac{9\theta-8}{6}\right)$. *Then for any fixed* $\varepsilon > 0$, *one has*

$$\sup_{\alpha\in\mathfrak{m}(P)} \left|f\left(\alpha; x, x^{\theta}\right)\right| \ll x^{\theta-\rho+\varepsilon} + x^{\theta+\varepsilon}P^{-1/2}.$$

To prove Theorem 1, we shall use Heath–Brown's identity for $\Lambda(n)$ to divide $f(\alpha; x, x^{\theta})$ to type I and type II sums. The refinement of our theorem comes mainly from the estimate for type I sums. Note that in [5, Lemma 3.2], Kumchev established the same conclusion as shown in Proposition 3.2 under the condition $M_1^5 \ll \delta x^{3-7\rho}$ (with $\delta = x^{\theta-1}$). In Proposition 3.2, we are able to enlarge the exponent of x to $3 - 6\rho$. In other words, our result is new in the case $\delta x^{3-7\rho} < M_1^5 \ll \delta x^{3-6\rho}$. With this refinement we achieve the bound $\frac{3\theta-2}{12}$ for ρ in the theorem, which improves Kumchev's result $\frac{2\theta-1}{14}$ in [5, Theorem 2]. See the argument in Sects. 3–4 for details of the proof.

In the special case $\theta = 1$, the theorem reduces to the following new bound of exponential sums over cubes of primes on sets of minor arcs, which improves Theorem 3 in [4].

Corollary 2 *For any fixed* $\varepsilon > 0$, *one has*

$$\sup_{\alpha\in\mathfrak{m}(P)} \left|\sum_{x<n\le 2x} \Lambda(n)e(n^3\alpha)\right| \ll x^{1-1/12+\varepsilon} + x^{1+\varepsilon}P^{-1/2}.$$

We add that, one can also combine [4, Theorem 2] or [10, Theorem 1.1] with [14, Lemma 8.5] to yield the bound in Corollary 2.

As an application of Theorem 1, we consider the representations of a large integer n as the sum of almost equal cubes of primes. Define the sets

$$\begin{aligned}
\mathcal{H}_5 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2,\ n \not\equiv 0, \pm 2 \pmod 9,\ n \not\equiv 0 \pmod 7\},\\
\mathcal{H}_6 &= \{n \in \mathbb{N} : n \equiv 0 \pmod 2,\ n \not\equiv \pm 1 \pmod 9\},\\
\mathcal{H}_7 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2,\ n \not\equiv 0 \pmod 9\},\\
\mathcal{H}_s &= \{n \in \mathbb{N} : n \equiv s \pmod 2\} \quad (s \ge 8).
\end{aligned}$$

For $s \ge 5$, we are interested in the representations of $n \in \mathcal{H}_s$ in the form

$$\begin{cases} n = p_1^3 + \cdots + p_s^3, \\ \left|p_j - (n/s)^{1/3}\right| \le H\ (j = 1, \ldots, s), \end{cases} \tag{1.2}$$

where $H = o(n^{1/3})$, and $p_1, \ldots, p_s$ are prime numbers. In this paper we focus on exploring bounds for the number of integers $n \in \mathcal{H}_s$, without representations as sums of s almost equal cubes of primes. For $5 \le s \le 8$ and $H = o(N^{1/3})$, we define

$$E_s(N, H) = \#\left\{n \in \mathcal{H}_s : |n - N| \le HN^{2/3} \text{ and (1.2) has no solution}\right\}.$$

Particularly, we are mainly dedicated to the case $s = 7$ and 8.

In [7], Liu and Sun established

$$E_7(N,H) \ll N^{\frac{1}{3}} H^{1-\varepsilon} \quad \text{for } H = N^{\frac{1}{3}-\frac{1}{150}+\varepsilon},$$
$$E_8(N,H) \ll H^{1-\varepsilon} \qquad \text{for } H = N^{\frac{1}{3}-\frac{1}{198}+\varepsilon}. \tag{1.3}$$

By applying the estimate of exponential sums in Theorem 1, we refine the above bounds of exceptional sets for sums of seven and eight cubes of primes. Precisely, we obtain the following results.

Theorem 3 *One has*

$$E_7(N,H) \ll N^{\frac{1}{3}} H^{1-\varepsilon} \quad \text{for } H = N^{\frac{1}{3}-\frac{1}{51}+\varepsilon}, \tag{1.4}$$
$$E_8(N,H) \ll H^{1-\varepsilon} \qquad \text{for } H = N^{\frac{1}{3}-\frac{1}{51}+\varepsilon}. \tag{1.5}$$

Moreover, we establish a new bound of exceptional sets for $s = 7$.

Theorem 4 *One has*

$$E_7(N,H) \ll H^{2-\varepsilon} \quad \text{for } H = N^{\frac{1}{3}-\frac{1}{51}+\varepsilon}. \tag{1.6}$$

Here we note that, with the same magnitude of H, the upper bound for $E_7(N,H)$ in (1.6) is smaller than its counterpart in (1.4), which appears to be first of its kind to the best of our knowledge. On the other hand, based on the relation of exceptional sets between seven and eight cubes, the informed reader may expect that, similar to the results of Liu and Sun, we should also be able to establish the bound $E_7(N,H) \ll N^{\frac{1}{3}} H^{1-\varepsilon}$ but with lower magnitude of H than $N^{\frac{1}{3}-\frac{1}{51}+\varepsilon}$. That, however, is not the case. Roughly speaking, the cause of such difference is closely connected with the treatment of the integrals over minor arcs, which is somewhat different from the situation when Liu and Sun met. These matters will be discussed in Sect. 5.

We also remark that the estimate for eight cubes (1.5) or seven cubes (1.6) implies the result of the second author [13], which states that all sufficiently large $n \in \mathcal{H}_9$ can be represented in the form (1.2) with $s = 9$ and $H = n^{\frac{1}{3}-\frac{1}{51}+\varepsilon}$. One can combine (1.5) or (1.6) with known results on the distribution of primes in short intervals to deduce the desired conclusion.

Actually, the interest in $E_s(N,H)$ is twofold. As an analogy, one shall also pursue a non-trivial bound of the form

$$E_s(N,H) \ll N^{\frac{2}{3}} H^{1-\varepsilon}, \tag{1.7}$$

which implies that almost all integers $n \in \mathcal{H}_s$ are representable in the form (1.2) with $H = o(n^{1/3})$. Thus, we are interested in bounds of the form (1.7) with H as small as possible. For example, Ren and the second author [11] showed that (1.7) holds for $H = N^{1/3-\theta_s+\varepsilon}$ with

$$\theta_5 = \frac{7}{261}, \quad \theta_6 = \frac{5}{159}, \quad \theta_7 = \frac{11}{333}, \quad \theta_8 = \frac{19}{561}.$$

As a comparison, in Theorems 3 and 4 we are not only interested in the size of H, but also concerned with the cardinality of $E_s(N,H)$. In other words, given a value of $H = o(N^{1/3})$, we want to minimize the upper bound of exceptional sets. As can be seen, the upper bounds of exceptional sets in Theorems 3 and 4 are much smaller than their counterparts in (1.7).

