Mathematische Zeitschrift

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The quaternionic Monge–Ampère operator and plurisubharmonic functions on the Heisenberg group

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Abstract

Many fundamental results of pluripotential theory on the quaternionic space $\mathbb{H}^n$ are extended to the Heisenberg group. We introduce notions of a plurisubharmonic function, the quaternionic Monge–Ampère operator, differential operators d_0 and d_1 and a closed positive current on the Heisenberg group. The quaternionic Monge–Ampère operator is the coefficient of $(d_0 d_1 u)^n$. We establish the Chern–Levine–Nirenberg type estimate, the existence of quaternionic Monge–Ampère measure for a continuous quaternionic plurisubharmonic function and the minimum principle for the quaternionic Monge–Ampère operator. Unlike the tangential Cauchy–Riemann operator $\overline{\partial}_b$ on the Heisenberg group which behaves badly as $\partial_b \overline{\partial}_b \neq -\overline{\partial}_b \partial_b$, the quaternionic counterpart d_0 and d_1 satisfy $d_0 d_1 = -d_1 d_0$. This is the main reason that we have a good theory for the quaternionic Monge–Ampère operator than $(\partial_b \overline{\partial}_b)^n$.

1 Introduction

The theory of subharmonic functions and potential theory has already been generalized to Carnot groups in terms of SubLaplacians (cf. e.g. [9,11] and references therein), and the generalized horizontal Monge–Ampère operator and H-convex functions on the Heisenberg group have been studied for more than a decade (cf. [7,11–13,15,17,19,22] and references therein). For the 3-dimensional Heisenberg group, Gutiérrez and Montanari [15] proved that the Monge–Ampère measure defined by

$$\int \det(Hess_X(u)) + 12(Tu)^2 \quad \text{for} \quad u \in C^2(\Omega), \tag{1.1}$$

can be extended to H-convex functions, where $Hess_X(u)$ is the symmetric 2×2-matrix

$$Hess_X(u) := \left(\frac{X_i X_j u + X_j X_i u}{2} \right) \tag{1.2}$$

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and X_1, X_2, T are standard left invariant vector fields on the 3-dimensional Heisenberg group. u is called *H-convex* on a domain Ω if for any $\xi, \eta \in \Omega$ such that $\xi^{-1}\eta \in H_0$ and $\xi\delta_r(\xi^{-1}\eta) \in \Omega$ for $r \in [0, 1]$, the function of one real variable $r \to u(\xi\delta_r(\xi^{-1}\eta))$ is convex in $[0, 1]$, where δ_r is the dilation and H_0 indicates the subset of horizontal directions through the origin. It was generalized to the 5-dimensional Heisenberg group by Garofalo and Tournier [13], and to k-Hessian measures for k-convex functions on any dimensional Heisenberg groups by Trudinger and Zhang [22].

In the theory of several complex variables, we have a powerful pluripotential theory about the complex Monge–Ampère operator $(\partial\overline{\partial})^n$ and closed positive currents, where $\overline{\partial}$ is the Cauchy-Riemann operator (cf. e.g. [18]). It is quite interesting to develop its CR version over the Heisenberg group. A natural CR generalization of the complex Monge–Ampère operator is $(\partial_b\overline{\partial}_b)^n$, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator. But unlike $\overline{\partial}\partial = -\partial\overline{\partial}$, it behaves badly as

$$\partial_b\overline{\partial}_b \neq -\overline{\partial}_b\partial_b, \tag{1.3}$$

because of the noncommutativity of horizontal vector fields (cf. Subsection 3.1). So it is very difficult to investigate the operator $(\partial_b\overline{\partial}_b)^n$, e.g. its regularity. On the other hand, pluripotential theory has been extended to the quaternionic space $\mathbb{H}^n$ (cf. [1–6,10,14,23–28,33] and references therein). If we equip the $(4n + 1)$-dimensional Heisenberg group a natural quaternionic strucuture on its horizontal subspace, we can introduce differential operators d_0, d_1 and $\triangle u = d_0 d_1 u$ in terms of complex horizontal vector fields, as the quaternionic counterpart of ∂_b, $\overline{\partial}_b$ and $\partial_b\overline{\partial}_b$. They behave so well that we can extend many fundamental results of quaternionic pluripotential theory on $\mathbb{H}^n$ to the Heisenberg group.

The $(4n + 1)$-dimensional *Heisenberg group* $\mathscr{H}$ is the vector space $\mathbb{R}^{4n+1}$ with the multiplication given by

$$(x, t) \cdot (y, s) = (x + y, t + s + 2\langle x, y \rangle), \quad \text{where} \quad \langle x, y \rangle := \sum_{l=1}^{2n}(x_{2l-1}y_{2l} - x_{2l}y_{2l-1}) \tag{1.4}$$

for $x, y \in \mathbb{R}^{4n}$, $t, s \in \mathbb{R}$. Here $\langle \cdot, \cdot \rangle$ is the standard symplectic form. We introduce a partial quaternionic structure on the Heisenberg group simply by identifying the underlying space of $\mathbb{H}^n$ with $\mathbb{R}^{4n}$. For a fixed $q \in \mathbb{H}^n$, consider a 5-dimensional real subspace

$$\mathscr{H}_q := \{(q\lambda, t) \in \mathscr{H}; \lambda \in \mathbb{H}, t \in \mathbb{R}\}, \tag{1.5}$$

which is a subgroup. $\mathscr{H}_q$ is nonabelian for all $q \in \mathbb{H}^n$ except for a $\mathrm{codim}_{\mathbb{R}}3$ quadratic cone $\mathfrak{D}$. For a point $\eta \in \mathscr{H}$, the left translate of the subgroup $\mathscr{H}_q$ by η,

$$\mathscr{H}_{\eta,q} := \eta\mathscr{H}_q,$$

is a 5-dimensional real hyperplane through η, called a *(right) quaternionic Heisenberg line*. A $[-\infty, \infty)$-valued upper semicontinuous function on $\mathscr{H}$ is said to be *plurisubharmonic* if it is L^1_{loc} and is subharmonic (in terms of the SubLaplacian) on each quaternionic Heisenberg line $\mathscr{H}_{\eta,q}$ for any $\eta \in \mathscr{H}$, $q \in \mathbb{H}^n \backslash \mathfrak{D}$.

Let $X_1, \ldots X_{4n}$ be the standard horizontal left invariant vector fields (2.2) on the Heisenberg group $\mathscr{H}$. Denote the *tangential Cauchy–Fueter operator* on $\mathscr{H}$ by

$$\overline{Q_l} := X_{4l+1} + \mathbf{i}X_{4l+2} + \mathbf{j}X_{4l+3} + \mathbf{k}X_{4l+4},$$

and its conjugate $Q_l = X_{4l+1} - \mathbf{i}X_{4l+2} - \mathbf{j}X_{4l+3} - \mathbf{k}X_{4l+4}$, $l = 0, \ldots, n-1$. See [32] for the Cauchy–Fueter operator on other nilpotent groups of step two. Compared to the Cauchy–Fueter operator on $\mathbb{H}^n$, the tangential Cauchy–Fueter operator $\overline{Q_l}$ is much more complicated because not only $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are noncommutative, but also X_a's are. In particular,

$$\overline{Q_l}Q_l = X_{4l+1}^2 + X_{4l+2}^2 + X_{4l+3}^2 + X_{4l+4}^2 - 8\mathbf{i}\partial_t, \tag{1.6}$$

is not real. But for a real C^2 function u, the $n \times n$ quaternionic matrix

$$\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right)$$

is hyperhermitian, called the *horizontal quaternionic Hessian*. It is nonnegative if u is plurisubharmonic. We define the *quaternionic Monge–Ampère operator* on the Heisenberg group as

$$\det\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right),$$

where det is the Moore determinant.

Alesker obtained Chern–Levine–Nirenberg estimate for the quaternionic Monge–Ampère operator on $\mathbb{H}^n$ [4]. We extend this estimate to the Heisenberg group, and obtain the following existence theorem of the quaternionic Monge–Ampère measure for a continuous plurisubharmonic function.

Theorem 1.1 *Let $\{u_j\}$ be a sequence of C^2 plurisubharmonic functions converging to u uniformly on compact subsets of a domain Ω in $\mathscr{H}$. Then u be a continuous plurisubharmonic function on Ω. Moreover,* $\det\left(\overline{Q_l}Q_m u_j + 8\delta_{lm}\mathbf{i}\partial_t u_j\right)$ *is a family of uniformly bounded measures on each compact subset K of Ω and weakly converges to a non-negative measure on Ω. This measure depends only on u and not on the choice of an approximating sequence $\{u_j\}$.*

It is worth mentioning that compared to the real case (1.2), our quaternionic Monge–Ampère operator need not to be symmetrized for off-diagonal entries and the Monge–Ampère measure does not have an extra term $(Tu)^2$ as in (1.1).

As in [20,27,29,30], motivated by the embedding of quaternionic algebra $\mathbb{H}$ into $\mathbb{C}^{2\times2}$:

$$x_1 + x_2\mathbf{i}_1 + x_3\mathbf{i}_2 + x_4\mathbf{i}_3 \mapsto \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix},$$

we consider complex left invariant vector fields

$$\begin{pmatrix} Z_{00'} & Z_{01'} \\ \vdots & \vdots \\ Z_{l0'} & Z_{l1'} \\ \vdots & \vdots \\ Z_{n0'} & Z_{n1'} \\ \vdots & \vdots \\ Z_{(n+l)0'} & Z_{(n+l)1'} \\ \vdots & \vdots \end{pmatrix} := \begin{pmatrix} X_1 + \mathbf{i}X_2 & -X_3 - \mathbf{i}X_4 \\ \vdots & \vdots \\ X_{4l+1} + \mathbf{i}X_{4l+2} & -X_{4l+3} - \mathbf{i}X_{4l+4} \\ \vdots & \vdots \\ X_3 - \mathbf{i}X_4 & X_1 - \mathbf{i}X_2 \\ \vdots & \vdots \\ X_{4l+3} - \mathbf{i}X_{4l+4} & X_{4l+1} - \mathbf{i}X_{4l+2} \\ \vdots & \vdots \end{pmatrix}, \tag{1.7}$$

where X_a's are the standard horizontal left invariant vector fields (2.2) on $\mathscr{H}$. Let $\wedge^p \mathbb{C}^{2n}$ be the complex exterior algebra generated by $\mathbb{C}^{2n}$, $p = 0, \ldots, 2n$. Denote by $\{\omega^0, \omega^1, \ldots, \omega^{2n-1}\}$ the standard basis of $\mathbb{C}^{2n}$. For a domain Ω in $\mathscr{H}$, we define differential operators $d_0, d_1 : C^1(\Omega, \wedge^p \mathbb{C}^{2n}) \to C(\Omega, \wedge^{p+1} \mathbb{C}^{2n})$ by

$$d_0 F := \sum_I \sum_{A=0}^{2n-1} Z_{A0'} f_I \, \omega^A \wedge \omega^I, \qquad d_1 F := \sum_I \sum_{A=0}^{2n-1} Z_{A1'} f_I \, \omega^A \wedge \omega^I, \qquad (1.8)$$

for $F = \sum_I f_I \omega^I \in C^1(\Omega, \wedge^p \mathbb{C}^{2n})$, where $\omega^I := \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$ for the multi-index $I = (i_1, \ldots, i_p)$. We call a form F closed if $d_0 F = d_1 F = 0$.

In contrast to the bad behaviour (1.3) of $\partial_b \overline{\partial}_b$, we have the following nice identities for d_0 and d_1:

$$d_0 d_1 = -d_1 d_0, \qquad (1.9)$$

which is the main reason that we could have a good theory for the quaternionic Monge–Ampère operator on the Heisenberg group.

Proposition 1.1   1. $d_0^2 = d_1^2 = 0$.
2. *The identity* (1.9) *holds.*
3. *For $F \in C^1(\Omega, \wedge^p \mathbb{C}^{2n})$, $G \in C^1(\Omega, \wedge^q \mathbb{C}^{2n})$, we have*

$$d_\alpha(F \wedge G) = d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G, \qquad \alpha = 0, 1.$$

We introduce a second-order differential operator $\triangle : C^2(\Omega, \wedge^p \mathbb{C}^{2n}) \to C(\Omega, \wedge^{p+2} \mathbb{C}^{2n})$ by

$$\triangle F := d_0 d_1 F, \qquad (1.10)$$

which behaves nicely as $\partial \overline{\partial}$ as in the following proposition.

Proposition 1.2 *For $u_1, \ldots, u_n \in C^2$,*

$$\triangle u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n = d_0(d_1 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n) = -d_1(d_0 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n)$$
$$= d_0 d_1(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n) = \triangle(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n).$$

The quaternionic Monge–Ampère operator can be expressed as the exterior product of $\triangle u$.

Theorem 1.2 *For a real C^2 function u on $\mathscr{H}$, we have*

$$\triangle u \wedge \cdots \wedge \triangle u = n! \, \det\left(\overline{Q_l} Q_m u + 8\delta_{lm} \mathbf{i} \partial_t u\right) \Omega_{2n}, \qquad (1.11)$$

where

$$\Omega_{2n} := \omega^0 \wedge \omega^n \cdots \wedge \omega^{n-1} \wedge \omega^{2n-1} \in \wedge_{\mathbb{R}+}^{2n} \mathbb{C}^{2n}. \qquad (1.12)$$

Theorem 1.3 (The minimum principle) *Let Ω be a bounded domain with smooth boundary in $\mathscr{H}$, and let u and v be continuous plurisubharmonic functions on Ω. Assume that $(\triangle u)^n \le (\triangle v)^n$. Then*

$$\min_{\overline{\Omega}} \{u - v\} = \min_{\partial \Omega} \{u - v\}.$$

An immediate corollary of this theorem is that the uniqueness of continuous solution to the Dirichlet problem for the quaternionic Monge–Ampère equation.

Originally, we define differential operators d_0 and d_1 and the quaternionic Monge–Ampère operator on the right quaternionic Heisenberg group. Later we find that these definitions also work on the Heisenberg group, on which the theory is simplified because its center is only

1-dimensional while the right quaternionic Heisenberg group has a 3-dimensional center. See [20,21] for the tangential k-Cauchy–Fueter complexes on the Heisenberg group and the right quaternionic Heisenberg group.

