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On a Rankin–Selberg integral of the *L*-function for $\widetilde{\mathrm{SL}}_2 \times \mathrm{GL}_2$

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Abstract
We present a Rankin–Selberg integral on the exceptional group G_2 which represents the L-function for generic cuspidal representations of $\widetilde{\mathrm{SL}}_2 \times \mathrm{GL}_2$. As an application, we show that certain Fourier–Jacobi type periods on G_2 are non-vanishing.

Keywords Rankin–Selberg integral · L-function · Exceptional group G_2 · Periods

Mathematics Subject Classification 2010 · 11F70

1 Introduction

Let F be a global field with the ring of adeles $\mathbb{A}$. We assume that the characteristics of F is not 2. We present in this paper a Shimura type integral on the exceptional group G_2 which represents the L-function

$$L(s, \widetilde{\pi} \times (\chi \otimes \tau))L(s, \widetilde{\pi} \otimes (\chi \otimes \omega_\tau)),$$

where $\widetilde{\pi}$ is an irreducible genuine cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, τ is an irreducible generic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ and χ is the quadratic character of $F^\times \backslash \mathbb{A}^\times$ defined by $\chi(a) = \prod_v (a_v, -1)_{F_v}$, where $a = (a_v)_v \in \mathbb{A}^\times$ and $(\ ,\ )_{F_v}$ is the Hilbert symbol on F_v.

To give more details about the integral, we introduce some notations. The group G_2 has two simple roots and we label the short root by α and the long root by β. Let $P = MV$ (resp. $P' = M'V'$) be the maximal parabolic subgroup of G_2 such that the root space of β is in the Levi M (resp. the root space of α is in the Levi M'). The Levi subgroups M and M' are isomorphic to GL_2. Let J be the subgroup of P which is isomorphic to $\mathrm{SL}_2 \ltimes V$. Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$. There is a Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ for a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let $\widetilde{\theta}_\phi$ be a corresponding theta series associated with a function $\phi \in \mathcal{S}(\mathbb{A})$. Let τ be an irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. For $f_s \in \mathrm{Ind}_{P'(\mathbb{A})}^{G_2(\mathbb{A})}(\tau \otimes \delta_{P'}^s)$, we can form an Eisenstein series $E(g, f_s)$ on $G_2(\mathbb{A})$. Let $\widetilde{\pi}$ be an irreducible genuine cuspidal automorphic forms of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. For a cusp form

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$\widetilde{\varphi} \in \widetilde{\pi}$, we consider the integral

$$I(\widetilde{\varphi}, \phi, f_s) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)E(vg, f_s)dvdg.$$

Our main result is the following

Theorem 1.1 *The above integral is absolutely convergent for* $\mathrm{Re}(s) \gg 0$ *and can be meromorphically continued to all* $s \in \mathbb{C}$. *When* $\mathrm{Re}(s) \gg 0$, *the integral* $I(\widetilde{\varphi}, \phi, f_s)$ *is Eulerian. Moreover, at an unramified place* v, *the local integral represents the* L*-function*

$$\frac{L(3s-1, \widetilde{\pi}_v \times (\chi_v \otimes \tau_v))L(6s-5/2, \widetilde{\pi}_v \otimes (\chi_v \otimes \omega_{\tau_v}))}{L(3s-1/2, \tau_v)L(6s-2, \omega_{\tau_v})L(9s-7/2, \tau_v \otimes \omega_{\tau_v})}.$$

This is Theorem 3.1 and Proposition 4.6. We remark that Ginzburg–Rallis–Soudry gave integral representations for L-functions of generic cuspidal representations of $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_m$ in [8] using symplectic groups. It is still interesting to have different integral representations. As an application of Theorem 1.1, we show that if $Wd_\psi(\widetilde{\pi}) = \chi \otimes \tau$, then a Shimura type period with respect to $\widetilde{\pi}$ and the residue of Eisenstein series on G_2 is non-vanishing, where Wd_ψ is the Shimura–Waldspurger lift. It is an interesting theme in number theory to investigate the relations between poles of L-functions and non-vanishing of automorphic periods. There are many examples of this kind relations. See [5,7,9] for some examples. The non-vanishing results of automorphic periods have many interesting applications in automorphic forms. We expect the non-vanishing period in our case would be useful on problems related to the residue spectrum of G_2.

There are several known Rankin–Selberg integrals on G_2 which represents different L-functions and have many applications, see [4–6] for example. The integral $I(\widetilde{\varphi}, \phi, f_s)$ can be viewed as a dual integral of the standard G_2 L-function integral in [5] in the following sense. The integral $I(\widetilde{\varphi}, \phi, f_s)$ is an integral of a triple product of a cusp form on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, a theta series and an Eisenstein series on $G_2(\mathbb{A})$, while the integral in [5] is an integral of a triple product of a cusp form on $G_2(\mathbb{A})$, a theta series and an Eisenstein series on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. The integral in [6] is also in a similar pattern, which is an integral of a triple product of a cusp form on $\mathrm{SL}_2(\mathbb{A})$, a theta series and an Eisenstein series on a cover of $G_2(\mathbb{A})$. The results presented here were known for D. Ginzburg. But we still think that it might be useful to write up the details.

2 The group G_2

2.1 Roots and Weyl group for G_2

Let G_2 be the split algebraic reductive group of type G_2 (defined over $\mathbb{Z}$). The group G_2 has two simple roots, the short root α and the long root β. The set of the positive roots is $\Sigma^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$. Let $(\ ,\ )$ be the inner product in the root system and $\langle\ ,\ \rangle$ be the pair defined by $\langle \gamma_1, \gamma_2 \rangle = \frac{2(\gamma_1,\gamma_2)}{(\gamma_2,\gamma_2)}$. For the root space G_2, we have the relations:

$$\langle \alpha, \beta \rangle = -1, \langle \beta, \alpha \rangle = -3.$$

For a root γ, let s_γ be the reflection defined by γ, i.e., $s_\gamma(\gamma') = \gamma' - \langle \gamma', \gamma \rangle \gamma$. We have the relation

$$s_\alpha(\beta) = 3\alpha + \beta, s_\beta(\alpha) = \alpha + \beta.$$

The Weyl group $\mathbf{W} = \mathbf{W}(G_2)$ of G_2 has 12 elements, which is explicitly given by

$$\mathbf{W} = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, (s_\alpha s_\beta)^2, (s_\beta s_\alpha)^2, s_\beta(s_\alpha s_\beta)^2, s_\alpha(s_\beta s_\alpha)^2, (s_\alpha s_\beta)^3\}.$$

For a root γ, let $U_\gamma \subset G$ be the root space of γ, and let $\mathbf{x}_\gamma : F \to U_\gamma$ be a fixed isomorphism which satisfies various Chevalley relations, see Chapter 3 of [14]. Among other things, $\mathbf{x}_\gamma$ satisfies the following commutator relations:

$$\begin{aligned}
[\mathbf{x}_\alpha(x), \mathbf{x}_\beta(y)] &= \mathbf{x}_{\alpha+\beta}(-xy)\mathbf{x}_{2\alpha+\beta}(-x^2y)\mathbf{x}_{3\alpha+\beta}(x^3y)\mathbf{x}_{3\alpha+2\beta}(-2x^3y^2)\\
[\mathbf{x}_\alpha(x), \mathbf{x}_{\alpha+\beta}(y)] &= \mathbf{x}_{2\alpha+\beta}(-2xy)\mathbf{x}_{3\alpha+\beta}(3x^2y)\mathbf{x}_{3\alpha+2\beta}(3xy^2)\\
[\mathbf{x}_\alpha(x), \mathbf{x}_{2\alpha+\beta}(y)] &= \mathbf{x}_{3\alpha+\beta}(3xy)\\
[\mathbf{x}_\beta(x), \mathbf{x}_{3\alpha+\beta}(y)] &= \mathbf{x}_{3\alpha+2\beta}(xy)\\
[\mathbf{x}_{\alpha+\beta}(x), \mathbf{x}_{2\alpha+\beta}(y)] &= \mathbf{x}_{3\alpha+2\beta}(3xy).
\end{aligned} \tag{2.1}$$

For all the other pairs of positive roots γ_1, γ_2, we have $[\mathbf{x}_{\gamma_1}(x), \mathbf{x}_{\gamma_2}(y)] = 1$. Here $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ for $g_1, g_2 \in G_2$. For these commutator relationships, see [12].

Following [14], we denote $w_\gamma(t) = \mathbf{x}_\gamma(t)\mathbf{x}_{-\gamma}(-t^{-1})\mathbf{x}_\gamma(t)$ and $w_\gamma = w_\gamma(1)$. Note that w_γ is a representative of s_γ. Let $h_\gamma(t) = w_\gamma(t)w_\gamma^{-1}$. Let T be the subgroup of G which consists of elements of the form $h_\alpha(t_1)h_\beta(t_2)$, $t_1, t_2 \in T$ and U be the subgroup of G_2 generated by U_γ for all $\gamma \in \Sigma^+$. Let $B = TU$, which is a Borel subgroup of G_2.