Throughout the paper, we write $(a,b) = \gcd(a,b)$. As usual, the letter p, with or without subscripts, is reserved for prime numbers. The letters ε and A denote positive constants which are arbitrarily small and sufficiently large, respectively.

2 Auxiliary lemmas

In this section, we present some estimates that will be involved in the proof of our theorems.

First we define the multiplicative function $w(q)$ by

$$w(p^{3u+v}) = \begin{cases} 3p^{-u-1/2} & \text{if } u \geq 0, v = 1, \\ p^{-u-1} & \text{if } u \geq 0, v = 2,\ 3. \end{cases}$$

Then one has

$$q^{-1/2} \leq w(q) \ll q^{-1/3}. \tag{2.1}$$

Moreover we need an auxiliary estimate for sums involving $w(q)$.

Lemma 2.1 *For any fixed* $\varepsilon > 0$ *and* $1 \leq j \leq 3$*, one has*

$$\sum_{n \sim N} w\left(\frac{q}{(q, n^j)}\right) \ll q^{\varepsilon} w(q) N. \tag{2.2}$$

Proof See Lemmas 2.3 in Kawada and Wooley [3]. □

The following result is due to Lemma 2.2 in [5] with $k = 3$.

Lemma 2.2 *Let* $0 < \rho \leq 1/4$*. Suppose that* $y \leq x$*,* $x^3 \leq y^{4-2\rho}$*, and* $\mathcal{I}$ *is a subinterval of* $(x, x+y]$*. If* α *is a real number satisfying that there exist integers* a *and* q *with*

$$(a, q) = 1, \quad 1 \leq q \leq y^{3\rho} \quad \text{and} \quad |q\alpha - a| \leq y^{3\rho-1}x^{-2}, \tag{2.3}$$

then one has

$$\sum_{n \in \mathcal{I}} e(n^3\alpha) \ll \frac{w(q)y}{1 + yx^2|\alpha - a/q|} + x^{3/2+\varepsilon}y^{-1}.$$

Otherwise, one has

$$\sum_{n \in \mathcal{I}} e(n^3\alpha) \ll y^{1-\rho+\varepsilon}.$$

The following lemma is a slight variation of [1, Lemma 6]. The proof is the same.

Lemma 2.3 *Let* q *and* X *be positive integers exceeding* 1 *and let* $0 < \Delta < \frac{1}{2}$*. Suppose that* $q \nmid a$ *and denote by* S *the number of integers* x *such that*

$$X \leq x < 2X, \quad (x, q) = 1, \quad \|ax^3/q\| < \Delta,$$

where $\|\alpha\| = \min_{n \in \mathbb{Z}} |\alpha - n|$*. Then*

$$S \ll \Delta q^{\varepsilon}(q + X).$$

We also quote the following estimate which is a variant of the main result in Liu, Lü and Zhan [6] with $k = 3$.

Lemma 2.4 *Let* $7/10 < \theta \leq 1$ *and* $0 < \rho \leq \min\{(8\theta - 5)/24,\ (10\theta - 7)/15\}$*. Suppose that* α *is real and that there exists integers* a *and* q *satisfying*

$$1 \leq q \leq x^{6\rho}, \quad (a, q) = 1 \quad \text{and} \quad |q\alpha - a| < x^{6\rho-2\theta-1}.$$

Then, for any fixed $\varepsilon > 0$*,*

$$f(\alpha; x, y) \ll x^{\theta-\rho+\varepsilon} + \frac{x^{\theta+\varepsilon}}{\sqrt{q + x^{2\theta+1}|q\alpha + a|}}.$$

For $\mathcal{A} \subseteq (x, x+y] \cap \mathbb{N}$, we define

$$g(\alpha) = g_{\mathcal{A}}(\alpha) = \sum_{n \in \mathcal{A}} (\log n) e(n^3 \alpha).$$

To deal with the mean values of the integral over minor arcs, we shall need the following two results which are Lemmas 2.1 and 2.3 in [13], respectively.

Lemma 2.5 *Let $\gamma \in \mathbb{R}$, $c,\ D > 0$, and $1 < M \le y \le x$. Then there exists a constant $c_0 > 0$ such that*

$$\sum_{q \le M} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{|\alpha - a/q| \le 1} \frac{w^2(q) d^c(q) \left| \sum_{x < p \le x+y} (\log p) e(p^3(\alpha+\gamma)) \right|^2}{1 + D|\alpha - a/q|} d\alpha \ll y^2 D^{-1} (\log x)^{c_0}.$$

Lemma 2.6 *Let ρ and y be defined as in Lemma 2.2. Let $\mathcal{M}$ be the set of $\alpha \in \mathbb{R}$ satisfying (2.3). Suppose that $G(\alpha)$ and $h(\alpha)$ are integrable functions of period one. Then for any measurable set $\mathfrak{w} \subseteq [0, 1]$, one has*

$$\int_{\mathfrak{w}} g(\alpha) G(\alpha) h(\alpha) d\alpha \ll y (\log x)^2 \mathcal{J}_0^{\frac{1}{4}} \left(\int_{\mathfrak{w}} |G(\alpha)|^2 d\alpha \right)^{\frac{1}{4}} J(\mathfrak{w})^{\frac{1}{2}} + y^{1 - \frac{\rho}{2} + \varepsilon} J(\mathfrak{w}),$$

where

$$J(\mathfrak{w}) = \int_{\mathfrak{w}} |G(\alpha) h(\alpha)| d\alpha, \quad \mathcal{J}_0 = \sup_{\beta \in [0,1]} \int_{\mathcal{M}} \frac{w^2(q) |h^2(\alpha + \beta)|}{\left(1 + x^2 y |\alpha - a/q|\right)^2} d\alpha.$$

3 Multilinear exponential sums

In this section, we obtain upper bounds for the exponential sums appearing in the proof of Theorem 1.

Let us write

$$\delta = x^{\theta - 1}, \quad L = \log x, \quad \mathcal{I} = (x, x+y],$$

and

$$\widehat{Q} = (\delta x^{3 - 2\rho})^{3/5}. \tag{3.1}$$

The following *Type II sum* estimate is Lemma 3.1 in [5] with $k = 3$.