This paper is arranged as follows. In Sect. 2, we give preliminaries on the Heisenberg group, the group structure of right quaternionic Heisenberg line $\mathscr{H}_q$, the SubLaplacian on $\mathscr{H}_q$ and its fundamental solution. After recall fundamental results on subharmonic functions on a Carnot group, we give basic properties of plurisubharmonic functions on the Heisenberg group. In Sect. 3, we discuss operators d_0, d_1 and nice behavior of brackets $[Z_{AA'}, Z_{BB'}]$, by which we can prove Proposition 1.1. Then we show that the horizontal quaternionic Hessian $(\overline{Q_l} Q_m u + 8\delta_{lm} \mathbf{i}\partial_t u)$ for a real C^2 function u is hyperhermitian, and prove the expression of the quaternionic Monge–Ampère operator in Theorem 1.2 by using linear algebra we developed before in [33]. In Sect. 4, we recall definitions of real forms and positive forms, and show that $\triangle u$ for a C^2 plurisubharmonic function u is a closed strongly positive 2-form. Then we introduce notions of a closed positive current and the "integral" of a positive $2n$-form current, and show that for any plurisubharmonic function u, $\triangle u$ is a closed positive 2-current. In Sect. 5, we give proofs of Chern-Levine-Nirenberg estimate, the existence of the quaternionic Monge–Ampère measure for a continuous plurisubharmonic function and the minimum principle.

2 Plurisubharmonic functions over the Heisenberg group

2.1 The Heisenberg group

We have the following conformal transformations on $\mathscr{H}$: (1) *dilations*: $\delta_r : (x, t) \longrightarrow (rx, r^2t)$, $r > 0$; (2) *left translations*: $\tau_{(y,s)} : (x, t) \longrightarrow (y, s) \cdot (x, t)$; (3) *rotations*: $R_U : (x, t) \longrightarrow (Ux, t)$, for $U \in U(n)$, where $U(n)$ is the unitary group; (4) the *inversion*: $R : (x, t) \longrightarrow \left(\frac{x}{|x|^2 + \mathbf{i}t}, \frac{t}{|x|^4 + |t|^2} \right)$. Define vector fields:

$$X_a u(x, t) := \frac{d}{d\varsigma} u((x, t)(\varsigma e_a, 0)) \Big|_{\varsigma=0}, \tag{2.1}$$

on the Heisenberg group $\mathscr{H}$, where $e_a = (\ldots, 0, 1, 0, \ldots) \in \mathbb{R}^{4n}$ with only the ath entry nonvanishing, $a = 1, 2, \ldots 4n$. It follows from the multiplication law (1.4) that

$$X_{2l-1} := \frac{\partial}{\partial x_{2l-1}} - 2x_{2l} \frac{\partial}{\partial t}, \qquad X_{2l} := \frac{\partial}{\partial x_{2l}} + 2x_{2l-1} \frac{\partial}{\partial t} \tag{2.2}$$

$l = 1, \cdots, 2n$, whose brackets are

$$[X_{2l-1}, X_{2l}] = 4\partial_t, \quad \text{and all other brackets vanish.} \tag{2.3}$$

X_a is *left invariant* in the sense that for any $(y, s) \in \mathscr{H}$,

$$\tau_{(y,s)*} X_a = X_a, \tag{2.4}$$

by definition (2.1), which means for fixed $(y, s) \in \mathscr{H}$,

$$X_a \left(\tau_{(y,s)}^* f \right) \Big|_{(x,t)} = (X_a f) \Big|_{(y,s)(x,t)}, \tag{2.5}$$

where the pull back function $(\tau_{(y,s)}^* f)(x,t) := f((y,s)(x,t))$. On the left hand side above, X_a is the differential operator in (2.2) with coefficients at point (x,t), while on the right hand side, X_a is the differential operator with coefficients at point $(y,s)(x,t)$,

2.2 Right quaternionic Heisenberg lines

For quaternionic numbers $q, p \in \mathbb{H}$, write

$$q = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4, \qquad p = y_1 + \mathbf{i}y_2 + \mathbf{j}y_3 + \mathbf{k}y_4.$$

Let $\widehat{p}$ be the column vector in $\mathbb{R}^4$ represented by p, i.e. $\widehat{p} := (y_1, y_2, y_3, y_4)^t$, and let $q^{\mathbb{R}}$ be the 4×4 matrix representing the transformation of left multiplying by q, i.e.

$$\widehat{qp} = q^{\mathbb{R}}\widehat{p}. \tag{2.6}$$

It is direct to check (cf. [31]) that

$$q^{\mathbb{R}} := \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}, \tag{2.7}$$

and

$$(q_1 q_2)^{\mathbb{R}} = q_1^{\mathbb{R}} q_2^{\mathbb{R}}, \qquad (\overline{q})^{\mathbb{R}} = (q^{\mathbb{R}})^t. \tag{2.8}$$

The multiplication law (1.4) of the Heisenberg group can be written as

$$(y,s) \cdot (x,t) = \left(y + x, s + t + 2 \sum_{l=0}^{n-1} \sum_{j,k=1}^{4} J_{kj} y_{4l+k} x_{4l+j} \right) \tag{2.9}$$

with

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.10}$$

The multiplication of the subgroup $\mathscr{H}_q$ is given by

$$(q\lambda, t)(q\lambda', t') = \left(q(\lambda + \lambda'), t + t' + 2 \sum_{l=0}^{n-1} \left(\widehat{q_l \lambda} \right)^t \widehat{Jq_l \lambda'} \right), \tag{2.11}$$

where

$$\sum_{l=0}^{n-1} \left(q_l^{\mathbb{R}} \widehat{\lambda} \right)^t Jq_l^{\mathbb{R}} \widehat{\lambda'} = \sum_{j,k=1}^{4} B_{kj}^q \lambda_k \lambda'_j, \qquad B^q := \sum_{l=0}^{n-1} (q_l^{\mathbb{R}})^t Jq_l^{\mathbb{R}} \tag{2.12}$$

for $\lambda = \lambda_1 + \mathbf{i}\lambda_2 + \mathbf{j}\lambda_3 + \mathbf{k}\lambda_4$, $\lambda' = \lambda'_1 + \mathbf{i}\lambda'_2 + \mathbf{j}\lambda'_3 + \mathbf{k}\lambda'_4 \in \mathbb{H}$. B^q is a 4×4 skew symmetric matrix. So if we consider the group $\mathscr{H}_q$ as the vector space $\mathbb{R}^5$ with the multiplication given by

$$(\lambda, t)(\lambda', t') = \left(\lambda + \lambda', t + t' + 2 \sum_{k,j=1}^{4} B_{kj}^q \lambda_k \lambda'_j \right), \tag{2.13}$$

we have the isomorphism of groups

$$\iota_q : \widetilde{\mathscr{H}_q} \longrightarrow \mathscr{H}_q, \qquad (\lambda, t) \mapsto (q\lambda, t). \tag{2.14}$$

$\widetilde{\mathscr{H}_q}$ is different from the 5-dimensional Heisenberg group in general. Note that the subgroup $\mathscr{H}_q$ of $\mathscr{H}$ is the same if q is replaced by qq_0 for $0 \neq q_0 \in \mathbb{H}$,

Write $\mathbf{i}_1 := 1$, $\mathbf{i}_2 := \mathbf{i}$, $\mathbf{i}_3 := \mathbf{j}$ and $\mathbf{i}_4 := \mathbf{k}$. Consider left invariant vector fields on $\widetilde{\mathscr{H}_q}$:
$\widetilde{X}_j u(\lambda, t) := \left. \frac{du}{d\varsigma}((\lambda, t)(\varsigma \mathbf{i}_j, 0)) \right|_{\varsigma=0}$ for $(\lambda, t) \in \widetilde{\mathscr{H}_q}$. Since

$$(\lambda, t)(\varsigma \mathbf{i}_j, 0) = \left(\cdots, \lambda_j + \varsigma, \cdots, t + 2\varsigma \sum_{k,j=1}^{4} B_{kj}^q \lambda_k \right),$$

we get

$$\widetilde{X}_j = \frac{\partial}{\partial \lambda_j} + 2 \sum_{k=1}^{4} B_{kj}^q \lambda_k \frac{\partial}{\partial t}. \tag{2.15}$$

Define the SubLaplacian on the right quaternionic Heisenberg line $\widetilde{\mathscr{H}_q}$ as

$$\widetilde{\triangle_q} := \sum_{j=1}^{4} \widetilde{X}_j^2.$$

Note that for $q \in \mathbb{H}$,

$$B^q = \sum_{l=0}^{n-1} \overline{q_l}^{\mathbb{R}} J q_l^{\mathbb{R}} = -\left(\sum_{l=0}^{n-1} \overline{q_l} \mathbf{i} q_l \right)^{\mathbb{R}}$$

by using (2.8) and $\mathbf{i}^{\mathbb{R}} = -J$ by (2.7). Then

$$B^q (B^q)^t = \Lambda_q^2 I_{4 \times 4}, \quad \text{where} \quad \Lambda_q := \left| \sum_{l=0}^{n-1} \overline{q_l} \mathbf{i} q_l \right|, \tag{2.16}$$

by (2.8) again. If write $q_l = x_{4l+1} + \mathbf{i} x_{4l+2} + \mathbf{j} x_{4l+3} + \mathbf{k} x_{4l+4}$, we have $\Lambda_q^2 := S_1^2 + S_2^2 + S_3^2$, where $S_1 := \sum_{l=0}^{n-1}(x_{4l+1}^2 + x_{4l+2}^2 - x_{4l+3}^2 - x_{4l+4}^2)$, $S_2 := 2\sum_{l=0}^{n-1}(-x_{4l+1}x_{4l+4} + x_{4l+2}x_{4l+3})$, $S_3 := 2\sum_{l=0}^{n-1}(x_{4l+1}x_{4l+3} + x_{4l+2}x_{4l+4})$, The *degenerate locus* $\mathfrak{D} := \{q \in \mathbb{H}^n; \Lambda_q = 0\}$ is the intersection of three quadratic hypersurfaces in $\mathbb{R}^{4n}$ given by $S_1 = S_2 = S_3 = 0$. Thus $\mathscr{H}_q$ is abelian if and only if $q \in \mathfrak{D}$.

Proposition 2.1 *For $q \in \mathbb{H}^n \setminus \mathfrak{D}$, the fundamental solution of $\widetilde{\triangle_q}$ on $\widetilde{\mathscr{H}_q}$ is*

$$\Gamma_q(\lambda, t) = -\frac{C_q}{\rho_q(\lambda, t)}, \quad \text{i.e.} \quad \widetilde{\triangle_q} \Gamma_q = \delta_0, \tag{2.17}$$

where

$$\rho_q(\lambda, t) = \Lambda_q^2 |\lambda|^4 + t^2, \quad C_q^{-1} := \int_{\widetilde{\mathscr{H}_q}} \frac{32\Lambda_q^2 |\lambda|^2}{(\rho_q(\lambda, t) + 1)^3} d\lambda dt.$$

Proof Note that for $\varepsilon > 0$, we have

$$\sum_{j=1}^{4} \widetilde{X}_j^2 \frac{-1}{\rho_q + \varepsilon} = \frac{\Sigma_{j=1}^{4} \widetilde{X}_j^2 \rho_q}{(\rho_q + \varepsilon)^2} - 2\frac{\Sigma_{j=1}^{4}(\widetilde{X}_j \rho_q)^2}{(\rho_q + \varepsilon)^3}. \tag{2.18}$$

It follows from the expression (2.15) of $\widetilde{X}_j$ that

$$\widetilde{X}_j \rho_q = 4\Lambda_q^2 |\lambda|^2 \lambda_j + 4 \sum_{k=1}^{4} B_{kj}^q \lambda_k t,$$

and

$$\sum_{j=1}^{4} \widetilde{X}_j^2 \rho_q = 4 \sum_{j=1}^{4} \Lambda_q^2 |\lambda|^2 + 8 \sum_{j=1}^{4} \Lambda_q^2 \lambda_j^2 + 8 \sum_{j=1}^{4} \left(\sum_{k=1}^{4} B_{kj}^q \lambda_k \right)^2$$

$$= 24\Lambda_q^2 |\lambda|^2 + 8 \langle B^q (B^q)^t \lambda, \lambda \rangle = 32\Lambda_q^2 |\lambda|^2, \qquad (2.19)$$

by skew symmetry of B^q and using (2.16). On the other hand, we have

$$\sum_{j=1}^{4} (\widetilde{X}_j \rho_q)^2 = 16\Lambda_q^4 |\lambda|^4 |\lambda|^2 + 16\Lambda_q^2 |\lambda|^2 t^2 = 16\Lambda_q^2 |\lambda|^2 \rho_q(\lambda, t) \qquad (2.20)$$

by $\sum_{j,k=1}^{4} B_{kj}^q \lambda_k \lambda_j = 0$. Substituting (2.19)–(2.20) into (2.18) to get

$$\sum_{j=1}^{4} \widetilde{X}_j^2 \frac{-1}{\rho_q + \varepsilon} = 32\Lambda_q^2 \frac{|\lambda|^2 \varepsilon}{(\rho_q + \varepsilon)^3}.$$

Then $\int \varphi \widetilde{\triangle}_b (\frac{-1}{\rho_q + \varepsilon}) \to C_q^{-1} \varphi(0,0)$ for $\varphi \in C_0^\infty(\widetilde{\mathscr{H}_q})$ by recaling and letting $\varepsilon \to 0+$. We get the result. □

2.3 Subharmonic functions on Carnot groups

A *Carnot group* $\mathbb{G}$ of step $r \geq 1$ is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is stratified, i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ and $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$. Let $Y_1, \cdots, Y_p$ are smooth left invariant vector fields on a Carnot group $\mathbb{G}$ and homogeneous of degree one with respect to the dilation group of $\mathbb{G}$, such that $\{Y_1, \cdots, Y_p\}$ is a basis of $\mathfrak{g}_1$. There exists a homogeneous norm $\| \cdot \|$ on a Carnot group $\mathbb{G}$ [9] such that

$$\Gamma(\xi, \eta) := -\frac{C_Q}{\|\xi^{-1}\eta\|^{Q-2}}, \qquad (2.21)$$

for some $Q > 0$ is a fundamental solution for the *SubLaplacian* $\triangle_{\mathbb{G}}$ given by $\triangle_{\mathbb{G}} = \sum_{j=1}^{p} Y_j^2$, (the fundamental solution used in [9] is different from the usual one (2.21) by a minus sign). $\triangle_{\mathbb{G}}$ is not elliptic except for $\mathbb{G}$ abelian. But it is hypoelliptic since vector fields $\{Y_1, \cdots, Y_p\}$ satisfy Hörmander's hypoellipticity condition.