For $t_1, t_2 \in \mathbb{G}_m$, denote $h(t_1, t_2) = h_\alpha(t_1t_2)h_\beta(t_1^2t_2)$. From the Chevalley relation $h_{\gamma_1}(t)\mathbf{x}_{\gamma_2}(r)h_{\gamma_1}(t)^{-1} = \mathbf{x}_{\gamma_2}(t^{\langle\gamma_2,\gamma_1\rangle}r)$ (see [14, Lemma 20, (c)]), we can check the following relations

$$\begin{aligned}
h^{-1}(t_1, t_2)\mathbf{x}_\alpha(r)h(t_1, t_2) &= \mathbf{x}_\alpha(t_2^{-1}r),\\
h^{-1}(t_1, t_2)\mathbf{x}_\beta(r)h(t_1, t_2) &= \mathbf{x}_\beta(t_1^{-1}t_2r)\\
h^{-1}(t_1, t_2)\mathbf{x}_{\alpha+\beta}(r)h(t_1, t_2) &= \mathbf{x}_{\alpha+\beta}(t_1^{-1}r),\\
h^{-1}(t_1, t_2)\mathbf{x}_{2\alpha+\beta}(r)h(t_1, t_2) &= \mathbf{x}_{2\alpha+\beta}(t_1^{-1}t_2^{-1}r)\\
h^{-1}(t_1, t_2)\mathbf{x}_{3\alpha+\beta}(r)h(t_1, t_2) &= \mathbf{x}_{3\alpha+\beta}(t_1^{-1}t_2^{-2}r),\\
h^{-1}(t_1, t_2)\mathbf{x}_{3\alpha+2\beta}(r)h(t_1, t_2) &= \mathbf{x}_{3\alpha+2\beta}(t_1^{-2}t_2^{-1}r).
\end{aligned} \tag{2.2}$$

Thus the notation $h(a, b)$ agrees with that of [5].

One can also check that

$$w_\alpha h(t_1, t_2)w_\alpha^{-1} = h(t_1t_2, t_2^{-1}), \quad w_\beta h(t_1, t_2)w_\beta^{-1} = h(t_2, t_1).$$

2.2 Subgroups

Let F be a field and denote $G = G_2(F)$. The group G has two proper parabolic subgroups. Let $P = M \ltimes V$ be the parabolic subgroup of G such that $U_\beta \subset M \cong \mathrm{GL}_2$. Thus the unipotent subgroup V is consisting of root spaces of $\alpha, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta$, and a typical element of V is of the form

$$\mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5), r_i \in F.$$

To ease the notation, we will write the above element as $[r_1, r_2, r_3, r_4, r_5]$. Denote by J the following subgroup of P

$$J = \mathrm{SL}_2(F) \ltimes V.$$

Let V_1 (resp. Z) be the subgroup of V which consists root spaces of $3\alpha+\beta$ and $3\alpha+2\beta$ (resp. $2\alpha+\beta$, $3\alpha+\beta$ and $3\alpha+2\beta$). Note that P and hence J normalizes V_1 and Z. We will always view $\mathrm{SL}_2(F)$ as a subgroup of G via the inclusion $\mathrm{SL}_2(F)\subset M$. Denote by A_{SL_2}, N_{SL_2} and B_{SL_2} the standard torus, the upper triangular unipotent subgroup and the upper triangular Borel subgroup of $\mathrm{SL}_2(F)$. Note that the torus element $h(a,b)$ can be identified with

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \in \mathrm{GL}_2(F)\cong M,$$

and thus $A_{\mathrm{SL}_2}=\left\{h(a,a^{-1})|a\in F^\times\right\}$ and $B_{\mathrm{SL}_2}=A_{\mathrm{SL}_2}\ltimes U_\beta$.

Let $P'=M'V'$ be the other maximal parabolic subgroup G with U_α in the Levi subgroup M'. The Levi M' is isomorphic to $\mathrm{GL}_2(F)$, and from relations in (2.2), one can check that one isomorphism $M'\cong\mathrm{GL}_2(F)$ can be determined by

$$\mathbf{x}_\alpha(r)\mapsto \begin{pmatrix} 1 & r\\ & 1 \end{pmatrix},$$
$$h(a,b)\mapsto \begin{pmatrix} ab & \\ & a \end{pmatrix}.$$

In particular, we see that $h(a,1)\in T\subset M'$ can be identified with $\mathrm{diag}(a,a)$. Let $\delta_{P'}$ be the modulus character of P'. One can check that $\delta_{P'}(m')=|\det(m')|^3$ for $m'\in M'$, where $\det(m')$ can be computed using the above isomorphism $M'\cong\mathrm{GL}_2(F)$.

2.3 Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})\ltimes V(\mathbb{A})$

In this subsection, we assume that F is a global field and $\mathbb{A}$ is its ring of adeles. In $\mathrm{SL}_2(F)$, we denote $t(a)=\mathrm{diag}(a,a^{-1})$, $a\in F^\times$ and

$$n(b)=\begin{pmatrix} 1 & b\\ & 1 \end{pmatrix}, b\in F.$$

Denote $w^1=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, which represents the unique nontrivial Weyl element of $\mathrm{SL}_2(F)$. Under the embedding $\mathrm{SL}_2(F)\subset M\subset G$, the element w^1 can be identified with w_β.

Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$. Then we have an exact sequence

$$0\to\mu_2\to\widetilde{\mathrm{SL}}_2(\mathbb{A})\to\mathrm{SL}_2(\mathbb{A})\to 0,$$

where $\mu_2=\{\pm1\}$.

We will identify $\mathrm{SL}_2(\mathbb{A})$ with the symplectic group of $\mathbb{A}^2$ with symplectic structure defined by

$$\langle (x_1,y_1),(x_2,y_2)\rangle=-2x_1y_2+2x_2y_1.$$

Let $\mathscr{H}(\mathbb{A})$ be the Heisenberg group of the symplectic space $(\mathbb{A}^2,\langle\ ,\ \rangle)$, i.e., $\mathscr{H}(\mathbb{A})=\mathbb{A}^3$ with group law

$$(x_1,y_1,z_1)(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2-x_1y_2+y_1x_2).$$

Let $\mathrm{SL}_2(\mathbb{A})$ act on $\mathscr{H}(\mathbb{A})$ from the right side by

$$(x_1,y_1,z_1).g=((x_1,y_1)g,z_1),\ g\in\mathrm{SL}_2(\mathbb{A}),$$

where $(x_1,y_1)g$ is the usual matrix multiplication.

We then can form the semi-direct product $\mathrm{SL}_2(\mathbb{A}) \ltimes \mathscr{H}(\mathbb{A})$, where the product is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, (h_1.g_2)h_2), g_i \in \mathrm{SL}_2(\mathbb{A}), h_i \in \mathscr{H}(\mathbb{A}), i = 1, 2.$$

Let ψ be a nontrivial additive character of $F\backslash\mathbb{A}$. Then there is a Weil representation ω_ψ of $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes \mathscr{H}(\mathbb{A})$. The space of ω_ψ is $\mathcal{S}(\mathbb{A})$, the Bruhat–Schwartz functions on $\mathbb{A}$.

For $\phi \in \mathcal{S}(\mathbb{A})$, we have the well-know formulas:

$$(\omega_\psi(n(b))\phi)(x) = \psi(bx^2)\phi(x), b \in \mathbb{A}$$
$$(\omega_\psi((r_1, r_2, r_3))\phi)(x) = \psi\,(r_3 - 2xr_2 - r_1 r_2)\,\phi(x + r_1), (r_1, r_2, r_3) \in \mathscr{H}(\mathbb{A}),$$

The above formulas could be found in [11].

Recall that for $r_1, r_2, r_3, r_4, r_5 \in \mathbb{A}$, the notation $[r_1, r_2, r_3, r_4, r_5]$ is an abbreviation of

$$\mathbf{x}_\alpha(r_1)\mathbf{x}_{\alpha+\beta}(r_2)\mathbf{x}_{2\alpha+\beta}(r_3)\mathbf{x}_{3\alpha+\beta}(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5) \in V(\mathbb{A}).$$

Define a map $\mathrm{pr} : V(\mathbb{A}) \to \mathscr{H}(\mathbb{A})$

$$\mathrm{pr}([r_1, r_2, r_3, r_4, r_5]) = (r_1, r_2, r_3 - r_1 r_2).$$

From the commutator relation (2.1), we can check that pr is a group homomorphism and defines an exact sequence

$$0 \to V_1(\mathbb{A}) \to V(\mathbb{A}) \to \mathscr{H}(\mathbb{A}) \to 0.$$

Recall that V_1 is the subgroup of V which is generated by the root space of $3\alpha+\beta$, $3\alpha+2\beta$. Note that there is a typo in the formula of the projection map pr in [5, p.316].

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \subset M$, we can check that

$$g^{-1}[r_1, r_2, r_3, 0, 0]g = [r_1', r_2', r_3', r_4', r_5'],$$

where $r_1' = ar_1 - cr_2$, $r_2' = -br_1 + dr_2$, $r_3' - r_1' r_2' = r_3 - r_1 r_2$.