Lemma 3.1 *Let $k \ge 3$ and $0 < \rho < 1/10$. Suppose that α is real and that there exist integers a and q such that (1.1) holds with $Q = \widehat{Q}$ given by (3.1). Let $|\xi_m| \le 1$, $|\eta_n| \le 1$, and define*

$$S(\alpha) = \sum_{m \sim M} \sum_{mn \in \mathcal{I}} \xi_m \eta_n e((mn)^3 \alpha).$$

Then one has

$$S(\alpha) \ll x^{\theta - \rho + \varepsilon} + \frac{w(q)^{1/2} x^{\theta + \varepsilon}}{(1 + \delta^2 x^3 |\alpha - a/q|)^{1/2}},$$

provided that

$$\delta^{-1} \max\left(x^{8\rho}, \delta^{-3} x^{4\rho}, (\delta^4 x^{2 + 12\rho})^{1/5}\right) \ll M \ll \delta x^{\theta - 2\rho}. \tag{3.2}$$

The main task of this section is to prove the following estimate for trilinear sums usually referred to as *Type I sums*.

Proposition 3.2 *Let* $0 < \rho < 1/4$. *Suppose that* α *is real and that there exist integers* a *and* q *such that* (1.1) *holds for certain positive* Q. *Let* $|\xi_{m_1,m_2}| \le 1$, *and define*

$$S(\alpha) = \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_1m_2n\in\mathcal{I}} \xi_{m_1,m_2} e((m_1m_2n)^3\alpha).$$

Then one has

$$S(\alpha) \ll x^{\theta-\rho+\varepsilon} + \frac{w(q)x^{\theta+\varepsilon}}{1+\delta x^3|\alpha - a/q|},$$

provided that

$$M_1^5 \ll \delta x^{3-6\rho}, \quad M_1M_2 \ll \min(\delta x^{1-4\rho}, \delta^4 x^{1-2\rho}), \quad M_1M_2^2 \ll \delta^{1/3}x^{1-2\rho}. \tag{3.3}$$

Proof Set $N = x(M_1M_2)^{-1}$ and $H = \delta N$, and define ν by $H^\nu = x^\rho L^{-1}$. Note that, for $m_1 \sim M_1,\ m_2 \sim M_2$, and $m_1m_2n \in \mathcal{I}$, we have

$$n \in \left(\frac{x}{M_1M_2}, \frac{x+y}{M_1M_2}\right]$$

with the length of

$$\frac{y}{M_1M_2} = x^{\theta-1}x(M_1M_2)^{-1} = \delta N = H.$$

Denote by $\mathcal{M}$ the set of pairs (m_1, m_2), with $m_1 \sim M_1$ and $m_2 \sim M_2$, for which there exist integers b_1 and r_1 with

$$1 \le r_1 \le H^{3\nu}, \quad (b_1, r_1) = 1, \quad |r_1(m_1m_2)^3\alpha - b_1| \le H^{3\nu}(\delta N^3)^{-1}. \tag{3.4}$$

Applying Lemma 2.2 to the summation over n, one has

$$\sum_{m_1m_2n\in\mathcal{I}} e((m_1m_2n)^3\alpha) \ll H^{1-\nu+\varepsilon} \quad \text{or} \quad \frac{w(r_1)H}{1+\delta N^3|(m_1m_2)^3\alpha - b_1/r_1|}.$$

Therefore,

$$S(\alpha) \ll \sum_{m_1\sim M_1}\sum_{m_2\sim M_2} H^{1-\nu+\varepsilon} + \sum_{(m_1,m_2)\in\mathcal{M}} \frac{w(r_1)H}{1+\delta N^3|(m_1m_2)^3\alpha - b_1/r_1|}.$$

One has

$$S(\alpha) \ll x^{\theta-\rho+\varepsilon} + T_1(\alpha),$$

where

$$T_1(\alpha) = \sum_{(m_1,m_2)\in\mathcal{M}} \frac{w(r_1)H}{1+\delta N^3|(m_1m_2)^3\alpha - b_1/r_1|}.$$

For each $m_1 \sim M_1$, we apply Dirichlet's theorem on Diophantine approximation to find integers b and r with

$$1 \le r \le x^{-3\rho}(\delta N^3), \quad (b, r) = 1, \quad |rm_1^3\alpha - b| \le x^{3\rho}(\delta N^3)^{-1}. \tag{3.5}$$

By (3.3), (3.4) and (3.5),

$$|b_1r - bm_2^3r_1| \le 2^3\delta^{-1}x^{-3+6\rho}L^{-3}(M_1M_2^2)^3 + L^{-3} < 1.$$

In the last step, we have used the third condition in (3.3). This gives that,

$$\frac{b_1}{r_1} = \frac{m_2^3b}{r}, \quad r_1 = \frac{r}{(r, m_2^3)}.$$

Thus, by (2.2),

$$\begin{aligned} T_1(\alpha) &\ll \sum_{m_1\sim M_1} \frac{H}{1+\delta(NM_2)^3|m_1^3\alpha - b/r|} \sum_{m_2\sim M_2} w\left(\frac{r}{(r, m_2^3)}\right) \\ &\ll \sum_{m_1\sim M_1} \frac{r^{\varepsilon}w(r)HM_2}{1+\delta(NM_2)^3|m_1^3\alpha - b/r|}. \end{aligned}$$

For each $r \in \mathbb{N}$, one has the unique decomposition $r = r_1r_2$, where r_1 is cube-free, and $r_2 = r_3^3$ is a cube. Throughout this section, the letter r_2 denotes a cube, r_1 is cube-free, and $r = r_1r_2$. Note that

$$w(r) \le r_1^{-1/2}r_2^{-1/3}r^{\varepsilon}.$$

Denote by $\mathcal{S}$ the set of $m_1 \sim M_1$ for which there exist integers b and r with

$$(b, r) = 1, \quad 1 \le r_1^{1/2}r_2^{1/3} \le H^{\nu} \quad \text{and} \quad |rm_1^3\alpha - b| \le r_1^{1/2}r_2^{2/3}H^{\nu}\delta^{-1}(M_2N)^{-3}. \tag{3.6}$$

Then one has

$$T_1(\alpha) \ll T_2(\alpha) + x^{\theta-\rho+\varepsilon},$$

where

$$T_2(\alpha) = \sum_{m_1\in\mathcal{S}} \frac{r^{\varepsilon}w(r)HM_2}{1+\delta(NM_2)^3|m_1^3\alpha - b/r|}.$$

By the dyadic argument, we have

$$T_2(\alpha) \ll (\log H)^2 \sup_{\substack{R_1, R_2 \\ R_1^{1/2}R_2^{1/3}\le H^{\nu}}} H(\alpha)$$

where

$$H(\alpha) = H_{R_1,R_2}(\alpha) = \sum_{m_1\in\mathcal{S}(R_1,R_2)} \frac{r^{\varepsilon}w(r)HM_2}{1+\delta(NM_2)^3|m_1^3\alpha - b/r|}.$$