We denote by $D(\xi, r)$ the *ball* of center ξ and radius r, i.e.

$$D(\xi, r) = \{\eta \in \mathbb{G} | \|\xi^{-1}\eta\| < r\}. \qquad (2.22)$$

Recall the *representation formulae* [9] for any smooth function u on $\mathbb{G}$:

$$u(\xi) = M_r^{\mathbb{G}}(u)(\xi) - N_r(\triangle_{\mathbb{G}} u)(\xi) = \mathscr{M}_r^{\mathbb{G}}(u)(\xi) - \mathscr{N}_r(\triangle_{\mathbb{G}} u)(\xi), \qquad (2.23)$$

for every $\xi \in \Omega$ and $r > 0$ such that $D(\xi, r) \subset \Omega$, where

$$M_r^{\mathbb{G}}(u)(\xi) := \frac{m_Q}{r^Q} \int_{D(\xi,r)} K(\xi^{-1}\eta)u(\eta)dV(\eta),$$

$$N_r^{\mathbb{G}}(u)(\xi) := \frac{n_Q}{r^Q} \int_0^r \rho^{Q-1}d\rho \int_{D(\xi,\rho)} \left(\frac{1}{\|\xi^{-1}\eta\|^{Q-2}} - \frac{1}{\rho^{Q-2}} \right) u(\eta)dV(\eta), \quad (2.24)$$

and

$$\mathscr{M}_r^{\mathbb{G}}(u)(\xi) := \int_{\partial D(\xi,r)} \mathscr{K}(\xi^{-1}\eta)u(\eta)dS(\eta),$$

$$\mathscr{N}_r^{\mathbb{G}}(u)(\xi) := C_Q \int_{D(\xi,r)} \left(\frac{1}{\|\xi^{-1}\eta\|^{Q-2}} - \frac{1}{\rho^{Q-2}} \right) u(\eta)dV(\eta), \quad (2.25)$$

for some positive constants m_Q, n_Q, and

$$K = |\nabla_{\mathbb{G}}d|^2, \quad \mathscr{K} = \frac{|\nabla_{\mathbb{G}}\Gamma|^2}{|\nabla\Gamma|}. \quad (2.26)$$

Here $\nabla_{\mathbb{G}}$ the vector valued differential operator $(Y_1, \ldots, Y_p)$ and ∇ is the usual gradient, $d(\xi) = \|\xi\|$, dV is the volume element and dS is the surface measure on $\partial D(\xi, r)$. Integrals $M_r^{\mathbb{G}}(u)$ and $\mathscr{M}_r^{\mathbb{G}}(u)$ are related by the coarea formula.

A function u on a domain $\Omega \subset \triangle_{\mathbb{G}}$ is called *harmonic* if $\triangle_{\mathbb{G}}u = 0$ in the sense of distributions. Then a harmonic function u in an open set Ω satisfies the *mean-value formula*

$$u(\xi) = \mathscr{M}_r^{\mathbb{G}}(u)(\xi) = M_r^{\mathbb{G}}(u)(\xi),$$

by (2.23). For an open set $\Omega \subset \mathbb{G}$, we say that an upper semicontinuous function function $u : \Omega \to [-\infty, \infty)$ is $\triangle_{\mathbb{G}}$-*subharmonic* if for every $\xi \in \Omega$ there exists $r_\xi > 0$ such that

$$u(\xi) \leq M_r^{\mathbb{G}}(u)(\xi) \quad \text{for} \quad r < r_\xi. \quad (2.27)$$

Proposition 2.2 (The maximum principle for the SubLaplacian [9]) *If $\Omega \subseteq \mathbb{G}$ is a bounded open set, for every $u \in C^2(\Omega)$ satisfying $\triangle_{\mathbb{G}}u \geq 0$ in Ω and $limsup_{\xi \to \eta}u(\xi) \leq 0$ for any $\eta \in \partial\Omega$, we have $u \leq 0$ in Ω.*

Theorem 2.1 (Theorem 4.3 in [9]) *Let Ω be an open set in $\mathbb{G}$ and $u : \Omega \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following statements are equivalent:*

(i) *u is subharmonic;*
(ii) *$u \in L_{loc}^1(\Omega)$, $u(\xi) = lim_{r \to 0+}M_r^{\mathbb{G}}(u)(\xi)$ for every $\xi \in \Omega$ and $\triangle_{\mathbb{G}}u \geq 0$ in Ω in the sense of distributions.*

When $\mathbb{G}$ is the Heisenberg group $\mathscr{H}$ in (1.4), the SubLaplacian is

$$\triangle_b = \sum_{a=1}^{4n} X_a^2,$$

where X_a's are given by (2.2). It is not elliptic, but is subelliptic. It is known that the fundamental solution of $\triangle_b$ is $-C_Q\| \cdot \|^{-Q+2}$ for some constant $C_Q > 0$ as in Proposition 2.1, with the norm given by

$$\|(x, t)\| := (|x|^4 + t^2)^{\frac{1}{4}}.$$

The invariant Haar measure on $\mathscr{H}$ is the usual Lebesgue measure $dxdt$ on $\mathbb{R}^{4n+3}$.

$$\mathscr{K}(x,t) = \frac{|\nabla_{\mathbb{G}}\Gamma|^2}{|\nabla\Gamma|}(x,t) = \frac{2C_Q(Q-2)}{\|(x,t)\|^{Q-2}} \frac{|x|^2}{\sqrt{4|x|^6+t^2}}, \tag{2.28}$$

in the mean-value formula, where $Q := 4n+2$, the *homogeneous dimension* of the $(4n+1)$-dimensional Heisenberg group $\mathscr{H}$.

When $\mathbb{G}$ is the group $\widetilde{\mathscr{H}_q}$ in (2.13), the SubLaplacian is $\widetilde{\triangle}_b$. Because of the fundamental solution of $\widetilde{\triangle}_b$ given in Proposition 2.1, its norm is given by

$$\|(\lambda,t)\|_q := (\Lambda_q^2|\lambda|^4 + t^2)^{\frac{1}{4}}.$$

The invariant Haar measure on $\widetilde{\mathscr{H}_q}$ is the usual Lebesgue measure $d\lambda dt$ on $\mathbb{R}^5$. Its homogeneous dimension is 6, and the mean-value formulae becomes

$$\mathscr{M}_r^q(u)(\eta) := \int_{\partial D_q(0,r)} \frac{|\nabla_q\Gamma_q|^2}{|\nabla\Gamma_q|}(\lambda',t')\iota_{\eta,q}^* u(\lambda',t')dS(\lambda',t'),$$

$$M_r^q(u)(\eta) := \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda',t')\iota_{\eta,q}^* u(\lambda',t')dV(\lambda',t'), \tag{2.29}$$

where $D_q(0,r)$ is the ball of radius r and centered at the origin in $\widetilde{\mathscr{H}_q}$ in terms of the norm $\|\cdot\|_q$, m_q is the constant in the representation formula (2.24) for the group $\widetilde{\mathscr{H}_q}$, and

$$K_q(\lambda,t) = \sum_{j=1}^{4}\left(\widetilde{X}_j\rho_q^{\frac{1}{4}}\right)^2 = \frac{\Lambda_q^2|\lambda|^2}{\|(\lambda,t)\|_q^2},$$

by (2.20), which is homogeneous of degree 0.

2.4 Plurisubharmonic functions on the Heisenberg group

Although $\mathscr{H}_{\eta,q}$ is not a subgroup, by the embedding

$$\iota_{\eta,q} : \widetilde{\mathscr{H}_q} \to \mathscr{H}_{\eta,q}, \quad (\lambda,t) \mapsto \eta(q\lambda,t), \tag{2.30}$$

we say that u is *subharmonic function on* $\mathscr{H}_{\eta,q}$ if $\iota_{\eta,q}^* u$ is $\widetilde{\triangle}_q$-subharmonic on $\widetilde{\mathscr{H}_q}$. Thus, a $[-\infty,\infty)$-valued upper semicontinuous function u on a domain $\Omega \subset \mathscr{H}$ is *plurisubharmonic* if u is $L^1_{\text{loc}}(\Omega)$ and $\iota_{\eta,q}^* u$ is $\widetilde{\triangle}_q$-subharmonic on $\iota_{\eta,q}^*\Omega \cap \widetilde{\mathscr{H}_q}$ for any $q \in \mathbb{H}^n\backslash\mathfrak{D}$ and $\eta \in \mathscr{H}^n$. Denote by $PSH(\Omega)$ the class of all plurisubharmonic functions on Ω.

Recall that the *convolution* of two functions u and v over $\mathscr{H}$ is defined as

$$u * v(x,t) = \int_{\mathscr{H}} u(y,s)v((y,s)^{-1}(x,t))dyds.$$

Then

$$Y(u * v) = u * Yv \tag{2.31}$$

for any left invariant vector field Y by (2.5), and

$$u * v(x,t) = \int_{\mathscr{H}} u((x,t)(y,s)^{-1})v(y,s)dyds.$$

by taking transformation $(y,s)^{-1}(x,t) \to (y,s)$ for fixed (x,t), whose Jacobian can be easily checked to be identity. By the non-commutativity

$$(x,t)(y,s)^{-1} \neq (y,s)^{-1}(x,t) \tag{2.32}$$

in general, we have $u * v \neq v * u$, and

$$\partial D(\xi, r) = \{\eta \in \mathbb{G} | \|\xi^{-1}\eta\| = r\} \neq \{\eta \in \mathbb{G} | \|\eta\xi^{-1}\| = r\}.$$

Consider the standard regularization given by the convolution $\chi_\varepsilon * u$ with

$$\chi_\varepsilon(\xi) := \frac{1}{\varepsilon^Q} \chi \left(\delta_{\frac{1}{\varepsilon}}(\xi) \right), \tag{2.33}$$

where $0 \leq \chi \in C_0^\infty(D(0, 1))$, $\int_{\mathscr{H}} \chi(\xi) dV(\xi) = 1$. Then $\chi_\varepsilon * u$ subharmonic if u is (cf. Proposition 2.3 (6)), but we do not know whether $\chi_\varepsilon * u$ is decreasing as ε decreasing to 0, which we could not prove as in the Euclidean case, because of the non-commutativity.

Remark 2.1 (1) It is a consequence of Theorem 2.1 that a function is in $L_{loc}^1(\Omega)$ if it is $\triangle_b$-subharmonic on $\Omega \subset \mathscr{H}$. But since $\mathscr{H}_q$ is different in general for different $q \in \mathbb{H}^n \setminus \mathfrak{D}$, we do not know wether a $PSH(\Omega)$ function is $\triangle_b$-subharmonic on Ω. So we require it as a condition in the definition.
(2) In the characterization of subharmonicity in Theorem 2.1 there is an additional condition $u(\xi) = \lim_{r \to 0+} M_r^{\mathbb{G}}(u)(\xi)$. We know that $M_r^{\mathbb{G}}(u)(\xi)$ is increasing in r if $\triangle_{\mathbb{G}}u \geq 0$.

The following basic properties of PSH functions also hold on the Heisenberg group.

Proposition 2.3 *Assume that Ω is a bounded domain in $\mathscr{H}$. Then we have that*

(1) *If $u, v \in PSH(\Omega)$, then $au + bv \in PSH(\Omega)$, for positive constants a, b;*
(2) *If $u, v \in PSH(\Omega)$, then $\max\{u, v\} \in PSH(\Omega)$;*
(3) *If $\{u_\alpha\}$ is a family of locally uniformly bounded functions in $PSH(\Omega)$, then the upper semicontinuous regularization $(\sup_\alpha u_\alpha)^*$ is a PSH function;*
(4) *If $\{u_n\}$ is a sequence of functions in $PSH(\Omega)$ such that u_n is decreasing to $u \in L_{loc}^1(\Omega)$, then $u \in PSH(\Omega)$;*
(5) *If $u \in PSH(\Omega)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is convex and nondecreasing, then $\gamma \circ u \in PSH(\Omega)$;*
(6) *If $u \in PSH(\Omega)$, then the regularization $\chi_\varepsilon * u(\xi)$ is also PSH on $\Omega' \subset \mathscr{H}$, where Ω' is subdomain such that $\Omega' D(0, \varepsilon) \subset \Omega$. Moreover, if u is also continuous, then $\chi_\varepsilon * u$ converges to u uniformly on any compact subset.*
(7) *If $\omega \Subset \Omega$, $u \in PSH(\Omega)$, $v \in PSH(\omega)$, and $\limsup_{\xi \to \eta} v(\xi) \leq u(\eta)$ for all $\eta \in \partial\omega$, then the function defined by*

$$\phi = \begin{cases} u, & \text{on } \Omega \setminus \omega, \\ \max\{u, v\}, & \text{on } \omega, \end{cases}$$

is PSH on Ω.

Proof (1)–(3) follows from definition trivially.
(4) It holds since for any fixed $q \in \mathbb{H}^n \setminus \mathfrak{D}$, $\eta \in \Omega$ and small $r > 0$,

$$u(\eta) = \lim_{n \to \infty} u_n(\eta) \leq \lim_{n \to \infty} \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) \iota_{\eta,q}^* u_n(\lambda, t) dV(\lambda, t)$$

$$= \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) \iota_{\eta,q}^* u(\lambda, t) dV(\lambda, t) = M_r^q(u)(\eta)$$

by the monotone convergence theorem.

(5) It holds since

$$M_r^q(\gamma \circ u)(\eta) = \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) \gamma(\iota_{\eta,q}^* u(\lambda, t)) dV(\lambda, t)$$

$$\geq \gamma \left(\frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) \iota_{\eta,q}^* u(\lambda, t) dV(\lambda, t) \right) \geq \gamma \left(\iota_{\eta,q}^* u(0) \right) = (\gamma \circ u)(\eta)$$

by Jensen's inequality for nondecreasing convex function γ, since $\frac{m_q}{r^6} K_q(\lambda, t)$ is nonnegative and its integral over $D_q(0, r)$ is 1. The latter fact follows from the mean value formula for the harmonic function $\equiv 1$.

