Mathematische Zeitschrift (2021) 298:821–837
https://doi.org/10.1007/s00209-020-02632-3

Mathematische Zeitschrift

Finite type in measure sense for self-similar sets with overlaps

Juan Deng[1] · **Zhiying Wen**[2] · **Lifeng Xi**[3,4]

Received: 29 September 2019 / Accepted: 8 October 2020 / Published online: 22 October 2020

Abstract
For self-similar sets with overlaps, we introduce a notion named the finite type in measure sense and reveal its intrinsic relationships with the weak separation condition and the generalized finite type.

Keywords Self-similar set · Finite type in measure sense · Weak separation condition · Generalized finite type

Mathematics Subject Classification 28A80

1 Introduction

The separation condition for the *iterated function system* (IFS) plays an important role in the study of self-similar fractals with overlaps. We have separation conditions or structures such as the open set condition (**OSC**) by Hutchinson [8], the weak separation condition (**WSC**) by Lau and Ngai [12], the finite type (**FT**) by Ngai and Wang [21], and the generalized finite type (**GFT**) by Jin and Yau [10] and Lau and Ngai [13] independently. As shown in the

Supported by National Natural Science Foundation of China (Nos. 11831007, 11771226, 11871098, 11371329, 11301346) and K.C. Wong Magna Fund in Ningbo University.

✉ Lifeng Xi
xilifengningbo@yahoo.com; xilifeng@nbu.edu.cn

Juan Deng
dengjuan@szu.edu.cn

Zhiying Wen
wenzy@tsinghua.edu.cn

1 Department of Mathematics, ShenZhen University, ShenZhen 518000, People's Republic of China

2 Department of Mathematics, Tsinghua University, Beijing 100084, People's Republic of China

3 Department of Mathematics, Ningbo University, Ningbo 315211, People's Republic of China

4 Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, Hunan, People's Republic of China

survey paper [4], it is known that

$$\text{OSC} \Rightarrow \text{GFT and FT} \Rightarrow \text{GFT} \Rightarrow \text{WSC}.$$

We refer the reader to [3,5,6,14,15,18–20,22,24] for the study on separation conditions and [9,11,23] for self-similar sets with overlaps respectively.

To characterize self-similar sets with overlaps, in this paper we introduce the separation condition named *finite type in measure sense* (Definition 1) in terms of Hausdorff measure rather than in topological and algebraic ways. We will study properties of this separation condition, and reveal its intrinsic relationships with WSC and GFT.

1.1 Main result

Let m be a positive integer and $\Phi = \{\phi_i(x)\}_{i=0}^m$ a family of contractive maps on $\mathbb{R}^n$ of the form

$$\phi_i(x) = \rho_i R_i x + b_i \text{ for } i = 0, 1, \ldots, m, \tag{1.1}$$

where $\rho_i \in (0, 1)$, R_i is orthogonal and $b_i \in \mathbb{R}^n$ for each i. Then Φ is an *iterated function system* (IFS) on $\mathbb{R}^n$ and the attractor of Φ is the unique compact set $K_\Phi \in \mathbb{R}$ satisfying

$$K = K_\Phi = \bigcup_{i=0}^{m} \phi_i(K_\Phi). \tag{1.2}$$

Without loss of generality, we always assume that K does not lie in any hyperplane. Denote $\rho = \min\{\rho_i : i = 0, 1, \ldots, m\}$. For any finite word $I = i_1 i_2 \cdots i_t \in \{0, 1, \ldots, m\}^t$, denote $|I|$ the length of the word, and write $\phi_I = \phi_{i_1} \circ \cdots \circ \phi_{i_t}$, $K_I = \phi_I(K)$ and $\phi_I x = \rho_I R_I x + b_I$.

The weak separation condition was first defined by Lau and Ngai in [12], see also [2,24] and [15]. Here we use an equivalent definition of the WSC from (3b) of Theorem 1 in [24]. An IFS Φ is said to satisfy the WSC if there exists a number $\epsilon > 0$ such that for all (I, J),

$$\text{either } \phi_I = \phi_J \text{ or } \mathrm{d}(\phi_I^{-1}\phi_J, \mathrm{id}) \geq \epsilon, \tag{1.3}$$

where $\mathrm{d}(a_1 Rx + d_1, a_2 Qx + d_2) = |a_1 - a_2| + \left\| RQ^{-1} - \mathrm{id} \right\| + |d_1 - d_2|$ for orthogonal matrices R and Q, $a_1, a_2 > 0$ and $d_1, d_2 \in \mathbb{R}^n$. Suppose Φ satisfies the WSC and $s = \dim_H K$, it is shown in [7] that $\mathcal{H}^s|_K$ is Ahlfors–David regular [17], i.e., there exists a constant $\xi > 0$ such that

$$\xi^{-1} r^s \leq \mathcal{H}^s|_K (B(x, r)) \leq \xi r^s \tag{1.4}$$

for all closed ball $B(x, r)$ centered at $x \in K$ with radius $r \leq \mathrm{diam}(K)$.

The generalized finite type was introduced by Lau and Ngai in [13]. A non-empty open set U is called an invariant open set, if $\bigcup_{i=0}^m \phi_i(U) \subset U$. Then Φ is of GFT if there exists an invariant open set U such that

$$\left\{\phi_I^{-1}\phi_J : \ \phi_I(U) \cap \phi_J(U) \neq \varnothing \text{ with } \rho_I^{-1}\rho_J \in (\rho, \rho^{-1})\right\} \text{ is a finite set.}$$

The finite type (**FT**) was introduced by Ngai and Wang [21]. When the contraction ratios of Φ are exponentially commensurable, then Φ is of FT ([21]) if and only if Φ is of GFT (Theorem 4.2 of [4]).

In this paper, we introduce the following definitions on separation conditions.

Definition 1 An IFS Φ is said to be of weak finite type in measure sense (***WFTM***), if for any $c > 0$, there is a finite set Π_c such that for any (I, J) with $\rho_I^{-1}\rho_J \in (\rho, \rho^{-1})$ and $\mathcal{H}^s(K_I \cap K_J) \geq c\mathcal{H}^s(K_I)$ with $s = \dim_H K$, we have

$$\phi_I^{-1}\phi_J \in \Pi_c.$$

An IFS Φ is said to be of finite type in measure sense (***FTM***), if there exists a finite set Δ such that

$$\phi_I^{-1}\phi_J \in \Delta \text{ for any } (I, J) \in \mathfrak{A},$$

where $\mathfrak{A} = \{(I, J)\colon I \neq J, \rho_I^{-1}\rho_J \in (\rho, \rho^{-1}) \text{ and } \mathcal{H}^s(K_I \cap K_J) > 0\}$ with $s = \dim_H K$.

Theorem 1 *Suppose the IFS Φ and the self-similar set K_Φ are defined in* (1.1) *and* (1.2). *Then*

(1) *WSC⇒WFTM;*
(2) *WSC⇒FTM if*

$$\inf_{(I,J)\in\mathfrak{A}} \frac{\mathcal{H}^s(K_I \cap K_J)}{\mathcal{H}^s(K_I)} > 0. \tag{1.5}$$

(3) *WSC+FTM⇔GFT and WSC+(1.5)⇔GFT.*

Remark 1 The result (3) of Theorem 1 shows that under the ***WSC***, the finite type in measure sense ***(FTM)*** is exactly the generalized finite type with respect to some invariant open set ***(GFT)***.

1.2 Invariant set [0, 1]

We will focus on the case that $K = K_\Phi = [0, 1]$, where the IFS $\Phi = \{\phi_i(x) = \rho_i x + b_i : [0, 1] \to [0, 1]\}_{i=0}^m$ satisfying

$$\rho_i \in (0, 1) \text{ and } b_0 = 0,\ b_m = 1 - \rho_m. \tag{1.6}$$

In this case $\mathfrak{A} = \{(I, J)\colon \rho_I^{-1}\rho_J \in (\rho, \rho^{-1}) \text{ and } \phi_I((0, 1)) \cap \phi_J((0, 1)) \neq \varnothing\}$. For two similitudes $\phi_I(x) = \rho_I x + \phi_I(0)$ and $\phi_J(x) = \rho_J x + \phi_J(0)$, we have

$$\phi_I^{-1}\phi_J(x) = \rho_I^{-1}\rho_J(x) + \rho_I^{-1}(\phi_J(0) - \phi_I(0)), \tag{1.7}$$

then $\phi_I^{-1}\phi_J$ includes the information of relative position $\rho_I^{-1}(\phi_J(0) - \phi_I(0))$ and relative size $\rho_I^{-1}\rho_J$.

In this paper, we will introduce the following Definition 2 according to (1.7).