Here $\mathcal{S}(R_1, R_2)$ denotes the set of $m_1 \sim M_1$ for which there exist integers b and r satisfying (3.6), $R_1 \le r_1 < 2R_1$ and $R_2 \le r_2 < 2R_2$. We need to prove

$$H(\alpha) \ll \frac{w(q)x^{\theta+\varepsilon}}{1+\delta x^3|\alpha - a/q|} + x^{\theta-\rho+\varepsilon} \tag{3.7}$$

for all pairs (R_1, R_2) with

$$R_1^{1/2}R_2^{1/3} \le H^{\nu}. \tag{3.8}$$

By Dirichlet's approximation theorem, there exist integers c and s satisfying

$$(c, s) = 1, \quad 1 \le s \le \widetilde{Q} \quad \text{and} \quad |s\alpha - c| \le \widetilde{Q}^{-1}$$

with

$$\widetilde{Q} = 64 M_1^3 R_1 R_2. \tag{3.9}$$

We will first consider the case that $s \le 5^{-1} H^{-\nu} \delta (M_2 N)^3 R_1^{-1/2} R_2^{-2/3}$. Then one has

$$|crm_1^3 - bs| \le \frac{1}{2} + \frac{2^{7/6}}{5} < 1.$$

It gives $crm_1^3 - bs = 0$. Thus we can obtain

$$\frac{b}{r} = \frac{cm_1^3}{s}, \quad r = \frac{s}{(s, m_1^3)}. \tag{3.10}$$

So by (2.2), we arrive at

$$\begin{aligned} H(\alpha) &\ll \sum_{m_1 \sim M_1} \frac{r^{\varepsilon} w(r) H M_2}{1 + \delta (N M_2)^3 m_1^3 |\alpha - c/s|} \\ &\ll \frac{H M_2}{1 + \delta (N M_2)^3 M_1^3 |\alpha - c/s|} \sum_{m_1 \sim M_1} w\left(\frac{s}{(s, m_1^3)}\right)\left(\frac{s}{(s, m_1^3)}\right)^{\varepsilon} \\ &\ll \frac{x^{\theta+\varepsilon}}{1 + \delta x^3 |\alpha - c/s|} w(s). \end{aligned}$$

If $s \le x^{3\rho}$ and $|\alpha - c/s| \le x^{3\rho}/(s\delta x^3)$, then we can get $c = a$ and $s = q$ by recalling (1.1) and (3.1), and thereby

$$H(\alpha) \ll \frac{x^{\theta+\varepsilon}}{1 + \delta x^3 |\alpha - c/s|} w(s).$$

Otherwise we have

$$H(\alpha) \ll x^{\theta - \rho + \varepsilon}.$$

Thus, the estimate (3.7) holds provided that $s \le 5^{-1} H^{-\nu} \delta (M_2 N)^3 R_1^{-1/2} R_2^{-2/3}$. Therefore, we now assume that

$$s > 5^{-1} H^{-\nu} \delta (M_2 N)^3 R_1^{-1/2} R_2^{-2/3}. \tag{3.11}$$

If $|rm_1^3 \alpha - b| < (4\widetilde{Q})^{-1}$, then

$$|crm_1^3 - bs| \le rm_1^3 |s\alpha - c| + s|rm_1^3 \alpha - b| \le rm_1^3 \widetilde{Q}^{-1} + s(4\widetilde{Q})^{-1} \le \frac{1}{2} + \frac{1}{4} < 1.$$

This implies (3.10). By repeating the argument after (3.10), we can again obtain the desired estimate (3.7).

It remains to treat the cases

$$(4\widetilde{Q})^{-1} \le |rm_1^3 \alpha - b| \le \left(5^{-1} H^{-\nu} \delta (M_2 N)^3 R_1^{-1/2} R_2^{-2/3}\right)^{-1}.$$

Let Z be some parameter satisfying $5^{-1}H^{-\nu}\delta(M_2N)^3R_1^{-1/2}R_2^{-2/3} \le Z \le 4\widetilde{Q}$ and $\mathcal{S}(R_1, R_2, Z)$ the subset of $\mathcal{S}(R_1, R_2)$ containing integers m_1 subject to $|rm_1^3\alpha - b| < Z^{-1}$. Define

$$H_0(\alpha) = H_{R_1,R_2,Z}(\alpha) = \frac{R_1^{-1/2}R_2^{-1/3}HM_2x^{\varepsilon}}{1+\delta(NM_2)^3R_1^{-1}R_2^{-1}Z^{-1}} \sum_{m_1\in\mathcal{S}(R_1,R_2,Z)} 1. \tag{3.12}$$

By the previous argument we have

$$H(\alpha) \ll \sup_{2^{-2}\delta(NM_2)^3H^{\nu}R_1^{-1/2}R_2^{-2/3}\le Z\le 4\widetilde{Q}} H_0(\alpha) + \frac{w(q)x^{\theta+\varepsilon}}{1+\delta x^3|\alpha - a/q|} + x^{\theta-\rho+\varepsilon}. \tag{3.13}$$

Note that

$$\sum_{m_1\in\mathcal{S}(R_1,R_2,Z)} 1 = \sum_{d|s} \sum_{m_1\in\mathcal{S}_d(R_1,R_2,Z)} 1, \tag{3.14}$$

where $\mathcal{S}_d(R_1, R_2, Z)$ is the subset of $\mathcal{S}(R_1, R_2, Z)$ containing integers m_1 subject to $(m_1, s) = d$. Let $\mathcal{S}_d^{(1)}(R_1, R_2, Z)$ and $\mathcal{S}_d^{(2)}(R_1, R_2, Z)$ denote the subsets of $\mathcal{S}_d(R_1, R_2, Z)$ subject to one more condition $(s, rd^3) < s$, and $(s, rd^3) = s$, respectively. If $(s, rd^3) = s$, then there exists an integer t, such that

$$d = (str^{-1})^{1/3} = (str_1^{-1}r_2^{-1})^{1/3},$$

which implies $d \gg (sR_1^{-1}R_2^{-1})^{1/3}$. Then one has

$$\sum_{m_1\in\mathcal{S}_d^{(2)}(R_1,R_2,Z)} 1 \ll \frac{M_1}{d} + 1 \ll \frac{M_1R_1^{1/3}R_2^{1/3}}{s^{1/3}} + 1. \tag{3.15}$$

