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The universal surface bundle over the Torelli space has no sections

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Abstract
For $g > 3$, we give two proofs of the fact that the *Birman exact sequence* for the Torelli group

$$1 \to \pi_1(S_g) \to \mathcal{I}_{g,1} \to \mathcal{I}_g \to 1$$

does not split. This result was claimed by Mess (Unit tangent bundle subgroups of the mapping class groups, MSRI Pre-print, 1990), but his proof has a critical and unrepairable error which will be discussed in the introduction. Let $\mathcal{UI}_{g,n} \xrightarrow{Tu'_{g,n}} \mathcal{BI}_{g,n}$ (resp. $\mathcal{UPI}_{g,n} \xrightarrow{Tu_{g,n}} \mathcal{BPI}_{g,n}$) denote the universal surface bundle over the Torelli space fixing n points as a set (resp. pointwise). We also deduce that $Tu'_{g,n}$ has no sections when $n > 1$ and that $Tu_{g,n}$ has precisely n distinct sections for $n \geq 0$ up to homotopy.

1 Introduction

It is a basic problem to understand when bundles have continuous sections, and the corresponding group theory problem as to when short exact sequences have splittings. These are equivalent problems when the fiber, the base and the total space are all $K(\pi, 1)$-spaces. In this article, we will discuss the "section problems" and the "splitting problems" in the setting of surface bundles. Here by *section* we mean continuous section.

Let $S_{g,n}$ be a closed orientable surface of genus g with n marked points. Let $\mathrm{Mod}(S_{g,n})$ (resp. $\mathrm{PMod}(S_{g,n})$) be the *mapping class group* of $S_{g,n}$, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of S_g fixing n points as a set (resp. pointwise). Both $\mathrm{Mod}(S_{g,n})$ and $\mathrm{PMod}(S_{g,n})$ act on $H^1(S_g;\mathbb{Z})$ leaving invariant the algebraic intersection numbers. Let $\mathcal{I}_{g,n}$ (resp. $\mathcal{PI}_{g,n}$) be the *Torelli group* (resp. *pure Torelli group*) of $S_{g,n}$, i.e., the subgroup of $\mathrm{Mod}(S_{g,n})$ (resp. $\mathrm{PMod}(S_{g,n})$) that acts trivially on $H^1(S_g;\mathbb{Z})$. We omit n when $n = 0$. The following *Birman exact sequence* for the Torelli group provides a relationship between $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$; e.g., see Farb and Margalit [7, Chapter 4.2]

$$1 \to \pi_1(S_g) \xrightarrow{\text{point pushing}} \mathcal{I}_{g,1} \xrightarrow{T\pi_{g,1}} \mathcal{I}_g \to 1. \quad (1)$$

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The main theorem of this paper is the following:

Theorem 1.1 (Nonsplitting of the Birman exact sequence for the Torelli group) *For $g > 3$, the Birman exact sequence for the Torelli group* (1) *does not split.*

Remark 1 Our proof needs the condition $g > 3$. By Mess [14, Proposition 4], $\mathcal{I}_2$ is a free group. So the Birman exact sequence for $\mathcal{I}_2$ splits. The case $g = 3$ is open.

Let $\mathcal{BPI}_{g,n} := K(\mathcal{PI}_{g,n}, 1)$ be the *pure universal Torelli space* fixing n marked points pointwise and let

$$S_g \to \mathcal{UPI}_{g,n} \xrightarrow{Tu_{g,n}} \mathcal{BPI}_{g,n} \tag{2}$$

be the *pure universal Torelli bundle*. Surface bundle (2) classifies smooth S_g-bundle equipped with a basis of $H^1(S_g;\mathbb{Z})$ and n ordered points on each fiber. Since $\mathcal{PI}_{g,n}$ fixes n points, there are n distinct sections $\{Ts_i | 1 \le i \le n\}$ of the universal Torelli bundle (2). Let $\mathcal{BI}_{g,n} := K(\mathcal{I}_{g,n}, 1)$ be the *universal Torelli space* fixing n marked points as a set and let

$$S_g \to \mathcal{UI}_{g,n} \xrightarrow{Tu'_{g,n}} \mathcal{BI}_{g,n} \tag{3}$$

be the *universal Torelli bundle*. This bundle classifies smooth S_g-bundles equipped with a basis of $H^1(S_g;\mathbb{Z})$ and n unordered points on each fiber. Theorem 1.1 says that $Tu_{g,0}$ has no sections. For $n \ge 0$, we have the following complete answer for sections of (refTUB2) and (3).

Theorem 1.2 (Classification of sections for the n-pointed Torelli bundles) *The following holds:*

(1) *For $n \ge 0$ and $g > 3$, every section of the universal Torelli bundle (2) is homotopic to Ts_i for some $i \in \{1, 2, ..., n\}$.*
(2) *For $n > 1$ and $g > 3$, the universal Torelli bundle (3) has no continuous sections.*

Let $\mathcal{M}_g := K(\mathrm{Mod}(S_g), 1)$. As is known, the universal bundle

$$S_g \to \mathcal{UM}_g \to \mathcal{M}_g$$

has no sections. This can be seen from the corresponding algebraic problem of finding splittings of the *Birman exact sequence*

$$1 \to \pi_1(S_g) \to \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g) \to 1.$$

An analysis of the finite subgroups $\mathrm{Mod}(S_g)$ and $\mathrm{Mod}(S_{g,1})$ shows that the sequence does not split: every finite subgroup of $\mathrm{Mod}(S_{g,1})$ is cyclic, but there are noncyclic finite subgroups of $\mathrm{Mod}(S_g)$; e.g., see Farb and Margalit [7, Corollary 5.11]. However, this method does not work for torsion-free subgroups of $\mathrm{Mod}(S_g)$ like Torelli groups. For any subgroup $G < \mathrm{Mod}(S_g)$, there is an extension Γ_G of G by $\pi_1(S_g)$ as the following short exact sequence

$$1 \to \pi_1(S_g) \to \Gamma_G \to G \to 1. \tag{4}$$

We call (4) the *Birman exact sequence* for G since it is induced from the Birman exact sequence. In a later paper of Chen and Salter [5], they establish the virtual non-splitting of Birman exact sequence for finite index subgroup of Torelli groups. This gives another proof of Theorem 1.1 and strengthens Theorem 1.1 as well.

Define $\pi^0 = \pi_1(S_g)$ and $\pi^{n+1} = [\pi^n, \pi^0]$. The kth term of the *Johnson filtration* subgroup of $\mathrm{Mod}(S_g)$, denoted by $\mathrm{Mod}_g(k)$, is the kernel of the action of $\mathrm{Mod}(S_g)$ on π^0/π^k. For example, $\mathrm{Mod}_g(1) = \mathcal{I}_g$. The Johnson filtration was defined by Johnson [11]. We pose the following open problem.

Problem (Splitting of Birman exact sequence for Johnson filtration subgroups) Do the Birman exact sequence split for Johnson filtration subgroups $\mathrm{Mod}_g(k)$ for $k \geq 2$?

Error in Mess [13, Proposition 2] In the unpublished paper of Mess [13, Proposition 2], he claimed that there are no splittings of the exact sequence (1). But his proof has a fatal error. Here is how the proof goes. Let C be a curve dividing S_g into 2 parts $S(1)$ and $S(2)$ of genus p and q, where $p, q \geq 2$. Let US_g be the unit tangent bundle of a surface of genus g. The unit tangent bundle subgroups of S_p and S_q amalgamate along the common boundary Dehn twist to give a subgroup $A \leq \mathrm{Mod}(S_g)$ satisfying the following short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(US_p) \times \pi_1(US_q) \to A \to 1.$$

In Case a) of Mess' proof, he tried to prove the following claim, which is the key of his proof. Without the claim, the argument appears to be unrepairable.