(6) For fixed $q \in \mathbb{H}^n \backslash \mathfrak{D}$ and $\eta \in \Omega$, $\chi_\varepsilon * u$ is PSH since it is smooth and

$$M_r^q(\chi_\varepsilon * u)(\eta) = \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) \iota_{\eta,q}^*(\chi_\varepsilon * u)(\lambda, t) dV(\lambda, t)$$

$$= \frac{m_q}{r^6} \int_{D_q(0,r)} K_q(\lambda, t) dV(\lambda, t) \int_{\mathscr{H}} \chi_\varepsilon(y, s) u \left((y, s)^{-1} \eta(q\lambda, t) \right) dy ds$$

$$= \int_{\mathscr{H}} \chi_\varepsilon(y, s) M_r^q \left(\iota_{(y,s)^{-1}\eta,q}^* u \right)(0) dy ds$$

$$\geq \int_{\mathscr{H}} \chi_\varepsilon(y, s) u \left((y, s)^{-1} \eta \right) dy ds = \chi_\varepsilon * u(\eta), \tag{2.34}$$

by Fubini's theorem and subharmonicity of u on the open subset $\Omega \cap \mathscr{H}_{(y,s)^{-1}\eta,q}$. The uniform convergence is trivial.

(7) ϕ is obviously in $L_{\mathrm{loc}}^1(\Omega)$, and is PSH on $\mathring{\omega}$ by (2). For $\eta \in \partial\omega$,

$$M_r^q(\phi)(\eta) \geq M_r^q(u)(\eta) \geq u(\eta) = \phi(\eta)$$

for small $r > 0$. □

Remark 2.2 Our notion of plurisubharmonic functions is different from that introduced by Harvey and Lawson [16] for calibrated geometries, i.e. an upper semicontinuous function u satisfies $\triangle u \geq 0$ on each calibrated submanifold in $\mathbb{R}^N$, where $\triangle$ is the Laplacian associated to the induced Riemannian metric on the calibrated submanifold. In our definition we require $\triangle_q u \geq 0$ for SubLaplacian $\triangle_q$, which is subelliptic, on each 5-dimensional real hyperplane $\mathscr{H}_{\eta,q}$ for $q \in \mathbb{H}^n \backslash \mathfrak{D}$.

3 Differential operators d_0, d_1, $\triangle$ and the quaternionic Monge–Ampère operator on the Heisenberg group

3.1 Differential operators d_0 and d_1

Denote $\overline{W}_j := X_{2j-1} + \mathbf{i} X_{2j}$, $W_j := X_{2j-1} - \mathbf{i} X_{2j}$, $j = 1, \ldots 2n$. Then

$$[W_j, \overline{W}_k] = 8\delta_{jk} \mathbf{i} \partial_t$$

and all other brackets vanish by (2.2). Let $\{\ldots, \theta^{\overline{j}}, \theta^j, \ldots, \theta\}$ be the basis dual to $\{\ldots, \overline{W}_j, W_j, \ldots, \partial_t\}$. The *tangential Cauchy-Riemann operator* is defined as $\overline{\partial}_b u = \sum_{j=1}^{2n} \overline{W}_j u \, \theta^{\overline{j}}$ for a function u and

$$\overline{\partial}_b \left(\sum u_{J\overline{K}} \theta^J \wedge \theta^{\overline{K}} \right) = \sum \overline{\partial}_b u_{J\overline{K}} \wedge \theta^J \wedge \theta^{\overline{K}} \tag{3.1}$$

where $\theta^J = \theta^{j_1} \wedge \cdots \wedge \theta^{j_l}$, $\theta^{\overline{K}} = \theta^{\overline{k}_1} \wedge \cdots \wedge \theta^{\overline{k}_m}$ for multi-indices $J = (j_1, \ldots, j_l)$, $\overline{K} = (\overline{k}_1, \ldots, \overline{k}_m)$. Similarly, $\partial_b u = \sum_{j=1}^{2n} W_j u \, \theta^j$ for a function u and is extended to forms as in (3.1). Then

$$\partial_b \overline{\partial}_b u = \sum_{j,k=1}^{2n} W_k \overline{W}_j u \, \theta^k \wedge \theta^{\overline{j}},$$

$$\overline{\partial}_b \partial_b u = \sum_{j,k=1}^{2n} \overline{W}_j W_k u \, \theta^{\overline{j}} \wedge \theta^k = - \sum_{j,k=1}^{2n} W_k \overline{W}_j u \theta^k \wedge \theta^{\overline{j}} + 8\mathbf{i}\partial_t u \sum_{k=1}^{2n} \theta^k \wedge \theta^{\overline{k}}.$$

Thus $\partial_b \overline{\partial}_b \neq -\overline{\partial}_b \partial_b$.

By the definition of the operator $\triangle$ in (1.10), we have

$$\triangle F = \frac{1}{2} \sum_{A,B,I} (Z_{A0'} Z_{B1'} - Z_{B0'} Z_{A1'}) f_I \, \omega^A \wedge \omega^B \wedge \omega^I, \tag{3.2}$$

for $F = \sum_I f_I \, \omega^I$. Now for a function $u \in C^2$ we define

$$\triangle_{AB} u := \frac{1}{2} (Z_{A0'} Z_{B1'} u - Z_{B0'} Z_{A1'} u). \tag{3.3}$$

$2\triangle_{AB}$ is the determinant of (2×2)-submatrix of Ath and Bth rows in (1.7). Note that $Z_{B0'} Z_{A1'} u$ in the above definition could not be replaced by $Z_{A1'} Z_{B0'} u$ in general because of noncommutativity. Then we can write

$$\triangle u = \sum_{A,B=0}^{2n-1} \triangle_{AB} u \, \omega^A \wedge \omega^B. \tag{3.4}$$

When $u_1 = \ldots = u_n = u$, $\triangle u_1 \wedge \cdots \wedge \triangle u_n$ coincides with $(\triangle u)^n := \wedge^n \triangle u$.

The following nice behavior of brackets plays a key role in the proof of properties of d_0, d_1.

Proposition 3.1 (1) *For fixed $A' = 0'$ or $1'$, we have $[Z_{AA'}, Z_{BA'}] = 0$ for any $A, B = 0, \ldots 2n - 1$, i.e. each column $\{Z_{0A'}, \ldots, Z_{(2n-1)A'}\}$ in (1.7) spans an abelian subalgebra.*
(2) *If $|A - B| \neq 0, n$, we have*

$$[Z_{A0'}, Z_{B1'}] = 0, \tag{3.5}$$

and

$$[Z_{l0'}, Z_{(n+l)1'}] = [Z_{(n+l)0'}, Z_{l1'}] = -8\mathbf{i}\partial_t, \tag{3.6}$$

for $l = 0, , \ldots n - 1$, and

$$2\triangle_{l(n+l)} = X_{4l+1}^2 + X_{4l+2}^2 + X_{4l+3}^2 + X_{4l+4}^2. \tag{3.7}$$

Proof Noting that by (1.7), $Z_{AA'}$ and $Z_{BB'}$ for $|A - B| \neq 0$ or n are linear combinations of X_{2l+j}'s, $j = 1, 2$, with different l, and so their bracket vanishes by (2.3). Thus (1) and (3.5) hold. (3.6) follows from brackets in (2.3) and the expression of $Z_{AA'}$'s in (1.7). (3.7) holds by

$$2\triangle_{l(n+l)} = (X_{4l+1} + \mathbf{i}X_{4l+2})(X_{4l+1} - \mathbf{i}X_{4l+2}) + (X_{4l+3} - \mathbf{i}X_{4l+4})(X_{4l+3} + \mathbf{i}X_{4l+4})$$
$$= X_{4l+1}^2 + X_{4l+2}^2 + X_{4l+3}^2 + X_{4l+4}^2 - \mathbf{i}[X_{4l+1}, X_{4l+2}] + \mathbf{i}[X_{4l+3}, X_{4l+4}]$$

and using (2.3). □

Proof of Proposition 1.1 (1) For any $F = \sum_I f_I \omega^I$, note that we have $Z_{A0'} Z_{B0'} f_I = Z_{B0'} Z_{A0'} f_I$ by Proposition 3.1 (1). So we have

$$d_0^2 F = \sum_I \sum_{A,B=0}^{2n-1} Z_{A0'} Z_{B0'} f_I \, \omega^A \wedge \omega^B \wedge \omega^I = 0,$$

by $\omega^A \wedge \omega^B = -\omega^B \wedge \omega^A$. It is similar for $d_1^2 = 0$.

(2) For any $F = \sum_I f_I \omega^I$, we have

$$
\begin{aligned}
d_0 d_1 F &= \sum_I \sum_{A,B} Z_{A0'} Z_{B1'} f_I \omega^A \wedge \omega^B \wedge \omega^I = \sum_I \sum_{|A-B|\neq 0,n} Z_{A0'} Z_{B1'} f_I \omega^A \wedge \omega^B \wedge \omega^I \\
&\quad + \sum_I \sum_{l=0}^{n-1} \left(Z_{l0'} Z_{(n+l)1'} - Z_{(n+l)0'} Z_{l1'} \right) f_I \omega^l \wedge \omega^{n+l} \wedge \omega^I \\
&= -\sum_I \sum_{|A-B|\neq 0,n} Z_{B1'} Z_{A0'} f_I \omega^B \wedge \omega^A \wedge \omega^I \\
&\quad - \sum_I \sum_{l=0}^{n-1} (Z_{l1'} Z_{(n+l)0'} - Z_{(n+l)1'} Z_{l0'}) f_I \omega^l \wedge \omega^{n+l} \wedge \omega^I \\
&= -\sum_{A,B,I} Z_{A1'} Z_{B0'} f_I \omega^A \wedge \omega^B \wedge \omega^I = -d_1 d_0 F,
\end{aligned}
$$

by using commutators (3.5)–(3.6) in Proposition 3.1 in the third identity.

(3) Write $G = \sum_J g_J \omega^J$. We have

$$
\begin{aligned}
d_\alpha(F \wedge G) &= \sum_{A,I,J} [Z_{A\alpha'}(f_I) g_J + f_I Z_{A\alpha'}(g_J)] \omega^A \wedge \omega^I \wedge \omega^J \\
&= \sum_{A,I} Z_{A\alpha'}(f_I) \omega^A \wedge \omega^I \wedge \sum_J g_J \omega^J + (-1)^p \sum_{A,I} f_I \omega^I \wedge \sum_J Z_{A\alpha'}(g_J) \omega^A \wedge \omega^J \\
&= d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G.
\end{aligned}
$$

by $\omega^A \wedge \omega^I = (-1)^p \omega^I \wedge \omega^A$.   □

Corollary 3.1 *For $u_1, \ldots, u_n \in C^3$, $\triangle u_1 \wedge \cdots \wedge \triangle u_k$ is closed, $k = 1, \ldots, n,$.*

Proof By Proposition 1.1 (3), we have

$$d_\alpha(\triangle u_1 \wedge \cdots \wedge \triangle u_k) = \sum_{j=1}^k \triangle u_1 \wedge \cdots \wedge d_\alpha(\triangle u_j) \wedge \cdots \wedge \triangle u_k,$$

for $\alpha = 0, 1$. Note that $d_0 \triangle = d_0^2 d_1 = 0$ and $d_1 \triangle = -d_1^2 d_0 = 0$ by using Proposition 1.1 (1)–(2). It follows that $d_\alpha(\triangle u_1 \wedge \cdots \wedge \triangle u_k) = 0$.   □

Proof of Proposition 1.2 It follows from Corollary 3.1 that

$$d_0(\triangle u_2 \wedge \cdots \wedge \triangle u_n) = d_1(\triangle u_2 \wedge \cdots \wedge \triangle u_n) = 0.$$

By Proposition 1.1 (3),

$$
\begin{aligned}
d_\alpha(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n) &= d_\alpha u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n + u_1 d_\alpha(\triangle u_2 \wedge \cdots \wedge \triangle u_n) \\
&= d_\alpha u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n.
\end{aligned}
$$

So we have

$$\triangle(u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n) = d_0 d_1 (u_1 \triangle u_2 \wedge \cdots \wedge \triangle u_n) = d_0 (d_1 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n)$$
$$= d_0 d_1 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n - d_1 u_1 \wedge d_0 (\triangle u_2 \wedge \cdots \wedge \triangle u_n)$$
$$= \triangle u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_n.$$

$\square$

3.2 The quaternionic Monge–Ampère operator on the Heisenberg groups

A quaternionic $(n \times n)$-matrix $(\mathcal{M}_{jk})$ is called *hyperhermitian* if $\mathcal{M}_{jk} = \overline{\mathcal{M}_{kj}}$.