Concerning the contribution from $\mathcal{S}_d^{(1)}(R_1, R_2, Z)$, we have

$$\sum_{m_1\in\mathcal{S}_d^{(1)}(R_1,R_2,Z)} 1 \le \sum_{R_1\le r_1<2R_1} \sum_{\substack{R_2\le r_2<2R_2\\(s,r_1r_2d^3)<s}} \mathcal{N}(r_1, r_2),$$

where $\mathcal{N}(r_1, r_2)$ is the number of integers $m_1 \sim M_1$ with $(m_1, s) = d$ for which there exists $b \in \mathbb{Z}$ such that

$$(b, r_1r_2) = 1 \quad \text{and} \quad |r_1r_2m_1^3\alpha - b| < Z^{-1}.$$

Note that

$$\mathcal{N}(r_1, r_2) \le \mathcal{N}_0(r_1, r_2),$$

where $\mathcal{N}_0(r_1, r_2)$ is the number of integers $m \sim M/d$ subject to

$$(m, s') = 1 \quad \text{and} \quad \left\| \frac{cr_1r_2d^2m^3}{s'} \right\| < \Delta$$

with $s' = s/d$ and

$$\Delta = Z^{-1} + 32R_1R_2M_1^3(s\widetilde{Q})^{-1}. \tag{3.16}$$

Since r_2 is a cube, we have

$$\sum_{R_1\le r_1<2R_1}\sum_{\substack{R_2\le r_2<2R_2\\(s,r_1r_2d^3)<s}}\mathcal{N}(r_1,r_2)\le\sum_{R_1\le r_1<2R_1}\sum_{\substack{R_2\le r_2<2R_2\\(s,r_1r_2d^3)<s}}\mathcal{N}_0(r_1,r_2)$$
$$\le\sum_{R_1\le r_1<2R_1}\sum_{\substack{r_3<(2R_2)^{1/3}\\r_3|s'^{\infty},\ (s',r_1r_3^3d^2)<s'}}$$
$$\times\sum_{\substack{R_2^{1/3}r_3^{-1}\le r_4<(2R_2)^{1/3}r_3^{-1}\\(s',r_4)=1}}\mathcal{N}_0(r_1,r_3^3r_4^3).$$

Recalling the definition of $\mathcal{N}_0(r_1,r_3r_4)$, we get

$$\sum_{m_1\in\mathcal{S}_d^{(1)}(R_1,R_2,Z)}1\le\sum_{R_1\le r_1<2R_1}\sum_{\substack{r_3<(2R_2)^{1/3}\\r_3|s'^{\infty},\ s'\nmid cr_1r_3^3d^2}}\mathcal{N}^+(r_1,r_3^3)x^{\varepsilon},\tag{3.17}$$

where $\mathcal{N}^+(r_1,r_3^3)$ is the number of integers m satisfying

$$M_1d^{-1}R_2^{1/3}r_3^{-1}\le m<4M_1d^{-1}R_2^{1/3}r_3^{-1}$$

and

$$(m,s')=1,\quad\left\|\frac{cr_1r_3^3d^2m^3}{s'}\right\|<\Delta.$$

Applying Lemma 2.3, and recalling (3.8) and (3.11), we have

$$\mathcal{N}^+(r_1,r_3^3)\ll\Delta s'^{\varepsilon}\left(s'+M_1d^{-1}R_2^{1/3}r_3^{-1}\right)$$
$$\ll\Delta s^{\varepsilon}d^{-1}\left(s+M_1R_2^{1/3}r_3^{-1}\right)\ll\Delta s^{\varepsilon}d^{-1}s\ll\Delta d^{-1}sx^{\varepsilon}.$$

Combining this with (3.17), we get

$$\sum_{m_1\in\mathcal{S}_d^{(1)}(R_1,R_2,Z)}1\ll R_1\Delta sx^{\varepsilon}d^{-1}.\tag{3.18}$$

Now we conclude from (3.12)–(3.15) and (3.18) that

$$H_0(\alpha)\ll\frac{R_1^{-1/2}R_2^{-1/3}HM_2x^{\varepsilon}}{1+\delta(NM_2)^3R_1^{-1}R_2^{-1}Z^{-1}}\left(R_1\Delta s+\frac{M_1R_1^{1/3}R_2^{1/3}}{s^{1/3}}+1\right).$$

By recalling (3.9), (3.11) and (3.16), we have

$$H_0(\alpha)\ll\frac{R_1^{-1/2}R_2^{-1/3}HM_2x^{\varepsilon}}{\delta(NM_2)^3R_1^{-1}R_2^{-1}Z^{-1}}R_1\big(Z^{-1}+32R_1R_2M_1^3(s\widetilde{Q})^{-1}\big)s$$
$$+\frac{R_1^{-1/2}R_2^{-1/3}HM_2x^{\varepsilon}}{1+\delta(NM_2)^3R_1^{-1}R_2^{-1}Z^{-1}}+\frac{R_1^{-1/2}R_2^{-1/3}HM_2x^{\varepsilon}}{1+\delta(NM_2)^3R_1^{-1}R_2^{-1}Z^{-1}}\frac{M_1R_1^{1/3}R_2^{1/3}}{s^{1/3}}.$$

We point out that the first term on the right side of the above estimate is $x^{\theta-\rho+\varepsilon}$ due to (3.8), (3.9), (3.11), (3.13) and $M^5\ll\delta x^{3-6\rho}$. After a brief argument, we can see that the second term is actually smaller than the first one, since the factor "$R_1\big(Z^{-1}+32R_1R_2M_1^3(s\widetilde{Q})^{-1}\big)s$"

is $\gg 1$. For the last term, we can obtain the desired bound in view of the condition $\rho < 1/4$, (3.8), (3.11) and (3.13). The proof is completed. □

4 Proof of Theorem 1

In this section, we prove Theorem 1 by employing Lemmas 2.4 and 3.1, Proposition 3.2 and Heath-Brown's identity for $\Lambda(n)$. We apply Heath-Brown's identity in the following form [2, Lemma 1]: if $n \le X$ and J is a positive integer, then

$$\Lambda(n) = \sum_{j=1}^{J} \binom{J}{j} (-1)^{-j} \sum_{\substack{n=n_1\cdots n_{2j} \\ n_1,\ldots,n_j \le x^{1/J}}} \mu(n_1)\cdots\mu(n_j)\log(n_{2j}). \tag{4.1}$$

Let $\alpha \in \mathfrak{m}(P)$. Recall that, by Dirichlet's theorem on Diophantine approximation, there exist integers a and q such that (1.1) holds with $Q = \widehat{Q}$ given by (3.1). Let β be defined as

$$x^{\beta} = \min\left(\delta^2 x^{1-10\rho},\ \delta^5 x^{1-6\rho},\ \left(\delta^6 x^{3-22\rho}\right)^{1/5}\right),$$

and suppose ρ and δ are chosen so that

$$\delta^{-1} x^{\beta+2\rho} \ge 2x^{1/3}, \quad x^{\beta} \ge \delta^{-1} x^{2\rho}. \tag{4.2}$$