Mess' Claim in the proof of Proposition 2 The Birman exact sequence for A does not split. □

However, this is a wrong claim. We construct a splitting of the Birman exact sequence for A as the following. Let $\mathrm{PConf}_2(S_p)$ be the *pure configuration space* of S_p, i.e., the space of ordered 2-tuples of distinct points on S_p. Define

$$\mathrm{PConf}_{1,1}(S_p) = \{(x, y, v) | x \neq y \in S_p \text{ and } v \in US_p \text{ a unit vector at} x\}.$$

We have the following pullback diagram

$$\begin{array}{ccc} \pi_1(\mathrm{PConf}_{1,1}(S_p)) & \longrightarrow & \pi_1(\mathrm{PConf}_2(S_p)) \\ \downarrow f & \lrcorner & \downarrow g \\ \pi_1(US_p) & \longrightarrow & \pi_1(S_p). \end{array} \tag{5}$$

The splitting of $\pi_1(US_p)$ should lie in $\pi_1(\mathrm{PConf}_{1,1}(S_p))$ instead of $\pi_1(US_{p,1})$ as Mess claimed in Case a) of Proposition 2. As long as we can find a section of f, we will find a section of A to $\mathcal{I}_{g,1}$. By the property of pullback diagrams, a section of g can induce a section of f in diagram (5). To negate the argument of Mess Proposition 2, we only need to construct a section of g. We simply need to find a self-map of S_g that has no fixed point. For example, the composition of a retraction of S_p onto a curve c and a rotation of c at any nontrivial angle does not have a fixed point. Therefore Mess' proof is invalid and does not seem to be repairable.

Strategy of the proof of Theorem 1.1 Our first strategy is the following: assume that we have a splitting $\phi : \mathcal{I}_g \to \mathcal{I}_{g,1}$ of (1). Let T_a be the Dehn twist about a simple closed curve a on S_g. The main result of Johnson [10] shows that all *bounding pair maps*, i.e., $T_a T_b^{-1}$ for a pair of non-separating curves a, b that bound a subsurface, generate $\mathcal{I}_g$. Firstly we need to understand $\phi(T_a T_b^{-1})$. We will show that $\phi(T_a T_b^{-1}) = T_{a'} T_{b'}^{-1}$ for a bounding pair a', b' on $S_{g,1}$ such that a' is homotopic to a and b' is homotopic to b after forgetting the puncture. Moreover, the curve a' does not depend on the choice of b. Then we use the lantern relation to derive a contradiction. Our main tool is the canonical reduction system for a mapping class, which in turn uses the Thurston classification of isotopy classes of diffeomorphisms of surfaces. This idea originated from Birman et al. [1].

We also give an algebraic proof using cohomology obstruction. The key tool is the classification theorem of Chen [4, Theorem 1.5] and the Johnson homomorphism.

Outline of the paper In Sect. 2, we give an introduction to canonical reduction systems and lantern relations. Then in Sect. 3, we give our first proof using those tools. In Sect. 4, we deal with the punctured case. Then in Sect. 5, we use Sect. 4 and homology to give the second proof of Theorem 1.1.

2 Canonical reduction systems and the lantern relation

In this section, we will give some background that will be used in the proof.

2.1 Canonical reduction systems

The central tool for the proof of Theorem 1.1 is the notion of a *canonical reduction system*, which can be viewed as an enhancement of the Nielsen–Thurston classification. We remind the reader that a curve $c \subset S$ is said to be *peripheral* if c is isotopic to a boundary component or surrounds a marked point of S.

The Nielsen–Thurston classification asserts that each nontrivial element $f \in \mathrm{Mod}(S)$ is of exactly one of the following types: *periodic*, *reducible*, or *pseudo-Anosov*. A mapping class f is *periodic* if $f^n = id$ for some $n \geq 1$, and is *reducible* if for some $n \geq 1$, there is some nonperipheral simple closed curve $c \subset S$ such that $f^n(c)$ is isotopic to c. If neither of these conditions are satisfied, f is said to be *pseudo-Anosov*. In this case, f is isotopic to a homeomorphism f' of a very special form. We will not need to delve into the theory of pseudo-Anosov mappings, and refer the interested reader to Farb and Margalit [7, Chapter 13] and Fathi et al. [6] for more details.

Definition 2.1 (*Reduction systems*) A *reduction system* of a reducible mapping class h in $\mathrm{Mod}(S)$ is a set of disjoint nonperipheral curves that h fixes as a set up to isotopy. A reduction system is *maximal* if it is maximal with respect to inclusion of reduction systems for h. The *canonical reduction system* $\mathrm{CRS}(h)$ is the intersection of all maximal reduction systems of h.

Canonical reduction systems allow for a refined version of the Nielsen–Thurston classification. For a reducible element f, there exists n such that f^n fixes each element in $\mathrm{CRS}(f)$ and after cutting out $\mathrm{CRS}(f)$, the restriction of f^n on each component is either identity or pseudo-Anosov. This is called the *canonical form* of f; e.g., see Farb and Margalit [7, Corollary 13.3]. In Propositions 2.2–2.6, we list some properties of the canonical reduction systems that will be used later.

Proposition 2.2 *$CRS(h^n) = CRS(h)$ for any n.*

Proof This is classical; see Farb and Margalit [7, Chapter 13]. □

For two curves a, b on a surface S, let $i(a, b)$ be the geometric intersection number of a and b. For two sets of curves P and Q, we say that P and Q *intersect* if there exist $a \in P$ and $b \in Q$ such that $i(a, b) \neq 0$. We emphasize that "intersection" here refers to the intersection of curves on S, and not the abstract set-theoretic intersection of P and Q as sets.

Proposition 2.3 *Let h be a reducible mapping class in Mod(S). If* $\{\gamma\}$ *and CRS(h) intersect, then no power of h fixes* γ.

Proof Suppose that h^n fixes γ. Therefore γ belongs to a maximal reduction system M. By definition, $\text{CRS}(h) \subset M$. However γ intersects some curve in $\text{CRS}(f)$; this contradicts the fact that M is a set of disjoint curves. □

Proposition 2.4 *Suppose that* $h, f \in Mod(S)$ *and* $fh = hf$. *Then CRS(h) and CRS(f) do not intersect.*

Proof Conjugating, $\text{CRS}(hfh^{-1}) = h(\text{CRS}(f))$. Since $hfh^{-1} = f$, it follows that $\text{CRS}(f) = h(\text{CRS}(f))$. Therefore h fixes the whole set $\text{CRS}(f)$. There is some $n \geq 1$ such that h^n fixes all curves element-wise in $\text{CRS}(f)$. By Proposition 2.3, curves in $\text{CRS}(h)$ do not intersect curves in $\text{CRS}(f)$. □

For a curve a on a surface S, denote by T_a the Dehn twist about a. More generally, a *Dehn multitwist* is any mapping class of the form

$$T := \prod T_{a_i}^{k_i}$$

for a collection of pairwise-disjoint simple closed curves $\{a_i\}$ and arbitrary integers k_i.

Proposition 2.5 *Let*

$$T := \prod T_{a_i}^{k_i}$$

be a Dehn multitwist. Then

$$\text{CRS}(T) = \{a_i\}.$$

Proof Firstly T cannot contain any simple closed curves b for which $i(b, a_i) \neq 0$, since no power of T preserves b. This can be seen from the equation

$$i\left(\prod T_{a_i}^{k_i}(b), b\right) = \sum |k_i| i(a_i, b) \neq 0 = i(b, b);$$

see Farb and Margalit [7, Proposition 3.2]. It follows that if S is any reduction system for T, then $S \cup \{a_i\}$ is also a reduction system, and hence that $\{a_i\} \subset \text{CRS}(T)$. If γ is disjoint from each element of $\{a_i\}$ but not equal to any a_i, then there exists some curve δ, also disjoint and distinct from each a_i, such that $i(\gamma, \delta) \neq 0$. As both $\{a_i\} \cup \{\gamma\}$ and $\{a_i\} \cup \{\delta\}$ are reduction systems for T, this shows that no such γ can be contained in $\text{CRS}(T)$ and hence that $\text{CRS}(T) = \{a_i\}$ as claimed. □

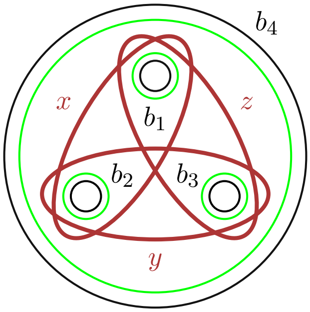
The final result we will require follows from the theory of pseudo-Anosov mapping. It appears in McCarthy [12, Theorem 1].