Consider the map $\overline{\mathrm{pr}} : J(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \ltimes V(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A}) \ltimes \mathscr{H}(\mathbb{A})$,

$$(g, v) \mapsto (g^*, \mathrm{pr}(v)), g \in \mathrm{SL}_2(\mathbb{A}), v \in V(\mathbb{A}).$$

where $g^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = d_1 g d_1^{-1}$, where $d_1 = \mathrm{diag}(1, -1) \in \mathrm{GL}_2(F)$. From the above discussion, the map $\overline{\mathrm{pr}}$ is a group homomorphism and its kernel is also $V_1(\mathbb{A})$. We will also view $\overline{\mathrm{pr}}$ as a homomorphism $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes V(\mathbb{A}) \to \widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes \mathscr{H}(\mathbb{A})$.

In the following, we will also view ω_ψ as a representation of $\widetilde{\mathrm{SL}_2}(\mathbb{A}) \ltimes V(\mathbb{A})$ via the projection map $\overline{\mathrm{pr}}$. For $\phi \in \mathcal{S}(\mathbb{A})$, we form the theta series

$$\widetilde{\theta}_\phi(vg) = \sum_{\xi \in F} \omega_\psi(vh)\phi(\xi), v \in V(\mathbb{A}), g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}).$$

Note that given a genuine cusp form $\widetilde{\varphi}$ on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, the product

$$\widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg), v \in V(\mathbb{A}), g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$$

can be viewed as a function on $J(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \ltimes V(\mathbb{A})$.

2.4 An Eisenstein series on G_2

In this subsection and in the rest of the paper, every representation appeared is assumed to be irreducible. Let τ be a cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A})$. We will view τ as a representation of $M'(\mathbb{A})$ via the identification $M' \cong \mathrm{GL}_2$. We then consider the induced representation $I(s,\tau) = \mathrm{Ind}_{P'(\mathbb{A})}^{G_2(\mathbb{A})}(\tau \otimes \delta_{P'}^s)$. A section $f_s \in I(s,\tau)$ is a smooth function satisfying

$$f_s(v'm'g) = \delta_{P'}(m')^s f_s(g), \forall v' \in V'(\mathbb{A}), m' \in M'(\mathbb{A}), g \in G_2(\mathbb{A}).$$

For $f_s \in I(s,\tau)$, we consider the Eisenstein series

$$E(g, f_s) = \sum_{\delta \in P'(F)\backslash G_2(F)} f_s(\delta g), g \in G_2(\mathbb{A}).$$

3 A global integral

Let $\widetilde{\pi}$ be a genuine cuspidal automorphic representation on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, and τ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. For $\widetilde{\varphi} \in V_\pi, \phi \in \mathcal{S}(\mathbb{A})$ and $f_s \in I(s,\tau)$, we consider the integral

$$I(\widetilde{\varphi}, \phi, f_s) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)E(vg, f_s)dvdg.$$

Let $\gamma = w_\beta w_\alpha w_\beta w_\alpha \in G_2(F)$.

Theorem 3.1 *The integral $I(\widetilde{\varphi}, \phi, f_s)$ is absolutely convergent when* $\mathrm{Re}(s) \gg 0$ *and can be meromorphically continued to all $s \in \mathbb{C}$. Moreover, when* $\mathrm{Re}(s) \gg 0$*, we have*

$$I(\tilde{\varphi}, \phi, f_s) = \int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(\mathbb{A})\backslash V(\mathbb{A})} W_{\widetilde{\varphi}}(g)\omega_\psi(vg)\phi(1)W_{f_s}(\gamma vg)dvdg,$$

where

$$W_{\widetilde{\varphi}}(g) = \int_{F\backslash \mathbb{A}} \widetilde{\varphi}(\mathbf{x}_\beta(r)g)\psi(r)dr,$$

and

$$W_{f_s}(\gamma vg) = \int_{F\backslash \mathbb{A}} f_s(\mathbf{x}_\alpha(r)\gamma vg)\psi(-2r)dr.$$

Proof The first assertion is standard. We only show that the above integral is Eulerian when $\mathrm{Re}(s) \gg 0$. Unfolding the Eisenstein series, we can get

$$I(\widetilde{\varphi}, \phi, f_s) = \sum_{\delta \in P'(F)\backslash G_2(F)/P(F)} \int_{\mathrm{SL}_2^\delta(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V^\delta(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)f_s(\delta vg)dvdg,$$

where $X^\delta = \delta^{-1}P'\delta \cap X$ for $X \subset G_2(F)$. We can check that a set of representatives of the double coset $P'(F)\backslash G_2(F)/P(F)$ can be taken as $\{1, w_\beta w_\alpha, \gamma = w_\beta w_\alpha w_\beta w_\alpha\}$. For $\delta = 1, w_\beta w_\alpha$, or $\gamma = w_\beta w_\alpha w_\beta w_\alpha$, denote

$$I_\delta = \int_{\mathrm{SL}_2^\delta(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V^\delta(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)f_s(\delta vg)dvdg.$$

If $\delta = 1$, the above integral I_δ has an inner integral

$$\int_{U_{2\alpha+\beta}(F)\backslash U_{2\alpha+\beta}(\mathbb{A})} \widetilde{\theta}_\phi(\mathbf{x}_{2\alpha+\beta}(r)vg) f_s(\mathbf{x}_{2\alpha+\beta}(r)vg)dr,$$

which is zero because $f_s(\mathbf{x}_{2\alpha+\beta}(r)vg) = f_s(vg), \widetilde{\theta}_\phi(\mathbf{x}_{2\alpha+\beta}(r)vg) = \psi(r)\tilde{\theta}_\phi(vg)$ and $\int_{F\backslash\mathbb{A}} \psi(r)dr = 0$. The last equation follows from the fact that ψ is non-trivial.

We next consider the term when $\delta = w_\beta w_\alpha$. We write

$$\widetilde{\theta}_\phi(vg) = \omega_\psi(vg)\phi(0) + \sum_{\xi\in F^\times} \omega_\psi(vg)\phi(\xi).$$

The contribution of the first term to the integral I_δ is

$$\int_{\mathrm{SL}_2^\delta(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V^\delta(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\omega_\psi(vg)\phi(0) f_s(\delta vg)dvdg.$$

Note that $\delta\mathbf{x}_\beta(r)\delta^{-1} \subset U_{2\alpha+\beta} \subset V'$, we have $f_s(\delta v\mathbf{x}_\beta(r)g) = f_s(\delta\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g)$. On the other hand, we have $\omega_\psi(\mathbf{x}_\beta(r)vg)\phi(0) = \omega_\psi(vg)\phi(0)$. After a changing variable on v, we can see that the above integral contains an inner integral

$$\int_{F\backslash\mathbb{A}} \widetilde{\varphi}(\mathbf{x}_\beta(r)vg)dr,$$

which is zero since $\widetilde{\varphi}$ is cuspidal. Thus the contribution of the term $\omega_\psi(vg)\phi(0)$ is zero when $\delta = w_\beta w_\alpha$. The contribution of $\sum_{\xi\in F^\times} \omega_\psi(vg)\phi(\xi)$ is

$$\int_{\mathrm{SL}_2^\delta(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V^\delta(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g) \sum_{\xi\in F^\times} \omega_\psi(vg)\phi(\xi) f_s(\delta vg)dvdg.$$

We consider the inner integral on $U_{\alpha+\beta}(F)\backslash U_{\alpha+\beta}(\mathbb{A})$. Note that $U_{\alpha+\beta} \subset V$ and $\delta U_{\alpha+\beta}\delta^{-1} = U_{2\alpha+\beta} \subset V'$, we get $f_s(\delta\mathbf{x}_{\alpha+\beta}(r)vg) = f_s(\delta vg)$. On the other hand, we have $\omega_\psi(\mathbf{x}_{\alpha+\beta}(r)vg)\phi(\xi) = \psi(-2r\xi)\omega_\psi(vg)\phi(\xi)$. Thus the above integral has an inner integral

$$\int_{F\backslash\mathbb{A}} \sum_{\xi\in F^\times} \psi(-2r\xi)\omega_\psi(vg)\phi(\xi)dr = \sum_{\xi\in F^\times} \omega_\psi(vg)\phi(\xi) \int_{F\backslash\mathbb{A}} \psi(-2r\xi)dr = 0.$$

Thus when $\delta = w_\beta w_\alpha$, the corresponding term is zero. Thus we get

$$I(\widetilde{\varphi}, \phi, f_s) = \int_{\mathrm{SL}_2^\gamma(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V^\gamma(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg) f_s(\gamma vg)dvdg.$$

We have $\mathrm{SL}_2^\gamma = B_{\mathrm{SL}_2}$ and $V^\gamma = U_{\alpha+\beta}$. We decompose $\widetilde{\theta}_\phi$ as

$$\widetilde{\theta}_\phi(vg) = \omega_\psi(vg)\phi(0) + \sum_{\xi\in F^\times} \omega_\psi(vg)\phi(\xi) = \omega_\psi(vg)\phi(0) + \sum_{a\in F^\times} \omega_\psi(t(a)vg)\phi(1).$$