We apply (4.1) with $X = x + x^{\theta}$ and $J \ge 3$ satisfying $x^{1/J} \le x^{\beta}$. After a standard splitting argument, we have

$$\sum_{n \in \mathcal{I}} \Lambda(n) e(\alpha n^3) \ll \sum_{\mathbf{N}} \left| \sum_{n \in \mathcal{I}} c(n; \mathbf{N}) e(\alpha n^3) \right|, \tag{4.3}$$

where $\mathbf{N}$ runs over $O(L^{2j-1})$ vectors $\mathbf{N} = (N_1, \ldots, N_{2j}),\ j \le J$, subject to

$$N_1, \ldots, N_j \ll x^{1/J}, \quad x \ll N_1 \cdots N_{2j} \ll x$$

and

$$c(n; \mathbf{N}) = \sum_{\substack{n=n_1\cdots n_{2j} \\ n_i \sim N_i}} \mu(n_1)\cdots\mu(n_j)\log(n_{2j}).$$

In fact, we can remove the coefficient $\log n_{2j}$ by partial summation and assume that

$$c(n; \mathbf{N}) = L \sum_{\substack{n=n_1\cdots n_{2j} \\ n_i \sim N_i'}} \mu(n_1)\cdots\mu(n_j),$$

where $N_i \le N_i' \le 2N_i$ (in reality, $N_i' = 2N_i$ except for $i = 2j$). We also assume (as we may) that the summation variables $n_{j+1}, \ldots, n_{2j}$ are labeled so that $N_{j+1} \le \cdots \le N_{2j}$. Next, we establish that each of the sums occurring on the right side of (4.3) has the upper bound

$$\sum_{n \in \mathcal{I}} c(n; \mathbf{N}) e(\alpha n^3) \ll x^{\theta-\rho+\varepsilon} + \frac{w(q)^{1/2} x^{\theta+\varepsilon}}{(1 + \delta^2 x^3 |\alpha - a/q|)^{1/2}}. \tag{4.4}$$

In the following argument, we give several cases depending on the sizes of $N_1, \ldots, N_{2j}$.

Case 1: $N_1 \cdots N_j \geq \delta^{-1}x^{2\rho}$. Since each of N_i $(1 \leq i \leq j)$ does not exceed x^β, there must be a set of indices $S \subset \{1, \ldots, j\}$ satisfying

$$\delta^{-1}x^{2\rho} \leq \prod_{i\in S} N_i \leq \delta^{-1}x^{\beta+2\rho}. \tag{4.5}$$

Hence, we can rewrite $c(n; \mathbf{N})$ in the form

$$c(n; \mathbf{N}) = \sum_{\substack{mr=n \\ m\asymp M}} \xi_m \eta_r, \tag{4.6}$$

where $|\xi_m| \ll m^\varepsilon$, $|\eta_r| \ll r^\varepsilon$ and $M = \prod_{i\notin S} N_i$. By the definition of β and (4.5), M satisfies (3.2), and thus (4.4) follows form Lemma 3.1.

Case 2: $N_1 \cdots N_j < \delta^{-1}x^{2\rho}$, $j \leq 2$.
When $j = 1$, we obtain (4.4) by Proposition 3.2 with $M_1 = N_1$, $M_2 = 1$ and $N = N_2$.
When $j = 2$, we have

$$N_3 \leq (x/N_1N_2)^{1/2} \leq x^{1/2}, \quad N_1N_2N_3 \leq (xN_1N_2)^{1/2} \leq \delta^{-1/2}x^{1/2+\rho},$$
$$(N_1N_2)^2N_3 \leq x^{1/2}(N_1N_2)^{3/2} \leq \delta^{-3/2}x^{1/2+3\rho}.$$

Hence, (4.4) follows from Proposition 3.2 with $M_1 = N_3$, $M_2 = N_1N_2$ and $N = N_4$, provided that

$$x^{5/2} \leq \delta x^{3-6\rho}, \quad \delta^{-3/2}x^{1/2+3\rho} \leq \delta^{1/3}x^{1-2\rho}, \quad \delta^{-1/2}x^{1/2+\rho} \leq \min\left(\delta x^{1-4\rho}, \delta^4x^{1-2\rho}\right). \tag{4.7}$$

Case 3: $N_1 \cdots N_{2j-2} < \delta^{-1}x^{2\rho}$, $j \geq 3$. This is a similar situation to Case 2 with $j = 2$ and with the product $N_1 \cdots N_{2j-2}$ playing the role of N_1N_2 in Case 2. Thus, we can again use Proposition 3.2 to obtain (4.4).

Case 4: $N_1 \cdots N_j < \delta^{-1}x^{2\rho} \leq N_1 \cdots N_{2j-2}$, $j \geq 3$. In this case, we have

$$N_{j+1}, \ldots, N_{2j-2} \leq 2x^{1/3} \leq \delta^{-1}x^{\beta+2\rho}.$$

If $N_{2j-2} \geq \delta^{-1}x^{2\rho}$, we can write $c(n; \mathbf{N})$ in the form (4.6) where $M = \prod_{i\neq 2j-2} N_i$. Then we appeal to Lemma 3.1 to show that (4.4) holds. On the other hand, if $N_{2j-2} < \delta^{-1}x^{2\rho}$, then $N_{j+1}, \ldots, N_{2j-2} \leq x^\beta$ (by (4.2)). Thus, we can use the product $N_1 \cdots N_{2j-2}$ in a similar fashion to the product $N_1 \cdots N_j$ in Case 1 to obtain a set of indices $S \subset \{1, 2, \ldots, 2j-2\}$ such that (4.5) holds. Hence, we can again represent $c(n; \mathbf{N})$ in the form (4.6) and then appeal to Lemma 3.1 to show that (4.4) holds one last time. By the above analysis,

$$\sum_{n\in\mathcal{I}} \Lambda(n)e(\alpha n^3) \ll x^{\theta-\rho+\varepsilon} + \frac{w(q)^{1/2}x^{\theta+\varepsilon}}{(1+\delta^2x^3|\alpha - a/q|)^{1/2}}, \tag{4.8}$$

provided that conditions (4.2) and (4.7) hold. Altogether, those conditions are equivalent to the following inequality

$$x^\rho \ll \min\Big(\delta^{1/8}x^{1/12}, \ \delta x^{1/6}, \ \delta^{1/12}x^{1/9}, \ \delta^{1/4}x^{1/12}, \ \delta^{3/4}x^{1/8}, \ \delta^{11/32}x^{3/32},$$
$$\delta^{1/6}x^{1/12}, \ \delta^{11/30}x^{1/10}, \ \delta^{3/10}x^{1/10}, \ \delta^{3/2}x^{1/6}\Big). \tag{4.9}$$