Proposition 2.6 (McCarthy) *Let S be a Riemann surface of finite type, and let* $f \in \text{Mod}(S)$ *be a pseudo-Anosov element. Then the centralizer subgroup of* f *in* Mod(S) *is virtually cyclic.*

2.2 The lantern relation

Now, we introduce a remarkable relation for Mod(S) that will be used in the proof.

Proposition 2.7 (The lantern relation) *Let $x, y, z, b_1, b_2, b_3, b_4$ be simple closed curves in $S = S_{0,4}$ that are arranged as the curves shown in the following figure.*



In Mod(S) we have the relation

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}.$$

Proof This is classical; see Farb and Margalit [7, Chapter 5.1].

3 The lantern relation proof of theorem 1.1

3.1 Images of bounding pair maps

Let $\{a, b\}$ be a *bounding pair*, i.e., a, b are non-separating curves such that a and b bounds a subsurface. In this subsection, we will determine $\phi(T_a T_b^{-1})$. For two curves c and d, denote by $i(c, d)$ the geometric intersection number of c and d. For a curve c' on $S_{g,1}$, when we say c' is isotopic to a curve c on S_g, we mean that c' is isotopic to c on S_g by forgetting the marked point.

Lemma 3.1 (1) *Let $\{a, b\}$ be a bounding pair, and fix $k > 0$ such that $(T_a T_b^{-1})^k \in \Gamma$. Up to a swap of a and b, we have that $\sigma((T_a T_b^{-1})^k) = (T_{a'} T_{b'}^{-1})^k (T_{a'}^{-1} T_{a''})^n$, where n is an integer and a', a'', b' are three disjoint curves on $\Sigma_{g,1}$ such that a', a'' are isotopic to a and b' is isotopic to b. Notice that n can be zero.*

(2) *Let c be a separating curve on Σ_g that divides Σ_g into two subsurfaces each of genus at least two. For any $k > 0$ such that $(T_c)^k \in \Gamma$, we have that $\sigma((T_c)^k) = (T_{c'})^k (T_{c'}^{-1} T_{c''})^n$ where n is an integer and c' and c'' are a pair of curves on $\Sigma_{g,1}$ that are both isotopic to c.*

Proof Let $(T_a T_b^{-1})^k \in \Gamma$ be a power of a bounding pair map. Since the centralizer of $(T_a T_b^{-1})^k$ contains a copy of $\mathbb{Z}^{2g-3}$ as a subgroup of $\mathcal{I}(\Sigma_g)$, the centralizer of $(T_a T_b^{-1})^k$ as a subgroup of Γ contains a copy of $\mathbb{Z}^{2g-3}$ as well. By the injectivity of σ, the centralizer of $\sigma(T_a T_b^{-1}) \in \mathcal{I}(\Sigma_{g,*})$ contains a copy of $\mathbb{Z}^{2g-3}$. When $g > 3$, we have that $2g - 3 > 3$. Therefore $\sigma((T_a T_b^{-1})^k) \in \mathcal{I}(\Sigma_{g,*})$ cannot be pseudo-Anosov because the centralizer of a pseudo-Anosov element is a virtually cyclic group by Proposition 2.6. For any curve γ' on $\Sigma_{g,*}$, denote by γ the same curve on Σ_g. We decompose the proof into the following three steps.

- **Step 1: CRS$(\sigma((T_aT_b^{-1})^k))$ only contains curves that are isotopic to a or b.** Suppose that there exists $\gamma' \in \mathrm{CRS}(\sigma((T_aT_b^{-1})^k))$ such that γ is not isotopic to a or b. There are two cases.
  - **Case 1: γ intersects a. or b.** Since a power of $\sigma((T_aT_b^{-1})^k)$ fixes γ', a power of $(T_aT_b^{-1})^k$ fixes γ. On the other hand, $\mathrm{CRS}((T_aT_b^{-1})^k) = \{a, b\}$. Combined with Lemma 2.3, this shows that $(T_aT_b^{-1})^k$ does not fix γ. This is a contradiction.
  - **Case 2: γ does not intersect a and b.** In this case by the change-of-coordinates principle, there exists a separating curve c on Σ_g such that $i(a, c) = 0$, $i(b, c) = 0$ and $i(c, \gamma) \neq 0$. Assume that $T_c^m \in \Gamma$. Since $(T_aT_b^{-1})^k$ and T_c^m commute in Γ, the two mapping classes $\sigma((T_aT_b^{-1})^k)$ and $\sigma(T_c^m)$ commute in $\mathcal{I}(\Sigma_{g,*})$. Therefore a power of $\sigma(T_c^m)$ fixes $\mathrm{CRS}(\sigma(T_aT_b^{-1}))$; more specifically a power of T_c^m fixes γ. However by Lemma 2.3, no power of T_c fixes γ. This is a contradiction.
- **Step 2: CRS$(\sigma((T_aT_b^{-1})^k))$ must contain curves a' and b' that are isotopic to a and b, respectively.**
  Suppose that $\mathrm{CRS}(\sigma((T_aT_b^{-1})^k))$ does not contain a curve a' isotopic to a. Then by Step 1, we have $\mathrm{CRS}(\sigma((T_aT_b^{-1})^k))$ either contains one curve b' isotopic to b or two curves b' and b'' both isotopic to b. After cutting $\Sigma_{g,*}$ along $\mathrm{CRS}(\phi((T_aT_b^{-1})^k))$, there is some component C that is not a punctured annulus. C is homeomorphic to the complement of b in Σ_g.
  By the Nielsen–Thurston classification, a power of $\sigma((T_aT_b^{-1})^k)$ is either pseudo-Anosov on C or else is the identity on C. If a power of $\sigma((T_aT_b^{-1})^k)$ is pseudo-Anosov on C, then the centralizer of $\sigma((T_aT_b^{-1})^k)|_C$ is virtually cyclic by Proposition 2.6. Combining with $T_{b'}$ and $T_{b''}$, the centralizer of $\sigma((T_aT_b^{-1})^k)$ in $\mathcal{I}(\Sigma_{g,*})$ is virtually an abelian group of rank at most 3. This contradicts the fact that the centralizer of $\sigma((T_aT_b^{-1})^k)$ contains a subgroup $\mathbb{Z}^{2g-3}$, since $g \geq 4$ and hence $2g - 3 > 3$. Therefore $\sigma((T_aT_b^{-1})^k)$ is the identity on C. However, viewing $C = \Sigma_g \backslash \{b\}$ as a subsurface of Σ_g that contains a, we see that $(T_aT_b^{-1})^k$ is actually not the identity on C; this is a contradiction.
- **Step 3:$\sigma((T_aT_b^{-1})^k) = (T_{a'}T_{b'}^{-1})^k(T_{a'}^{-1}T_{a''})^n$, where n is an integer and a', a'', b' are three disjoint curves on $\Sigma_{g,*}$ such that a', a'' are isotopic to a and b' is isotopic to b.** Suppose that $\sigma((T_aT_b^{-1})^k)$ is pseudo-Anosov on some component C of

  $$\Sigma_{g,*} \backslash \mathrm{CRS}(\sigma((T_aT_b^{-1})^k))$$

  Since the genus $g(C) \geq 1$, there exists a separating curve s on C such that $\sigma(T_s^m)$ commutes with $\sigma((T_aT_b^{-1})^k)$ in $\sigma(\Gamma)$. Therefore, some power of $\sigma((T_aT_b^{-1})^k)$ fixes $\mathrm{CRS}(\sigma(T_s^m))$, which is either one curve or two curves isotopic to s. Thus a power of $\sigma((T_aT_b^{-1})^k)$ fixes some curve on C, which means that $\sigma((T_aT_b^{-1})^k)$ is not pseudo-Anosov on C. It follows that a power of $\sigma((T_aT_b^{-1})^k)$ must be a product of Dehn twists about the curves in $\mathrm{CRS}(\sigma((T_aT_b^{-1})^k))$. Since $\sigma((T_aT_b^{-1})^k)$ is a lift of $(T_aT_b^{-1})^k$, the lemma holds.