Recall that $t(a) = \mathrm{diag}(a, a^{-1})$. Since $\gamma U_\beta\gamma^{-1} \subset U_{3\alpha+\beta} \subset V'$, we have

$$f_s(\gamma v\mathbf{x}_\beta(r)g) = f_s(\gamma\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g).$$

On the other hand we have $\omega_\psi(v\mathbf{x}_\beta(r)g)\phi(0) = \omega_\psi(\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g)\phi(0)$. Thus after a changing variable on v, we can get that the contribution of $\omega_\psi(vg)\phi(0)$ to $I(\widetilde{\varphi}, \phi, f_s)$ has an inner integral

$$\int_{F\backslash\mathbb{A}} \widetilde{\varphi}(\mathbf{x}_\beta(r)g)dr,$$

which is zero by the cuspidality of $\widetilde{\varphi}$. Thus we get

$$I(\widetilde{\varphi},\phi,f_s) = \int_{B_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g) \sum_{a\in F^\times} \omega_\psi(t(a)vg)\phi(1) f_s(\gamma vg)dvdg.$$

Collapsing the summation with the integration, we then get

$$\begin{aligned}
&I(\widetilde{\varphi},\phi,f_s)\\
&= \int_{N_{\mathrm{SL}_2}(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\omega_\psi(vg)\phi(1) f_s(\gamma vg)dvdg\\
&= \int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(F)\backslash V(\mathbb{A})} \int_{F\backslash\mathbb{A}} \widetilde{\varphi}(\mathbf{x}_\beta(r)g)\omega_\psi(v\mathbf{x}_\beta(r)g)\phi(1) f_s(\gamma v\mathbf{x}_\beta(r)g)drdvdg.
\end{aligned}$$

Note that we have $\omega_\psi(v\mathbf{x}_\beta(r)g)\phi(1) = \omega_\psi(\mathbf{x}_\beta(r)\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g)\phi(1) = \psi(r)$ $\omega_\psi(\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g)\phi(1)$. On the other hand, we have $\gamma\mathbf{x}_\beta(r)\gamma^{-1} \subset U_{3\alpha+\beta} \subset V'$. Thus $f_s(\gamma v\mathbf{x}_\beta(r)g) = f_s(\gamma\mathbf{x}_\beta(-r)v\mathbf{x}_\beta(r)g)$. After a changing of variable on v, we get

$$I(\widetilde{\varphi},\phi,f_s) = \int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(F)\backslash V(\mathbb{A})} W_{\widetilde{\varphi}}(g)\omega_\psi(vg)\phi(1) f_s(\gamma vg)dvdg,$$

where

$$W_{\widetilde{\varphi}}(g) = \int_{F\backslash\mathbb{A}} \tilde{\varphi}(\mathbf{x}_\beta(r)g)\psi(r)dr.$$

We can further decompose the above integral as

$$\begin{aligned}
I(\widetilde{\varphi},\phi,f_s) = &\int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(\mathbb{A})\backslash V(\mathbb{A})}\\
&\int_{F\backslash\mathbb{A}} W_{\widetilde{\varphi}}(g)\omega_\psi(\mathbf{x}_{\alpha+\beta}(r)vg)\phi(1) f_s(\gamma\mathbf{x}_{\alpha+\beta}(r)vg)drdvdg.
\end{aligned}$$

Note that $\omega_\psi(\mathbf{x}_{\alpha+\beta}(r)vg)\phi(1) = \psi(-2r)\omega_\psi(vg)\phi(1)$ and $f_s(\gamma\mathbf{x}_{\alpha+\beta}(r)vg) = f_s$ $(\mathbf{x}_\alpha(r)\gamma vg)$ since $\gamma\mathbf{x}_{\alpha+\beta}(r)\gamma^{-1} = \mathbf{x}_\alpha(r)$. We then get

$$I(\widetilde{\varphi},\phi,f_s) = \int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(\mathbb{A})\backslash V(\mathbb{A})} W_{\widetilde{\varphi}}(g)\omega_\psi(vg)\phi(1) W_{f_s}(\gamma vg)dvdg,$$

where

$$W_{f_s}(\gamma vg) = \int_{F\backslash\mathbb{A}} f_s(\mathbf{x}_\alpha(r)\gamma vg)\psi(-2r)dr.$$

This concludes the proof. □

4 Unramified calculation

In this section, let F be a p-adic field with $p \neq 2$. Let $\mathfrak{o}$ be the ring of integers of F, and let p be a uniformizer of $\mathfrak{o}$ by abuse of notation. Let q be the cardinality of the residue field $\mathfrak{o}/(p)$.

4.1 Local Weil representations

Let ψ be an additive character of F and let $\gamma(\psi)$ be the Weil index and let $\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}$. Let ω_ψ be the Weil representation of $\widetilde{\mathrm{SL}}_2(F) \ltimes V$ on $\mathcal{S}(F)$ via the projection $\widetilde{\mathrm{SL}}_2(F) \ltimes V \to \widetilde{\mathrm{SL}}_2(F) \ltimes \mathscr{H}$. For $\phi \in \mathcal{S}(F)$, we have the well-know formulas:

$$\begin{aligned}
(\omega_\psi(w^1)\phi)(x) &= \gamma(\psi)\hat{\phi}(x),\\
(\omega_\psi(n(b))\phi)(x) &= \psi(bx^2)\phi(x), b \in F\\
(\omega_\psi(t(a))\phi)(x) &= |a|^{1/2}\mu_\psi(a)\phi(ax), a \in F^\times\\
(\omega_\psi((r_1, r_2, r_3))\phi)(x) &= \psi\,(r_3 - 2xr_2 - r_1r_2)\,\phi(x + r_1), (r_1, r_2, r_3) \in \mathscr{H}(F).
\end{aligned}$$

where $\hat{\phi}(x) = \int_F \phi(y)\psi(2xy)dy$ is the Fourier transform of ϕ with respect to ψ. Note that under the embedding $\mathrm{SL}_2(F) \hookrightarrow G_2(F)$, we have $w^1 = w_\beta, n(b) = \mathbf{x}_\beta(b)$ and $t(a) = h(a, a^{-1})$.

4.2 Unramified calculation

In this subsection, we compute the local integral in last section. The strategy is similar to the unramified calculation in [6].

Let $\tilde{\pi}$ be an unramified genuine representation of $\widetilde{\mathrm{SL}}_2(F)$ with Satake parameter a, and let τ be an unramified irreducible representation of $\mathrm{GL}_2(F)$ with Satake parameters b_1, b_2. Let $\widetilde{W} \in \mathcal{W}(\tilde{\pi}, \psi)$ with $\widetilde{W}(1) = 1$. Let $v_0 \in V_\tau$ be an unramified vector and $\lambda \in \mathrm{Hom}_N(V_\tau, \psi)$ such that $\lambda(v_0) = 1$. Let $f_s : G_2 \to V_\tau$ be the unramified section in $I(s, \tau)$ with $f_s(e) = v_0$. Let

$$W_{f_s} : G_2 \times \mathrm{GL}_2(F) \to \mathbb{C}$$

be the function $W_{f_s}(g, a) = \lambda(\tau(a) f_s(g))$. We will write $W_{f_s}(g)$ for $W_{f_s}(g, 1)$ in the following. By assumption and Shintani formula, we have

$$\begin{aligned}
W_{f_s}(h(p^k, p^l)) &= q^{-3s(2k+l)}\lambda(\tau(\mathrm{diag}(p^{k+l}, p^k))v_0)\\
&= q^{-3s(2k+l)} W_{v_0}(\mathrm{diag}(p^{k+l}, p^k))\\
&= \begin{cases} q^{-3s(2k+l)} \frac{(b_1b_2)^k q^{-l/2}}{b_1 - b_2}(b_1^{l+1} - b_2^{l+1}), & \text{if } l \geq 0,\\ 0, & \text{if } l < 0.\end{cases}
\end{aligned} \tag{4.1}$$

Let $\phi \in \mathcal{S}(F)$ be the characteristic function of $\mathfrak{o}$. We need to compute the integral

$$I(\widetilde{W}, W_{f_s}, \phi) = \int_{N_2\backslash \mathrm{SL}_2(F)} \int_{U_{\alpha+\beta}\backslash V} \widetilde{W}(g)\omega_\psi(vg)\phi(1) W_{f_s}(\gamma vg) dv dg.$$

In the following, we fix the Haar measure such that $\mathrm{vol}(dr, \mathfrak{o}) = 1$. Thus $\mathrm{vol}(d^*r, \mathfrak{o}^\times) = 1 - q^{-1}$.