Note that $\delta = x^{\theta-1} \le 1$, we have

$$\delta^{11/30}x^{1/10} \le \delta^{3/10}x^{1/10}, \quad \delta^{3/2}x^{1/6} \le \delta x^{1/6}, \quad \delta^{1/4}x^{1/12} \le \delta^{1/6}x^{1/12} \le \delta^{1/8}x^{1/12}.$$

Furthermore,

$$\begin{aligned}
&\min\left(\delta^{3/2}x^{1/6},\ \delta^{1/4}x^{1/12}\right) \le \left(\delta^{3/2}x^{1/6}\right)^{2/5}\left(\delta^{1/4}x^{1/12}\right)^{3/5} = \delta^{3/4}x^{7/60} < \delta^{3/4}x^{1/8},\\
&\min\left(\delta^{3/2}x^{1/6},\ \delta^{1/4}x^{1/12}\right) \le \left(\delta^{3/2}x^{1/6}\right)^{7/75}\left(\delta^{1/4}x^{1/12}\right)^{68/75} = \delta^{11/30}x^{41/450} < \delta^{11/30}x^{1/10},\\
&\min\left(\delta^{3/2}x^{1/6},\ \delta^{1/4}x^{1/12}\right) \le \left(\delta^{3/2}x^{1/6}\right)^{1/3}\left(\delta^{1/4}x^{1/12}\right)^{2/3} = \delta^{2/3}x^{1/9} < \delta^{1/12}x^{1/9}.
\end{aligned}$$

For $\delta \ge x^{-1/9}$, it follows that

$$\delta^{1/4}x^{1/12} \le \delta^{11/32}x^{3/32}.$$

Hence, in this case, (4.9) is equivalent to

$$\rho \le \min\left(\frac{3\theta-2}{12}, \frac{9\theta-8}{6}\right).$$

If either $q \ge x^{6\rho}$ or $|q\alpha - a| \ge \delta^{-2}x^{-3+6\rho}$, we can use (2.1) to show that the second term on the right side of (4.8) is smaller than the first. Thus,

$$\sup_{\alpha\in\mathfrak{m}(x^{6\rho})} |f(\alpha; x, x^{\theta})| \ll x^{\theta-\rho+\varepsilon}. \tag{4.10}$$

This establishes the theorem when $q \ge x^{6\rho}$. When $q \le x^{6\rho}$, we combine (4.10) with the inequality

$$\sup_{\alpha\in\mathfrak{m}(P)\cap\mathfrak{M}(x^{6\rho})} |f(\alpha; x, x^{\theta})| \ll x^{\theta-\rho+\varepsilon} + x^{\theta+\varepsilon}P^{-1/2},$$

which follows from Lemma 2.4, provided that $\rho \le \frac{8\theta-5}{24}$. To complete the proof, we note that the last condition on ρ is implied by the hypotheses of the theorem.

5 Proof of Theorems 3 and 4

Write

$$X = \sqrt[3]{N/s}, \quad \mathcal{I} = (X-H, X+H], \quad L = \log N.$$

Recall the definition of $E_s(N,H)$ with $H = X^{16/17+\varepsilon}$. Denote by $\mathcal{Z}_s$ the set of integers counted by $E_s(N,H)$, and set $I_s(N,H) = [N - 3s^{1/3}N^{2/3}H, N + 3s^{1/3}N^{2/3}H]$. Consider the sum

$$R_s(n) = \sum_{\substack{n=p_1^2+\cdots+p_s^2\\ p_i\in\mathcal{I}}} (\log p_1)\cdots(\log p_s).$$

We note that $R_s(n) = 0$ for all $n \in \mathcal{Z}_s$. Define

$$T(\alpha) = \sum_{p\in\mathcal{I}} (\log p)e(p^3\alpha).$$

Recall the definition of the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ in Sect. 1, with $P,\ Q$ given by

$$P = H^2 X^{-19/12-\varepsilon}, \quad Q = X^{31/12+\varepsilon}. \tag{5.1}$$

Then we have

$$R_s(n) = \int_{1/Q}^{1+1/Q} T^s(\alpha)e(-n\alpha)d\alpha = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) T^s(\alpha)e(-n\alpha)d\alpha.$$

For the contribution of the integral over major arcs, we quote the following lemma which is [11, Proposition 1]. Indeed the asymptotic formula (5.2) was established in [11] for $s \le 8$. However, one can verify that the case $s = 9$ is also valid with the choice of P and Q as given in (5.1).

Lemma 5.1 *Let the major arcs $\mathfrak{M}$ be defined as above, with P, Q given by* (5.1). *Then for $n \in I_s(N, H)$ with $5 \le s \le 9$ and any $A > 0$, one has*

$$\int_{\mathfrak{M}} T^s(\alpha)e(-n\alpha)d\alpha = \frac{1}{3^s}\mathfrak{S}_s(n)\mathfrak{J}_s(n) + O\left(H^{s-1}N^{-2/3}L^{-A}\right), \tag{5.2}$$

where $\mathfrak{S}_s(n)$ is the corresponding singular series satisfying $\mathfrak{S}_s(n) \gg 1$, and $\mathfrak{J}_s(n)$ is the singular integral satisfying $\mathfrak{J}_s(n) \asymp H^{s-1}N^{-2/3}$.

Next we shall focus on the estimates over minor arcs. First note that with the parameter P given in (5.1), we have the upper bound estimate of exponential sums over minor arcs

$$\sup_{\alpha\in\mathfrak{m}} |T(\alpha)| \ll H^{3/4+\varepsilon}X^{1/6} \tag{5.3}$$

with

$$H = X^\theta \quad \text{and} \quad \theta > \frac{14}{15}. \tag{5.4}$$

For an integer $s > 0$, define

$$I(s) = \int_{\mathfrak{m}} |T(\alpha)|^s d\alpha.$$

In order to evaluate the contribution from minor arcs, we need to deal with $I(9)$ and $I(10)$. Indeed $I(9)$ has been investigated in the proof of Proposition 2 in [13], which shows that

$$I(9) \ll H^{5+\varepsilon}X^{-1/2} \sup_{\alpha\in\mathfrak{m}} |T(\alpha)|^{3/2} + H^{47/8+\varepsilon}. \tag{5.5}$$

Then by (5.3), one can get, for H satisfying (5.4),

$$I(9) \ll H^{47/8+\varepsilon}. \tag{5.6}$$

Now we can treat $I(10)$. Precisely we establish the following sharp bound.