The same argument works for $T_c^m \in \Gamma$ the Dehn twist about a separating curve c as long as both components of $\Sigma_g \backslash \{c\}$ have genus two or greater. □

In $\phi(T_aT_b^{-1}) = (T_{a'})^n(T_{a''})^{1-n}T_{b'}$, denote $(T_{a'})^n(T_{a''})^{1-n}$ by the *a component of* $\phi(T_aT_b^{-1})$. Notice that without loss of generality, the b component of $\phi(T_aT_b^{-1})$ could also be a product of Dehn twists. In the following lemma, we will prove that the a component of $\phi(T_aT_b^{-1})$ does not depend on the choice of b.

Lemma 3.2 *For two bounding pairs* $\{a, b\}$ *and* $\{a, c\}$*, the a component of* $\phi(T_aT_b^{-1})$ *is the same as the a component of* $\phi(T_aT_c^{-1})$.

Proof For b, c disjoint, let us first work on the case when $\phi(T_aT_b^{-1}) = (T_{a'})^n(T_{a''})^{1-n}T_{b'}^{-1}$ and $n \neq 0, 1$. Then by Proposition 2.5, we know that $\mathrm{CRS}(\phi(T_aT_b^{-1})) = \{a', a'', b'\}$. Since $T_aT_b^{-1}$ and $T_aT_c^{-1}$ commutes, we know that $\mathrm{CRS}(\phi(T_aT_b^{-1}))$ and $\mathrm{CRS}(\phi(T_aT_c^{-1}))$ should disjoint by Proposition 2.4. Therefore $\phi(T_aT_c^{-1}) = (T_{a'})^k(T_{a''})^{1-k}T_{c'}^{-1}$ since a', a'' are the only curves that are isotopic to a after forgetting and disjoint from a', a''. However then

$$\phi(T_cT_b^{-1}) = \phi(T_aT_b^{-1}))\phi(T_aT_c^{-1})^{-1} = T_{c'}T_{b'}^{-1}(T_{a'})^{n-k}(T_{a''})^{n-k}.$$

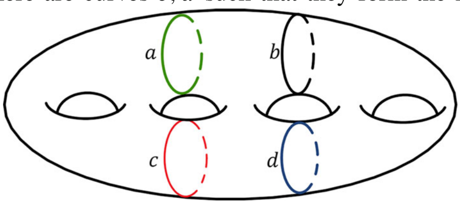
This contradicts Lemma 3.1. We can do a similar argument for case $n = 0$.

When b, c intersect, there are a series of curves $\{b_1 = b, b_2, ..., b_n = c\}$ such that b_i, b_{i+1} are disjoint for $1 \leq i \leq n-1$ and a, b_i form a bounding pair for $1 \leq i \leq n$; see Putman [15, Theorem 1.9]. Therefore, the a components of $\phi(T_aT_b^{-1})$ and $\phi(T_aT_c^{-1})$ are the same by applying the above argument $n-1$ times. □

We denote by the capital letter A the subset of curves in $\mathrm{CRS}(\phi(T_aT_b^{-1}))$ that are isotopic to a. By Lemma 3.2, A only depends on the curve a. It can be a one-element set or a two-element set.

Lemma 3.3 *For* $g \geq 4$ *and two curves* a, b *on* S_g *such that* $i(a, b) = 0$*, we have that* A *is disjoint from* B.

Proof Suppose that a, b are non-separating. The case of separating curves are the same. If a, b bound a subsurface, then by Lemma 3.1, we know that A and B are disjoint. If a, b do not bound, then there are curves c, d such that they form the following configuration.



Notice that $g \geq 4$ is needed here. Since $\phi(T_aT_c^{-1})$ and $\phi(T_bT_d^{-1})$ commute, their canonical reduction systems do not intersect by Corollary 2.4. Therefore A and B are disjoint. □

3.2 A nonsplitting lemma for the braid groups

Let $\mathbb{D}_n$ be a 2-disk with n marked points. The n-strand *pure braid group* is denoted by PB_n, i.e., the pure mapping class group of $\mathbb{D}_n$ fixing the n marked points pointwise. The center Z_n of PB_n is the Dehn twist about the curve surrounding all marked points. Let $\mathcal{F}_n : PB_{n+1} \to PB_n$ be the natural forgetful map forgetting a marked point. In this subsection, we prove a nonsplitting lemma for $\mathcal{F}_3$ that will be used in the proof of Theorem 1.1.

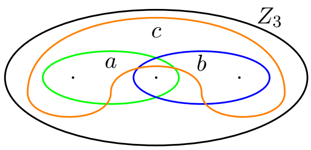


Fig. 1 $T_a T_b T_c = T_{Z_3}$

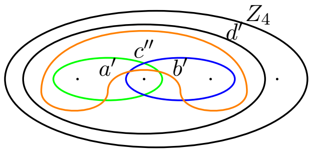


Fig. 2 Case 1

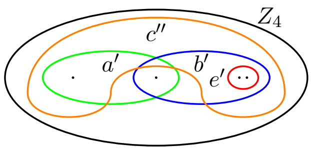


Fig. 3 Case 2

Lemma 3.4 *There is no homomorphism $\mathcal{S} : PB_3 \to PB_4$ such that Dehn twists are mapped to Dehn twists, $\mathcal{S}(T_{Z_3}) = T_{Z_4}$ and $\mathcal{F}_3 \circ \mathcal{S} = id$.*

Proof Suppose the opposite that we have $\mathcal{S} : PB_3 \to PB_4$ such that Dehn twists map to Dehn twists, $\mathcal{S}(Z_3) = Z_4$ and $\mathcal{F}_3 \circ \mathcal{S} = id$. Let c be a simple closed curve on $\mathbb{D}_3$ and we call c' the curve on $\mathbb{D}_4$ such that $\mathcal{S}(T_c) = T_{c'}$. In Fig. 1, the lantern relation gives $T_a T_b T_c = T_{Z_3} \in PB_3$. Since $\mathcal{S}(T_{Z_3}) = T_{Z_4}$, we have $T_{a'} T_{b'} T_{c'} = T_{Z_4} \in PB_4$. If $i(a', b') > 2$, then $T_{a'} T_{b'}$ is pseudo-Anosov on a subsurface of $\mathbb{D}_4$ by Thurston's construction, which contradicts that $T_{a'} T_{b'} = T_{Z_4} T_{c'}^{-1}$ a multitwist; e.g., see Chen [3, Proposition 2.13] for Thurston's construction. So $i(a', b') = 2$. There are several cases we need to concern about the number of points a' and b' surround.

- **Case 1: a' bounds 2 points and b' bounds 2 points.** Then we have $T_{a'} T_{b'} = T_{d'} T_{c''}^{-1}$ for some curve c', d' by lantern relation as in Fig. 2 such that c'' surrounds 2 points and d' surrounds 3 points. By applying ϕ to the original lantern relation $T_a T_b T_c = T_{Z_3}$, we obtain $T_{a'} T_{b'} T_{c'} = T_{Z_4}$. This gives us that $T_{d'} T_{c''}^{-1} = T_{Z_4} T_{c'}^{-1}$. Thus we have $c'' = c'$ and $d' = Z_4$, which contradicts the fact that d' only surrounds 3 points.
- **Case 2: a' bounds 2 points and b' bounds 3 points or a' bounds 3 points and. b' bounds 3 points** By symmetry, we only consider the case that a' bounds 2 points and b' bounds 3 points. Then we have $T_{a'} T_{b'} = T_{Z_4} T_{e'} T_{c''}^{-1}$ by the lantern relation as is shown in Fig. 3 where c'' surrounds 3 points and e' surrounds 2 points. By applying ϕ to the original lantern relation $T_a T_b T_c = T_{Z_3}$, we obtain $T_{a'} T_{b'} T_{c'} = T_{Z_4}$. Therefore $T_{Z_4} T_{e'} T_{c''}^{-1} = T_{Z_4} T_{c'}^{-1}$, which is not possible.

Without loss of generality, we have exhausted all possibilities. □

3.3 Proof of Theorem 1.1

In this proof, we do a case study on the possibilities of $\phi(T_a T_e^{-1})$ for a bounding pair map $T_a T_e^{-1}$. Case 1 and 2 are when the a component is not a single Dehn twist. Case 3 is when the component of every curve is a single Dehn twist.