Proposition 3.2 (Claim 1.1.4, 1.1.7 in [1]) *For a hyperhermitian $(n \times n)$-matrix $\mathcal{M}$, there exists a unitary matrix $\mathcal{U}$ such that $\mathcal{U}^* \mathcal{M} \mathcal{U}$ is diagonal and real.*

Proposition 3.3 (Theorem 1.1.9 in [1])

(1) *The Moore determinant of any complex hermitian matrix considered as a quaternionic hyperhermitian matrix is equal to its usual determinant.*
(2) *For any quaternionic hyperhermitian $(n \times n)$-matrix $\mathcal{M}$ and any quaternionic $(n \times n)$-matrix $\mathcal{C}$*
$$\det(\mathcal{C}^* \mathcal{M} \mathcal{C}) = \det(\mathcal{A}) \det(\mathcal{C}^* \mathcal{C}).$$

Proposition 3.4 *For a real C^2 function u, the horizontal quaternionic Hessian $(\overline{Q_l} Q_m u + 8\delta_{lm} \mathbf{i} \partial_t u)$ is hyperhermitian.*

Proof It follows from definition (1.7) of $Z_{AA'}$'s that

$$\mathbf{j} Z_{(n+m)0'} = -Z_{m1'} \mathbf{j}, \qquad \mathbf{j} Z_{(n+m)1'} = Z_{m0'} \mathbf{j} \tag{3.8}$$

and so

$$\overline{Q_l} Q_m = (X_{4l+1} + \mathbf{i} X_{4l+2} + \mathbf{j} X_{4l+3} + \mathbf{k} X_{4l+4}) (X_{4m+1} - \mathbf{i} X_{4m+2} - \mathbf{j} X_{4m+3} - \mathbf{k} X_{4m+4})$$
$$= (Z_{l0'} - Z_{l1'} \mathbf{j}) (Z_{(n+m)1'} - \mathbf{j} Z_{(n+m)0'})$$
$$= (Z_{l0'} Z_{(n+m)1'} - Z_{l1'} Z_{(n+m)0'}) + (Z_{l0'} Z_{m1'} - Z_{l1'} Z_{m0'}) \mathbf{j}. \tag{3.9}$$

When $l = m$, it follows that

$$\overline{Q_l} Q_l u = 2 \triangle_{l(n+l)} u - [Z_{l1'}, Z_{(n+l)0'}] u + [Z_{l0'}, Z_{l1'}] u \mathbf{j} = 2 \triangle_{l(n+l)} u - 8 \mathbf{i} \partial_t u \tag{3.10}$$

by using (3.5)–(3.6). Thus $\overline{Q_l} Q_l u + 8 \mathbf{i} \partial_t u$ is real by (3.7). If $l \neq m$, we have

$$\overline{Q_l} Q_m u = (Z_{l0'} Z_{(n+m)1'} - Z_{(n+m)0'} Z_{l1'}) u + (Z_{l0'} Z_{m1'} - Z_{m0} Z_{l1'}) u \mathbf{j}$$
$$= 2 (\triangle_{l(n+m)} u + \triangle_{lm} u \mathbf{j}), \tag{3.11}$$

by using commutators

$$[Z_{l1'}, Z_{(n+m)0'}] = 0 \quad \text{and} \quad [Z_{m0'}, Z_{l1'}] = 0, \quad \text{for} \quad l \neq m, \tag{3.12}$$

by Proposition 3.1.

To see the horizontal quaternionic Hessian to be hyperhermitian, note that for $l \neq m$

$$\overline{\overline{Q_l} Q_m u} = 2 \left(\overline{\triangle_{l(n+m)} u} - \mathbf{j} \overline{\triangle_{lm} u} \right), \tag{3.13}$$

and

$$\overline{\triangle_{l(n+m)}u} = \overline{Z_{l0'}Z_{(n+m)1'}}u - \overline{Z_{(n+m)0'}Z_{l1'}}u = Z_{(n+l)1'}Z_{m0'}u - Z_{m1'}Z_{(n+l)0'}u$$
$$= Z_{m0'}Z_{(n+l)1'}u - Z_{(n+l)0'}Z_{m1'}u = \triangle_{m(n+l)}u \qquad (3.14)$$

by the conjugate of $Z_{AA'}$'s in (1.7) and (3.12). Similarly, for any l, m, we have

$$\overline{\triangle_{lm}u} = (\overline{Z_{l0'}Z_{m1'}} - \overline{Z_{m0'}Z_{l1'}})u = (-Z_{(n+l)1'}Z_{(n+m)0'} + Z_{(n+m)1'}Z_{(n+l)0'})u, \qquad (3.15)$$

and so

$$\mathbf{j}\overline{\triangle_{lm}u} = (Z_{l0'}Z_{m1'} - Z_{m0'}Z_{l1'})u\,\mathbf{j} = -\triangle_{ml}u\,\mathbf{j} \qquad (3.16)$$

by using (3.8) and (3.12). Now substitute (3.14) and (3.16) into (3.13) to get

$$\overline{\overline{Q_l}Q_m u} = 2\left(\triangle_{m(n+l)}u + \triangle_{ml}u\mathbf{j}\right) = \overline{Q_m}Q_l u$$

for $l \neq m$. This together with the reality of $\overline{Q_l}Q_l u + 8\mathbf{i}\partial_t u$ implies that the quaternionic Hessian $(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u)$ is hyperhermitian. $\qquad\square$

As in [33], denote by $M_{\mathbb{F}}(p, m)$ the space of $\mathbb{F}$-valued $(p \times m)$-matrices, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. For a quaternionic $p \times m$-matrix $\mathcal{M}$, write $\mathcal{M} = a + b\mathbf{j}$ for some complex matrices $a, b \in M_{\mathbb{C}}(p, m)$. Then we define the $\tau(\mathcal{M})$ as the complex $(2p \times 2m)$-matrix

$$\tau(\mathcal{M}) := \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad (3.17)$$

Recall that for skew symmetric matrices $M_\alpha = (M_{\alpha;AB}) \in M_{\mathbb{C}}(2n, 2n)$, $\alpha = 1, \ldots, n$, such that 2-forms $\omega_\alpha = \sum_{i,j} M_{\alpha;AB}\omega^A \wedge \omega^B$ are real, define

$$\omega_1 \wedge \cdots \wedge \omega_n = \triangle_n(M_1, \ldots, M_n)\Omega_{2n}, \qquad (3.18)$$

Consider the homogeneous polynomial $\det(\lambda_1 \mathcal{M}_1 + \ldots + \lambda_n \mathcal{M}_n)$ in real variables $\lambda_1, \ldots, \lambda_n$ of degree n. The coefficient of the monomial $\lambda_1 \ldots \lambda_n$ divided by $n!$ is called the *mixed discriminant* of the matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n$, and it is denoted by $\det(\mathcal{M}_1, \ldots, \mathcal{M}_n)$. In particular, when $\mathcal{M}_1 = \ldots = \mathcal{M}_n = \mathcal{M}$, $\det(\mathcal{M}_1, \ldots, \mathcal{M}_n) = \det(\mathcal{M})$.

Theorem 3.1 (Theorem 1.2 in [33]) *For hyperhermitian matrices* $\mathcal{M}_1, \ldots, \mathcal{M}_n \in M_{\mathbb{H}}(n, n)$, *we have*

$$2^n n!\, det\,(\mathcal{M}_1, \ldots, \mathcal{M}_n) = \triangle_n\,(\tau(\mathcal{M}_1)\mathbb{J}, \ldots, \tau(\mathcal{M}_n)\mathbb{J}), \qquad (3.19)$$

where

$$\mathbb{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \qquad (3.20)$$

Proof of Theorem 1.2 The proof is similar to that of Theorem 1.3 in [33] except that $Z_{AA'}$'s are noncommutative. (3.10)–(3.11) implies that the quaternionic Hessian can be written as

$$\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right) = a + b\mathbf{j},$$

with $n \times n$ complex matrices

$$a = 2\left(\triangle_{l(n+m)}u\right), \qquad b = 2\left(\triangle_{lm}u\right).$$

Thus

$$\tau\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right)\mathbb{J} = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}\mathbb{J} = \begin{pmatrix} b & a \\ -\bar{a} & \bar{b} \end{pmatrix}$$
$$= 2\begin{pmatrix} \triangle_{lm}u & \triangle_{l(n+m)}u \\ -\overline{\triangle_{l(n+m)}u} & \overline{\triangle_{lm}u} \end{pmatrix}, \qquad (3.21)$$

Note that $\triangle_{ll}u = \triangle_{(n+l)(n+l)}u = 0$ by definition. For $l \neq m$,

$$\overline{\triangle_{l(n+m)}} = Z_{m0'}Z_{(n+l)1'} - Z_{(n+l)0'}Z_{m1'} = -\triangle_{(n+l)m}$$

by (3.14), while for $l = m$ we also have

$$\overline{\triangle_{l(n+l)}u} = Z_{(n+l)1'}Z_{l0'}u - Z_{l1'}Z_{(n+l)0'}u = Z_{l0'}Z_{(n+l)1'}u - Z_{(n+l)0'}Z_{l1'}u = -\triangle_{(n+l)l}u, \tag{3.22}$$

by using Proposition 3.1 (2). Moreover,

$$\overline{\triangle_{lm}u} = \triangle_{(n+l)(n+m)}u,$$

which follows from (3.15). Therefore we have

$$\tau\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right)\mathbb{J} = 2(\triangle_{AB}u). \tag{3.23}$$

Then the result follows from applying Theorem 3.1 to matrices $\mathcal{M}_j = 2\left(\overline{Q_l}Q_m u_j + 8\delta_{lm}\mathbf{i}\partial_t u_j\right)$. $\qquad\square$

4 Closed positive currents on the quaternionic Heisenberg group

4.1 Positive 2k-forms

Now let us recall definitions of real forms and positive $2k$-forms (cf. [4,27,33] and references therein). Let $\{\omega^0, \omega^1, \ldots, \omega^{2n-1}\}$ be the standard basis of $\mathbb{C}^{2n}$ and

$$\beta_n := \sum_{l=0}^{n-1} \omega^l \wedge \omega^{n+l}. \tag{4.1}$$

Then $\beta_n^n = \wedge^n \beta_n = n! \, \Omega_{2n}$, where Ω_{2n} is given by (1.12). For $\mathcal{A} \in \mathrm{GL}_{\mathbb{H}}(n)$, define the *induced $\mathbb{C}$-linear transformation* of $\mathcal{A}$ on $\mathbb{C}^{2n}$ as $\mathcal{A}.\omega^p = \tau(\mathcal{A}).\omega^p$ with

$$M.\omega^p = \sum_{j=0}^{2n-1} M_{jp}\omega^j, \tag{4.2}$$

for $M \in M_{\mathbb{C}}(2n, 2n)$, and define the *induced $\mathbb{C}$-linear transformation* of $\mathcal{A}$ on $\wedge^{2k}\mathbb{C}^{2n}$ as

$$\mathcal{A}.(\omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^{2k-1}) = \mathcal{A}.\omega^0 \wedge \mathcal{A}.\omega^1 \wedge \ldots \wedge \mathcal{A}.\omega^{2k-1}.$$

Therefore for $A \in \mathrm{U}_{\mathbb{H}}(n)$, $\mathcal{A}.\beta_n = \beta_n,$. Consequently $\mathcal{A}.(\wedge^n \beta_n) = \wedge^n \beta_n$, i.e., $\mathcal{A}.\Omega_{2n} = \Omega_{2n}$.

$\mathbf{j}$ defines a real linear map

$$\rho(\mathbf{j}) : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}, \qquad \rho(\mathbf{j})(z\omega^k) = \bar{z}\mathbb{J}.\omega^k, \tag{4.3}$$

which is not $\mathbb{C}$-linear, where $\mathbb{J}$ is given by (3.20). Also the right multiplying of $\mathbf{i}$: $(q_1, \ldots, q_n) \mapsto (q_1\mathbf{i}, \ldots, q_n\mathbf{i})$ induces

$$\rho(\mathbf{i}) : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}, \qquad \rho(\mathbf{i})(z\omega^k) = z\mathbf{i}\omega^k.$$

Thus ρ defines $GL_{\mathbb{H}}(1)$-action on $\mathbb{C}^{2n}$. The actions of $GL_{\mathbb{H}}(1)$ and $GL_{\mathbb{H}}(n)$ on $\mathbb{C}^{2n}$ are commutative, and equip $\mathbb{C}^{2n}$ a structure of $GL_{\mathbb{H}}(n)GL_{\mathbb{H}}(1)$-module. This is because $(MN).\omega^p = M.(N.\omega^p)$ by definition and $\rho(\mathbf{j})\rho(\mathbf{i}) = -\rho(\mathbf{i})\rho(\mathbf{j})$. This action extends to $\wedge^{2k}\mathbb{C}^{2n}$ naturally.

The real action (4.3) of $\rho(\mathbf{j})$ on $\mathbb{C}^{2n}$ naturally induces an action on $\wedge^{2k}\mathbb{C}^{2n}$. An element φ of $\wedge^{2k}\mathbb{C}^{2n}$ is called *real* if $\rho(\mathbf{j})\varphi = \varphi$. Denote by $\wedge_{\mathbb{R}}^{2k}\mathbb{C}^{2n}$ the subspace of all real elements in $\wedge^{2k}\mathbb{C}^{2n}$. These forms are counterparts of $(k, k)-$forms in complex analysis.

A right $\mathbb{H}$-linear map $g : \mathbb{H}^k \to \mathbb{H}^m$ induces a $\mathbb{C}$-linear map $\tau(g) : \mathbb{C}^{2k} \to \mathbb{C}^{2m}$. If we write $g = (g_{jl})_{m \times k}$ with $g_{jl} \in \mathbb{H}$, then $\tau(g)$ is the complex $(2m \times 2k)$-matrix given by (3.17). The induced $\mathbb{C}$-linear pulling back transformation of $g^* : \mathbb{C}^{2m} \to \mathbb{C}^{2k}$ is defined as:

$$g^*\widetilde{\omega}^p = \tau(g)^t.\omega^p = \sum_{j=0}^{2k-1} \tau(g)_{pj}\omega^j, \qquad p = 0, \ldots, 2m-1, \tag{4.4}$$

where $\{\widetilde{\omega}^0, \ldots, \widetilde{\omega}^{2m-1}\}$ is the standard basis of $\mathbb{C}^{2m}$ and $\{\omega^0, \ldots, \omega^{2k-1}\}$ is the standard basis of $\mathbb{C}^{2k}$. It induces a $\mathbb{C}$-linear pulling back transformation on $\wedge^{2k}\mathbb{C}^{2m}$ given by $g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta$ inductively.

