



Some remarks on Hermitian manifolds satisfying Kähler-like conditions

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Abstract

We study Hermitian metrics whose Bismut connection ∇^B satisfies the first Bianchi identity in relation to the SKT condition and the parallelism of the torsion of the Bismut connection. We obtain a characterization of complex surfaces admitting Hermitian metrics whose Bismut connection satisfy the first Bianchi identity and the condition $R^B(x, y, z, w) = R^B(Jx, Jy, z, w)$, for every tangent vectors x, y, z, w , in terms of Vaisman metrics. These conditions, also called Bismut Kähler-like, have been recently studied in Angella et al. (Commun Anal Geom, to appear, 2018), Yau et al. (2019) and Zhao and Zheng (2019). Using the characterization of SKT almost abelian Lie groups in Arroyo and Lafuente (Proc Lond Math Soc (3) 119:266–289, 2019), we construct new examples of Hermitian manifolds satisfying the Bismut Kähler-like condition. Moreover, we prove some results in relation to the pluriclosed flow on complex surfaces and on almost abelian Lie groups. In particular, we show that, if the initial metric has constant scalar curvature, then the pluriclosed flow preserves the Vaisman condition on complex surfaces.

Keywords Bismut connection · Curvature · Hermitian metric

Mathematics Subject Classification 53C55 · 53C05 · 22E25 · 53C30 · 53C44

1 Introduction

Given a Hermitian manifold (X, J, g) , the Bismut connection ∇^B is the unique connection on X that is Hermitian (i.e. such that $\nabla^B g = 0$ and $\nabla^B J = 0$) and has totally skew-symmetric torsion tensor (cf. [8]). If the torsion 3-form of ∇^B is closed, the Hermitian metric g is called

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SKT or pluriclosed. Examples of SKT manifolds are given by Lie groups (and its compact quotients) endowed with left-invariant SKT Hermitian structures (see for instance [11,24], [12]). A characterization of almost abelian Lie groups, i.e. Lie groups whose Lie algebra has a codimension- one abelian ideal, admitting left-invariant SKT metrics has been recently obtained in [5].

In [27] the authors studied Hermitian metrics whose Levi-Civita and Chern connection have curvature tensors satisfying all the symmetry conditions of a Kähler metric. Hermitian metrics with the Bismut connection being “Kähler-like”, namely, satisfying the first Bianchi identity and the condition $R^B(x, y, z, w) = R^B(Jx, Jy, z, w)$, for every tangent vectors x, y, z, w , have been studied, in [4], investigating this property on 6-dimensional solvmanifolds with holomorphically trivial canonical bundle.

In [30] Zhao and Zheng show that if the curvature tensor of the Bismut connection satisfies the symmetry conditions

$$R^B(x, y, z, w) = R^B(z, y, x, w), \quad R^B(x, y, Jz, Jw) = R^B(x, y, z, w) \quad (1)$$

for any tangent vectors x, y, z, w in X , then the Hermitian metric must be SKT. In [28] a classification for compact non-Kähler Hermitian manifolds whose Bismut connection is Kähler-like in complex dimension 3 and those with degenerate torsion in higher dimensions is given.

An evolution equation of SKT metrics is given by the pluriclosed flow, introduced by Streets and Tian in [19]. In recent years the pluriclosed flow has been an active subject of study, and already many regularity and convergence results have been proved in [20,21]. A natural question is to see if the Bismut Kähler-like condition is preserved by the flow.

In this paper we study Hermitian metrics whose Bismut connection ∇^B satisfies the first Bianchi identity in relation to the SKT condition and the parallelism of the torsion of the Bismut connection. In particular, as a consequence of Theorems 3.1 and 3.2 we show that, if X is a complex manifold and g is a Hermitian metric such that the Bismut connection satisfies the first Bianchi identity, then,

$$\nabla^B T^B = 0 \iff g \text{ is SKT.}$$

Moreover, we show in Proposition 3.11 that the existence of these metrics is not open under small deformations of the complex structure.

Specializing to complex dimension 2 we are able to characterize these metrics as follows

Theorem A *Let X be a complex surface and g be a Hermitian metric. Then, g is Vaisman if and only if g is a SKT metric and the Bismut connection satisfies the first Bianchi identity.*

Where we recall that a Vaisman metric ω on a complex manifold X is a Hermitian metric satisfying $d\omega = \theta \wedge \omega$ for some d -closed 1-form θ with $\nabla^{LC}\theta = 0$.