Using the Iwasawa decomposition $\mathrm{SL}_2(F) = N_2(F)A_2(F)\mathrm{SL}_2(\mathfrak{o})$, we have

$$\begin{aligned} & I(\widetilde{W}, W_{f_s}, \phi) \\ &= \int_{F^\times}\int_{F^4} \widetilde{W}(t(a))\omega_\psi([r_1, 0, r_3]t(a))\phi(1)W_{f_s} \\ &\quad (\gamma(r_1, 0, r_3, r_4, r_5)t(a))|a|^{-2}dr_1dr_3dr_4dr_5d^\times a \\ &= \int_{F^\times}\int_{F^4} \widetilde{W}(t(a))\omega_\psi(t(a)[r_1, 0, r_3])\phi(1)W_{f_s} \\ &\quad (\gamma t(a)(r_1, 0, r_3, r_4, r_5))|a|^{-3}dr_1dr_3dr_4dr_5d^\times a \end{aligned}$$

If $\widetilde{W}(t(a)) \neq 0$, then $|a| \leq 1$. On the other hand, we have

$$\omega_\psi(t(a)[r_1, 0, r_3])\phi(1) = \mu_\psi(a)|a|^{1/2}\psi(r_3)\phi(a + r_1).$$

If $\phi(a + r_1) \neq 0$ and $a \in \mathfrak{o}$, then $r_1 \in \mathfrak{o}$. Thus the domain for a and r_1 in the above integral is $\{a \in F^\times \cap \mathfrak{o}, r_1 \in \mathfrak{o}\}$. Note that $\gamma t(a) = h(1, a)\gamma = h(1, a)w_\beta w_\alpha w_\beta w_\alpha$. Thus, if we conjugate $w_\alpha \mathbf{x}_\alpha(r_1)$ to the right side, we can get

$$h(1, a)\gamma[r_1, 0, r_3, r_4, r_5] = h(1, a)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(-r_3)\mathbf{x}_\beta(-r_4 - 3r_1r_3)\mathbf{x}_{3\alpha+2\beta}(r_5)w_\alpha \mathbf{x}_\alpha(r_1).$$

Since $w_\alpha \mathbf{x}_\alpha(r_1) \in K$ for $r_1 \in \mathfrak{o}$, by changing of variables, we get

$$\begin{aligned} & I(\widetilde{W}, W_{f_s}, \phi) \\ &= \int_{|a|\leq 1} \widetilde{W}(t(a))|a|^{-5/2}\mu_\psi(a) \\ &\quad \cdot \int_{F^3} W_{f_s}(h(1, a)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(r_3)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-r_3)dr_3dr_4dr_5d^*a \\ &= \sum_{n\geq 0} \widetilde{W}(t(p^n))q^{5n/2}\mu_\psi(p^n)J(n), \end{aligned}$$

where

$$J(n) = \int_{F^3} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(r_3)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-r_3)dr_3dr_4dr_5.$$

By dividing the domain of r_3 into two parts, we can write $J(n) = J_1(n) + J_2(n)$, where

$$\begin{aligned} J_1(n) &= \int_{|r_3|\leq 1}\int_{F^2} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(r_3)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-r_3)dr_3dr_4dr_5 \\ &= \int_{F^2} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))dr_4dr_5, \end{aligned}$$

and

$$J_2(n) = \int_{|r_3|>1}\int_{F^2} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(r_3)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-r_3)dr_3dr_4dr_5.$$

Lemma 4.1 *Set*

$$I(n) = \int_F W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) dr.$$

Then

$$I(n) = \frac{q^{-(3s+1/2)n}}{b_1 - b_2}\Big[(b_1^{n+1} - b_2^{n+1}) \\ + (1-q^{-1})\frac{b_1 b_2 X}{(1-b_1X)(1-b_2X)}, \\ (b_1^n - b_2^n - b_1^{n+1}X + b_2^{n+1}X + b_1X(b_1b_2X)^n - b_2X(b_1b_2X)^n)\Big],$$

where $X = q^{-(3s-3/2)}$.

Proof We have

$$\begin{aligned} I(n) &= \int_F W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) dr \\ &= \int_{|r|\le 1} W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) dr \\ &\quad + \int_{|r|>1} W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) dr \\ &= W_{f_s}(h(1, p^n)) + \int_{|r|>1} W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) dr. \end{aligned}$$

To deal with the integral when $|r| > 1$, we consider the following Iwasawa decomposition of $w_\beta \mathbf{x}_\beta(r)$:

$$w_\beta \mathbf{x}_\beta(r) = \mathbf{x}_\beta(-r^{-1}) h(-r^{-1}, -r) \mathbf{x}_{-\beta}(r^{-1}).$$

Since $\mathbf{x}_{-\beta}(r^{-1})$ is in the maximal compact subgroup for $|r| > 1$, we have

$$W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r)) = W_{f_s}(h(1, p^n) \mathbf{x}_\beta(-r^{-1}) h(-r^{-1}, -r)) = W_{f_s}(h(1, p^n) h(r^{-1}, r)),$$

where we used $U_\beta \subset V'$. For $|r| > 1$, we can write $r = p^{-m}u$ for some $m \ge 1$ and $u \in \mathfrak{o}^\times$. We then have $dr = q^m du$. Note that $\mathrm{vol}(\mathfrak{o}^\times) = 1 - q^{-1}$. Thus we have

$$I(n) = W_{f_s}(h(1, p^n)) + \sum_{m\ge 1} (1-q^{-1}) q^m W_{f_s}(h(p^m, p^{n-m})).$$

Note that $h(p^m, 1) \mapsto \mathrm{diag}(p^m, p^m)$ under the isomorphism $M' \cong \mathrm{GL}_2$. Thus we have

$$W_{f_s}(h(p^m, 1) h(1, p^{n-m})) = q^{-6sm} \omega_\tau(p)^m W_{f_s}(h(1, p^{n-m})).$$

Thus we get

$$I(n) = W_{f_s}(h(1, p^n)) + \sum_{m\ge 1} (1-q^{-1}) q^{(-6s+1)m} \omega_\tau(p)^m W_{f_s}(h(1, p^{n-m})).$$

By (4.1), we have

$$W_{f_s}(h(1, p^{n-m})) = \begin{cases} \frac{q^{-3s(n-m)-(n-m)/2}}{b_1 - b_2}(b_1^{n-m+1} - b_2^{n-m+1}), & \text{if } n \ge m, \\ 0, & \text{if } n < m. \end{cases}$$

Thus for $n \geq 1$, we have

$$I(n) = \frac{q^{-(3s+1/2)n}}{b_1 - b_2} \left((b_1^{n+1} - b_2^{n+1}) + \sum_{m=1}^{n} (1 - q^{-1}) q^{-(3s-3/2)m} (b_1^{n+1} b_2^m - b_2^{n+1} b_1^m) \right).$$

Thus result can be computed using the geometric summation formula. One can check that the given formula also satisfies $I(0) = 1$. □

Lemma 4.2 *We have*

$$J_1(n) = \frac{1 - q^{-6s+1} b_1 b_2}{1 - q^{-6s+2} b_1 b_2} I(n).$$

Proof To compute $J_1(n)$, we break up the domain of integration in r_4 and get

$$\begin{aligned} J_1(n) &= \int_F \int_{|r_4| \leq 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_\beta(r_4) \mathbf{x}_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \\ &\quad + \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_\beta(r_4) \mathbf{x}_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \\ &:= J_{11}(n) + J_{12}(n), \end{aligned}$$

where

$$\begin{aligned} J_{11}(n) &= \int_F \int_{|r_4| \leq 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_\beta(r_4) \mathbf{x}_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \\ &= \int_F \int_{|r_4| \leq 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_{3\alpha+2\beta}(r_5) w_\beta^{-1} w_\alpha^{-1} w_\alpha w_\beta \mathbf{x}_\beta(r_4)) dr_4 dr_5 \\ &= \int_F W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r_5)) dr_5 \\ &= I(n), \end{aligned}$$

and

$$\begin{aligned} J_{12}(n) &= \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_\beta(r_4) \mathbf{x}_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \\ &= \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_{3\alpha+2\beta}(r_5) w_\beta^{-1} w_\alpha^{-1} w_\alpha w_\beta \mathbf{x}_\beta(r_4)) dr_4 dr_5 \\ &= \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r_5) w_\alpha w_\beta \mathbf{x}_\beta(r_4)) dr_4 dr_5. \end{aligned}$$

We have the Iwasawa decomposition of $w_\beta \mathbf{x}_\beta(r_4)$:

$$w_\beta \mathbf{x}_\beta(r_4) = \mathbf{x}_\beta(-r_4^{-1}) h(-r_4^{-1}, -r_4) \mathbf{x}_{-\beta}(r_4^{-1}).$$

Since $\mathbf{x}_{-\beta}(r_4^{-1})$ is in the maximal compact subgroup for $|r_4| > 1$, we then get

$$\begin{aligned} J_{12}(n) &= \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) w_\beta \mathbf{x}_\beta(r_5) w_\alpha \mathbf{x}_\beta(-r_4^{-1}) h(r_4^{-1}, r_4)) dr_4 dr_5 \\ &= \int_F \int_{|r_4| > 1} W_{f_s}(h(1, p^n) h(r_4^{-1}, 1) w_\beta \mathbf{x}_\beta(r_4^{-1} r_5)) dr_4 dr_5 \end{aligned}$$

$$= \int_F \int_{|r_4|>1} |r_4| W_{f_s}(h(1, p^n)h(r_4^{-1}, 1)w_\beta \mathbf{x}_\beta(r_5)) dr_4 dr_5$$
$$= \sum_{m\geq 1}(1-q^{-1})q^{2m} \int_F W_{f_s}(h(p^m, 1)h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)) dr_5,$$

where in the second equality, we conjugated $\mathbf{x}_\beta(-r_4^{-1})h(r_4^{-1}, r_4)$ to the left, and in the third equality, we wrote $r_4 = p^{-m}u$ for $m \geq 1$, $u \in \mathfrak{o}^\times$ and used $dr_4 = q^m du$, $\mathrm{vol}(\mathfrak{o}^\times) = 1-q^{-1}$. Note that $h(p^m, 1)$ is in the center of M', and thus

$$W_{f_s}(h(p^m, 1)g) = q^{-6sm}\omega_\tau(p)^m W_{f_s}(g),$$

we get

$$J_{12}(n) = (1-q^{-1}) \sum_{m\geq 1} q^{-6sm+2m}\omega_\tau(p)^m \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)) dr_5.$$