An element $\omega \in \wedge_{\mathbb{R}}^{2k}\mathbb{C}^{2n}$ is said to be *elementary strongly positive* if there exist linearly independent right $\mathbb{H}$-linear mappings $\eta_j : \mathbb{H}^n \to \mathbb{H}$, $j = 1, \ldots, k$, such that

$$\omega = \eta_1^*\widetilde{\omega}^0 \wedge \eta_1^*\widetilde{\omega}^1 \wedge \cdots \wedge \eta_k^*\widetilde{\omega}^0 \wedge \eta_k^*\widetilde{\omega}^1,$$

where $\{\widetilde{\omega}^0, \widetilde{\omega}^1\}$ is a basis of $\mathbb{C}^2$ and $\eta_j^* : \mathbb{C}^2 \to \mathbb{C}^{2n}$ is the induced $\mathbb{C}$-linear pulling back transformation of η_j. The definition in the case $k = 0$ is obvious: $\wedge_{\mathbb{R}}^0\mathbb{C}^{2n} = \mathbb{R}$ and the positive elements are the usual ones. For $k = n$, $\dim_{\mathbb{C}} \wedge^{2n} \mathbb{C}^{2n} = 1$, Ω_{2n} defined by (1.12) is an element of $\wedge_{\mathbb{R}}^{2n}\mathbb{C}^{2n}$ ($\rho(\mathbf{j})\beta_n = \beta_n$) and spans it. An element $\eta \in \wedge_{\mathbb{R}}^{2n}\mathbb{C}^{2n}$ is called *positive* if $\eta = \kappa \, \Omega_{2n}$ for some non-negative number κ. By definition, $\omega \in \wedge_{\mathbb{R}}^{2k}\mathbb{C}^{2n}$ is elementary strongly positive if and only if

$$\omega = \mathcal{M}.(\omega^0 \wedge \omega^n \wedge \cdots \wedge \omega^{k-1} \wedge \omega^{n+k-1}) \tag{4.5}$$

for some quaternionic matrix $\mathcal{M} \in M_{\mathbb{H}}(n, k)$ of rank k.

An element $\omega \in \wedge_{\mathbb{R}}^{2k}\mathbb{C}^{2n}$ is called *strongly positive* if it belongs to the convex cone $\mathrm{SP}^{2k}\mathbb{C}^{2n}$ in $\wedge_{\mathbb{R}}^{2k}\mathbb{C}^{2n}$ generated by elementary strongly positive $2k$-elements; that is, $\omega = \sum_{l=1}^m \lambda_l \xi_l$ for some non-negative numbers $\lambda_1, \ldots, \lambda_m$ and some elementary strongly positive elements $\xi_1, \ldots, \xi_m$. An $2k$-element ω is said to be *positive* if for any strongly positive element $\eta \in \mathrm{SP}^{2n-2k}\mathbb{C}^{2n}$, $\omega \wedge \eta$ is positive. We will denote the set of all positive $2k$-elements by $\wedge_{\mathbb{R}+}^{2k}\mathbb{C}^{2n}$. Any $2k$ element is a $\mathbb{C}$-linear combination of strongly positive $2k$ elements by Proposition 5.2 in [4], i.e. $\mathrm{span}_{\mathbb{C}}\{\varphi; \ \varphi \in \wedge_{\mathbb{R}+}^{2k}\mathbb{C}^{2n}\} = span_{\mathbb{C}}\{\varphi; \ \varphi \in \mathrm{SP}^{2k}\mathbb{C}^{2n}\} = \wedge^{2k}\mathbb{C}^{2n}$. By definition, β_n is a strongly positive 2-form, and $\beta_n^n = \wedge^n \beta_n = n! \, \Omega_{2n}$ is a positive $2n$-form.

For a domain Ω in $\mathscr{H}$, let $\mathcal{D}_0^p(\Omega) = C_0(\Omega, \wedge^p\mathbb{C}^{2n})$ and $\mathcal{D}^p(\Omega) = C_0^\infty(\Omega, \wedge^p\mathbb{C}^{2n})$. An element of the latter one is called a *test p-form*. An element $\eta \in \mathcal{D}_0^{2k}(\Omega)$ is called a *positive $2k$-form* (respectively, *strongly positive $2k$-form*) if for any $q \in \Omega$, $\eta(q)$ is a positive (respectively, strongly positive) element.

Theorem 1.1 in [33] and its proof implies the following result.

Proposition 4.1 *For a hyperhermitian $n \times n$-matrix $\mathcal{M} = (\mathcal{M}_{jk})$, there exists a quaternionic unitary matrix $\mathcal{E} \in U_{\mathbb{H}}(n)$ such that $\mathcal{E}^*\mathcal{M}\mathcal{E} = \mathrm{diag}(\nu_0, \ldots, \nu_{n-1})$. Then the 2-form*

$$\omega = \sum_{A,B=0}^{2n-1} M_{AB}\,\omega^A \wedge \omega^B, \tag{4.6}$$

with $M = \tau(\mathcal{M})\mathbb{J}$, *can be normalized as*

$$\omega = 2 \sum_{l=0}^{n-1} \nu_l \widetilde{\omega}^l \wedge \widetilde{\omega}^{l+n} \tag{4.7}$$

with $\widetilde{\omega}^A = \mathcal{E}^*.\omega^A$. *In particular,* ω *is strongly positive if and only if* $\mathcal{M}$ *is nonnegative.*

Proposition 4.2 *For any* C^1 *real function* u, $d_0 u \wedge d_1 u$ *is elementary strongly positive if grad* $u \neq 0$.

Proof Let $p := (p_1, \ldots, p_n) \in \mathbb{H}^n$ with $p_l = X_{4l+1}u + \mathbf{i}X_{4l+2}u + \mathbf{j}X_{4l+3}u + \mathbf{k}X_{4l+4}u$. Then as (3.9), we have

$$\overline{p_l}\, p_m = \widetilde{\triangle}_{l(n+m)} + \widetilde{\triangle}_{lm}\mathbf{j}, \tag{4.8}$$

where

$$\widetilde{\triangle}_{AB} := Z_{A0'}u Z_{B1'}u - Z_{B1'}u Z_{A0'}u.$$

Denote $n \times n$ quaternionic matrix $\widetilde{\mathcal{M}} := (\overline{p_l}\, p_m)$. Then $\widetilde{\mathcal{M}} = a + b\mathbf{j}$ with $n \times n$ complex matrices $a = (\widetilde{\triangle}_{l(n+m)}u)$, $b = (\widetilde{\triangle}_{lm}u)$. Thus

$$\tau\left(\widetilde{\mathcal{M}}\right)\mathbb{J} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\mathbb{J} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} = \begin{pmatrix} \widetilde{\triangle}_{lm} & \widetilde{\triangle}_{l(n+m)} \\ -\overline{\widetilde{\triangle}_{l(n+m)}} & \overline{\widetilde{\triangle}_{lm}} \end{pmatrix} = (\widetilde{\triangle}_{AB}), \tag{4.9}$$

since we can easily check

$$\overline{\widetilde{\triangle}_{l(n+m)}} = -\widetilde{\triangle}_{(n+l)m}, \qquad \overline{\widetilde{\triangle}_{lm}} = \widetilde{\triangle}_{(n+l)(n+m)}.$$

Since $\mathcal{M}$ has eigenvalues $|p|^2, 0, \ldots, 0$, we see that

$$d_0 u \wedge d_1 u = \sum_{A,B=0}^{2n-1} Z_{A0'}u Z_{B1'}u\, \omega^A \wedge \omega^B = \sum_{A,B=0}^{2n-1} \widetilde{\triangle}_{AB}\, \omega^A \wedge \omega^B.$$

is elementary strongly positive by Proposition 4.1. $\qquad\square$

See [28, Proposition 3.3] for this proposition for $\mathbb{H}^n$ with a different proof.

4.2 The closed strongly positive 2-form given by a smooth PSH

Proposition 4.3 *For* $u \in C^2(\Omega)$, u *is PSH if and only if the hyperhermitian matrix* $(\overline{Q_l}Q_m u - 8\delta_{lm}\mathbf{i}\partial_t u)$ *is nonnegative.*

The tangential mapping $\iota_{\eta,q*}$ maps horizontal left invariant vector fields on $\widetilde{\mathscr{H}_q}$ to that on the quaternionic Heisenberg line $\mathscr{H}_{\eta,q}$. In particular, we have

Proposition 4.4 *For* $q \in \mathbb{H}^n \backslash \mathfrak{D}$,

$$\iota_{\eta,q*}\widetilde{X}_j = \sum_{l=0}^{n-1} \sum_{k=1}^{4} \left(\overline{q_l}^{\mathbb{R}}\right)_{jk} X_{4l+k} \tag{4.10}$$

Proof Since $\iota_{\eta,q} = \tau_\eta \circ \iota_q$ and X_j's are invariant under τ_η, it sufficient to prove (4.10) for $\eta = 0$. For fixed $j = 1, 2, 3, 4$ and $l = 1, \ldots, n$, note that

$$\widehat{q_l \mathbf{i}_j} = q_l^{\mathbb{R}} \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} (q_l^{\mathbb{R}})_{1j} \\ \vdots \\ (q_l^{\mathbb{R}})_{4j} \end{pmatrix}.$$

by (2.6). Thus for $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$ and $\varsigma \in \mathbb{R}$, if we write $\iota_q(\lambda, t) = (q\lambda, t) = (x, t)$, we get

$$\iota_q\{(\lambda, t)(\varsigma \mathbf{i}_j, 0)\} = \left(q(\lambda + \varsigma \mathbf{i}_j), t + 2\varsigma \sum_{k=1}^{4} B_{kj}^q \lambda_k \right)$$

$$= \left(\ldots, x_{4l+i} + \varsigma \left(q_l^{\mathbb{R}} \right)_{ij}, \ldots, t + 2\varsigma \sum_{l=0}^{n-1} \sum_{k,i=1}^{4} J_{ki} x_{4l+k} (q_l^{\mathbb{R}})_{ij} \right),$$

by the multiplication (2.13) of the group $\widetilde{\mathscr{H}_q}$ and B^q in (2.12). So

$$\left(\iota_{q*} \widetilde{X}_j f \right)(x, t) = \left. \frac{d}{d\varsigma} \right|_{\varsigma=0} f \left(\iota_q\{(\lambda, t)(\varsigma \mathbf{i}_j, 0)\} \right)$$

$$= \sum_{l=0}^{n-1} \sum_{i=1}^{4} \left(q_l^{\mathbb{R}} \right)_{ij} \frac{\partial f}{\partial x_{4l+i}} + 2 \sum_{l=0}^{n-1} \sum_{k,i=1}^{4} J_{ki} x_{4l+k} \left(q_l^{\mathbb{R}} \right)_{ij} \frac{\partial f}{\partial t}$$

$$= \sum_{l=0}^{n-1} \sum_{i=1}^{4} \left(\overline{q_l^{\mathbb{R}}} \right)_{ji} X_{4l+i} f(x, t)$$

by (2.8). $\qquad\qquad\square$

Proof of Proposition 4.3 Denote $\overline{\widetilde{Q}} := \widetilde{X}_1 + \mathbf{i}\widetilde{X}_2 + \mathbf{j}\widetilde{X}_3 + \mathbf{k}\widetilde{X}_4$. Then we have

$$\iota_{\eta,q*} \overline{\widetilde{Q}} = \sum_{j=1}^{4} \iota_{\eta,q*} \widetilde{X}_j \mathbf{i}_j = \sum_{l=0}^{n-1} \sum_{j,k=1}^{4} \left(\overline{q_l^{\mathbb{R}}} \right)_{jk} X_{4l+k} \mathbf{i}_j = \sum_{l=0}^{n-1} \overline{q_l} \overline{Q_l} \qquad (4.11)$$

by Proposition 4.4, (2.6) and definition of $q^{\mathbb{R}}$ in (2.7), and $\iota_{\eta,q*} \widetilde{Q} = \sum_{l=0}^{n-1} Q_l q_l$ by taking conjugate. Therefore for real u, we have

$$\iota_{\eta,q*} \left(\widetilde{X}_1^2 + \widetilde{X}_2^2 + \widetilde{X}_3^2 + \widetilde{X}_4^2 \right) u = \mathrm{Re} \left(\iota_{\eta,q*} \overline{\widetilde{Q}} \cdot \iota_{\eta,q*} \widetilde{Q} u \right) = \mathrm{Re} \left(\sum_{l,m=0}^{n-1} \overline{q_l} \cdot \overline{Q_l} Q_m u \cdot q_m \right).$$
$$(4.12)$$

On the other hand, we have

$$\sum_{l,m=0}^{n-1} \overline{q_l} \left(\overline{Q_l} Q_m u + 8\delta_{lm} \mathbf{i}\partial_t u \right) q_m = \left(\sum_{l=0}^{n-1} \overline{q_l} \overline{Q_l} \right) \left(\sum_{m=0}^{n-1} Q_m q_m \right) u + 8 \sum_{l=0}^{n-1} \overline{q_l} \mathbf{i} q_l \partial_t u.$$

Since the horizontal quaternionic Hessian $(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u)$ is hyperhermitian by Proposition 3.4, we see that the above quadratic form is real for any q. Note that $\overline{p}\mathbf{i}p \in \text{Im}\,\mathbb{H}$ for any $0 \neq p \in \mathbb{H}$. Therefore, we get

$$\sum_{l,m=0}^{n-1} \overline{q_l}\left(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u\right)q_m = \text{Re}\left(\sum_{l,m=0}^{n-1} \overline{q_l}\cdot\overline{Q_l}Q_m u \cdot q_m\right) = (\iota_{\eta,q*}\widetilde{\triangle_q})u \tag{4.13}$$

for $q \in \mathbb{H}^n\backslash\mathfrak{D}$ by (4.12).

Now if u is PSH, then $\widetilde{\triangle_q}(\iota_{\eta,q}^* u)$ is nonnegative by applying Theorem 2.1 to the group $\widetilde{\mathscr{H}_q}$ for $q \in \mathbb{H}^n\backslash\mathfrak{D}$. Consequently, (4.13) holds for any $q \in \mathbb{H}^n$ by continuity, i.e. the hyperhermitian matrix $(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u)$ is nonnegative. Conversely, if the hyperhermitian matrix is nonnegative, we get u is is subharmonic on each quaternionic Heisenberg line $\mathscr{H}_{\eta,q}$ for any $q \in \mathbb{H}^n\backslash\mathfrak{D}$ and $\eta \in \mathscr{H}^n$ by applying Theorem 2.1 again. □

Corollary 4.1 *For $u \in PSH \cap C^2(\Omega)$, $\triangle u$ is a closed strongly positive 2-form.*

Proof It follows from applying Proposition 4.1 to nonnegative $\mathcal{M} = (\overline{Q_l}Q_m u - 8\delta_{lm}\mathbf{i}\partial_t u)$ and using (3.23). □

Corollary 4.2 *A C^2 function u is pluriharmonic if and only if $\triangle u = 0$.*

Proof u is pluriharmonic means that $\widetilde{\triangle_q}\iota_{\eta,q}^* u = 0$ on the quaternionic Heisenberg line $\widetilde{\mathscr{H}_q}$ for any $\eta \in \mathscr{H}$ and $q \in \mathbb{H}^n\backslash\mathfrak{D}$. It holds if and only if

$$\sum_{l,m} \overline{q_l}(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u)q_m = 0$$

for any $q \in \mathbb{H}^n$ by (4.13), i.e. $(\overline{Q_l}Q_m u + 8\delta_{lm}\mathbf{i}\partial_t u) = 0$, which equivalent to $\triangle u = 0$ by (3.23). □

Recall that the *tangential 1-Cauchy–Fueter operator* on a domain Ω in the Heisenberg group $\mathscr{H}$ is $\mathscr{D} : C^1(\Omega, \mathbb{C}^2) \to C^0(\Omega, \mathbb{C}^{2n})$ [20] given by

$$(\mathscr{D}f)_A = \sum_{A'=0',1'} Z_A^{A'} f_{A'}, \qquad A = 0,\ldots,2n-1,$$

where $Z_A^{0'} = Z_{A1'}$ and $Z_A^{1'} = -Z_{A0'}$. A $\mathbb{C}^2$-valued function $f = (f_{0'}, f_{1'}) = (f_1 + \mathbf{i}f_2, f_3 + \mathbf{i}f_4)$ is called 1-*CF* if $\mathscr{D}f = 0$.