In [6] a generalization of Vaisman metrics, called metrics with Lee potential, is introduced, and in Sect. 3.1 we study this condition in relation with the SKT condition.

In Sect. 4 we construct new examples of Hermitian manifolds satisfying the Bismut Kähler-like condition, using the characterization of SKT simply connected almost abelian Lie groups. In order to do this we compute explicitly the components of the Bismut connection. Moreover, we give conditions on the structure equations of almost abelian Lie groups in order to have Kähler and flat Hermitian metrics.

In the last Section we study the Bismut Kähler-like condition in relation to the pluriclosed flow, discussing its behavior on complex dimension 2 and on almost abelian Lie groups. In particular, we prove the following

Theorem B *Let X be a compact complex surface admitting a Vaisman metric ω_0 with constant scalar curvature, then the pluriclosed flow starting with ω_0 preserves the Vaisman condition.*

2 Preliminaries

Let (X, J) be a complex manifold of complex dimension n and let g be a Hermitian metric on X with associated fundamental form $\omega(\cdot, \cdot) = g(\cdot, J \cdot)$. An affine connection is called Hermitian if it preserves the metric g and the complex structure J . In particular, Gauduchon in [14] proved that there exists an affine line $\{\nabla^t\}_{t \in \mathbb{R}}$ of canonical Hermitian connections, passing through the Chern connection and the Bismut connection; these connections are completely determined by their torsion. Let ∇ be a Hermitian connection and $T(x, y) = \nabla_x y - \nabla_y x - [x, y]$ be its torsion, we denote with the same symbol

$$T(x, y, z) := g(T(x, y), z).$$

Then the Chern connection ∇^{Ch} is the unique Hermitian connection whose torsion has trivial $(1, 1)$ -component and the Bismut connection (also called Strominger connection) ∇^B is the unique Hermitian connection with totally skew-symmetric torsion. In particular, the torsion of the Bismut connection satisfies

$$T^B(x, y, z) = d^c \omega(x, y, z)$$

where $d^c = -J^{-1}dJ$.

A Hermitian metric ω is called *strong Kähler with torsion (SKT for brevity)* or *pluriclosed* if T^B is a closed 3-form, namely $dT^B = 0$, or equivalently $dd^c \omega = 0$.

Recall that the trace of the torsion of the Chern connection is equal to the Lee form of ω (cf. [13]), that is the 1-form defined by

$$\theta = Jd^* \omega$$

where d^* is the adjoint of the exterior derivative d with respect to ω , or equivalently θ is the unique 1-form satisfying

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

A Hermitian metric ω is called *Gauduchon* if $dd^c \omega^{n-1} = 0$, or equivalently $d^* \theta = 0$. In particular, in dimension 2 Gauduchon and SKT metrics coincide. We recall the following

Definition 2.1 A Hermitian metric ω on X is called *locally conformally Kähler (lck for brevity)* if

$$d\omega = \alpha \wedge \omega$$

where α is a d -closed 1-form. In particular, $\alpha = \frac{1}{n-1}d\theta$ and θ is d -closed.

A locally conformally Kähler metric is called *Vaisman* if the Lee form is parallel with respect to the Levi-Civita connection ∇^{LC} , namely

$$\nabla^{LC} \theta = 0.$$

In particular, it is immediate to see that Vaisman metrics are Gauduchon and the norm of the Lee form $|\theta|$ with respect to ω is constant.

The Chern and Bismut connections are related to the Levi-Civita connection ∇^{LC} by

$$\begin{aligned}g(\nabla_x^B y, z) &= g(\nabla_x^{LC} y, z) + \frac{1}{2}d^c\omega(x, y, z), \\g(\nabla_x^{Ch} y, z) &= g(\nabla_x^{LC} y, z) + \frac{1}{2}d\omega(Jx, y, z).\end{aligned}$$