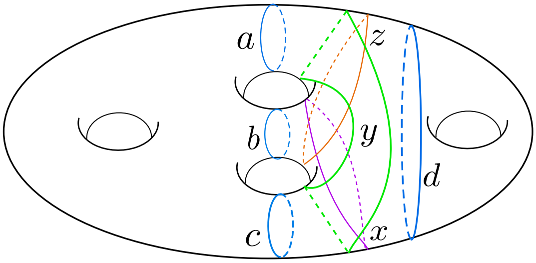


Fig. 4 On S_g

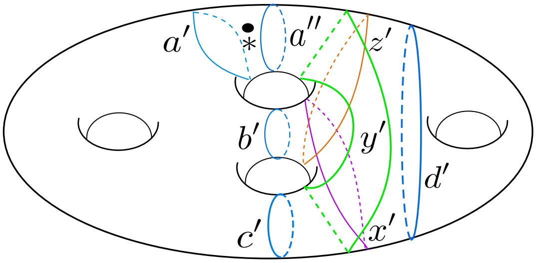


Fig. 5 On $S_{g,1}$

Proof We break our discussion into the following two cases.

Case 1: There is a bounding pair map $T_aT_e^{-1}$ such that the a component is a multi-twist about a', a''. Then there exist curves b, c, d such that a, b, c, d bound a sphere with 4 boundary components as Fig. 4. We need $g \geq 4$ here.

There are curves x, y, z such that we have the lantern relation $T_aT_x^{-1}T_bT_y^{-1}T_cT_z^{-1}T_d = 1$ as in Fig. 4. The curves $\{b', c', d', x', y', z'\}$ of $\{b, c, d, x, y, z\}$ do not intersect a', a'' as in Fig. 5 by Lemma 3.1. The complement of a in S_g and the complement of the annulus formed by a', a'' in $S_{g,1}$ are the same surfaces. Therefore there is an identification of curves on those subsurfaces. We call the correspondent curves b', c', d', x', y', z'. By observation, they should satisfy lantern relation

$$T_{a'}T_{x'}^{-1}T_{b'}T_{y'}^{-1}T_{c'}T_{z'}^{-1}T_{d'} = 1.$$

After applying ϕ, we have

$$(T_{a'})^n(T_{a''})^{1-n}T_{x'}^{-1}T_{b'}T_{y'}^{-1}T_{c'}T_{z'}^{-1}T_{d'} = 1.$$

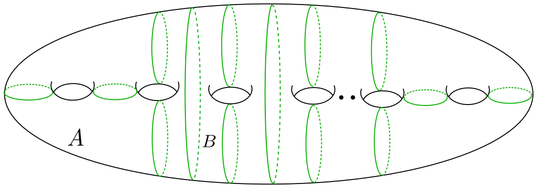
By computation, we have $(T_{a'})^n(T_{a''})^{1-n} = T_{a'}$, which contradicts the assumption on the components of a.

Case 2: There is a bounding curve c such that the c component has 2 curves c', c''. The proof of Case 1 works for this case as well.

Case 3: For any bounding pair map $T_aT_e^{-1}$, we have $\phi(T_aT_e^{-1}) = T_{a'}T_{e'}^{-1}$ and for any Dehn twist T_s about a separating curve s, we have $\phi(T_s) = T_{s'}$. Let $S_{g,p}^b$ be a genus g surface with p marked points and b boundary components. In this case, firstly we need to locate $*$. Let us decompose the surface into pair of pants as the following Fig. 6.

With the pants decomposition in Fig. 6, the point $*$ lies in a pair of pants P that is either like A where all three boundary curves are non-separating or like B where one of the boundary curve c is separating and the subsurface cutting out by c containing P is of genus at least 2. We will use the following procedure to find a subsurface S in two different situations.

Fig. 6 A decomposition



- When the boundary curves of P are all non-separating, there is a separating curve d bounding a subsurface S_1 of genus 2 containing P. Find a curve c on $S_1 - P$ such that c, t form a bounding pair. Then a, b, c, d bound a subsurface S which is a sphere with 4 boundary components.
- When one of the boundary curve d of P is separating and the subsurface S_1 cutting out by d containing P is of genus at least 2, the other two boundary curves a, t of P form a bounding pair. Find two other non-separating curves b, c on $S_1 - P$ such that t, b, c form a pair of pants. Then a, b, c, d bound a subsurface S which is a sphere with 4 boundary components.

Then S satisfies the following:

- $\partial S = a \cup b \cup c \cup d$ such that d is separating and any two of a, b, c does not separate S_g
- The lifts a', b', c', d' are 4 disjoint simple closed curves on $S_{g,1}$ such that d' is separating and $a' \cup b' \cup c' \cup d'$ bounds a sphere with 4-boundary components and a puncture $*$ in $S' \approx S_{0,1}^4 \subset S_{g,1}$.

Let W be the subgroup of $\mathcal{I}_g$ generated by bounding pair maps about curves inside S. Let W' be the subgroup of $\mathcal{I}_{g,1}$ generated by bounding pair maps about curves inside S'. After gluing punctured disks to the boundaries a, b, c and a', b', c' there are forgetful homomorphisms $\mu : W \to PB_3$ and $\mu' : W' \to PB_4$. The following claim and Lemma 3.4 concludes the proof of Theorem 1.1.

Claim 3.5 μ and μ' are isomorphisms and $\phi : W \cong PB_3 \to W' \cong PB_4$ is a section of the forgetful map $\mathcal{F}_3 : PB_4 \to PB_3$ satisfying that Dehn twists are mapped to Dehn twists, $\mathcal{S}(T_{Z_3}) = T_{Z_4}$ and $\mathcal{F}_3 \circ \mathcal{S} = id$.

We need to show that μ is injective and surjective. Firstly, PB_3 is generated by Dehn twist about simple closed curves. Every simple closed curves surrounds 2 boundary components in $\mathbb{D}_3$, so forming a bounding pair with the 3rd boundary component. This shows surjectivity.

For injectivity, if $f \in \ker(\mu)$ as a mapping class on S, then f is either trivial or equal to a product of Dehn twists on a, b, c because μ is forgetful map. However, we claim that a nontrivial product of Dehn twists on a, b, c is never in the Torelli group, which shows that μ is injective. Assume the opposite and without loss of generality assume that $f = T_a^m T_b^n T_c^l \in \mathcal{I}_g$ for $l \neq 0$. Let x, y be two curves or two elements in $H_1(S_g; \mathbb{Z})$ and denote by $I(x, y)$ the *algebraic intersection number* of x and y. For $x \in H_1(S_g; \mathbb{Z})$, we have that

$$T_a^m T_b^n T_c^l(x) = mI(a, x)a + nI(b, x)b + lI(c, x)c + x.$$

The fact that $T_a^m T_b^n T_c^l$ is in $\mathcal{I}_g$ implies that $mI(a, x)a + nI(b, x)b + lI(c, x)c = 0$ for any x. Since $[a]$, $[b]$ are independent in $H_1(S_g; \mathbb{Z})$, there exists an element x such that $I(a, x) = 0$ and $I(b, x) = 1$. Since $a \cup b \cup c$ separates the surface, we have that $a + b + c = 0$ and $I(c, x) = -1$. Then

$$mI(a, x)a + nI(b, x)b + lI(c, x)c = nb - lc = 0$$

However this contradicts the fact that $[b], [c] \in H_1(S_g; \mathbb{Z})$ are independent and $l \neq 0$.

For the same proof, we can show that μ' is injective. For surjectivity, any simple closed curve γ on $\mathbb{D}_4$ that surrounds 2 punctures has 2 possibilities without loss of generality: (1) γ surrounds a, b; (2) γ surrounds $a, *$. For (1), we have that $T_\gamma T_c^{-1} \in \mathcal{I}_{g,1}$; for (2), we have that $T_\gamma T_a^{-1} \in \mathcal{I}_{g,1}$. For both cases, T_γ is in the image of μ'. Since PB_4 is generated by Dehn twists about simple closed curves surrounding 2 punctures, we know that μ' is surjective. The condition on map ϕ is given by the assumption of Case 3. □ □

4 The *n*-pointed torelli spaces

In this section, we will prove Theorem 4.1. The main tool is Chen [4, Theorem 1.5] and a technical lemma about Torelli action on $\pi_1(S_g)$, which is proved by Johnson homomorphism.