Thus we get

$$J_1(n) = I(n) + \sum_{m\geq 1}(1-q^{-1})q^{(-6s+2)m}(b_1b_2)^m I(n).$$

A simple calculation gives the formula of $J_1(n)$. □

We next consider the term

$$J_2(n) = \int_{|r_3|>1}\int_{F^2} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(r_3)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-r_3) dr_3 dr_4 dr_5.$$

For $|r_3| > 1$, we can write $r_3 \in p^{-m}u$ with $m \geq 1$, $u \in \mathfrak{o}^\times$. We then have,

$$J_2(n)$$
$$= \int_{F^2}\sum_{m\geq 1} q^m W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-m}u)\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-p^{-m}u) du dr_4 dr_5.$$

Write $\mathbf{x}_{\alpha+\beta}(p^{-m}u) = h(u, u^{-1})\mathbf{x}_{\alpha+\beta}(p^{-m})h(u^{-1}, u)$, and by conjugation and changing of variables, we get

$$J_2(n)$$
$$= \int_{F^2}\sum_{m\geq 1} q^m W_{f_s}(h(u^{-1}, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-m})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-p^{-m}u) du dr_4 dr_5,$$

where we used $h(u, u^{-1})$ is in the maximal compact subgroup of $G_2(F)$. Since $h(u^{-1}, 1)$ maps to the center of M' and $|\omega_\tau(u)| = 1$, we have

$$W_{f_s}(h(u^{-1}, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-m})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))$$
$$= W_{f_s}(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-m})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5)).$$

Thus we get

$$J_2(n)$$
$$= \int_{F^2}\sum_{m\geq 1} q^m W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-m})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))\psi(-p^{-m}u) du dr_4 dr_5.$$

Since

$$\int_{\mathfrak{o}^\times} \psi(p^k u)du = \begin{cases} 1-q^{-1}, & \text{if } k \geq 0, \\ -q^{-1}, & \text{if } k=-1, \\ 0, & \text{if } k \leq -2, \end{cases}$$

we get $J_2(n) = -R(n)$, where

$$R(n) = \int_{F^2} W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))dr_4dr_5.$$

To evaluate $R(n)$, we split the domain of r_4, and write $R(n) = R_1(n) + R_2(n)$, where

$$\begin{aligned} R_1(n) &= \int_{|r_4|\leq 1}\int_F W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))dr_4dr_5, \\ &= \int_F W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1})\mathbf{x}_{3\alpha+2\beta}(r_5))dr_5, \end{aligned}$$

and

$$R_2(n) = \int_{|r_4|>1}\int_F W_{f_s}(h(1, p^n)w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1})\mathbf{x}_\beta(r_4)\mathbf{x}_{3\alpha+2\beta}(r_5))dr_4dr_5.$$

We now compute $R_1(n)$. We conjugate $w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1})$ to the right and then get

$$\begin{aligned} R_1(n) &= \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1}))dr_5 \\ &= \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)w_\alpha \mathbf{x}_\alpha(-p^{-1}))dr_5 \end{aligned}$$

Next, we use the Iwasawa decomposition of $w_\alpha \mathbf{x}_\alpha(p^{-1})$:

$$w_\alpha \mathbf{x}_\alpha(-p^{-1}) = \mathbf{x}_\alpha(p)h(p^{-1}, p^2)\mathbf{x}_{-\alpha}(-p)$$

to get

$$R_1(n) = \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)\mathbf{x}_\alpha(p)h(p^{-1}, p^2))dr_5.$$

Next, we use the commutator relation

$$\mathbf{x}_\beta(r_5)\mathbf{x}_\alpha(p) = \mathbf{x}_{\alpha+\beta}(pr_5)u\mathbf{x}_\alpha(p)\mathbf{x}_\beta(r_5),$$

where u is in the root space of $2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta$. Then we get

$$R_1(n) = \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_{\alpha+\beta}(pr_5)u\mathbf{x}_\alpha(p)\mathbf{x}_\beta(r_5)h(p^{-1}, p^2))dr_5.$$

Note that $w_\beta u \mathbf{x}_\alpha(r) w_\beta(1) \in V'$, and $h(1, p^n)w_\beta \mathbf{x}_{\alpha+\beta}(pr_5)(h(1, p^n)w_\beta)^{-1} = \mathbf{x}_\alpha(-p^{n+1}r_5)$, and $W_{f_s}(\mathbf{x}_\alpha(r)g) = \psi(2r)W_{f_s}(g)$, we get

$$\begin{aligned} R_1(n) &= \int_F W_{f_s}(h(1, p^n)w_\beta \mathbf{x}_\beta(r_5)h(p^{-1}, p^2))\psi(-2p^{n+1}r_5)dr_5 \\ &= \int_F W_{f_s}(h(p^2, 1)h(1, p^{n-1})w_\beta \mathbf{x}_\beta(p^3r_5))\psi(-2p^{n+1}r_5)dr_5 \end{aligned}$$

$$= q^{-12s+3}\omega_\tau(p^2)\int_F W_{f_s}(h(1, p^{n-1})w_\beta \mathbf{x}_\beta(r_5))\psi(-2p^{n-2}r_5)dr_5,$$

where the last equality comes from a changing of variable on r_5 and the fact that $h(p^2, 1) \mapsto \mathrm{diag}(p^2, p^2)$ under the isomorphism $M' \cong \mathrm{GL}_2$. We next break up the integral on r_5 and get

$$R_1(n) = q^{-12s+3}\omega_\tau(p^2)W_{f_s}(h(1, p^{n-1}))\int_{|r_5|\le 1}\psi(-2p^{n-2}r_5)dr_5 + q^{-12s+3}\omega_\tau(p^2)\int_{|r_5|>1} W_{f_s}(h(1, p^{n-1})w_\beta \mathbf{x}_\beta(r_5))\psi(-2p^{n-2}r_5)dr_5.$$

Using the Iwasawa decomposition of $w_\beta \mathbf{x}_\beta(r_5)$, we have

$$R_1(n) = q^{-12s+3}\omega_\tau(p^2) \left(W_{f_s}(h(1, p^{n-1}))\int_{|r_5|\le 1}\psi(-2p^{n-2}r_5)dr_5 + \sum_{m=1}^{\infty} W_{f_s}(h(p^m, p^{n-m-1}))q^m \int_{\mathfrak{o}^\times}\psi(-2p^{n-m-2}u)du\right).$$

Lemma 4.3 *We have $R_1(n) = 0$ if $n \le 1$, and*

$$R_1(n) = q^{-12s+3}\omega_\tau(p)^2 I(n-1) - q^{-6s(n+1)+n+2}\omega_\tau(p)^{n+1},$$

for $n \ge 2$.

Proof Note that $\int_{|r|\le 1}\psi(p^k r)dr = 0$ if $k < 0$ and $\int_{|r|\le 1}\psi(p^k r)dr = 1$ if $k \ge 0$. Moreover, we have

$$\int_{\mathfrak{o}^\times}\psi(p^k u)du = \begin{cases} 1-q^{-1}, & \text{if } k \ge 0,\\ -q^{-1}, & \text{if } k = -1,\\ 0, & \text{if } k \le -2.\end{cases}$$

Thus we get $R_1(n) = 0$ for $n \le 1$. For $n \ge 2$, we have

$$\begin{aligned} R_1(n) &= q^{-12s+3}\omega_\tau(p^2) \cdot \left(W_{f_s}(h(1, p^{n-1})) + \sum_{m=1}^{n-2}(1-q^{-1})q^m W_{f_s}(h(p^m, p^{n-m-1}))\right.\\ &\left. -q^{-1}q^{n-1}W_{f_s}(h(p^{(n-1)}, 1))\right) = q^{-12s+3}\omega_\tau(p^2)\\ &\cdot \left(W_{f_s}(h(1, p^{n-1})) + \sum_{m=1}^{n-1}(1-q^{-1})q^m W_{f_s}(h(p^m, p^{n-m-1}))\right.\\ &\left. -q^{n-1}W_{f_s}(h(p^{(n-1)}, 1))\right) = q^{-12s+3}\\ &\omega_\tau(p)^2 I(n-1) - q^{-12s+3+n-1}\omega_\tau(p)^2 W_{f_s}(h(p^{n-1}, 1)),\end{aligned}$$

where in the last equation, we used the formula in the computation of $I(n)$. Since $h(p^{n-1}, 1)$ is in the center of M', we have $W_{f_s}(h(p^{n-1}, 1)) = q^{-6s(n-1)}\omega_\tau(p)^{n-1}$. The result follows. □