Proposition 4.5 *Each real component of a 1-CF function $f : \mathscr{H}^n \to \mathbb{C}^2$ is pluriharmonic.*

Proof Note that $\sum_{A'=0',1'} Z_A^{A'} f_{A'} = 0$ is equivalent to $\sum_A \sum_{A'=0',1'} Z_A^{A'} f_{A'}\omega^A = 0$, which can be written as

$$d_1 f_{0'} - d_0 f_{1'} = 0.$$

Apply d_0 on both sides to get $d_0 d_1 f_{0'} = 0$ since $d_0^2 = 0$. Similarly, we get $d_0 d_1 f_{1'} = 0$. Writing $f_{0'} = f_1 + \mathbf{i}f_2$ for some real functions f_1 and f_2, we have

$$\triangle f_1 + \mathbf{i}\triangle f_2 = 0.$$

Note that for a real valued function u, $\triangle u$ is a real 2-form by (3.23) and Proposition 4.1, i.e. $\rho(\mathbf{j})\triangle u = \triangle u$. We get

$$\triangle f_1 - \mathbf{i}\triangle f_2 = 0.$$

Thus $\triangle f_1 = 0 = \triangle f_2$. Similarly, we have $\triangle f_3 = 0 = \triangle f_4$. □

See Corollary 2.1 in [31] for this Proposition on the quaternionic space $\mathbb{H}^n$. Since 1-regular functions are abundant, so are pluriharmonic functions on the Heisenberg group.

4.3 Closed positive currents

An element of the dual space $(\mathcal{D}^{2n-p}(\Omega))'$ is called a *p-current*. A $2k$-current T is said to be *positive* if we have $T(\eta) \geq 0$ for any strongly positive form $\eta \in \mathcal{D}^{2n-2k}(\Omega)$. Although a $2n$-form is not an authentic differential form and we cannot integrate it, we can define

$$\int_\Omega F := \int_\Omega f \, dV, \tag{4.14}$$

if we write $F = f\,\Omega_{2n} \in L^1(\Omega, \wedge^{2n}\mathbb{C}^{2n})$, where dV is the Lebesgue measure. In general, for a $2n$-current $F = \mu\,\Omega_{2n}$ with the coefficient to be a measure μ, define

$$\int_\Omega F := \int_\Omega \mu. \tag{4.15}$$

Now for the p-current F, we define a $(p+1)$-current $d_\alpha F$ as

$$(d_\alpha F)(\eta) := -F(d_\alpha \eta), \qquad \alpha = 0, 1, \tag{4.16}$$

for any test $(2n - p - 1)$-form η. We say a current F is *closed* if $d_0 F = d_1 F = 0$.

An element of the dual space $(\mathcal{D}_0^{2n-p}(\Omega))'$ are called a *p-current of order zero*. Obviously, a $2n$-current is just a distribution on Ω, whereas a $2n$-current of order zero is a Radon measure on Ω. Let ψ be a p-form whose coefficients are locally integrable in Ω. One can associate with ψ the p-current T_ψ defined by

$$T_\psi(\varphi) = \int_\Omega \psi \wedge \varphi, \quad \text{for any} \quad \varphi \in \mathcal{D}^{2n-p}(\Omega).$$

If T is a $2k$-current on Ω, ψ is a $2l$-form on Ω with coefficients in $C^\infty(\Omega)$, and $k + l \leq n$, then the formula

$$(T \wedge \psi)(\varphi) = T(\psi \wedge \varphi) \qquad \text{for} \quad \varphi \in \mathcal{D}^{2n-2k-2l}(\Omega) \tag{4.17}$$

defines a $(2k + 2l)$-current. In particular, if ψ is a smooth function, $\psi T(\varphi) = T(\psi\varphi)$.

A $2k$-current T is said to be *positive* if we have $T(\eta) \geq 0$ for any $\eta \in C_0^\infty(\Omega, SP^{2n-2k}\mathbb{C}^{2n})$. Namely, T is positive if for any $\eta \in C_0^\infty(\Omega, SP^{2n-2k}\mathbb{C}^{2n})$, $T \wedge \eta = \mu\,\Omega_{2n}$ for some positive distribution μ (and hence a measure).

Let $I = (i_1, \ldots, i_{2k})$ be a multi-index such that $1 \leq i_1 < \ldots < i_{2k} \leq n$. Denote by $\widehat{I} = (l_1, \ldots, l_{2n-2k})$ the *increasing complements* to I in the set $\{0, 1, \ldots, 2n - 1\}$, i.e., $\{i_1, \ldots, i_{2k}\} \cup \{l_1, \ldots, l_{2n-2k}\} = \{0, 1, \ldots, 2n-1\}$. For a $2k$-current T in Ω and multi-index I, define distributions T_I by $T_I(f) := \varepsilon_I T(f\omega^{\widehat{I}})$ for $f \in C_0^\infty(\Omega)$, where $\varepsilon_I = \pm 1$ is so chosen that

$$\varepsilon_I \omega^I \wedge \omega^{\widehat{I}} = \Omega_{2n}. \tag{4.18}$$

If T is a current of order 0, the distributions T_I are Radon measures and

$$T(\varphi) = \sum_I \varepsilon_I T_I(\varphi_{\widehat{I}}), \tag{4.19}$$

for $\varphi = \sum_{\widehat{I}} \varphi_{\widehat{I}} \omega^{\widehat{I}} \in \mathcal{D}^{2n-2k}(\Omega)$, where I and $\widehat{I}$ are increasing. Namely,

$$T = \sum_I T_I \omega^I, \tag{4.20}$$

where the summation is taken over increasing multi-indices of length $2k$, holds in the sense that if we write $T \wedge \varphi = \mu \, \Omega_{2n}$ for some Radon measure μ, then we have

$$T(\varphi) = \int_\Omega \mu = \int_\Omega T \wedge \varphi. \tag{4.21}$$

Proposition 4.6 *Any positive $2k$-current T on Ω has measure coefficients (i.e. is of order zero), and we can write $T = \sum_I T_I \omega^I$ for some complex Radon measures T_I, where the summation is taken over all increasing multi-indices I.*

Proof By Proposition 5.4 in [4], we can find $\{\varphi_L\} \subseteq SP^{2n-2k}\mathbb{C}^{2n}$ such that any $\eta \in \wedge^{2n-2k}\mathbb{C}^{2n}$ is a $\mathbb{C}$-linear combination of φ_L, i.e., $\eta = \sum \lambda_L \varphi_L$ for some $\lambda_L \in \mathbb{C}$. Let $\{\widetilde{\varphi_L}\}$ be a basis of $\wedge^{2k}\mathbb{C}^{2n}$ which is dual to $\{\varphi_L\}$. Then $T = \sum T_L \widetilde{\varphi_L}$ with distributional coefficients T_L as (4.20). If ψ is a nonnegative test function, $\psi \varphi_L \in C_0^\infty(\Omega, SP^{2n-2k}\mathbb{C}^{2n})$. Then $T_L(\psi) = T(\psi \varphi_L) \geq 0$ by definition. It follows that T_L is a positive distribution, and so is a nonnegative measure. □

The following Proposition is obvious and will be used frequently.

Proposition 4.7 (1) *(linearity) For $2n$-currents T_1 and T_2 with (Radon) measure coefficients, we have*

$$\int_\Omega \alpha T_1 + \beta T_2 = \alpha \int_\Omega T_1 + \beta \int_\Omega T_2.$$

(2) *If $T_1 \leq T_2$ as positive $2n$-currents (i.e. $\mu_1 \leq \mu_2$ if we write $T_j = \mu_j \Omega_{2n}$, $j = 1, 2$), then $\int_\Omega T_1 \leq \int_\Omega T_2$.*

Lemma 4.1 (Stokes-type formula) *Let Ω be a bounded domain with smooth boundary and defining function ρ (i.e. $\rho = 0$ on $\partial\Omega$ and $\rho < 0$ in Ω) such that $|\mathrm{grad}\rho| = 1$. Assume that $T = \sum_A T_A \omega^{\widehat{A}}$ is a smooth $(2n-1)$-form in Ω, where $\omega^{\widehat{A}} = \omega^A \rfloor \Omega_{2n}$. Then for $h \in C^1(\overline{\Omega})$, we have*

$$\int_\Omega h d_\alpha T = -\int_\Omega d_\alpha h \wedge T + \sum_{A=0}^{2n-1} \int_{\partial\Omega} h T_A Z_{A\alpha'} \rho \, dS, \tag{4.22}$$

where dS denotes the surface measure of $\partial\Omega$. In particular, if $h = 0$ on $\partial\Omega$, we have

$$\int_\Omega h d_\alpha T = -\int_\Omega d_\alpha h \wedge T, \quad \alpha = 0, 1, \tag{4.23}$$

Proof Note that

$$d_\alpha(hT) = \sum_{B,A} Z_{B\alpha'}(hT_A)\omega^B \wedge \omega^{\widehat{A}} = \sum_A Z_{A\alpha'}(hT_A)\Omega_{2n}.$$

Then

$$\int_\Omega d_\alpha(hT) = \int_\Omega \sum_A Z_{A\alpha'}(hT_A)dV = \int_{\partial\Omega} \sum_A hT_A Z_{A\alpha'} \rho \, dS,$$

by definition (4.14) and integration by part,

$$\int_\Omega X_j f \, dV = \int_{\partial\Omega} f X_j \rho \, dS, \tag{4.24}$$

for $j = 1, \dots 4n$. (4.24) holds because the coefficient of ∂_t is independent of t. (4.22) follows from the above formula and $d_\alpha(hT) = d_\alpha h \wedge T + h d_\alpha T$. □

Now let us show that $d_\alpha F$ in the generalized sense (4.16), coincides with the original definition when F is a smooth $2k$-form. Let η be arbitrary $(2n - 2k - 1)$-test form compactly supported in Ω. It follows from Lemma 4.1 that $\int_\Omega d_\alpha(F \wedge \eta) = 0$. By Proposition 1.1 (3), $d_\alpha(F \wedge \eta) = d_\alpha F \wedge \eta + F \wedge d_\alpha \eta$. We have

$$- \int_\Omega F \wedge d_\alpha \eta = \int_\Omega d_\alpha F \wedge \eta, \qquad i.e., \qquad (d_\alpha F)(\eta) = -F(d_\alpha \eta). \qquad (4.25)$$

We also define $\triangle F$ in the generalized sense, i.e., for each test $(2n - 2k - 2)$-form η,

$$(\triangle F)(\eta) := F(\triangle \eta). \qquad (4.26)$$

As a corollary, $\triangle F$ in the generalized sense coincides with the original definition when F is a smooth $2k$-form:

$$\int \triangle F \wedge \eta = \int F \wedge \triangle \eta.$$

Corollary 4.3 *For $u \in PSH(\Omega)$, $\triangle u$ is a closed positive 2-current.*

Proof If u is smooth, $\triangle u$ is a closed strongly positive 2-form by Corollary 4.1. When u is not smooth, consider regularization $u_\varepsilon = \chi_\varepsilon * u$ as in Proposition 2.3 (6). It suffices to show that coefficients $\triangle_{AB} u_\varepsilon \to \triangle_{AB} u$ in the sense of weak convergence of distributions. For any $\varphi \in C_0^\infty(\Omega)$,

$$\int \triangle_{AB} u_\varepsilon \cdot \varphi = \int u_\varepsilon \cdot \triangle_{AB} \varphi \to \int u \cdot \triangle_{AB} \varphi = (\triangle_{AB} u)(\varphi)$$

as $\varepsilon \to 0$, by using integration by part (4.24) and the standard fact that $\chi_\varepsilon * u \to u$ in $L^1_{\text{loc}}(\Omega)$ if $u \in L^1_{\text{loc}}(\Omega)$ [19]. It follows that the currents $\triangle u_\varepsilon$ converge to $\triangle u$, and so the current $\triangle u$ is positive. For any test form η,

$$(d_\alpha \triangle u)(\eta) = -\triangle u(d_\alpha \eta) = -\lim_{\varepsilon \to 0} \triangle u_\varepsilon(d_\alpha \eta) = \lim_{\varepsilon \to 0}(d_\alpha \triangle u_\varepsilon)(\eta) = 0,$$

$\alpha = 0, 1$, where the last identity follows from Corollary 3.1. Here u_ε is smooth, and $d_\alpha \triangle u_\varepsilon$ coincides with its usual definition. $\qquad \square$

5 The quaternionic Monge–Ampère measure over the Heisenberg group

For positive $(2n-2p)$-form T and an arbitrary compact subset K, define $\|T\|_K := \int_K T \wedge \beta_n^p$, where β_n is given by (4.1). In particular, if T is a positive $2n$-current, $\|T\|_K$ coincides with $\int_K T$ defined by (4.15). Let $\| \cdot \|$ be a norm on $\wedge^{2k} \mathbb{C}^{2n}$.