4.1 Translation to a group-theoretic problem

We first translate the "section problem" of the universal Torelli surface bundle into a group-theoretic statement. As is discussed in Chen [4, Section 2.1], we have the following correspondence when $g > 1$:

$$\left\{\begin{array}{c}\text{Conjugacy classes of}\\ \text{representations}\\ \rho : \pi_1(B) \to \mathrm{Mod}(S_g)\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of oriented}\\ S_g - \text{bundles over} B\end{array}\right\}. \tag{6}$$

Let $f : E \to B$ be a surface bundle determined by $\rho : \pi_1(B) \to \mathrm{Mod}(S_g)$. Let $f_* : \pi_1(E) \to \pi_1(B)$ be the map on the fundamental groups. By the property of the pullback diagram, finding a splitting of f_* is the same as finding a homomorphism p that makes the following diagram commute, i.e., $\pi_{g,1} \circ p = \rho$.

$$\begin{array}{ccc}\pi_1(E) & \longrightarrow & \mathrm{Mod}(S_{g,1})\\ \downarrow f_* & \nearrow p & \downarrow \pi_{g,1}\\ \pi_1(B) & \xrightarrow{\rho} & \mathrm{Mod}(S_g).\end{array} \tag{7}$$

Now we have a new correspondence:

$$\left\{\begin{array}{c}\text{Homotopy classes of}\\ \text{continuous sections of}\\ S_g \to E \xrightarrow{f} B\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{Homomorphisms} p \text{satisfying diagram (7) up}\\ \text{to conjugacy by an element in}\\ \mathrm{Ker}(\pi_{g,1}) \cong \pi_1(S_g)\end{array}\right\}. \tag{8}$$

Let $\mathcal{PI}_{g,n} \xrightarrow{T\pi_{g,n}} \mathcal{I}_g$ and $\mathcal{I}_{g,n} \xrightarrow{T\pi'_{g,n}} \mathcal{I}_g$ be the forgetful maps forgetting the marked points. Let $\mathcal{I}_{g,n} \xrightarrow{Tp_{g,n,i}} \mathrm{Mod}(S_{g,1})$ be the forgetful homomorphism forgetting the fixed points $\{x_1, ..., \hat{x}_i, ..., x_n\}$. Let $PB_n(S_g)$ (resp. $B_n(S_g)$) be the *n-strand surface braid group*, i.e., $PB_n := \mathrm{PConf}_n(S_g)$ (resp. $B_n := \mathrm{Conf}_n(S_g)$), where $\mathrm{PConf}_n(S_g)$ (resp. $\mathrm{Conf}_n(S_g)$) denotes the space of ordered (resp. unordered) n-tuples of distinct points on S_g. By the *generalized Birman exact sequence* (e.g., see Farb and Margalit [7, Theorem 9.1]), we have that $Ker(T\pi_{g,n}) \cong PB_n(S_g)$ and $Ker(T\pi'_{g,n}) \cong B_n(S_g)$. By the correspondence (8), we can translate Theorem 1.2 into the following group-theoretic statement.

Proposition 4.1 *For $g > 1$ and $n \geq 0$. The following holds:*

(1) *Every homomorphism p satisfying the following diagram is either conjugate to a forgetful homomorphism $Tp_{g,n,i}$ by an element in $\mathcal{PI}_{g,n}$, or else factors through $T\pi_{g,n}$, i.e., there exists $f : \mathcal{I}_g \to \text{Mod}(S_{g,1})$ such that $p = f \circ T\pi_{g,n}$;*

$$\begin{array}{ccccccc} 1 \to PB_n(S_g) & \longrightarrow & \mathcal{PI}_{g,n} & \xrightarrow{T\pi_{g,n}} & \mathcal{I}_g & \longrightarrow & 1 \\ \downarrow R & & \downarrow p & & \downarrow & & \\ 1 \to \pi_1(S_g) & \longrightarrow & Mod(S_{g,1}) & \xrightarrow{\pi_{g,1}} & Mod(S_g) & \longrightarrow & 1. \end{array} \tag{9}$$

(2) *For $n > 1$, every homomorphism p' satisfying the following diagram factors through $T\pi'_{g,n}$, i.e., there exists $f' : \mathcal{I}_g \to \text{Mod}(S_{g,1})$ such that $p' = f' \circ T\pi'_{g,n}$*

$$\begin{array}{ccccccc} 1 \to B_n(S_g) & \longrightarrow & \mathcal{I}_{g,n} & \xrightarrow{T\pi'_{g,n}} & \mathcal{I}_g & \longrightarrow & 1 \\ \downarrow R' & & \downarrow p' & & \downarrow & & \\ 1 \to \pi_1(S_g) & \longrightarrow & Mod(S_{g,1}) & \xrightarrow{\pi_{g,1}} & Mod(S_g) & \longrightarrow & 1. \end{array} \tag{10}$$

We will prove Proposition 4.1 in the next subsection. Now we finish the proof of Theorem 1.2 using Proposition 4.1.

Proof of Theorem 1.2 assuming Proposition 4.1 The monodromies of bundles

$$Tu_{g,n} : \mathcal{UPI}_{g,n} \to \mathcal{PI}_{g,n} \text{ and } Tu'_{g,n} : \mathcal{UI}_{g,n} \to \mathcal{BI}_{g,n}$$

are the natural projections

$$\mathcal{PI}_{g,n} \to \mathcal{I}_g \to \text{Mod}(S_g) \text{ and } \mathcal{I}_{g,n} \to \mathcal{I}_g \to \text{Mod}(S_g).$$

By correspondence (8) and Proposition 4.1, every section of $Tu_{g,n}$ is either a pullback of a section of Tu_g or one of the canonical sections; every section of $Tu'_{g,n}$ for $n \geq 2$ is a pullback of a section of Tu_g. However, by Theorem 1.1, the short exact sequence

$$1 \to \pi_1(S_g) \to \mathcal{I}_{g,1} \xrightarrow{\pi_{g,1}} \mathcal{I}_g \to 1$$

has no section, which implies that Tu_g has no section. This concludes the proof of Theorem 1.2 □

4.2 The proof of Proposition 4.1

The top exact sequence of diagram (9) gives us a representation

$$\rho_T : \mathcal{I}_g \to \text{Out}(PB_n(S_g)).$$

The following lemma describes a property of ρ_T. Let $p_i : PB_n(S_g) \to \pi_1(S_g)$ be the induced map on the fundamental groups of the forgetful map forgetting all points except the ith point. The following lemma says that even though $\mathcal{I}_g$ preserves all surjections $\pi_1(S_g) \to \mathbb{Z}$ but $\mathcal{I}_g$ does not preserve any surjection $\pi_1(S_g) \to F_h$ for $h > 1$, where F_h denotes the free group with h generators.

Lemma 4.2 *Let $h > 1$. For any surjective homomorphism $\phi : PB_n(S_g) \to F_h$, there exists an element $t \in \mathcal{I}_g$ such that $t(Ker(\phi)) \neq Ker(\phi)$.*

Proof By Chen [4, Lemma 3.5], any surjective homomorphism $\phi : PB_n(S_g) \to F_h$ factors through some p_i. Thus we only need to deal with the case $n = 1$. We will prove the lemma by contradiction.