Definition 2 Suppose Φ is an *IFS* satisfying $K = K_\Phi = [0, 1]$ and (1.6). We say that Φ satisfies the finiteness of relative positions (***FP***), if there exists a finite set $\Gamma \subset [0, 1)$ such that

$$\rho_I^{-1}(\phi_J(0) - \phi_I(0)) \in \Gamma \text{ for all } (I, J) \in \mathfrak{A} \text{ with } \phi_J(0) \geq \phi_I(0). \tag{1.8}$$

An IFS Φ is said to satisfy the finiteness of relative sizes (***FS***), if there is a finite set Λ such that

$$\rho_I^{-1}\rho_J \in \Lambda \text{ for all } (I, J) \in \mathfrak{A}.$$

Notice that FTM⇔FP+FS in this case. In fact, we have

Theorem 2 *Let* $\Phi = \{\phi_i(x) = \rho_i x + b_i : [0,1] \to [0,1]\}_{i=0}^m$ *be an IFS satisfying* (1.6) *and* $K_\Phi = [0,1]$. *Then*

$$WSC \Leftrightarrow FTM \Leftrightarrow GFT \Leftrightarrow FP.$$

Remark 2 In Theorem 2 when $\rho_0 = \cdots = \rho_m$, Feng [5] have obtained WSC⇔FT.

Remark 3 We note that FS $\not\Leftrightarrow$ WSC. For example, let $m = 1$ and $\rho_0 = \rho_1 = \rho > 1/2$, by using a result of Akiyama and Komornik [1], Feng [5] proved that Φ is of FT (and the WSC holds) if and only if ρ^{-1} is a Pisot number.

Using the FS, we can discuss the rational dependence for ratio's logarithm. Firstly, we see the following two examples.

Example 1 As in the left part of Fig. 1, take $\lambda \in (0,1)$ such that $\log\lambda/\log(1-\lambda) \notin \mathbb{Q}$, let $\phi_0(x) = \lambda x$ and $\phi_1(x) = (1-\lambda)x + \lambda$. Then $K_\Phi = [0,1]$ and $\{\phi_i\}_{i=0}^1$ is of GFT.

Example 2 As in the right part of Fig. 1, take $\rho < 1/2$ such that $\log\frac{1-2\rho}{1-\rho}/\log\rho \notin \mathbb{Q}$. Let $\phi_0(x) = \rho_0 x,\ \phi_1(x) = \rho x + \rho_0(1-\rho),\ \phi_2(x) = \rho x + (1-\rho)$. where $\rho_0 = \frac{1-2\rho}{1-\rho}$. Then $\{\phi_i\}_{i=0}^2$ is of GFT but $\log\rho_0/\log\rho \notin \mathbb{Q}$.

Note that $\lambda \notin \phi_0((0,1)) \cup \phi_1((0,1))$ in Example 1 and $(1-\rho) \notin \phi_1((0,1)) \cup \phi_2((0,1))$ in Example 2. Now we add a natural assumption as in Fig. 2:

$$\text{For any } x \in (0,1) \text{ there is a letter } i \text{ such that } x \in \phi_i((0,1)). \tag{1.9}$$

Under assumption (1.9), we have the rational dependence of ratio's logarithm.

Theorem 3 *Let* $\Phi = \{\phi_i(x) = \rho_i x + b_i\}_{i=0}^m$ *be an IFS satisfying* $K_\Phi = [0,1]$, (1.6) *and* (1.9). *If the WSC holds, then there exists a non-zero vector* $(n_0, n_1, \ldots, n_m) \in \mathbb{Z}^{m+1}$ *such that*

$$\sum_{i=0}^m n_i \log\rho_i = 0. \tag{1.10}$$

In particular, if $m = 1$, *then* $\frac{\log\rho_0}{\log\rho_1} \in \mathbb{Q}$. *Moreover, when* $m = 2$, *if the WSC holds and* Φ *is non-degenerate to a sub-IFS with invariant set* $[0,1]$ *satisfying* (1.9), *then* $\frac{\log\rho_0}{\log\rho_2} \in \mathbb{Q}$.

Remark 4 Example 2 shows that we need the assumption (1.9) in Theorem 3 for $m = 2$.

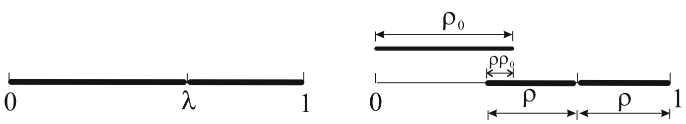

Fig. 1 Examples 1 and 2

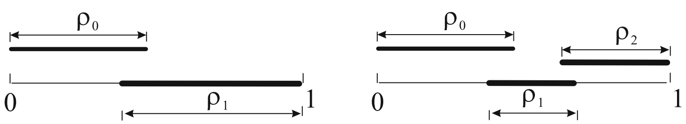

Fig. 2 Assumption (1.9) with $m = 1$ or 2

As shown by Zerner in Proposition 1 of [24], if Φ satisfies the WSC but the OSC fails, then Φ satisfies the complete overlap condition (**COC**). The following example shows

$$\text{COC} \nRightarrow \text{WSC}.$$

Example 3 Let $\Phi = \{\phi_0(x) = \frac{4}{5}x, \phi_1(x) = \frac{5}{6}x + \frac{1}{6}\}$ be an IFS in $\mathbb{R}$. Then the attractor $K_\Phi = [0, 1]$. Furthermore, we can check that

$$\phi_{011100} = \phi_{101001} = \frac{8}{27}x + \frac{91}{270},$$

i.e., Φ satisfies the complete overlap condition. But by Theorem 3, we note that Φ is not of GFT and the WSC fails since $\frac{\log(4/5)}{\log(5/6)} \notin \mathbb{Q}$.

The paper is organized as follows. In Sect. 2, combining the key approach in [5] and the Ahlfors-David regularity of Hausdorff measure on the self-similar set, we obtain a result on the separation condition in measure sense (Theorem 1). In fact, our proof of Theorem 1 (on Hausdorff measure) is also inspired by the dichotomy on self-similar measure in [16] and the construction of "minimal" invariant open set in [4]. In Sect. 3, we will prove Theorem 2. Since the invariant set is a closed interval and GFT$\Leftrightarrow$FTM in this case, we need to verify $\inf_{(I,J)\in\mathfrak{A}} \frac{\mathcal{H}^s(K_I\cap K_J)}{\mathcal{H}^s(K_I)} > 0$ under the WSC. In the last section, we give the rational dependence of ratio's logarithm (Theorem 3).

2 Weak finite type in measure sense

This section is devoted to the proof of Theorem 1. Keep notations in Sect. 1.

Suppose the IFS Φ satisfies the WSC. For a word τ, let

$$M_\tau = \{\phi : \phi = \phi_I \text{ for some word } I \text{ s.t. } K_I \cap K_\tau \neq \varnothing, \rho_I \in [\rho\rho_\tau, \rho_\tau)\}.$$

Here $K_I = \phi_I(K)$. Note that if $\phi_I = \phi_{I'} = \phi$ with $\phi(x) = \bar{\rho}\bar{R}x + \bar{b}$, then $\rho_I = \rho_{I'} = \bar{\rho}$, $R_I = R_{I'} = \bar{R}$ and $b_I = b_{I'} = \bar{b}$. We have the following lemma enlightened by the ideas in [5,6].

Lemma 1 $\sup_\tau \sharp M_\tau < \infty$.

Proof Given a word τ, let $f_\tau : M_\tau \to \mathbb{R} \times O(n) \times \mathbb{R}^n$ be defined by

$$f_\tau(\phi_I) = (\rho_I/\rho_\tau, R_I R_\tau^{-1}, \rho_\tau^{-1}(b_I - b_\tau)) \text{ for } \phi_I(x) = \rho_I x + b_I.$$

We note that the metric on $\mathbb{R} \times O(n) \times \mathbb{R}^n$ is $\mathrm{d}((a_1, R, d_1), (a_2, Q, d_2)) = |a_1 - a_2| + \|RQ^{-1} - id\| + |d_1 - d_2|$ for any $(a_1, R, d_1), (a_2, Q, d_2) \in \mathbb{R} \times O(n) \times \mathbb{R}^n$. Take $a \in K$ and $b \in K_\tau \cap K_I (\neq \varnothing)$, we have $\phi_I(a) \in K_I$ and $\phi_\tau(a) \in K_\tau$, by $\rho_I < \rho_\tau$ we have

$$\begin{aligned}|\rho_\tau^{-1} R_I^{-1}(b_I - b_\tau)| &= \rho_\tau^{-1}|\phi_I(0) - \phi_\tau(0)| \\ &\leq \frac{|\phi_I(a) - b| + |b - \phi_\tau(a)| + |\phi_I(a) - \phi_I(0)| + |\phi_\tau(a) - \phi_\tau(0)|}{\rho_\tau} \\ &\leq \frac{\mathrm{diam}(K_I) + \mathrm{diam}(K_\tau) + \rho_I|a| + \rho_\tau|a|}{\rho_\tau} \leq 2(\mathrm{diam}(K) + |a|).\end{aligned}$$