We next consider

$$R_2(n) = \int_{|r_4|>1} \int_F W_{f_s}(h(1, p^n) w_\beta w_\alpha w_\beta \mathbf{x}_{\alpha+\beta}(p^{-1}) \mathbf{x}_\beta(r_4) \mathbf{x}_{3\alpha+2\beta}(r_5)) dr_4 dr_5.$$

Conjugating w_β to the right side and using the Iwasawa decomposition of $w_\beta \mathbf{x}_\beta(r_4)$, we can get

$$R_2(n) = \int_F \int_{|r_4|>1} W_{f_s}(h(1, p^n) w_\beta w_\alpha \mathbf{x}_\alpha(p^{-1}) \mathbf{x}_{3\alpha+\beta}(r_5) \mathbf{x}_\beta(r_4^{-1}) h(r_4^{-1}, r_4)) dr_4 dr_5.$$

From the commutator relation, we have

$$\mathbf{x}_\alpha(p^{-1}) \mathbf{x}_\beta(r_4^{-1}) = \mathbf{x}_\beta(r_4^{-1}) \mathbf{x}_\alpha(p^{-1}) \mathbf{x}_{2\alpha+\beta}(p^{-2} r_4^{-1}) u,$$

for some u in the group generated by roots subgroups of $\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$. Like in the computation of $R_1(n)$, we have

$$\begin{aligned}
R_2(n) &= \int_F \int_{|r_4|>1} W_{f_s}(h(1, p^n) w_\beta w_\alpha \mathbf{x}_\alpha(p^{-1}) \mathbf{x}_{3\alpha+\beta}(r_5) h(r_4^{-1}, r_4)) \psi \\
&\quad (-2p^{n-2} r_4^{-1}) dr_4 dr_5 \\
&= \int_F \int_{|r_4|>1} W_{f_s}(h(1, p^n) h(r_4^{-1}, 1) w_\beta \mathbf{x}_\beta(r_5 r_4^{-1}) w_\alpha \mathbf{x}_\alpha(p^{-1} r_4^{-1})) \psi(-2p^{n-2} r_4^{-1}) \\
&\quad dr_4 dr_5 \\
&= \int_F \int_{|r_4|>1} |r_4| W_{f_s}(h(1, p^n) h(r_4^{-1}, 1) w_\beta \mathbf{x}_\beta(r)) \psi(-2p^{n-2} r_4^{-1}) dr_4 dr \\
&= I(n) \int_{|r_4|>1} |r_4|^{-6s+1} \omega_\tau(r_4^{-1}) \psi(-2p^{n-2} r_4^{-1}) dr_4 \\
&= I(n) \sum_{m=1}^{\infty} q^{(-6s+2)m} \omega_\tau(p)^m \int_{\mathfrak{o}^\times} \psi(-2p^{m+n-2} u) du.
\end{aligned}$$

Lemma 4.4 *We have*

$$R_2(n) = \begin{cases} I(0) q^{-6s+2} \omega_\tau(p) \left(-q^{-1} + (1 - q^{-1}) \frac{q^{-6s+2} \omega_\tau(p)}{1 - q^{-6s+2} \omega_\tau(p)} \right), & n = 0, \\ I(n)(1 - q^{-1}) \frac{q^{-6s+2} \omega_\tau(p)}{1 - q^{-6s+2} \omega_\tau(p)}, & n \geq 1 \end{cases}$$

Proof If $n \geq 1$, then $\int_{\mathfrak{o}^\times} \psi(p^{m+n-2} u) du = (1 - q^{-1})$ for $m \geq 1$. Thus, we have

$$\begin{aligned}
R_2(n) &= I(n) \sum_{m=1}^{\infty} q^{(-6s+2)m} \omega_\tau(p)^m (1 - q^{-1}) \\
&= I(n)(1 - q^{-1}) \frac{q^{-6s+2} \omega_\tau(p)}{1 - q^{-6s+2} \omega_\tau(p)}.
\end{aligned}$$

If $n = 0$, then $\int_{\mathfrak{o}^\times} \psi(p^{m+n-2} u) du = (1 - q^{-1})$ for $m \geq 2$, and $\int_{\mathfrak{o}^\times} \psi(p^{m+n-2} u) du = -q^{-1}$ for $m = 1$. Thus, we have

$$R_2(0) = I(0)(-q^{-1} q^{-6s+2} \omega_\tau(p) + (1 - q^{-1}) \sum_{m=2}^{\infty} q^{(-6s+2)m} \omega_\tau(p)^m)$$

$$= I(0)q^{-6s+2}\omega_\tau(p)\left(-q^{-1} + (1-q^{-1})\frac{q^{-6s+2}\omega_\tau(p)}{1-q^{-6s+2}\omega_\tau(p)}\right).$$

The completes the proof of the lemma. □

Combining the above results, we get the following

Lemma 4.5 *We have*

$$R(n) = \begin{cases} -I(0)q^{-6s+1}\omega_\tau(p)\frac{1-q^{-6s+3}\omega_\tau(p)}{1-q^{-6s+2}\omega_\tau(p)}, & n=0, \\ I(1)(1-q^{-1})\frac{q^{-6s+2}\omega_\tau(p)}{1-q^{-6s+2}\omega_\tau(p)}, & n=1, \\ q^{-12s+3}\omega_\tau(p)^2 I(n-1) - q^{-6s(n+1)+n+2}\omega_\tau(p)^{n+1} & \\ \qquad +I(n)(1-q^{-1})\frac{q^{-6s+2}\omega_\tau(p)}{1-q^{-6s+2}\omega_\tau(p)}, & n \geq 2, \end{cases}$$

and

$$\begin{aligned} J(n) &= J_1(n) - R(n) \\ &= \begin{cases} 1+Y, & n=0 \\ I(1), & n=1, \\ I(n) - q^{-1}Y^2 I(n-1) + q^{-n}Y^{n+1}, & n \geq 2. \end{cases} \end{aligned}$$

where $Y = q^{-6s+2}\omega_\tau(p)$

By the main result of [1], we have

$$\widetilde{W}(t(p^n)) = \frac{\mu_\psi(p^n)q^{-n}}{a-a^{-1}}\left((1-\chi(p)q^{-1/2}a^{-1})a^{n+1} - (1-\chi(p)q^{-1/2}a)a^{-(n+1)}\right),$$

where $\chi(p) = (p,p)_F = (p,-1)_F$. Note that the notation $\gamma(a)$ in [1] is our $\mu_\psi(a)^{-1}$. Note that $\mu_\psi(p^n)\mu_\psi(p^n) = (p^n,p^n)_F = \chi(p)^n$. Thus

$$\begin{aligned} I(\widetilde{W}, W_{f_s}, \phi) = \sum_{n\geq 0} \frac{q^{3n/2}\chi(p)^n}{a-a^{-1}} \\ \left((1-\chi(p)q^{-1/2}a^{-1})a^{n+1} - (1-\chi(p)q^{-1/2}a)a^{-(n+1)}\right) J(n). \end{aligned}$$

Plugging the formula $J(n)$ into the above equation, we can get that

$$\begin{aligned} &I(\widetilde{W}, W_f, \phi) \\ &= \frac{(1-b_1q^{-1}X)(1-b_2q^{-1}X)(1-b_1b_2q^{-1}X^2)(1-b_1^2b_2q^{-1}X^3)(1-b_1b_2^2q^{-1}X^3)}{(1-\chi(p)a^{-1}b_1b_2q^{-1/2}X^2)(1-\chi(p)ab_1b_2q^{-1/2}X^2)} \\ &\cdot \frac{1}{\prod_{i=1}^2(1-\chi(p)a^{-1}b_iq^{-1/2}X)\prod_{i=1}^2(1-\chi(p)ab_iq^{-1/2}X)} \\ &= \frac{L(3s-1,\widetilde{\pi}\times(\chi\otimes\tau))L(6s-5/2,\widetilde{\pi}\otimes(\chi\otimes\omega_\tau))}{L(3s-1/2,\tau)L(6s-2,\omega_\tau)L(9s-7/2,\tau\otimes\omega_\tau)}. \end{aligned}$$

Here

$$L(s,\widetilde{\pi}\otimes(\chi\otimes\omega_\tau)) = \frac{1}{(1-a\chi(p)b_1b_2q^{-s})((1-a^{-1}\chi(p)b_1b_2q^{-s}))}$$

is the L function of $\widetilde{\pi}$ twisted by the character $\chi\otimes\omega_\tau$, and

$$L(s,\widetilde{\pi}\times(\chi\otimes\tau)) = \frac{1}{\prod_{i=1}^2(1-\chi(p)a^{-1}b_iq^{-s})\prod_{i=1}^2(1-\chi(p)ab_iq^{-s})}$$

is the Rankin–Selberg L-function of $\widetilde{\pi}$ twisted by $\chi \otimes \tau$. We record the above calculation in the following

Proposition 4.6 *Let $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi)$ be the normalized unramified Whittaker function, f_s be the normalized unramified section in $I(s, \tau)$ and $\phi \in \mathcal{S}(F)$ is the characteristic function of $\mathfrak{o}$, we have*

$$I(\widetilde{W}, W_{f_s}, \phi) = \frac{L(3s-1, \widetilde{\pi} \times (\chi \otimes \tau))L(6s-5/2, \widetilde{\pi} \otimes (\chi \otimes \omega_\tau))}{L(3s-1/2, \tau)L(6s-2, \omega_\tau)L(9s-7/2, \tau \otimes \omega_\tau)}.$$

5 Some local theory

In this section, let F be a local field, which can be archimedean or non-archimedean. If F is non-archimedean, let $\mathfrak{o}$ be the ring of integers of F, p be a uniformizer of $\mathfrak{o}$ and $q = \mathfrak{o}/(p)$. Let $\widetilde{\pi}$ be an irreducible genuine generic representation of $\widetilde{\mathrm{SL}}_2(F)$, τ be an irreducible generic representation of $\mathrm{GL}_2(F)$. Let ψ be a nontrivial additive character of F.