Suppose the opposite that there exists a surjective homomorphism $\phi : \pi_1(S_g) \to F_h$ such that for any element $e \in \mathcal{I}_g$, we have $e(\mathrm{Ker}(\phi)) = \mathrm{Ker}(\phi)$. Since ϕ is surjective, the induced map on $H_1(\_\_, \mathbb{Z})$ is also surjective. Let $a_1, a_2, ..., a_h \in \pi_1(S_g)$ be group elements such that $\phi(a_1), ..., \phi(a_h)$ generate F_h. Since the cup product $H^1(F_h, \mathbb{Z}) \otimes H^1(F_h, \mathbb{Z}) \xrightarrow{cup} H^2(F_h, \mathbb{Z})$ is trivial, the image of $\phi^* : H^1(F_h; \mathbb{Z}) \to H^1(S_g; \mathbb{Z})$ is an isotropic subspace with dimension at most g. Thus we can find $b \in \pi_1(S_g)$ such that $\phi(b) = 1$ and $[b] \neq 0 \in H_1(S_g; \mathbb{Z})$. It is clear that $[b]$ and $\{[a_1], ..., [a_h]\}$ are linearly independent in $H_1(S_g; \mathbb{Z})$. Setting $\pi^0 = \pi_1(S_g)$, and $\pi^{n+1} = [\pi^n, \pi^0]$, we have the following exact sequence.

$$1 \to \pi^1/\pi^2 \to \pi^0/\pi^2 \to \pi^0/\pi^1 \to 1.$$

Define $H := H_1(S_g; \mathbb{Z})$ and $\omega = \sum_{j=1}^{g} a_j \wedge b_j$. For the following discussion, we refer the reader to Johnson [9] and Farb and Margalit [7, Section 6.6] for more details. We know that $\pi^1/\pi^2 \cong \wedge^2 H/\mathbb{Z}\omega$, where the identification is given by $[x, y] \to x \wedge y$. Notice that $\mathcal{I}_g$ acts trivially on both π^1/π^2 and π^0/π^1 but nontrivially on π^0/π^2, which is measured by the Johnson homomorphism $\tau : \mathcal{I}_g \to \mathrm{Hom}(H, \wedge^2 H/\mathbb{Z}\omega)$. For $t \in \mathcal{I}_g$ and $x \in H$, the Johnson homomorphism is defined by $\tau(t)(x) = t(\tilde{x})\tilde{x}^{-1} \in \pi^1/\pi^2$, where $\tilde{x} \in \pi^0$ is any element such that $[\tilde{x}] = x$. It is standard to check that $\tau(t)$ does not depend on the choice of $\tilde{x}$.

D. Johnson showed that $\tau(\mathcal{I}_g)$ is a subspace of $\mathrm{Hom}(H, \wedge^2 H/\mathbb{Z}\omega)$ that is isomorphic to $\wedge^3 H/H$. Under this isomorphism, an element $t \in \mathcal{I}_g$ such that $\tau(t) = b \wedge a_1 \wedge a_2 \in \wedge^3 H/H$ corresponds to an element $\tau(t) \in \mathrm{Hom}(H, \wedge^2 H/\mathbb{Z}\omega)$ satisfying $\tau(t)(b) = a_1 \wedge a_2$. By the definition of the Johnson homomorphism, we have that $t(b)b^{-1} = [a_1, a_2]T$, where $T \in \pi^2$. Since $\phi(b) = 1$, we have that $\phi(t(b)) = 1$ by the assumption that $t(\mathrm{Ker}(\phi)) = \mathrm{Ker}(\phi)$. As a result, $\phi([a_1, a_2])\phi(T) = 1$. Let $F_h^1 = [F_h, F_h]$ and $F_h^{n+1} = [F_h^n, F_h]$. We have that $\phi(\pi^n) \subset F_h^n$, which implies that $\phi(T) \in F_h^2$. However $\phi([a_1, a_2]) \neq 1 \in F_h^1/F_h^2$, which contradicts the fact that $\phi([a_1, a_2]) = \phi(T)^{-1}$. □

A natural question follows Lemma 4.2:

Problem Does any of the Johnson filtration subgroups $\mathrm{Mod}_g(k)$ (as defined in the introduction) preserve a surjective homomorphism $\phi : \pi_1(S_g) \to F_h$ for $h > 1$?

We need the following lemma from Handel and Thurston [8, Lemma 2.2].

Lemma 4.3 (Handel and Thurston [8]) *For $g > 1$, a pseudo-Anosov element of $Mod(S_{g,n})$ does not fix any nonperipheral isotopy class of curves including nonsimple curves. Equivalently, viewing a mapping class as an outer automorphism of $\pi_1(S_{g,n})$, a pseudo-Anosov mapping class does not preserve any nontrivial conjugacy class in $\pi_1(S_{g,n})$.*

To prove statement (2) in Proposition 4.1, we need the following lemma.

Lemma 4.4 *For $n > 1$, the image of any homomorphism $B_n(S_g) \to \pi_1(S_g)$ is a free group.*

Proof Suppose that there exists a homomorphism $\Phi : B_n(S_g) \to \pi_1(S_g)$ such that the image is not a free group. By the classification of subgroups of $\pi_1(S_g)$, we have $\mathrm{Image}(\Phi) \cong \pi_1(S_h)$ is a finite index subgroup of $\pi_1(S_g)$. Since $PB_n(S_g)$ is a finite index subgroup of $B_n(S_g)$, we know that $\Phi' = \Phi|_{PB_n(S_g)} : PB_n(S_g) \to \pi_1(S_g)$ has image a nontrivial finite index subgroup in $\pi_1(S_g)$. By Chen [4, Theorem 1.5], the map Φ' factors through some p_i. Since the betti number of a finite index subgroup of $\pi_1(S_g)$ is bigger than the betti number of $\pi_1(S_g)$, there is no surjection from $\pi_1(S_g)$ to a nontrivial finite index subgroup of $\pi_1(S_g)$. This is a contradiction. □

Now we start the proof of Proposition 4.1.

Proof of Proposition 4.1 For any $p : \mathcal{PI}_{g,n} \to \mathrm{Mod}(S_{g,1})$ satisfying diagram (9) and for $e \in \mathcal{PI}_{g,n}$, $x \in PB_n(S_g)$, the restriction map R satisfies

$$R(exe^{-1}) = p(exe^{-1}) = p(e)R(x)p(e)^{-1}.$$

Denote by C_e the conjugation by e in any group. This induces the following diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{C_e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \pi_1(S_g) & \xrightarrow{C_{p(e)}} & \pi_1(S_g). \end{array} \tag{11}$$

By Chen [4, Theorem 1.5], a homomorphism $R : PB_n(S_g) \to \pi_1(S_g)$ either factors through a forgetful homomorphism p_i or has cyclic image. By Lemma 4.4, we know that $R' : B_n(S_g) \to \pi_1(S_g)$ is not a surjection. Therefore, the image of R' is either cyclic or a noncyclic free group. We will only discuss R in the following, but exact same reasoning works for R'. We break our discussion into the following four cases.

- **Case 1: Im$(R) = 1$** In this case, p factors through $T\pi_{g,n}$.
- **Case 2: Image$(R) \cong \mathbb{Z}$** In diagram (11), the conjugation map $C_{p(e)}$ corresponds to the outer automorphism induced by e. Then $C_{p(e)}$ preserves Image(R) for any e. However, it is known that $\mathcal{I}_g$ contains pseudo-Anosov elements, which does not preserve Image(R) by Lemma 4.3; e.g., see Farb and Margalit [7, Corollary 14.3] for the fact that the Torelli group contains pseudo-Anosov maps.
- **Case 3: Im$(R) = F_h$ for $h > 1$** We have the following diagram.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{C_e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ F_h & \xrightarrow{C_{p(e)}} & F_h \end{array} \tag{12}$$

  Therefore $p(e)$ has to preserves the kernel of the surjection R for any $e \in \mathcal{I}_g$, which contradicts Lemma 4.2.
- **Case 4: Im$(R) = \pi_1(S_g)$** Therefore $R = A \circ p_i$ for $A \in \mathrm{Aut}(\pi_1(S_g)) \cong \mathrm{Mod}(S_{g,1})$ by Dehn–Nielsen–Baer Theorem (e.g., see Farb and Margalit [7, Theorem 8.8]). We claim that then $p = A \circ p_i$ and $A \in \pi_1(S_g)$ for $g > 2$, which concludes this case. For $e \in \mathcal{I}_{g,n}$ and $f \in PB_n(S_g)$,

$$A \circ p_i(efe^{-1}) = p(efe^{-1}) = p(e)A(p_i(f))p(e)^{-1}.$$

  We know that

$$A \circ p_i(efe^{-1}) = A(p_i(e)p_i(f)p_i(e)^{-1}) = A(p_i(e))A(p_i(f))A(p_i(e)^{-1})$$

  Therefore $A(p_i(e))p(e)^{-1}$ is in the centralizer of $A(p_i(f))$ in $\mathrm{Mod}(S_{g,1})$, which is trivial for $g \geq 3$. So $A(p_i(e))p(e)^{-1} = i$ the trivial element. So $A(p_i(e)) = p(e)$. As an element in $\mathrm{Mod}(S_{g,1})$, we have that $A(p_i(e)) = Ap_i(e)A^{-1}$. For $x \in \mathrm{Mod}(S_{g,1})$, let $\overline{x}$ be its image in $\mathrm{Mod}(S_g)$ under the forgetful map. Therefore we have that

$$\overline{Ap_i(e)A}^{-1} = \overline{p_i(e)} \in \mathrm{Mod}(S_g)$$

because $A(p_i(e))$ and $p_i(e)$ have to be equal in Mod(S_g) for any e. Therefore, we have

$$\overline{A} \in \text{Center}(\text{Mod}(S_g)).$$

For $g > 2$, Center(Mod$(S_g)) = 1$, therefore we have $A \in \pi_1(S_g)$; e.g., see Farb and Margalit [7, Chapter 3.4].