If we denote with

$$\begin{aligned}R(x, y)z &= \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]}z, \\R(x, y, z, u) &= g(R(x, y)z, u),\end{aligned}$$

the curvature tensor of type (1, 3) and (0, 4), respectively, of a connection ∇ then we have the following identities involving the torsion and the curvature of the Bismut connection (cf. [15]) which will be useful in the following (cf. [15, Formulas (3.20), (3.21)])

$$\begin{aligned}dT^B(x, y, z, u) &= \sigma_{x, y, z} \left\{ (\nabla_x^B T^B)(y, z, u) + 2g \left(T^B(x, y), T^B(z, u) \right) \right. \\&\quad \left. - (\nabla_u^B T^B)(x, y, z) \right\},\end{aligned}\tag{2}$$

$$\begin{aligned}\sigma_{x, y, z} R^B(x, y, z, u) &= dT^B(x, y, z, u) + (\nabla_u^B T^B)(x, y, z) \\&\quad - \sigma_{x, y, z} g \left(T^B(x, y), T^B(z, u) \right).\end{aligned}\tag{3}$$

In particular, this shows that the Bismut connection does not satisfy the first Bianchi identity in general. We recall the following definition (cf. [4, Definition 4], [27,30])

Definition 2.2 The Bismut connection ∇^B is called *Kähler-like* if it satisfies the first Bianchi identity

$$\sigma_{x, y, z} R^B(x, y, z) = 0\tag{4}$$

and the type condition

$$R^B(x, y, z, w) = R^B(Jx, Jy, z, w).\tag{5}$$

In [4] the authors study these two conditions for the canonical connections considered by Gauduchon on 6-dimensional solvmanifolds with invariant complex structures, trivial canonical bundle and invariant Hermitian metrics.

In [30] Zhao and Zheng show that if the curvature tensor of the Bismut connection satisfies the symmetry conditions

$$R^B(x, y, z, w) = R^B(z, y, x, w), \quad R^B(x, y, Jz, Jw) = R^B(x, y, z, w)$$

for any tangent vectors x, y, z, w in X , then the Hermitian metric must be SKT. We will show that the previous condition for the curvature of the Bismut connection to be symmetric when the first and the third position are interchanged is stronger than the first Bianchi identity. In fact, in a private communication Zhao and Zheng showed that those two conditions are equivalent to the vanishing of the curvature R^B .

3 Bismut Kähler-like condition

In this section we investigate the properties (4) and (5) in the definition of Bismut Kähler like Hermitian metrics. In particular, we focus on the relations between the first Bianchi identity for the Bismut connection, the SKT condition and the parallelism of the torsion of the Bismut

connection, specializing then our considerations to dimension 2, giving a characterization for Vaisman metrics.

Theorem 3.1 *Let X be a complex manifold with a SKT Hermitian metric g such that the Bismut connection satisfies the first Bianchi identity. Then,*

$$\nabla^B T^B = 0.$$

Proof By hypothesis, $dT^B = 0$ and the first Bianchi-identity holds, hence by Formula (3) one has that for any tangent vectors x, y, z, u ,

$$(\nabla_u T^B)(x, y, z) = \sigma_{x,y,z}g(T^B(x, y), T^B(z, u))$$

and by Formula (2)

$$\sigma_{x,y,z}(\nabla_x^B T^B)(y, z, u) + \sigma_{x,y,z}2g(T^B(x, y), T^B(z, u)) - (\nabla_u^B T^B)(x, y, z) = 0.$$

Hence, one gets

$$\sigma_{x,y,z}(\nabla_x^B T^B)(y, z, u) = -(\nabla_u^B T^B)(x, y, z).$$

Now notice that both sides of the equality are tensorial and on the left hand side the expression is symmetric in x, y, z while on the right hand side is antisymmetric in x, y, z . Therefore in order to be equal they must vanish. Hence, $\nabla^B T^B = 0$. □

Theorem 3.2 *Let X be a complex manifold and let g be a Hermitian metric such that $\nabla^B T^B = 0$. Then, the Bismut connection satisfies the first Bianchi identity if and only if g is SKT.*

Proof If $\nabla^B T^B = 0$ then by Formula (2)

$$dT^B(x, y, z, u) = \sigma_{x,y,z}2g(T^B(x, y), T^B(z, u))$$

and so by Formula (3)

$$\sigma_{x,y,z}R^B(x, y, z, u) = dT^B(x, y, z, u) - \sigma_{x,y,z}g(T^B(x, y), T^B(z, u)) = \frac{1}{2}dT^B(x, y, z, u).$$

Therefore, the Bismut connection satisfies the first Bianchi identity if and only if g is SKT. □