Lemma 5.2 *Let H be defined as* (5.4), *one has*

$$I(10) \ll H^{27/4+\varepsilon}.$$

Proof We take $\rho = \frac{1}{4}$ in Lemma 2.6, and choose $G(\alpha) = |T(\alpha)|^8$, $h(\alpha) = T(-\alpha)$ and $g(\alpha) = T(\alpha)$. Then one gets

$$I(10) \ll HL^2\mathcal{J}_0^{1/4}I(16)^{1/4}I(9)^{\frac{1}{2}} + H^{7/8+\varepsilon}I(9),$$

where

$$\mathcal{J}_0 = \sup_{\beta \in [0,1]} \sum_{q \le H^{3/4}} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{\mathcal{M}(q,a)} \frac{w^2(q)|h^2(\alpha+\beta)|}{\left(1 + X^2 H|\alpha - a/q|\right)^2} d\alpha$$

with $\mathcal{M}(q,a) = \{\alpha : |q\alpha - a| \le X^{-2}H^{-1/4}\}$. Employing Lemma 2.5 with $M = H^{3/4}$ and $D = X^2 H$, one has $\mathcal{J}_0 \ll X^{-2+\varepsilon}H$. Together with (5.3) and (5.6), one then obtain

$$\begin{aligned} I(10) &\ll H^{67/16+\varepsilon} X^{-1/2} \sup_{\alpha \in \mathfrak{m}} |T(\alpha)|^{3/2} I(10)^{1/4} + H^{27/4+\varepsilon} \\ &\ll H^{87/16+\varepsilon} X^{-3/8} I(10)^{1/4} + H^{27/4+\varepsilon}. \end{aligned}$$

Then the conclusion holds after a simple calculation. □

Remark 1 One may note that the first term on the right side of (5.5) seems to vanish. Actually, it converts exactly into the second term $H^{\frac{47}{8}+\varepsilon}$ under the present condition. On the other hand, if we make a comparison with the situation faced in [13], where the previous bound $\frac{2\theta-1}{14}$ (who is now improved to $\frac{3\theta-2}{12}$ in Theorem 1) for exponential sums was employed and hence the exponential sum was bounded by

$$\sup_{\alpha \in \mathfrak{m}} |T(\alpha)| \ll H^{6/7+\varepsilon} X^{1/14}, \tag{5.7}$$

we can find that

$$I(9) \ll H^{44/7+\varepsilon} X^{-11/28} + H^{47/8+\varepsilon}. \tag{5.8}$$

Under this stage, one needs to judge between the size of the former and latter terms on the right side, according to whether θ is less than $\frac{22}{23}$ or not. With such an estimate of $I(9)$ together with the bound of exponential sums $T(\alpha)$ in (5.7), one needs to evaluate $I(10)$ whose upper bound would also include two terms similar to (5.8). Along this way, it will clearly require extra efforts to control the bound of exceptional sets $E_s(N,H)$ and of H.

Write

$$D_s(\alpha) = \sum_{n \in \mathcal{Z}_s} e(-n\alpha).$$

We need to quote the following lemma, which is a short interval variation of Wooley's argument [12, Lemmas 5.1 & 6.2].

Lemma 5.3 *For* $s_1 = 4,\ 6$ *and* $s_2 = 7,\ 8$*, one has*

$$\int_0^1 \left|T^{s_1}(\alpha) D_{s_2}^2(\alpha)\right| d\alpha \ll H^{\varepsilon}\left(H^{s_1-3}E_{s_2}^2 + H^{(s_1+2)/2}E_{s_2}\right).$$

We are ready to present the proof of the theorems.

Proof of Theorems 3 and 4 For $n \in \mathcal{Z}_s$, one has

$$0 = \sum_{n \in \mathcal{Z}_s} \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right) T^s(\alpha) e(-n\alpha) d\alpha.$$

Then applying Lemma 5.1,

$$\left|\sum_{n\in\mathcal{Z}_s}\int_{\mathfrak{m}} T^s(\alpha)e(-n\alpha)d\alpha\right| = \left|\sum_{n\in\mathcal{Z}_s}\int_{\mathfrak{M}} T^s(\alpha)e(-n\alpha)d\alpha\right| \gg E_s H^{s-1}X^{-2}, \tag{5.9}$$

where $E_s = E_s(N, H)$. On the other hand, by Cauchy's inequality and Lemma 5.3, one has

$$\begin{aligned}\left|\sum_{n\in\mathcal{Z}_s}\int_{\mathfrak{m}} T^s(\alpha)e(-n\alpha)d\alpha\right| &= \left|\int_{\mathfrak{m}} T^s(\alpha)D_s(\alpha)d\alpha\right| \\ &\ll I(10)^{1/2}\left(\int_0^1 T^{2s-10}(\alpha)D_s^2(\alpha)d\alpha\right)^{1/2} \\ &\ll H^{(8s-25)/8+\varepsilon}E_s + H^{(4s+11)/8+\varepsilon}E_s^{1/2}.\end{aligned}$$

It therefore follows from the above estimate and (5.9) that

$$E_s H^{s-1}X^{-2} \ll H^{(4s+11)/8+\varepsilon}E_s^{1/2} + H^{(8s-25)/8+\varepsilon}E_s. \tag{5.10}$$

To obtain a nontrivial estimate of the exceptional sets, we need to make sure that the last term on the right side of (5.10) is smaller than the left side, for which the condition

$$H \gg X^{16/17+\varepsilon} \tag{5.11}$$

would be required, and therefore we must have $E_s H^{s-1}X^{-2} \ll H^{(4s+11)/8+\varepsilon}E_s^{1/2}$, that is

$$E_s \ll X^4 H^{19/4-s+\varepsilon} \ll N^{4/3}H^{19/4-s+\varepsilon}.$$

Theorems 3 and 4 clearly follow from this as required. □

Remark 2 In the last step above, it deduces exactly the same restriction for H as (5.11) when evaluating (1.5) and (1.6). While in the case of (1.4), only a weaker restriction $H \gg X^{12/13+\varepsilon}$ is required. To sum up, the three cases are restricted by (5.11) without exception. This is indeed why we gain the same size of H for all three cases in Theorems 3 and 4, regardless of the upper bound of exceptional sets.

However, when Liu and Sun established (1.3) in [7], they employed a previous estimate for exponential sums over minor arcs due to Meng [9], and consequently the size of H did not come from the restriction that the last term on the right side of (5.10) should be smaller than the left side, but from the restriction related to the first term on the right side of (5.10). Namely, the restriction from the last term is weaker than from the former one. Hence they were able to establish the bound $E_7(N, H) \ll N^{1/3}H^{1-\varepsilon}$ with $H = N^{1/3-1/150+\varepsilon}$, which is of lower magnitude than the case of eight cubes in (1.3).

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