For any *distinct* elements $\phi(= \phi_I), \phi'(= \phi_J) \in M_\tau$, we have $\rho_I^{-1}\rho_J \in (\rho, \rho^{-1})$ since $\rho_I, \rho_J \in [\rho\rho_\tau, \rho_\tau)$. Notice that

$$\phi^{-1}\phi'(x) = \phi_I^{-1}\phi_J(x) = \rho_I^{-1}R_I^{-1}(\rho_J R_J x + b_J - b_I)$$

$$= (\rho_I^{-1}\rho_J)(R_I^{-1}R_J)x + \rho_I^{-1}R_I^{-1}(b_J - b_I),$$

which implies

$$\begin{aligned} \mathrm{d}(\phi^{-1}\phi', \mathrm{id}) &= \mathrm{d}(\phi_I^{-1}\phi_J, \mathrm{id}) \\ &= |\rho_I^{-1}\rho_J - 1| + \left\| R_I^{-1}R_J - \mathrm{id} \right\| + |\rho_I^{-1}R_I^{-1}(b_J - b_I)| \\ &= \rho_I^{-1}|\rho_I - \rho_J| + \left\| R_I^{-1}R_J - \mathrm{id} \right\| + \rho_I^{-1}|b_I - b_J|. \end{aligned}$$

By (1.3), we have

$$\rho_I^{-1}|\rho_I - \rho_J| + \left\| R_I^{-1}R_J - \mathrm{id} \right\| + \rho_I^{-1}|b_I - b_J| > \epsilon. \tag{2.1}$$

Since $\rho_I \geq \rho_\tau \rho$, i.e., $\rho_\tau^{-1} \geq \rho\rho_I^{-1}$, by (2.1) we have

$$\begin{aligned} \mathrm{d}(f_\tau(\phi), f_\tau(\phi')) &= \mathrm{d}(f_\tau(\phi_I), f_\tau(\phi_J)) \\ &= \rho_\tau^{-1}|\rho_I - \rho_J| + \rho_\tau^{-1}|b_I - b_J| + \left\| R_I^{-1}R_J - \mathrm{id} \right\| \\ &\geq \rho(\rho_I^{-1}|\rho_I - \rho_J| + \rho_I^{-1}|b_I - b_J| + \left\| R_I^{-1}R_J - \mathrm{id} \right\|) > \rho\epsilon. \end{aligned}$$

Notice that $f_\tau(M_\tau) \subset [\rho, 1] \times O(n) \times B(0, 2(\mathrm{diam}(K) + |a|))$ which is a compact subspace, then for this compact set we can take a finite covering of open balls $U_1, \ldots, U_\varsigma$ with radius $\rho\epsilon/2$, now each open ball contains at most one element of $f_\tau(M_\tau)$ due to $\mathrm{d}(f_\tau(\phi), f_\tau(\phi')) > \rho\epsilon$, Hence $\sharp M_\tau \leq \varsigma$ for all τ, i.e., $\sup_\tau \sharp M_\tau \leq \varsigma$. □

Take τ_0 such that

$$\sharp M_{\tau_0} = \sup_\tau \sharp M_\tau = L. \tag{2.2}$$

Let $M_{\tau_0} = \{\phi_{V_1}, \ldots, \phi_{V_L}\}$, where $V_1, \ldots, V_L$ are some words. Suppose Φ satisfies the WSC, it is shown in [7] that $\mathcal{H}^s|_{K_\Phi}$ is Ahlfors-David regular with $s = \dim_H K_\Phi$, i.e., the inequality (1.4) holds.

Lemma 2 *Given any $c \in (0, 1)$, there is an integer N depending only on c such that for any Borel set $K' \subset K$ with $\mathcal{H}^s(K') \geq c\mathcal{H}^s(K)$, we have a word β of length less than N satisfying*

$$K' \cap K_{\beta\tau_0} \neq \varnothing,$$

where τ_0 is given in (2.2).

Proof Let $E_n = K \setminus \left(\bigcup_{\beta \in \{0,\ldots,m\}^p, p \leq n-1} K_{\beta\tau_0} \right)$ and $E = \bigcap_{n=1}^\infty E_n$.

We only need to show that

$$\mathcal{H}^s(E) = \lim_{n\to\infty} \mathcal{H}^s(E_n) = 0. \tag{2.3}$$

In fact, take N such that $\mathcal{H}^s(E_N) < c\mathcal{H}^s(K)$, by $\mathcal{H}^s(K') \geq c\mathcal{H}^s(K)$ we have $K' \cap K_{\beta\tau_0} \neq \varnothing$ for some word β of length less than N.

To verify (2.3), we suppose on the contrary that

$$\mathcal{H}^s(E) > 0.$$

Since $\mathcal{H}^s|_K$ is a Borel regular and locally finite, $\mathcal{H}^s|_K$ is a Radon measure. Using density theorem of Radon measure (e.g. see Corollary 2.14(1) of [17]), we have

$$\lim_{r\to 0}\frac{\mathcal{H}^s|_K(E\cap B(x,r))}{\mathcal{H}^s|_K(B(x,r))}=1 \text{ for } \mathcal{H}^s|_K\text{-a.e. } x\in E. \tag{2.4}$$

Given such a point x satisfying (2.4), for every $B(x,r)$, we can take

$$(x\in)K_\beta\subset B(x,r) \text{ with } \rho r\leq \operatorname{diam}(K_\beta)=\rho_\beta \operatorname{diam}(K)<r.$$

Now, $K_{\beta\tau_0}\subset B(x,r)\backslash E$. Hence

$$\begin{aligned}\mathcal{H}^s|_K\left(E\cap B(x,r)\right)&\leq \mathcal{H}^s|_K\left(B(x,r)\right)-\mathcal{H}^s\left(K_{\beta\tau_0}\right)\\ &=\mathcal{H}^s|_K\left(B(x,r)\right)-\rho_\beta^s\rho_{\tau_0}^s\mathcal{H}^s\left(K\right)\end{aligned}$$

where

$$\frac{\rho_\beta^s\rho_{\tau_0}^s\mathcal{H}^s\left(K\right)}{\mathcal{H}^s|_K\left(B(x,r)\right)}\geq\frac{\left(\mathcal{H}^s\left(K\right)\frac{\rho^s}{\operatorname{diam}(K)^s}\right)r^s}{\xi r^s}=\frac{\mathcal{H}^s\left(K\right)\rho^s}{\xi \operatorname{diam}(K)^s}.$$

Therefore we obtain that

$$\frac{\mathcal{H}^s|_K\left(E\cap B(x,r)\right)}{\mathcal{H}^s|_K\left(B(x,r)\right)}\leq 1-\frac{\mathcal{H}^s\left(K\right)\rho^s}{\xi \operatorname{diam}(K)^s}(<1).$$

This is a contradiction. Then $\mathcal{H}^s\left(E\right)=0$. □

Proof of Theorem 1 (1) Now, consider (I,J) with $\rho_I^{-1}\rho_J\in(\rho,\rho^{-1})$ and $\mathcal{H}^s(K_I\cap K_J)\geq c\mathcal{H}^s(K_I)$. Since $\mathcal{H}^s(\phi_I^{-1}(K_I\cap K_J))\geq c\mathcal{H}^s(K)$, by Lemma 2, we obtain a word β of length less than N satisfying

$$\phi_I^{-1}(K_I\cap K_J)\cap K_{\beta\tau_0}\neq\varnothing.$$

Then take $y\in K_{I\beta\tau_0}\cap K_J$ and let $y\in K_{JJ'}$ with $\rho_{JJ'}\in[\rho\rho_{I\beta\tau_0},\rho_{I\beta\tau_0})$. Note that

$$\rho_{J'}=\frac{\rho_{JJ'}}{\rho_J}\geq\frac{\rho\rho_I\rho_\beta\rho_{\tau_0}}{\rho_J}\geq\rho^2\rho^{|\beta|}\rho_{\tau_0}\geq\rho^{N+2}\rho_{\tau_0}.$$

Now we have $K_{JJ'}\cap K_{I\beta\tau_0}\neq\varnothing$ and $\rho_{JJ'}\in[\rho\rho_{I\beta\tau_0},\rho_{I\beta\tau_0})$, which implies

$$\phi_{JJ'}\in M_{I\beta\tau_0}=\phi_{I\beta}\{\phi_{V_1},\ldots,\phi_{V_L}\}$$

due to the choice of M_{τ_0}. For some $1\leq i\leq L$, we have

$$\phi_{JJ'}=\phi_{I\beta V_i}$$

i.e., $\phi_J\circ\phi_{J'}=\phi_I\circ\phi_{\beta V_i}$ which implies

$$\phi_I^{-1}\circ\phi_J=\phi_{\beta V_i}\circ\phi_{J'}^{-1}$$

Therefore we obtain that

$$\phi_I^{-1}\phi_J\in\{\phi_\beta\phi_{V_i}\phi_{J'}^{-1}:|\beta|<N,\ 1\leq i\leq L \text{ and } \rho_{J'}\geq\rho^{N+2}\rho_{\tau_0}\}\hat{=}\Pi_c.$$

(2) This result follows from (1.5) and result (1) of Theorem 1 directly.