Hence, by putting together Theorems 3.1 and 3.2 we obtain the following

Corollary 3.3 *Let X be a complex manifold and let g be a Hermitian metric such that the Bismut connection satisfies the first Bianchi identity. Then,*

$$\nabla^B T^B = 0 \iff g \text{ is SKT.}$$

Now we consider some relations with respect to the Levi–Civita connection

Theorem 3.4 *Let X be a complex manifold and g a Hermitian metric. If the Bismut connection satisfies the first Bianchi identity and g is SKT then*

$$\nabla^{LC} T^B = 0.$$

Proof If the Bismut connection satisfies the first Bianchi identity and g is SKT ($dT^B = 0$), then by Theorem 3.1 we have $\nabla^B T^B = 0$ and so by Formula (3)

$$0 = (\nabla_u^B T^B)(x, y, z) = \sigma_{x,y,z}g(T^B(x, y), T^B(z, u)),$$

then by [15, Formula (3.18)]

$$(\nabla_x^{LC} T^B)(y, z, u) = (\nabla_x^B T^B)(y, z, u) + \frac{1}{2} \sigma_{x,y,z} g(T^B(x, y), T^B(z, u)) = 0.$$

□

Notice that if the dimension of X is 2, then (cf. [15, (2.14)])

$$T^B = - * \theta$$

hence, on a complex surface

$$\nabla^{LC} T^B = 0 \iff \nabla^{LC} \theta = 0.$$

Then as a consequence of Theorem 3.4 one has

Corollary 3.5 *Let X be a complex surface and g be a Gauduchon metric such that the Bismut connection satisfies the first Bianchi identity. Then, g is Vaisman.*

In fact, we can prove that also its converse is true and show Theorem A.

Proof of Theorem A Since Vaisman metrics are Gauduchon (or equivalently SKT in complex dimension 2), we just need to prove that if g is Vaisman then the Bismut connection satisfies the first Bianchi identity. Recall that on complex surfaces by [1, Appendix A]

$$\nabla^B T^B = \nabla^{LC} T^B.$$

Since, by hypothesis $\nabla^{LC} \theta = 0$ then $\nabla^B T^B = \nabla^{LC} T^B = 0$, hence by Theorem 3.2 the Bismut connection satisfies the first Bianchi identity. □

Notice that, up to now we have not used the second part of the definition of Bismut Kähler-like metrics. First of all, notice that if

$$R^B(x, y, z, w) = R^B(Jx, Jy, z, w)$$

then for the Ricci form of the Bismut connection we have

$$\rho^B(x, y) = \frac{1}{2} \sum R^B(x, y, e_i, J e_i) = \frac{1}{2} \sum R^B(Jx, Jy, e_i, J e_i) = \rho^B(Jx, Jy)$$

where $\{e_i\}$ denotes an orthonormal basis of the tangent space. Namely, ρ^B is a $(1, 1)$ -form. In particular,

$$d(\rho^B)^{1,1} = d\rho^B = 0$$

where $(\beta)^{1,1}$ denotes the $(1, 1)$ component of a 2-form β . Hence, since

$$(\rho^B)^{1,1} = \rho^{Ch} + \frac{dJ\theta + JdJ\theta}{2}$$

where ρ^{Ch} is the Ricci form of the Chern connection, one has that

$$dd^c \theta = 0.$$

As a consequence,

Proposition 3.6 *Let X be a complex surface and let g be a Hermitian metric such that*

$$R^B(x, y, z, w) = R^B(Jx, Jy, z, w)$$

for every tangent vectors x, y, z, w . Then, g is locally conformally Kähler.

Proof Let ω be the fundamental form associated to g . Then, $d\omega = \theta \wedge \omega$ and so we need to prove that $d\theta = 0$.

First of all, notice that in any dimension $d\theta$ is a primitive 2-form, indeed differentiating $d\omega^{n-1} = \theta \wedge \omega^{n-1}$ one has

$$0 = d\theta \wedge \omega^{n-1} = L^{n-1}d\theta .$$

where $L = \omega \wedge -$ is the operator of multiplication by ω acting on forms. Therefore, we compute

$$d^*d\theta = - * d(*d\theta) = *JJ^{-1}dJd\theta = *Jdd^c\theta ,$$

where in the second equality we have used for instance [25, Formula (A8)]. Then,

$$dd^c\theta = 0 \iff d^*d\theta = 0 \iff d\theta = 0 .$$

□

3.1 SKT metrics with Lee potential

In [6] a generalization of lck metrics with potential (and so also of Vaisman metrics) is introduced and it is shown that these metrics exist on Calabi-Eckmann manifolds.