Lemma 5.1 *Let $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi)$, $f_s \in I(s, \tau)$, $\phi \in \mathcal{S}(F)$, then the integral $I(\widetilde{W}, W_{f_s}, \phi)$ converges absolutely for* $\mathrm{Re}(s)$ *large and has a meromorphic continuation to the whole s-plane. Moreover, if F is a p-adic field, then $I(\widetilde{W}, W_{f_s}, \phi)$ is a rational function in q^{-s}.*

The proof is similar to [5, Lemma 4.2–4.7] and [6, Lemma 3.10, Lemma 3.3]. We omit the details.

Lemma 5.2 *Let $s_0 \in \mathbb{C}$. Then there exists $\widetilde{W} \in \mathcal{W}(\widetilde{\pi}, \psi)$, $f_{s_0} \in I(s_0, \tau)$, $\phi \in \mathcal{S}(F)$ such that $I(\widetilde{W}, W_{f_{s_0}}, \phi) \neq 0$.*

Proof The proof is similar to the proof of [5, Lemma 4.4,4.7], [6, Proposition 3.4]. We omit the details. □

6 Nonvanishing of certain periods on G_2

6.1 Poles of Eisenstein series on G_2

Let τ be a cuspidal unitary representation of $\mathrm{GL}_2(\mathbb{A}) \cong M'(\mathbb{A})$. Let K be a maximal compact subgroup of $G_2(\mathbb{A})$. Given a $K \cap \mathrm{GL}_2(\mathbb{A})$-finite cusp form f in τ, we can extend f to a function $\widetilde{f} : G_2(\mathbb{A}) \to \mathbb{C}$ as in [13, §2]. We then define

$$\Phi_{\widetilde{f},s}(g) = \widetilde{f}(g)\delta_{P'}(m')^{s/3+1/2},$$

for $g = v'm'k$ with $v' \in V'(\mathbb{A}), m' \in M'(\mathbb{A}), k \in K$. Then $\Phi_{\widetilde{f},s}$ is well-defined and $\Phi_{\widetilde{f},s} \in I(\frac{s}{3} + \frac{1}{2}, \tau)$. Then we can consider the Eisenstein series

$$E(s, \widetilde{f}, g) = \sum_{P'(F)\backslash G_2(F)} \Phi_{\widetilde{f},s}(\gamma g).$$

Proposition 6.1 *The Eisenstein series $E(s, \widetilde{f}, g)$ has a pole on the half plane* $\mathrm{Re}(s) > 0$ *if and only if $s = \frac{1}{2}$, $\omega_\tau = 1$ and $L(\frac{1}{2}, \tau) \neq 0$.*

For a proof of the above proposition, see [16, §1] or [10, §5]. If $\omega_\tau = 1$ and $L(\frac{1}{2}, \tau) \neq 0$, denote by $\mathcal{R}(\frac{1}{2}, \tau)$ the space generated by the residues of Eisenstein series $E(s, \widetilde{f}, g)$ defined as above. Note that an element $R \in \mathcal{R}(\frac{1}{2}, \tau)$ is an automorphic form on $G_2(\mathbb{A})$.

6.2 On the Shimura–Waldspurger lift

Let $\widetilde{\pi}$ be a genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. Let $Wd_\psi(\widetilde{\pi})$ be the Shimura–Waldspurger lift of $\widetilde{\pi}$. Then $Wd_\psi(\widetilde{\pi})$ is a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$. A cuspidal automorphic representation τ is in the image of Wd_ψ if and only if $L(\frac{1}{2}, \tau) \neq 0$. Moreover, the correspondence $\widetilde{\pi} \mapsto Wd_\psi(\widetilde{\pi})$ respects the Rankin-Selberg L-functions. For these assertions, see [15] or [2].

6.3 A period on G_2

Theorem 6.2 *Let $\widetilde{\pi}$ be a genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ and τ be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Assume that $\omega_\tau = 1$ and $L(\frac{1}{2}, \tau) \neq 0$. In particular, τ can be viewed as a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. If $Wd_\psi(\widetilde{\pi}) = \chi \otimes \tau$, then there exists $\widetilde{\varphi} \in V_{\widetilde{\pi}}$, $\phi \in \mathcal{S}(\mathbb{A})$, $R \in \mathcal{S}(\frac{1}{2}, \tau)$ such that the period*

$$\mathcal{P}(\widetilde{\varphi}, \widetilde{\theta}_\phi, R) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)R(vg)dvdg$$

is non-vanishing.

Proof For $\widetilde{\varphi} \in V_\pi$, $\phi \in \mathcal{S}(\mathbb{A})$ and a good section $\Phi_{\widetilde{f},s}$ as in Sect. 6.1, by Theorem 3.1 and Proposition 4.6, we have

$$\begin{aligned} I(\widetilde{\varphi}, \phi, \widetilde{f}, s) &= \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)E(vg, \Phi_{\widetilde{f},s})dvdg \\ &= \int_{N_{\mathrm{SL}_2}(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{U_{\alpha+\beta}(\mathbb{A})\backslash V(\mathbb{A})} W_{\widetilde{\varphi}}(g)\omega_\psi(vg)\phi(1)W_{\Phi_{\widetilde{f},s}}(\gamma vg)dvdg \\ &= I_S \cdot \frac{L^S(s+\frac{1}{2}, \widetilde{\pi} \times (\chi \otimes \tau))L^S(2s+\frac{1}{2}, \widetilde{\pi} \otimes (\chi \otimes \omega_\tau))}{L^S(s+1, \tau)L^S(2s+1, \omega_\tau)L^S(3s+1, \tau \otimes \omega_\tau)}. \end{aligned}$$

Here S is a finite set of places of F such that for $v \notin S$, π_v, τ_v are unramified, and I_S is the product of the local zeta integrals over all places $v \in S$ and L^S denotes the partial L-function which is the product of all local L-function as the place v runs over $v \notin S$. Note that $\tau \cong \tau^\vee$ since $\omega_\tau = 1$. Suppose that $Wd_\psi(\widetilde{\pi}) = \chi \otimes \tau = \chi \otimes \tau^\vee$, then $L^S(s+1/2, \widetilde{\pi} \times (\chi \otimes \tau))$ has a pole at $s = 1/2$. Note that at $s = \frac{1}{2}$, $L^S(2s+1/2, \widetilde{\pi} \otimes (\chi \otimes \omega_\tau))$ is holomorphic and nonzero, while $L^S(s+1, \tau)L^S(2s+1, \omega_\tau)L^S(3s+1, \tau \otimes \omega_\tau)$ is holomorphic. Moreover, I_S can be chosen to be nonzero. Thus we get that $I(\widetilde{\varphi}, \phi, \widetilde{f}, s)$ has a pole at $s = 1/2$, which means that there exists a residue $R(g)$ of $E(s, \widetilde{f}, g)$ such that

$$\mathcal{P}(\widetilde{\varphi}, \theta_\phi, R) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\theta}_\phi(vg)R(vg)dvdg \neq 0.$$

This completes the proof. □

Remark 6.3 For an L^2-automorphic form $\eta \in L^2(G_2(F)\backslash G_2(\mathbb{A}))$, one can form the period

$$\eta_{\widetilde{\phi}, \widetilde{\theta}_\phi}(g) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \widetilde{\varphi}(h)\widetilde{\theta}_\phi(vh)\eta(vhg)dvdh.$$

Theorem 6.2 says that if $\eta \in \mathcal{S}(\frac{1}{2}, \tau)$, then under the condition $Wd_{\psi}(\widetilde{\pi}) = \chi \otimes \tau$, the period $\eta_{\widetilde{\varphi}, \widetilde{\theta}_\phi}$ is non-vanishing for certain $\widetilde{\varphi}$ and ϕ. For general η, one can ask under what conditions $\eta_{\widetilde{\varphi}, \widetilde{\theta}_\phi}$ is not identically zero as $\widetilde{\varphi}$ varies in $\widetilde{\pi}$ and $\phi \in \mathcal{S}(\mathbb{A})$. In the classical group case, this is the global Gan–Gross–Prasad conjecture for Fourier–Jacobi case, see [3]. It is natural to ask if it is possible to extend the GGP-conjecture to the G_2-case.

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