(3) Note that $\frac{\mathcal{H}^s(K_I\cap K_J)}{\mathcal{H}^s(K_I)}=\frac{\mathcal{H}^s(K\cap(\phi_I^{-1}\phi_J(K))}{\mathcal{H}^s(K)}$, it suffices to show that

$$\text{WSC}+\text{FTM}\Leftrightarrow\text{GFT}.$$

It is known that GFT⇒WSC. Under the WSC, we need following two lemmas to show FTM⇒GFT and GFT⇒FTM.

□

Lemma 3 *Suppose K is the self-similar set of an IFS Φ satisfying the WSC. Then there is an invariant open set U with $U\cap K\neq\varnothing$ and*

$$\mathcal{H}^s|_K(U)=\mathcal{H}^s(K) \text{ with } s=\dim_H K, \tag{2.5}$$

such that for any (I,J) with $\rho_I^{-1}\rho_J\in(\rho,\rho^{-1})$ and any invariant open set V,

$$\phi_I(U)\cap\phi_J(U)\neq\varnothing\Longrightarrow\phi_I(V)\cap\phi_J(V)\neq\varnothing. \tag{2.6}$$

Proof The existence of invariant open set U satisfying (2.6) and $U\cap K\neq\varnothing$ has been proved in Proposition 6.5 of [4], we only need to show (2.5).

Suppose on the contrary that $\mathcal{H}^s(K\backslash U)>0$. Using density theorem of Radon measure again, for $F=K\backslash U$ we have

$$\lim_{r\to0}\frac{\mathcal{H}^s|_K(F\cap B(x,r))}{\mathcal{H}^s|_K(B(x,r))}=1 \text{ for } \mathcal{H}^s|_K\text{-a.e. } x\in F. \tag{2.7}$$

Given such a point x satisfying (2.7), for every $B(x,r)$, we can take a word β^* such that

$$(x\in)K_{\beta^*}\subset B(x,r) \text{ with } \rho r\leq\operatorname{diam}(K_{\beta^*})=\rho_{\beta^*}\operatorname{diam}(K)<r.$$

Since $U\cap K\neq\varnothing$, we can find a finite word σ such that $K_\sigma\subset U$. Notice that U is invariant, then for the above word β^*, we have $\phi_{\beta^*}(U)\subset U$ and thus $K_{\beta^*\sigma}\subset U$. Hence

$$\begin{aligned}\mathcal{H}^s|_K\left(F\cap B(x,r)\right)&\leq\mathcal{H}^s|_K\left(B(x,r)\right)-\mathcal{H}^s\left(K_{\beta^*\sigma}\right)\\&=\mathcal{H}^s|_K\left(B(x,r)\right)-\rho_{\beta^*}^s\rho_\sigma^s\mathcal{H}^s\left(K\right)\end{aligned}$$

where

$$\frac{\rho_{\beta^*}^s\rho_\sigma^s\mathcal{H}^s\left(K\right)}{\mathcal{H}^s|_K\left(B(x,r)\right)}\geq\frac{\left(\rho_\sigma^s\mathcal{H}^s\left(K\right)\frac{\rho^s}{\operatorname{diam}(K)^s}\right)r^s}{\xi r^s}=\frac{\rho_\sigma^s\rho^s\mathcal{H}^s\left(K\right)}{\xi\operatorname{diam}(K)^s}.$$

Therefore we obtain

$$\frac{\mathcal{H}^s|_K\left(F\cap B(x,r)\right)}{\mathcal{H}^s|_K\left(B(x,r)\right)}\leq1-\frac{\rho_\sigma^s\rho^s\mathcal{H}^s\left(K\right)}{\xi\operatorname{diam}(K)^s}(<1).$$

This is a contradiction. Hence $\mathcal{H}^s\left(F\right)=0$ and (2.5) follows. □

Motivated by [4], we let $U_\varepsilon=\{x:\mathrm{d}(x,K)<\varepsilon\}$ for some fixed $\varepsilon>0$ and denote

$$U=\bigcup\nolimits_{I\in\{0,\ldots,m\}^t,\ t\geq0}\phi_I(\phi_\sigma U_\varepsilon), \tag{2.8}$$

where $n_I=\sharp\Lambda_I$ with $\Lambda_I=\{\phi_J:\rho_I^{-1}\rho_J\in(\rho,\rho^{-1}),\phi_I(U_\varepsilon)\cap\phi_J(U_\varepsilon)\neq\varnothing\}$ and σ is a word such that

$$n_\sigma=\max_{I\in\{0,\ldots,m\}^t,\ t\geq0}n_I<\infty. \tag{2.9}$$

We also refer the reader to the proof of Lemma 1 for $\max_{I\in\{0,\ldots,m\}^t,\ t\geq0}n_I<\infty$.

By Lemma 3 we focus on the Hausdorff measure and obtain

Lemma 4 *Suppose the WSC holds and U is the invariant open set in Lemma 3. If $\rho_I^{-1}\rho_J \in (\rho, \rho^{-1})$, then $\mathcal{H}^s(K_I \cap K_J) > 0 \Leftrightarrow \phi_I(U) \cap \phi_J(U) \neq \varnothing$.*

Proof "$\Longrightarrow$" Since $\phi_I(K \cap U) \cap \phi_J(K \cap U) \subset K \cap (\phi_I(U) \cap \phi_J(U))$, we obtain that

$$\begin{aligned} \mathcal{H}^s|_K(\phi_I(U) \cap \phi_J(U)) &= \mathcal{H}^s(K \cap (\phi_I(U) \cap \phi_J(U))) \\ &\geq \mathcal{H}^s(\phi_I(K \cap U) \cap \phi_J(K \cap U)). \end{aligned} \tag{2.10}$$

By Lemma 3 we have $\mathcal{H}^s(K \backslash U) = 0$, then $\mathcal{H}^s(\phi_I(K \backslash U)) = \mathcal{H}^s(\phi_J(K \backslash U)) = 0$ since ϕ_I, ϕ_J are Lipschitz maps. Hence

$$\mathcal{H}^s(\phi_I(K \cap U) \cap \phi_J(K \cap U)) = \mathcal{H}^s(K_I \cap K_J). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\mathcal{H}^s|_K(\phi_I(U) \cap \phi_J(U)) \geq \mathcal{H}^s(\phi_I(K \cap U) \cap \phi_J(K \cap U)) = \mathcal{H}^s(K_I \cap K_J) > 0,$$

which implies $\phi_I(U) \cap \phi_J(U) \neq \varnothing$.

"$\Longleftarrow$" Let U be the open set defined in (2.8) and σ the word satisfying (2.9). Since $n_\sigma = n_{I\sigma}$ for any I, we may assume that the length $|\sigma|$ is so large that $\rho_\sigma < \rho^2$.

Now we assume that $\phi_I(U) \cap \phi_J(U) \neq \varnothing$ with $\rho_I^{-1}\rho_J \in (\rho, \rho^{-1})$, by the structure of U, we can find two words β and τ such that $\phi_{I\beta\sigma}(U_\varepsilon) \cap \phi_{J\tau\sigma}(U_\varepsilon) \neq \varnothing$. Without loss of generality, we obtain that $\rho_{I\beta\sigma} \leq \rho_{J\tau\sigma}$, then we can find a prefix κ of $I\beta\sigma$ such that $\rho_\kappa^{-1}\rho_{J\tau\sigma} \in (\rho, \rho^{-1})$ and

$$\phi_\kappa(U_\varepsilon) \cap \phi_{J\tau\sigma}(U_\varepsilon) \neq \varnothing.$$

Notice that $\rho_\kappa < \rho^{-1}\rho_{J\tau\sigma} \leq \rho^{-1}\rho_J\rho_\sigma < \rho\rho_J < \rho_I$, which implies that I is a prefix of κ, says $\kappa = I\kappa'$ for some word κ'. On the other hand, let $\Lambda_\sigma = \{\phi_{\alpha_1}, \ldots, \phi_{\alpha_{n_\sigma}}\}$. Then by the choice of σ, we have

$$\phi_\kappa = \phi_I \circ \phi_{\kappa'} \in \phi_J \circ \phi_\tau \circ \{\phi_{\alpha_1}, \ldots, \phi_{\alpha_{n_\sigma}}\},$$

i.e., there exists an index $1 \leq i \leq n_\sigma$ such that $\phi_I \circ \phi_{\kappa'} = \phi_J \circ \phi_\tau \circ \phi_{\alpha_i}$, that means

$$K_{I\kappa'} = K_{J\tau\alpha_i} \text{ with } \mathcal{H}^s(K_I \cap K_J) \geq \mathcal{H}^s(K_{I\kappa'}) > 0$$

under the WSC. □

By Lemma 4 we obtain FTM⇒GFT (with the above U in Lemma 3) under the WSC directly.