Lemma 5.1 (Lemma 3.3 in [27]) *For $\eta \in \wedge_{\mathbb{R}}^{2k} \mathbb{C}^{2n}$ with $\|\eta\| \leq 1$, $\beta_n^k \pm \varepsilon \eta$ is a positive $2k$-form for some sufficiently small $\varepsilon > 0$.*

Proposition 5.1 (Chern–Levine–Nirenberg type estimate) *Let Ω be a domain in $\mathscr{H}^n$. Let K and L be compact subsets of Ω such that L is contained in the interior of K. Then there exists a constant C depending only on K, L such that for any $u_1, \ldots u_k \in PSH(\Omega) \cap C^2(\Omega)$, we have*

$$\|\triangle u_1 \wedge \cdots \wedge \triangle u_k\|_L \leq C \prod_{i=1}^k \|u_i\|_{C^0(K)}. \qquad (5.1)$$

Proof By Corollary 4.1, $\triangle u_1 \wedge \cdots \wedge \triangle u_k$ is already closed and strongly positive. Since L is compact, there is a covering of L by a family of balls $D'_j \Subset D_j \subseteq K$. Let $\chi \geq 0$ be a smooth function equals to 1 on $\overline{D'_j}$ with support in D_j. For a closed smooth $(2n - 2p)$-form T, we have

$$
\int_\Omega \chi \triangle u_1 \wedge \cdots \wedge \triangle u_p \wedge T = - \int_\Omega d_0 \chi \wedge d_1 u_1 \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_p \wedge T
$$

$$
= - \int_\Omega u_1 d_1 d_0 \chi \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_p \wedge T \qquad (5.2)
$$

$$
= \int_\Omega u_1 \triangle \chi \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_p \wedge T
$$

by using Stokes-type formula (4.23) and Proposition 1.2. Then

$$
\| \triangle u_1 \wedge \cdots \wedge \triangle u_k \|_{L \cap \overline{D'_j}} = \int_{L \cap \overline{D'_j}} \triangle u_1 \wedge \cdots \wedge \triangle u_k \wedge \beta_n^{n-k} \leq \int_{D_j} \chi \triangle u_1 \wedge \cdots \wedge \triangle u_k \wedge \beta_n^{n-k}
$$

$$
= \int_{D_j} u_1 \triangle \chi \wedge \triangle u_2 \wedge \cdots \wedge \triangle u_k \wedge \beta_n^{n-k}
$$

$$
\leq \frac{1}{\varepsilon} \| u_1 \|_{L^\infty(K)} \| \triangle \chi \| \int_{D_j} \triangle u_2 \wedge \cdots \wedge \triangle u_k \wedge \beta_n^{n-k+1},
$$

by using (5.2) and Lemma 5.1. The result follows by repeating this procedure. $\qquad \square$

Proof of Theorem 1.1 It is sufficient to prove for any compactly supported continuous function χ, the sequence $\int_\Omega \chi (\triangle u_j)^n$ is a Cauchy sequence. We can assume $\chi \in C_0^\infty(\Omega)$. Note the following identity

$$
(\triangle v)^n - (\triangle u)^n = \sum_{p=1}^{n} \left\{ (\triangle v)^p \wedge (\triangle u)^{n-p} - (\triangle v)^{p-1} \wedge (\triangle u)^{n-p+1} \right\}
$$

$$
= \sum_{p=1}^{n} (\triangle v)^{p-1} \wedge \triangle (v - u) \wedge (\triangle u)^{n-p}. \qquad (5.3)
$$

Then we have

$$
\left| \int_\Omega \chi (\triangle u_j)^n - \int_\Omega \chi (\triangle u_k)^n \right| \leq \sum_{p=1}^{n} \left| \int_K \chi \triangle u_j \wedge \cdots \wedge \triangle (u_j - u_k) \wedge \triangle u_k \wedge \cdots \wedge \triangle u_k \right|
$$

$$
= \sum_{p=1}^{n} \left| \int_K (u_j - u_k) \triangle u_j \wedge \cdots \wedge \triangle \chi \wedge \triangle u_k \wedge \cdots \wedge \triangle u_k \right|
$$

$$
\leq \frac{\| \triangle \chi \|}{\varepsilon} \| u_j - u_k \|_\infty \sum_{p=1}^{n} \int_K \triangle u_j \wedge \cdots \wedge \beta_n \wedge \triangle u_k \wedge \cdots \wedge \triangle u_k \leq C \| u_j - u_k \|_\infty.
$$

as in the proof of Proposition 5.1, where C depends on the uniform upper bound of $\| u_j \|_\infty$. $\qquad \square$

Proposition 5.2 *Let* $u, v \in C(\overline{\Omega})$ *be plurisubharmonic functions. Then* $(\triangle(u + v))^n \geq (\triangle u)^n + (\triangle v)^n$.

Proof For smooth PSH $u_\varepsilon = \chi_\varepsilon * u$, we have

$$(\triangle(u_\varepsilon + v_\varepsilon))^n = (\triangle u_\varepsilon)^n + (\triangle v_\varepsilon)^n + \sum_{j=1}^{n-1} C_n^j (\triangle u_\varepsilon)^j \wedge (\triangle v_\varepsilon)^{n-j} \geq (\triangle u_\varepsilon)^n + (\triangle v_\varepsilon)^n.$$

The result follows by taking limit $\varepsilon \to 0$ and using the convergence of the quaternionic Monge–Ampère measure in Theorem 1.1. □

We need the following proposition to prove the minimum principle.

Proposition 5.3 *Let Ω be a bounded domain with smooth boundary in $\mathscr{H}$, and let $u, v \in C^2(\overline{\Omega})$ be plurisubharmonic functions on Ω. If $u = v$ on $\partial\Omega$ and $u \geq v$ in Ω, then*

$$\int_\Omega (\triangle u)^n \leq \int_\Omega (\triangle v)^n. \tag{5.4}$$

Proof We have

$$\begin{aligned}
\int_\Omega (\triangle v)^n - \int_\Omega (\triangle u)^n &= \sum_{p=1}^n \int_\Omega d_0 \left\{ d_1 (v - u) \wedge (\triangle v)^{p-1} \wedge (\triangle u)^{n-p} \right\} \\
&= \sum_{p=1}^n \sum_{A=0}^{2n-1} \int_{\partial\Omega} T_A^p \cdot Z_{A0'} \rho \cdot dS
\end{aligned} \tag{5.5}$$

by using (5.3) and Stokes-type formula (4.22), if we write

$$d_1 (v - u) \wedge (\triangle v)^{p-1} \wedge (\triangle u)^{n-p} =: \sum_A T_A^p \, \omega^{\widehat{A}},$$

where ρ is a defining function of Ω with $|\text{grad}\rho| = 1$, and $\omega^{\widehat{A}} = \omega^A \rfloor \Omega_{2n}$. Note that we have

$$\sum_{A=0}^{2n-1} T_A^p \cdot Z_{A0'} \rho(\xi) \cdot \Omega_{2n} = d_0 \rho(\xi) \wedge d_1 (v - u) \wedge (\triangle v)^{p-1} \wedge (\triangle u)^{n-p}. \tag{5.6}$$

Since $u = v$ on $\partial\Omega$ and $u \geq v$ in Ω, for a point $\xi \in \partial\Omega$ with $\text{grad}(v - u)(\xi) \neq 0$, we can write $v - u = h\rho$ in a neighborhood of ξ for some positive smooth function h. Consequently, we have $\text{grad}(v - u)(\xi) = h(\xi)\text{grad}\rho$, and so $Z_{A1'}(v - u)(\xi) = h(\xi)Z_{A1'}\rho(\xi)$ on $\partial\Omega$. Thus,

$$d_0\rho(\xi) \wedge d_1 (v - u) (\xi) = h(\xi)d_0\rho(\xi) \wedge d_1\rho(\xi),$$

which is strongly positive by Proposition 4.2. Moreover, both $\triangle v$ and $\triangle u$ are strongly positive for C^2 plurisubharmonic functions u and v on Ω by Proposition 4.1. We find that the right hand of (5.6) is a positive $2n$-form, and so the integrant in the right hand of (5.5) on $\partial\Omega$ is nonnegative if $\text{grad}(v - u)(\xi) \neq 0$, while if $\text{grad}(v - u)(\xi) = 0$, the integrant at ξ in (5.5) vanishes. Therefore the difference in (5.5) is nonnegative. □

The proof of the minimum principle is similar to the complex case [8] and the quaternionic case [1], but we need some modifications because we do not know whether the regularization $\chi_\varepsilon * u$ of a PSH function u on the the Heisenberg group is decreasing as $\varepsilon \to 0+$.

Proof of Theorem 1.3 Without loss of generality, we may assume $\min_{\partial\Omega}\{u - v\} = 0$. Suppose that there exists a point $(x_0, t_0) \in \Omega$ such that $u(x_0, t_0) < v(x_0, t_0)$. Denote

$\eta_0 = \frac{1}{2}[v(x_0, t_0) - u(x_0, t_0)]$. Then for each $0 < \eta < \eta_0$, the set $G(\eta) := \{(x, t) \in \Omega; u(x, t) + \eta < v(x, t)\}$ is a non-empt, open, relatively compact subset of Ω. Now consider

$$G(\eta, \delta) := \{(x, t) \in \Omega; u(x, t) + \eta < v(x, t) + \delta|x - x_0|^2\}.$$

There exists an increasing function $\delta(\eta)$ such that $G(\eta, \delta)$ for $0 < \delta < \delta(\eta)$ is a non-empt, open, relatively compact subset of Ω. On the other hand, there exists small $\alpha(\eta, \delta)$ such that for $0 < \alpha < \alpha(\eta, \delta)$, we have $\{\xi \in \Omega; \text{dist}(\xi, \partial\Omega) > \alpha\} =: \Omega_\alpha \supset G(\eta, \delta)$ for $0 < \delta < \delta(\eta/2)$, where $\text{dist}(\xi, \zeta) = \|\xi^{-1}\zeta\|$.

We hope to apply Proposition 5.3 to $G(\eta, \delta)$ to get a contradict, but its boundary may not be smooth. We need to regularize them. Recall that $u_\varepsilon \to u$ and $v_\varepsilon \to v$ uniformly as $\varepsilon \to 0+$ on any compact subset of Ω. Define

$$G(\eta, \delta, \varepsilon) := \{(x, t) \in \Omega; u(x, t) + \eta < v_\varepsilon(x, t) + \delta|x - x_0|^2\},$$

which satisfies $G(\eta, \delta, \varepsilon) \subset G(3\eta/4, \delta) \subset G(\eta/2, \delta)$ if $0 < \varepsilon < \alpha(\eta, \delta)$ is sufficiently small, since $|v(x, t) - v_\varepsilon(x, t)| \le \eta/4$ for $(x, t) \in G(\eta/2, \delta)$. Now choose τ so small that

$$G(\eta, \delta, \varepsilon, \tau) := \{(x, t) \in \Omega; u_\tau(x, t) + \eta < v_\varepsilon(x, t) + \delta|x - x_0|^2\}$$

is a non-empt, open, relatively compact subset of Ω. At last we can choose positive numbers $\eta_1 < \eta_2, \delta_0, \varepsilon_0, \tau_0$ such that for any $\eta \in [\eta_1, \eta_2], 0 < \varepsilon < \varepsilon_0, 0 < \tau < \tau_0, G(\eta, \delta_0, \varepsilon, \tau)$ is a non-empt, open, relatively compact subset of Ω.

For fixed ε, τ, by Sard's theorem, almost all values of the C^∞ function $v_\varepsilon(x, t) + \delta_0|x - x_0|^2 - u_\tau(x, t)$ are regular, i.e. $G(\eta, \delta_0, \varepsilon, \tau)$ has smooth boundary for almost all η. Consequently, we can take sequence of numbers $\tau_k \to 0$ and $\varepsilon_k \to 0$ such that $G(\eta, \delta_0, \varepsilon_k, \tau_k)$ has a smooth boundary for each k and almost all $\eta \in [\eta_1, \eta_2]$. Now apply Proposition 5.3 to the domain $G(\eta, \delta_0, \varepsilon_k, \tau_k)$ to get

$$\int (\triangle u_{\tau_k})^n \ge \int (\triangle(v_\varepsilon + \delta_0|x - x_0|^2))^n \ge \int (\triangle v_{\varepsilon_k})^n + \delta_0^n \int (\triangle|x - x_0|^2)^n$$
$$= \int (\triangle v_{\varepsilon_k})^n + 4^n n! \delta_0^n vol(G(\eta, \delta_0, \varepsilon_k, \tau_k))$$

(5.7)

by using Proposition 5.2 (2), where integrals are taken over $G(\eta, \delta_0, \varepsilon_k, \tau_k)$, and

$$(\triangle|x - x_0|^2)^n = \left(\sum_{l=0}^{n-1} \triangle_{l(n+l)}|x - x_0|^2 \omega^l \wedge \omega^{n+l}\right)^n = 4^n n! \Omega_{2n},$$

by the expression of $\triangle_{l(n+l)}$ in (3.7). Since $(\triangle u)^n \le (\triangle v)^n$ and $\eta \to (\triangle v)^n(G(\eta, \delta_0))$ is decreasing in η, we can choose a continuous point η such that $G(\eta, \delta_0, \varepsilon_k, \tau_k)$ has a smooth boundary. For any $\eta_1 < \eta' < \eta < \eta'' < \eta_2$, $G(\eta', \delta_0) \supset G(\eta, \delta_0, \varepsilon_k, \tau_k) \supset G(\eta'', \delta_0)$ for large k. So we have

$$\int_{G(\eta', \delta_0)} (\triangle u_{\tau_k})^n \ge \int_{G(\eta'', \delta_0)} (\triangle v_{\varepsilon_k})^n + (4\delta_0)^n n! vol(G(\eta'', \delta_0))$$

(5.8)

by (5.7). Thus,

$$(\triangle u)^n(G(\eta', \delta_0)) \ge (\triangle v)^n(G(\eta'', \delta_0)) + (4\delta_0)^n n! vol(G(\eta'', \delta_0)),$$

by convergence of quaternionic Monge–Ampère measures by Theorem 1.1. At the continuous point η, we have

$$(\triangle v)^n(G(\eta, \delta_0)) \geq (\triangle v)^n(G(\eta, \delta_0)) + (4\delta_0)^n n! vol(G(\eta'', \delta_0)).$$

This is a contradict since $G(\eta'', \delta_0)$ is a nonempt open subset of Ω for η'' close to η. $\qquad\square$

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