□

5 Cohomological proof of theorem 1.1

5.1 A nonsplitting statement

In this subsection, we will prove the following. Notice that the proof only depends on Proposition 4.1 and Lefschetz fixed point theorem, which does not use Theorem 1.1. Let $F_i : \mathcal{PI}_{g,2} \to \mathcal{PI}_{g,1}$ be the forgetful map forgetting the ith point.

Lemma 5.1 *For $g > 1$, the forgetful map F_2 does not has a section.*

Proof Assume that $s : \mathcal{PI}_{g,1} \to \mathcal{PI}_{g,2}$ is a section of F_2. After composing with F_1, we obtain a map $F_1 \circ s : \mathcal{PI}_{g,1} \to \mathcal{PI}_{g,1}$. This map $F_1 \circ s$ satisfies the condition of Proposition 4.1 for $n = 1$. Then either $F_1 \circ s$ factors through $\mathcal{I}_g$ or $F_1 \circ s$ is conjugate to a forgetful homomorphism.

Since S_g is a $K(\pi, 1)$-space, any group homomorphism of $\eta : \pi_1(S_g) \to \pi_1(S_g)$ will induce a geometric map $\widetilde{\eta} : S_g \to S_g$. Since the map $s(\pi_1(S_g)) \subset PB_2(S_g)$, then some geometric map $\widetilde{F_1 \circ s|_{\pi_1(S_g)}} : S_g \to S_g$ of $F_1 \circ s|_{\pi_1(S_g)}$ has no fixed point. If $F_1 \circ s$ factors through $\mathcal{I}_g$, then $F_1 \circ s|_{\pi_1(S_g)} = i$ the trivial map. The Lefschetz number of the trivial map is 1, which contradicts the fact that $\widetilde{F_1 \circ s|_{\pi_1(S_g)}}$ is fixed point free by Lefschetz fixed point theorem. When $F_1 \circ s$ is conjugate to identity by an element in $\pi_1(S_g)$, the map $F_1 \circ s$ is conjugate to identity by an element in $\pi_1(S_g)$. This means that a geometric map $\widetilde{F_1 \circ s|_{\pi_1(S_g)}} : S_g \to S_g$ is the identity map. The Lefschetz number of the trivial map is $2-2g \neq 0$, which contradicts the fact that $\widetilde{F_1 \circ s|_{\pi_1(S_g)}}$ is fixed point free by Lefschetz fixed point theorem. □

5.2 Second proof of Theorem 1.1

We want to point out here that the marked pointed case can help us with the case of no marked point, i.e., Lemma 5.1 can give us another proof of Theorem 1.1. Notice that the proof of Lemma 5.1 does not depend on Theorem 1.1. Let $\mathcal{I}^b_{g,p}$ be the Torelli group of $S^b_{g,p}$, i.e., the subgroup of Mod$(S^b_{g,p})$ that acts trivially on $H^1(S_g; \mathbb{Z})$.

Second proof of Theorem 1.1 Again assume $g > 3$. Assume that the exact sequence (1) has a splitting which is denoted by ϕ such that $F \circ \phi = id$. By Lemma 3.1, the image $\phi(T_s)$ of T_s the Dehn twist about a separating curve s is $T^n_{s'} T^{1-n}_{s''}$ where s' and s'' are curves on $S_{g,1}$ that are isotopic to s. Let US_g be the unit tangent bundle of genus g surface. Let s be a separating curve that separates S_g into two parts $C_1 \cong S^1_p$ and $C_2 \cong S^1_q$ such that $p, q \geq 2$. We denote by $C(s) \subset \mathcal{I}_g$ the stabilizer of s by the action of $\mathcal{I}_g$ on curves, which satisfies the following:

$$1 \to \mathbb{Z} \xrightarrow{(T_s, T_s^{-1})} \mathcal{I}^1_p \times \mathcal{I}^1_q \to C(s) \to 1.$$

The disk pushing subgroup is $\pi_1(US_p) \to \mathcal{I}_p^1$; e.g., see Farb and Margalit [7, Page 118]. The amalgamation of disk pushing subgroups of C_1 and C_2 gives us a subgroup $A \subset C(s)$ satisfying the following short exact sequence, which has already been encountered when we describe the work of Mess.

$$1 \to \mathbb{Z} \xrightarrow{(T_s, T_s^{-1})} \pi_1(US_p) \times \pi_1(US_q) \to A \to 1 \tag{13}$$

Claim 5.2 $\phi(T_s) = T_{s'}$ for a curve s' on $S_{g,1}$ that is isotopic to s.

Proof We have already proved this result in the proof of Case 1 of Theorem 1.1. Here we give another proof. If the s component contains 2 curves, then sections on A in Birman exact sequence will induces a section of (13). We will prove that (13) does not split using Euler class and group cohomology. For a $\mathbb{Z}$-central extension of a group T

$$1 \to \mathbb{Z} \to \widetilde{T} \xrightarrow{\alpha} T \to 1, \tag{14}$$

there is an associated *Euler class* $Eu(\alpha) \in H^2(T;\mathbb{Z})$. The extension α splits if and only if $Eu(\alpha)$ vanishes; see Brown [2, Chapter 4.3]. $Eu(\alpha)$ can be constructed using the Lyndon-Hochschild-Serre spectral sequence of (14), by taking $Eu(\alpha) = d_2(1)$. Here d_2 is the differential $d_2 : \mathbb{Z} \to H^2(T;\mathbb{Z})$ on the E_2 page. The (rational) Betti number $b_1(\tilde{T})$ can be computed from the spectral sequence as

$$b_1(\widetilde{T}) = b_1(T) + \dim(\ker(d_2)).$$

Therefore $Eu(\alpha) \neq 0$ is nonvanishing if and only if $b_1(\widetilde{T}) = b_1(T)$. By the above discussion, we only need to show that $b_1(A) = b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q))$. However, since $p \geq 2$ and $q \geq 2$ by assumption,

$$b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q)) = b_1(\pi_1(\Sigma_p) \times \pi_1(\Sigma_q)).$$

Since $A \to \pi_1(\Sigma_p) \times \pi_1(\Sigma_q)$ is surjective, it follows that $b_1(A) \geq b_1(\pi_1(\Sigma_p) \times \pi_1(\Sigma_q))$, and so $b_1(A) = b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q))$ as desired. ◻

We denote by $C(s') \subset \mathcal{I}_{g,1}$ the stabilizer of s' by the action of $\mathcal{I}_{g,1}$ on curves. Since $\mathcal{I}_{g,1}$ fixes homology, $C(s')$ fixes the two components of $S_{g,1} - s'$. Therefore $C(s')$ satisfies the following exact sequence

$$1 \to \mathbb{Z} \to \mathcal{I}_p^1 \times \mathcal{I}_{q,1}^1 \to C(s') \to 1.$$

So we have a section of $\mathcal{F} : \mathcal{I}_{q,1}^1 \to \mathcal{I}_q^1$ which maps T_s to $T_{s'}$, which implies that this section gives a section of $\mathcal{F}_{q,2,1}$ in the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{I}_{q,1}^1 & \longrightarrow & \mathcal{PI}_{q,2} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \mathcal{F} & & \downarrow \mathcal{F}_{q,2,1} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{I}_q^1 & \longrightarrow & \mathcal{I}_{q,1} & \longrightarrow & 1. \end{array} \tag{15}$$

However, we already prove that $\mathcal{F}_{q,2,1}$ does not have a section in Lemma 5.1, which implies that $\mathcal{F}$ does not have a section. The statement follows. ◻

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