For GFT⇒FTM, we notice that if Φ is of GFT for some invariant open set V, by $\mathcal{H}^s(K_I \cap K_J) > 0 \Rightarrow \phi_I(U) \cap \phi_J(U) \neq \varnothing$ and (2.6) we obtain that $\phi_I(V) \cap \phi_J(V) \neq \varnothing$, which implies $\phi_I^{-1}\phi_J$ belongs to a fixed finite set.

Then part (3) of Theorem 1 is proved.

3 Invariant set [0, 1]

This section is devoted to the proof of Theorem 2. It is known that GFT ⇒ WSC and FTM⇔FS+FP. By part (3) of Theorem 1, we only need to show WSC⇒GFT and FP⇒FS.

3.1 WSC⇒GFT

Suppose the WSC holds. By Theorem 1, to obtain the GFT we only need to verify

$$\inf_{(I,J)\in\mathfrak{A}} \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} > 0,$$

where $s = 1$ and $\mathcal{H}^s = \mathcal{L}$. Let $T = \left\lceil \frac{\log \rho}{\log \max_i(\rho_i)} \right\rceil$ where $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$.

Before the proof, we need

Claim 1 *Suppose* $\rho_I \rho_J^{-1} \in (\rho, \rho^{-1})$ *with* $|I| > 1$ *and* $|J| > 1$*. Then we can find words* σ *and* τ*, with* $1 \leq |\sigma|, |\tau| < 3T$ *such that* $I = I^-\sigma,\ J = J^-\tau$ *and* $\rho_{I^-}\rho_{J^-}^{-1} \in (\rho, \rho^{-1})$.

Remark 5 I^- and J^- are prefixes of I and J respectively, which are not traditional notations standing for the father words of I and J.

Let $c = \rho c_1$ with $c_1 = \min(\rho^{3T+1}, \min\{b_\sigma > 0 : |\sigma| < 3T\})$. By Theorem 1, there is a corresponding finite set Π_c. Let

$$\Pi = \bigcup_{1\leq|\sigma|,|\tau|<3T} \phi_\sigma^{-1}\Pi_c\phi_\tau \text{ and } d = \min_{\psi\in\Pi\cap\mathcal{PM}} \frac{\mathcal{L}(K \cap \psi(K))}{\mathcal{L}(K)} > 0, \tag{3.1}$$

where $\mathcal{PM} = \{\psi = \phi_I^{-1}\phi_J : \mathcal{L}(K \cap \psi(K)) > 0\}$.

For WSC$\Longrightarrow$GFT, we only need to show

Lemma 5 *Suppose* $\Phi = \{\phi_i(x) = \rho_i x + b_i\}_{i=0}^m$ *such that* $K = K_\Phi = [0, 1]$*. If* Φ *satisfies the weak separation condition, then*

$$\inf_{(I,J)\in\mathfrak{A}} \frac{\mathcal{H}^s(K_I \cap K_J)}{\mathcal{H}^s(K_I)} = \inf_{(I,J)\in\mathfrak{A}} \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} \geq d.$$

Proof Suppose on the contrary there exist $\varepsilon \in (0, d]$ and $(I, J) \in \mathfrak{A}$ such that

$$0 < \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} < \varepsilon(\leq d). \tag{3.2}$$

Assume that the above (I, J) is the pair of shortest words satisfying (3.2).

We divide the proof into two steps.

Step 1. We will prove that

$$\frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{I^-})} \geq c. \tag{3.3}$$

Let $K_I = [A_I, B_I]$, $K_J = [A_J, B_J]$. Without of generality, we assume K_J intersects K_I only around A_I as in Fig. 3. We use Cases (1)–(3) to prove inequality (3.3).

Case (1). If A_I is an inner point of K_{I^-}, i.e., $K_{I^-} = [\alpha, \beta]$ with

$$\alpha < A_I < \beta.$$

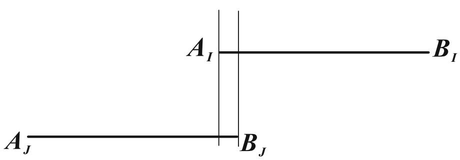

Fig. 3 K_J intersects K_I

Therefore, we have

$$\begin{aligned}\mathcal{L}(K_{I^-} \cap K_{J^-}) &\geq \mathcal{L}([\alpha, \beta] \cap [A_J^1, B_J^1]) \\ &\geq \mathcal{L}([\max(\alpha, A_J),\ B_J]) \\ &\geq \min(\mathcal{L}([\alpha, A_I],\ \mathcal{L}([A_J, B_J])).\end{aligned}$$

Let $I = I^-\sigma^*$. We have $\mathcal{L}([\alpha, A_I]) = |A_I - \alpha| = b_{\sigma^*}\rho_{I^-} \geq c_1\rho_{I^-} = c_1\mathcal{L}(K_{I^-})$ and $\mathcal{L}([A_J, B_J]) = \mathcal{L}(K_J) \geq \frac{\mathcal{L}(K_J)}{\mathcal{L}(K_I)}\frac{\mathcal{L}(K_I)}{\mathcal{L}(K_{I^-})}\mathcal{L}(K_{I^-}) \geq \rho^{3T+1}\mathcal{L}(K_{I^-}) \geq c_1\mathcal{L}(K_{I^-})$. Hence

$$\frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{I^-})} \geq c_1(> c).$$

Case (2). If B_J is an inner point of the convex hull of K_{J^-}, in the same way as above we also have

$$\frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{J^-})} \geq c_1,$$

and thus

$$\frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{I^-})} = \frac{\mathcal{L}(K_{J^-})}{\mathcal{L}(K_{I^-})} \cdot \frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{J^-})} \geq \rho \cdot c_1 = c.$$

Case (3). Now, we may assume that A_I is the left end-point of the convex hull of K_{I^-} and B_J is the right end-point of the convex hull of K_{J^-}, then

$$\frac{\mathcal{L}(K_{I^-} \cap K_{J^-})}{\mathcal{L}(K_{I^-})} = \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_{I^-})} < \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} < \varepsilon,$$

which is a contradiction to the shortest choice of (I, J).

Then inequality (3.3) follows.

Step 2. We will show that

$$\frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} \geq d,$$

which is contradictory to (3.2). Now, by (3.3) and Theorem 1, there is a finite set Π_c such that

$$\phi_{I^-}^{-1}\phi_{J^-} \in \Pi_c,$$

which implies

$$\phi_I^{-1}\phi_J \in \bigcup_{1\leq|\sigma|,|\tau|<3T} \phi_\sigma^{-1}\Pi_c\phi_\tau.$$

Let Π and d be defined in (3.1). Suppose $\phi_I^{-1}\phi_J = \psi^* \in \Pi$, we have

$$0 < \frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} = \frac{\mathcal{L}(\phi_I(K) \cap \phi_J(K))}{\mathcal{L}(\phi_I(K))} = \frac{\rho_I\mathcal{L}(K \cap \phi_I^{-1}\phi_J(K))}{\rho_I\mathcal{L}(K)} = \frac{\mathcal{L}(K \cap \psi^*(K))}{\mathcal{L}(K)}.$$

That means $\psi^* \in \mathcal{PM}$ and

$$\frac{\mathcal{L}(K_I \cap K_J)}{\mathcal{L}(K_I)} \geq \min_{\psi\in\Pi\cap\mathcal{PM}} \frac{\mathcal{H}^s(K \cap \psi(K))}{\mathcal{L}(K)} = d > 0.$$

$\square$

3.2 FP ⇒ FS

Suppose Φ satisfies the FP. we will show that Φ satisfies the FS, and thus Φ is of GFT.

At first, we will deal with the case $\phi_I(0) = \phi_J(0)$. Denote $[i]^k = \underbrace{i \cdots i}_{k}$.