Definition 3.7 [6,28] A Hermitian manifold (X, J, g) is called *LP* (or equivalently it has *Lee potential*) if the $(1, 0)$ -part of the Lee form θ of g , namely $\eta := \theta^{1,0}$, satisfies

$$\eta \neq 0, \quad \partial\eta = 0, \quad \partial\omega = c\eta \wedge \partial\bar{\eta} ,$$

for some non-zero constant c .

If in addition $\nabla^B T^B = 0$ the metric g is called *Generalized Calabi-Eckmann* (*GCE* for short).

As a consequence of Theorem 3.2 the Bismut connection of a Generalized Calabi-Eckmann SKT metric satisfies the first Bianchi identity.

We show that if a SKT metric is LP then all its powers are $\partial\bar{\partial}$ -closed.

Proposition 3.8 *Let X be a complex manifold of complex dimension n endowed with a SKT metric ω with Lee potential. Then,*

$$\partial\bar{\partial}\omega^k = 0, \quad \forall 1 \leq k \leq n - 1 .$$

In particular, ω is k^{th} -Gauduchon for every $1 \leq k \leq n - 1$ i.e.,

$$\partial\bar{\partial}(\omega^k) \wedge \omega^{n-k-1} = 0,$$

and astheno-Kähler, i.e., $\partial\bar{\partial}\omega^{n-2} = 0$.

Proof First of all, notice that for any k

$$\partial\bar{\partial}\omega^k = k(\partial\bar{\partial}\omega \wedge \omega - (k - 1)\bar{\partial}\omega \wedge \partial\omega) \wedge \omega^{k-2}$$

Since, ω is SKT and LP one has that

$$\partial\bar{\partial}\omega = 0, \quad \bar{\partial}\omega \wedge \partial\omega = |c|^2\bar{\eta} \wedge \bar{\partial}\eta \wedge \eta \wedge \partial\bar{\eta} ,$$

hence

$$\partial\bar{\partial}\omega^k = -k(k - 1)|c|^2(\bar{\eta} \wedge \bar{\partial}\eta \wedge \eta \wedge \partial\bar{\eta}) \wedge \omega^{k-2} .$$

Moreover, by the SKT and LP conditions one has also

$$\partial\eta = 0, \quad 0 = \partial\bar{\partial}\omega = \bar{c}\partial\bar{\eta} \wedge \bar{\partial}\eta$$

concluding the proof. \square

Remark 3.9 Observe that in [28, Remark 2] it was proved that a Hermitian metric that is both SKT and Gauduchon then it is k^{th} -Gauduchon.

The LP assumption is fundamental in the previous Proposition, indeed in the following example we exhibit a manifold with a SKT metric ω which is not LP and $\partial\bar{\partial}\omega^{n-2} \neq 0$, namely it is not astheno-Kähler.

Example 3.10 Let (G, J) be the 4-dimensional nilpotent Lie group equipped with a left invariant complex structure J with structure equations

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = 0 \\ d\varphi^3 = \lambda_1\varphi^{1\bar{1}} + i a\varphi^{2\bar{2}} \\ d\varphi^4 = \lambda_2\varphi^{2\bar{2}}. \end{cases}$$

with respect to a left invariant unitary coframe $\{\varphi^i\}_{i=1,\dots,4}$, where λ_1, λ_2, a are real numbers and $\lambda_1, \lambda_2 > 0$. It is easy to see that the left-invariant Hermitian metric $\omega := \frac{i}{2} \sum_{j=1}^4 \varphi^j \wedge \bar{\varphi}^j$ is SKT. Now we show that it is not LP.

By explicit computation one can show that the Gauduchon torsion $(1, 0)$ -form η is

$$\eta = \lambda_2\varphi^4 + (\lambda_1 - ia)\varphi^3.$$

In particular, computing

$$\begin{aligned} \eta \wedge \partial\bar{\eta} &= -\lambda_2(-\lambda_2^2 + ia\lambda_1 - a^2)\varphi^{24\bar{2}} \\ &\quad + \lambda_1(\lambda_1 + ia)\varphi^{14\bar{1}} - (\lambda_1 - ia)(-\lambda_2^2 + ia\lambda_1 - a^2)\varphi^{23\bar{2}} + \\ &\quad + \lambda_1(\lambda_1 - ia)(\lambda_1 + ia)\varphi^{13\bar{1}} \end{aligned}$$

and

$$\partial\omega = -\lambda_1\varphi^{13\bar{1}} + ia\varphi^{23\bar{2}} - \lambda_2\varphi^{24\bar{2}}$$

One gets that $\partial\omega = c\eta \wedge \partial\bar{\eta}$ for some non-zero constant c if and only if $\lambda_1 = -ia = 0$ but $\lambda_1 > 0$ so the metric ω is not LP.