Lemma 6 *There exists a finite set Λ_1 such that if $\phi_I(0) = \phi_J(0)$ (i.e., $b_I = b_J$), with $\rho_I \rho_J^{-1} \in (\rho, \rho^{-1})$, then*

$$\rho_I \rho_J^{-1} \in \Lambda_1.$$

Proof Suppose that $\rho_I \le \rho_J$. We have $\phi_J^{-1}\phi_I(x) = (\rho_I x + b_I - b_J)/\rho_J = (\rho_J^{-1}\rho_I)x$ due to $b_I = b_J$, and thus

$$\phi_J^{-1}\phi_I(1) = \rho_J^{-1}\rho_I \in [0, 1].$$

Since $K_\Phi = [0, 1]$, there is an infinite sequence $i_1 \cdots i_k \cdots$ in $\{0, \ldots, m\}^\infty$ such that $\{\phi_J^{-1}\phi_I(1)\} = \bigcap_{k=1}^{\infty} \phi_{i_1 \cdots i_k \cdots}([0, 1])$, which implies that we can find an integer k such that $\frac{\rho_{i_1 \cdots i_k}}{\rho_J^{-1}\rho_I} \in [\rho \cdot \rho_m^{2T}, \rho_m^{2T})$ where $(\rho_J^{-1}\rho_I)\rho_m^{2T} \le \rho_m^{2T} \le \rho$ and $\frac{\rho_{i_1 \cdots i_{t+1}}}{\rho_{i_1 \cdots i_t}} \ge \rho$ for all t. Let $\sigma = i_1 \cdots i_k$, we obtain

$$\rho \cdot \rho_m^{2T} \le \frac{\rho_\sigma}{\rho_J^{-1}\rho_I} < \rho_m^{2T}, \tag{3.4}$$

and thus

$$\rho_\sigma \ge (\rho_J^{-1}\rho_I)(\rho \cdot \rho_m^{2T}) \ge \rho^2 \cdot \rho_m^{2T}.$$

Therefore we have

$$\phi_J \phi_\sigma([0, 1]) \cap \phi_{I[m]^{2T}}([0, 1]) \ne \varnothing.$$

We will distinguish the following two cases.

Case 1. If $\phi_J^{-1}\phi_I(1) = \phi_\sigma(0)$ or $\phi_\sigma(1)$, then

$$\rho_J^{-1}\rho_I \in \{\phi_\sigma(\theta) : \theta = 0 \text{ or } 1 \text{ and } \rho_\sigma \ge \rho^2 \cdot \rho_m^{2T}\},$$

which is a finite set.

Case 2. If $\phi_J^{-1}\phi_I(1) \in (\phi_\sigma(0), \phi_\sigma(1))$, then we have

$$\phi_{I[m]^{2T}}((0, 1)) \cap \phi_{J\sigma}((0, 1)) \ne \varnothing$$

and $\phi_{I[m]^{2T}}(0) < \phi_{J\sigma}(0)$ by (3.4) which implies

$$\phi_J^{-1}\phi_{I[m]^{2T}}(0) < \phi_\sigma(0). \tag{3.5}$$

Suppose Γ is the finite set with respect to the FP as in (1.8), we have

$$\rho_{I[m]^{2T}}^{-1}(\phi_{J\sigma}(0) - \phi_{I[m]^{2T}}(0)) \in \Gamma, \tag{3.6}$$

where $\phi_{J\sigma}(0) = \phi_J(0) + \rho_J\phi_\sigma(0)$ and $\phi_{I[m]^{2T}}(0) = \phi_I(0) + \rho_I\phi_{[m]^{2T}}(0) = \phi_I(0) + \rho_I(1 - \rho_m^{2T})$ with $\phi_J(0) = \phi_I(0)$. Notice that

$$\rho_{I[m]^{2T}}^{-1}(\phi_{J\sigma}(0) - \phi_{I[m]^{2T}}(0)) = \rho_m^{-2T}(\rho_I^{-1}\rho_J\phi_\sigma(0) - (1 - \rho_m^{2T})). \tag{3.7}$$

By (3.6) and (3.7), we obtain that

$$\rho_I^{-1}\rho_J \in \frac{\rho_m^{2T}\Gamma + (1-\rho_m^{2T})}{\phi_\sigma(0)},$$

where $\phi_\sigma(0) > \phi_J^{-1}\phi_{I[m]^{2T}}(0) = \rho_J^{-1}\rho_I(1-\rho_m^{2T}) > 0$ due to (3.5). Therefore,

$$\rho_I^{-1}\rho_J \in \bigcup_{\rho_\sigma \geq \rho^2\cdot\rho_m^{2T}} \frac{\rho_m^{2T}\Gamma + (1-\rho_m^{2T})}{\phi_\sigma(0)}$$

which is a finite set. □

We will show that there is a finite set Λ such that

$$\rho_I^{-1}\rho_J \in \Lambda \text{ for all } (I, J) \in \mathfrak{A}.$$

Since $\phi_J(0) \geq \phi_I(0)$, by Lemma 6, we may assume that

$$\phi_J(0) > \phi_I(0). \tag{3.8}$$

Lemma 7 *There exists a finite set Ω such that if $(I^{(0)}, J^{(0)}) \in \mathfrak{A}$ with $\phi_{J^{(0)}}(0) > \phi_{I^{(0)}}(0)$, we can find (I, J) satisfying $0 \leq \rho_I^{-1}(\phi_J(0) - \phi_I(0)) \leq 1$ and $\rho_I\rho_J^{-1} \in (\rho, \rho^{-1})$ such that*

$$\frac{\rho_I}{\rho_{I^{(0)}}}, \frac{\rho_J}{\rho_{J^{(0)}}} \in \Omega$$

and the last letter of J is m.

Proof Since $(I^{(0)}, J^{(0)}) \in \mathfrak{A}$, we have $\rho_{I^{(0)}}^{-1}(\phi_{J^{(0)}}(0) - \phi_{I^{(0)}}(0)) \leq \max_{\gamma\in\Gamma\setminus\{1\}}\gamma < 1$ due to the discreteness of Γ. Then

$$\rho_{I^{(0)}}^{-1}(\phi_{I^{(0)}}(1) - \phi_{J^{(0)}}(0)) = \rho_{I^{(0)}}^{-1}(\rho_{I^{(0)}} - (\phi_{J^{(0)}}(0) - \phi_{I^{(0)}}(0)) \geq 1 - \max_{\gamma\in\Gamma\setminus\{1\}}\gamma.$$

Let $c = 1 - \max_{\gamma\in\Gamma\setminus\{1\}}\gamma > 0$. Take an integer

$$k = \left[\frac{\log(\rho c)}{\log\rho_0}\right] + 1,$$

then $(\rho_0)^k < c\rho$ and thus

$$(\rho_0)^k\rho_{J^{(0)}} < c\rho(\rho_{J^{(0)}}\rho_{I^{(0)}}^{-1})\rho_{I^{(0)}} < c\rho_{I^{(0)}} \leq |\phi_{I^{(0)}}(1) - \phi_{J^{(0)}}(0)|.$$

That means $\phi_J([0,1]) \subset [\phi_{J^{(0)}}(0), \phi_{I^{(0)}}(1)]$ for $J = J^{(0)}[0]^k m$. Suppose the left endpoint $\phi_J(0)$ belongs to $\phi_{I^{(0)}\sigma}([0,1])$ with its length

$$\rho\cdot\rho_J \leq |\phi_{I^{(0)}\sigma}([0,1])| < \rho_J$$

where

$$\rho_J = (\rho_0)^k\rho_{J^{(0)}}\rho_m < (c\rho)(\rho_{J^{(0)}}\rho_{I^{(0)}}^{-1})\rho_{I^{(0)}}\rho_m < \rho_{I^{(0)}}\rho_m < \rho_{I^{(0)}}.$$

We also note that

$$\rho_\sigma \geq (\rho_{I^{(0)}}^{-1}\rho_{J^{(0)}})\rho_m\rho_0^k \geq \rho\rho_m\rho_0^k.$$

Take $I = I^{(0)}\sigma$. Now, since $\phi_J(0) \in \phi_J([0,1]) \cap \phi_I([0,1]) \neq \varnothing$, we have $0 \leq \rho_I^{-1}(\phi_J(0) - \phi_I(0)) \leq 1$ and $\rho_I\rho_J^{-1} \in (\rho, \rho^{-1})$. Let

$$\Omega = \{\rho_\sigma : \rho_\sigma \geq \rho\rho_m\rho_0^k\} \cup \{\rho_m\rho_0^k\}$$

with $k = \left[\frac{\log(\rho c)}{\log \rho_0}\right] + 1$. Then

$$\frac{\rho_I}{\rho_{I^{(0)}}}, \frac{\rho_J}{\rho_{J^{(0)}}} \in \Omega.$$

□

By Lemmas 6 and 7, we only need to deal with the finiteness of the values $\rho_I \rho_J^{-1}$ under the conditions:

(1) $0 < \rho_I^{-1}(\phi_J(0) - \phi_I(0)) \leq 1$;
(2) $\rho_I \rho_J^{-1} \in (\rho, \rho^{-1})$;
(3) the last letter of J is m with $b_m > 0$.
For notational convenience, we add 1 to Γ, i.e., $1 \in \Gamma$, then the above condition (1) implies

$$\rho_I^{-1}(\phi_J(0) - \phi_I(0)) \in \Gamma.$$

Using Claim 1, we can find words σ and τ, with $1 \leq |\sigma|, |\tau| < 3T$ such that $I = I'\sigma$, $J = J'\tau$ and $\rho_{I'} \rho_{J'}^{-1} \in (\rho, \rho^{-1})$. Now,

$$\phi_I(0) = \phi_{I'}(0) + \rho_{I'} b_\sigma, \ \phi_J(0) = \phi_{J'}(0) + \rho_{J'} b_\tau.$$

Therefore,

$$\phi_{J'}(0) - \phi_{I'}(0) = \phi_J(0) - \phi_I(0) + \rho_{I'} b_\sigma - \rho_{J'} b_\tau. \tag{3.9}$$

Using Lemma 7, we only to need to show $\rho_I \rho_J^{-1}$ belongs to a finite set. Since the last letter of J is m, we have $b_\tau > 0$.

Case A. If $\phi_{J'}(0) > \phi_{I'}(0)$, then

$$\rho_{I'}^{-1}(\phi_{J'}(0) - \phi_{I'}(0)) = (\rho_\sigma)(\rho_I^{-1}(\phi_J(0) - \phi_I(0))) + b_\sigma - (\rho_{I'}^{-1}\rho_{J'}) b_\tau.$$

Since $b_\tau \neq 0$, then

$$\rho_{I'}^{-1}\rho_{J'} = \frac{(\rho_\sigma)(\rho_I^{-1}(\phi_J(0) - \phi_I(0))) + b_\sigma - \rho_{I'}^{-1}(\phi_{J'}(0) - \phi_{I'}(0))}{b_\tau} \in \frac{\rho_\sigma \Gamma + b_\sigma - \Gamma}{b_\tau}.$$

Hence

$$\rho_I^{-1}\rho_J \in \bigcup_{1 \leq |\sigma|, |\tau| < 3T, \ b_\tau > 0} \rho_\sigma^{-1} \rho_\tau \left(\frac{\rho_\sigma \Gamma + b_\sigma - \Gamma}{b_\tau}\right)$$

which is a finite set.

Case B. If $\phi_{J'}(0) \leq \phi_{I'}(0)$, then

$$\rho_{J'}^{-1}(\phi_{J'}(0) - \phi_{I'}(0)) = (\rho_J^{-1}\rho_I)\rho_\tau(\rho_I^{-1}(\phi_J(0) - \phi_I(0))) + (\rho_J^{-1}\rho_I)\rho_\tau \rho_\sigma^{-1} b_\sigma - b_\tau.$$

That is

$$(\rho_J^{-1}\rho_I)(\rho_\tau(\rho_I^{-1}(\phi_J(0) - \phi_I(0)) + \rho_\sigma^{-1} b_\sigma)) = \rho_{J'}^{-1}(\phi_{J'}(0) - \phi_{I'}(0)) + b_\tau \in \Gamma + b_\tau,$$

and $\rho_\tau(\rho_I^{-1}(\phi_J(0) - \phi_I(0)) \in \Gamma \backslash \{0\}$. Since

$$\rho_I^{-1}(\phi_J(0) - \phi_I(0)) + \rho_\sigma^{-1} b_\sigma \geq \min_{\gamma \in \Gamma \backslash \{0\}} \gamma > 0,$$

we have

$$\rho_J^{-1}\rho_I \in \bigcup_{1\le|\sigma|,|\tau|<3T} \frac{\Gamma + b_\tau}{\rho_\tau(\Gamma\backslash\{0\} + \rho_\sigma^{-1}b_\sigma)}$$

which is a finite set.

4 Rational dependence of ratio's logarithm

Let $\Phi = \{\phi_i(x) = \rho_i x + b_i\}_{i=0}^m$ be an IFS satisfying (1.6) and $K = K_\Phi = [0, 1]$.

In this section, we always assume that Φ satisfies the WSC, i.e., Φ is of GFT according to Theorem 2. In particular, Φ satisfies the finiteness of relative sizes (FS). We will discuss the rational dependence of ratio's logarithm by using the FS.

For any infinite word $\sigma = i_1 \cdots i_j \cdots$, denote the starting finite word with length j by $\sigma|_j = i_1 \cdots i_j$.

Under the assumption of Theorem 3, we obtain

Lemma 8 *For any* $0 < x < 1$*, there exists an infinite word* $\sigma = i_1 \cdots i_j \cdots$ *such that*

$$\{x\} = \bigcap_{j=1}^{\infty} \phi_{\sigma|_j}((0, 1)).$$

Proof Prove it by induction. Firstly, by the assumption (1.9) of Theorem 3, for any $0 < x_0 < 1$, there exists a letter $i \in \{0, 1, \ldots, m\}$ such that $x_0 \in \phi_i((0, 1))$. Assume there exists a finite word $i_1 \cdots i_k$ such that $x \in \bigcap_{j=1}^{k} \phi_{i_1\cdots i_j}((0, 1))$. Now $\phi_{i_1\cdots i_k}^{-1}(x) := x_0 \in (0, 1)$, by the above discussion, $x_0 \in \phi_i((0, 1))$ for some $i \in \{0, 1, \ldots, m\}$. Then take $i_{k+1} = i$, we have $x \in \phi_{i_1\cdots i_k i_{k+1}}((0, 1))$. □

Let

$$\Omega = \{(I, J) : I \neq J \text{ s.t. } \phi_I((0, 1)) \cap \phi_J((0, 1)) \neq \varnothing \text{ and } \rho_I\rho_J^{-1} \in (\rho, \rho^{-1})\}. \quad (4.1)$$

Given a word $I = i_1 \cdots i_k$, denote $\sharp i_I = \sharp\{t \le k : i_t = i\}$.

Lemma 9 *Assume* Φ *satisfies the FS. If* Φ *is rationally independent for ratio's logarithm, that is*

$$\sum_{i=0}^{m} n_i \log \rho_i \neq 0, \quad \forall (n_0, n_1, \ldots, n_m) \in \mathbb{Z}^{m+1}\backslash\{0\}. \quad (4.2)$$

then there is an integer $N \in \mathbb{N}$ *such that for all* $(I, J) \in \Omega$ *and all* $i = 0, \ldots, m$*,*

$$|\sharp i_I - \sharp i_J| \le N, \quad (4.3)$$

Proof Since $\rho_I\rho_J^{-1} = \prod_{i=0}^{m} \rho_i^{\sharp i_I - \sharp i_J} \in \Lambda$, by (4.2) we obtain the existence of N. □

Proof Theorem 3 To show (1.10), we suppose on the contrary that Φ is rationally independent for ratio's logarithm. We assume that $\phi_1(0) \in \phi_0((0, 1))$. Otherwise, we can take a sub-IFS $\Phi' = \{\phi_0', \phi_1', \ldots\} \subset \Phi$ such that the assumption in Theorem 3 holds for Φ' where $\phi_0'(0) = 0 < \phi_1'(0)$ and $\phi_1'(0) \in \phi_0'((0, 1))$.

Let $x_0 = \phi_0^{-1}\phi_1(0) \in (0, 1)$. By Lemma 8, there exists an infinite word σ such that $\{x_0\} = \bigcap_{j=1}^{\infty} \phi_{\sigma|_j}((0, 1))$. Since x_0 is not the left endpoint of $\phi_{\sigma|_j}((0, 1))$ for any j, then we can find out a letter $i^* \neq 0$ such that i^* appears in $\sigma = u_1 \cdots u_k \cdots$ for infinitely many times. Suppose $\{t+1 : u_t = i^*\} = \{k_1 < k_2 < \cdots\}$. Let $J_t = 0u_1 \cdots u_{k_t}$, and take

$$I_t = 1[0]^{g(t)} \text{ with } g(t) \in \{0\} \cup \mathbb{N}$$

such that $\rho_{I_t}\rho_{J_t}^{-1} \in (\rho, \rho^{-1})$. It is clear that $\phi_{I_t}((0,1)) \cap \phi_{J_t}((0,1)) \neq \varnothing$. Now, for any integer $t > 1$,

$$\sharp i^*_{I_t} \leq 1 \text{ and } \sharp i^*_{J_t} = t,$$

that means

$$|\sharp i^*_{I_t} - \sharp i^*_{J_t}| \geq t - 1, \ \forall t > 1. \tag{4.4}$$

By Lemma 9, there exists a positive integer N such that $|\sharp i^*_{I_t} - \sharp i^*_{J_t}| \leq N$ for any $t > 0$, this is a contradiction. Then (1.10) is proved.

In particular, if $m = 1$, then $\log \rho_0 / \log \rho_1 \in \mathbb{Q}$ follows from (1.10) directly.

Suppose $m = 2$ and the IFS Φ is non-degenerate to the case $m = 1$, we can assume that $\phi_1(0) \in \phi_0((0,1))$ without loss of generality. Now we will show that $\log \rho_0 / \log \rho_2 \in \mathbb{Q}$. Notice that $K_\Phi = [0,1]$. Denote

$$x = \phi_0^{-1}\phi_1(0) \in (0,1).$$

Then $\phi_1(0) = \phi_0(x)$. By Lemma 8, there exists an infinite word $\sigma = u_1 \cdots u_k \cdots\cdots$ such that $\{x\} = \bigcap_{k=1}^\infty \phi_{\sigma|_k}((0,1))$. Since x is not the left endpoint of $\phi_{\sigma|_t}((0,1))$ for any t, we can find out $i^* \in \{1,2\}$ such that i^* appears in $\sigma = u_1 \cdots u_k \cdots$ for infinitely many times. Suppose $\{t+1 : u_t = i^*\} = \{k_1 < k_2 < \cdots\}$. Let $J_t = 0u_1 \cdots u_{k_t}$, and $I_t = 1[0]^{T(t)}$ with $T(t) \in \{0\} \cup \mathbb{N}$ such that $\rho_{I_t}\rho_{J_t}^{-1} \in (\rho, \rho^{-1})$. Since $\phi_0(x) \in \phi_{0\sigma|_{k_i}}((0,1))$, we have

$$\phi_{I_t}((0,1)) \cap \phi_{J_t}((0,1)) \neq \varnothing \text{ for all } t.$$

Since Φ is of GFT, the FS holds, i.e., there exists a finite set Λ such that $\rho_{I_t}\rho_{J_t}^{-1} \in \Lambda$ for all t. Using the finiteness of Λ, we can find $k_i < k_j$,

$$\frac{\rho_{1[0]^{T(k_i)}}}{\rho_{0\sigma|_{k_i}}} = \frac{\rho_{1[0]^{T(k_j)}}}{\rho_{0\sigma|_{k_j}}} \in \Lambda,$$

and thus $(\rho_0)^{T(k_j)-T(k_i)} = \rho_{u_{k_i+1}} \cdots \rho_{u_{k_j}} = \rho_0^l \rho_1^u \rho_2^v$ with $l, u, v \in \{0\} \cup \mathbb{N}$ and $u+v > 0$ due to the existence of $i^* \in \{1,2\}$. We notice that $l < T(k_j) - T(k_i)$. In fact $(\rho_0)^{T(k_j)-T(k_i)-l} = \rho_1^u \rho_2^v < 1$ due to $u + v > 0$. Let $k = T(k_j) - T(k_i) - l > 0$. Now, we have

$$\rho_1^u \rho_2^v = \rho_0^k. \tag{4.5}$$

In the same way, when considering the point $\phi_2^{-1}\phi_1(1)$ and the right endpoint, we obtain

$$\rho_0^{u'} \rho_1^{v'} = \rho_2^{k'} \tag{4.6}$$

with integers $k' > 0$ and $u', v' \geq 0$ with $u' + v' > 0$.

Suppose on the contrary that $\log \rho_0 / \log \rho_2 \notin \mathbb{Q}$. Without loss of generality, we assume that $u > 0$ and $v' > 0$. Otherwise, for example we suppose $u = 0$, by (4.5), we have $\rho_2^v = \rho_0^k$ with $k > 0$.

Now $u > 0$ and $v' > 0$. Suppose $\rho_1 = \rho_0^a \rho_2^b$ with $a, b \in \mathbb{Q}$. Using (4.5), we have $\rho_2^{bu+v} = \rho_0^{k-ua}$ which implies $bu + v = k - ua = 0$. Hence

$$b = -v/u \leq 0. \tag{4.7}$$

Using (4.6), we have $\rho_0^{u'+av} = \rho_2^{k'-bv'}$, which implies $u' + av = k' - bv' = 0$. Hence

$$b = k'/v' > 0. \tag{4.8}$$

Then (4.7) and (4.8) are contradictory. That means $\log \rho_0 / \log \rho_2 \in \mathbb{Q}$. □

Acknowledgements The authors thank Professor dejun Feng for helpful discussion.

References

1. Akiyama, S., Komornik, V.: Discrete spectra and Pisot numbers. J. Number Theory **133**, 375–390 (2013)
2. Bandt, C., Graf, S.: Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorr measure. Proc. Am. Math. Soc **114**, 995–1001 (1992)
3. Das, M., Edgar, G.A.: Finite type, open set condition and weak separation condition. Nonlinearity. **9**, 2489–2503 (2011)
4. Deng, Q.-R., Lau, K.-S., Ngai, S.-M.: Separation conditions for iterated function systems with overlaps, Fractal geometry and dynamical systems in pure and applied mathematics. I. Fractals in pure mathematics, 1–20, Contemp. Math., 600, Amer. Math. Soc., Providence, RI, (2013)
5. Feng, D.J.: On the topology of polynomial with bounded integer coefficients. J. Eur. Math. Soc. **18**(1), 181–193 (2016)
6. Feng, D.-J., Lau, K.-S.: Multifractal formalism for self-similar measures with weak separation condition. J. Math. Pures. Appl. **92**, 407–428 (2009)
7. Fraser, J.M., Henderson, A.M., Olson, E.J., Robinson, J.C.: On the Assouad dimension of self-similar sets with overlaps. Adv. Math. **273**, 188–214 (2015)
8. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**(5), 713–747 (1981)
9. Hochman, M.: On self-similar sets with overlaps and inverse theorems for entropy. Ann. Math. **180**(2), 773–822 (2014)
10. Jin, N., Yau, Stephen S.T.: General finite type IFS and M-matrix. Comm. Anal. Geom **13**(4), 821–843 (2005)
11. Kenyon, R.: Projecting the one dimensional Sierpinski gasket. Israel J. Math. **97**, 221–238 (1997)
12. Lau, K.-S., Ngai, S.-M.: Multifractal measures and a weak seperation condition. Adv. Math. **141**(1), 45–96 (1999)
13. Lau, K.-S., Ngai, S.-M.: A generalalized finite type condition for iterated function systems. Adv. Math. **208**(2), 647–671 (2007)
14. Lau, K.-S., Ngai, S.-M., Rao, H.: Iterated function systems with overlaps and self-similar measures. J. Lond. Math. Soc.(2) **63**(1), 99–116 (2001)
15. Lau, K.-S., Ngai, S.-M., Wang, X.-Y.: Separation conditions for conformal iterated function systems. Monatsh. Math. **156**(4), 325–355 (2009)
16. Lau, K.-S., Wang, J.: Mean quadratic variations and Fourier asymptotics of self-similar measures. Monatsh. Math. **115**(1–2), 99–132 (1993)
17. Mattila, P.: Geometry of sets and measures in Euclidean spaces. Cambridge University Press, Cambridge (1995)
18. Nguyen, N.: Iterated function systems of finite type and the weak separation property. Proc. Am. Math. Soc. **130**(2), 483–487 (2001)
19. Ni, T.-J., Wen, Z.-Y.: Open set condition for graph directed self-similar structure. Math. Z. **276**(1–2), 243–260 (2014)
20. Ngai, S.-M., Wang, F., Dong, X.: Graph-directed iterated function systems satisfying the generalized finite type condition. Nonlinearity **23**(9), 2333–2350 (2010)
21. Ngai, S.-M., Wang, Y.: Hausdorff dimension of self-similar sets with overlaps. J. Lond. Math. Soc. **63**, 655–672 (2001)
22. Rao, H., Wen, Z.-Y.: A class of self-similar fractals with overlap structure. Adv. Appl. Math. **20**(1), 50–72 (1998)
23. Solomyak, B.: On the random series $\sum \pm\lambda^n$ (an Erdős problem). Ann. Math.(2) **142**(3), 611–625 (1995)
24. Zerner, M.P.W.: Weak separation properties for self-similar sets. Proc. Am. Math. Soc. **124**(11), 3529–3539 (1996)