Moreover, notice that the metric ω is not astheno-Kähler, indeed one has

$$\partial\bar{\partial}\omega^2 = 2\lambda_1\lambda_2\varphi^{123\bar{1}\bar{2}\bar{4}} + 2\lambda_1\lambda_2\varphi^{124\bar{1}\bar{2}\bar{3}} \neq 0.$$

3.2 Behavior under small deformations

In this section we partially answer to a question proposed in [4] about the stability of the Bismut Kähler-like property under small deformations of the complex structure. In particular, we show the following

Proposition 3.11 *The existence of a SKT metric with Bismut connection satisfying the first Bianchi identity and with $\nabla^B T^B = 0$ on compact complex manifolds is not an open property under small deformations of the complex structure.*

Proof Let $X = \mathbb{S}^3 \times \mathbb{S}^3$ and $\mathbb{S}^3 \simeq \text{SU}(2)$ be the Lie group of special unitary 2×2 matrices and denote by $\mathfrak{su}(2)$ its Lie algebra. Denote by $\{e_1, e_2, e_3\}, \{f_1, f_2, f_3\}$ a basis of the first copy of $\mathfrak{su}(2)$, respectively of the second copy of $\mathfrak{su}(2)$ and by $\{e^1, e^2, e^3\}, \{f^1, f^2, f^3\}$ the corresponding dual co-frames. Then we have the following commutation relations:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1,$$

and the corresponding Cartan structure equations

$$\begin{cases} de^1 = -2e^2 \wedge e^3 \\ de^2 = 2e^1 \wedge e^3 \\ de^3 = -2e^1 \wedge e^2 \\ df^1 = -2f^2 \wedge f^3 \\ df^2 = 2f^1 \wedge f^3 \\ df^3 = -2f^1 \wedge f^2 \end{cases} \quad (6)$$

Define a complex structure J on X by setting

$$Je_1 = e_2, \quad Jf_1 = f_2, \quad Je_3 = f_3.$$

Therefore a complex co-frame of $(1, 0)$ -forms for J is given by

$$\begin{cases} \varphi^1 := e^1 + i e^2 \\ \varphi^2 := f^1 + i f^2 \\ \varphi^3 := e^3 + i f^3 \end{cases} \quad (7)$$

In particular the complex structure equations are given by

$$\begin{cases} d\varphi^1 = i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \bar{\varphi}^3 \\ d\varphi^2 = \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \bar{\varphi}^3 \\ d\varphi^3 = -i\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2 \end{cases}, \quad (8)$$

Note that (X, J) is a central Calabi-Eckmann threefold and in [26] it is showed that this complex manifold admits a Bismut-flat Hermitian metric, which in particular is SKT and satisfies the first Bianchi identity.

Now let J_t be the almost complex structure on X considered in [22] defined as

$$\begin{cases} \varphi_t^1 := \varphi^1 \\ \varphi_t^2 := \varphi^2 \\ \varphi_t^3 := \varphi^3 - t\bar{\varphi}^3 \end{cases};$$

then, using the structure equations (8), a straightforward computation yields to

$$\begin{cases} d\varphi_t^1 = \frac{i(\bar{t}+1)}{1-|t|^2} \varphi_t^{13} + \frac{i(t+1)}{1-|t|^2} \varphi_t^{1\bar{3}} \\ d\varphi_t^2 = \frac{1-\bar{t}}{1-|t|^2} \varphi_t^{23} + \frac{t-1}{1-|t|^2} \varphi_t^{2\bar{3}} \\ d\varphi_t^3 = i(t-1)\varphi_t^{1\bar{1}} + (t+1)\varphi_t^{2\bar{2}} \end{cases}$$

and consequently J_t is integrable. Set $X_t = (X, J_t)$, in [22, Remark 3.6] it was proven that when $|t|^2 + \text{Re } t - \text{Im } t \neq 0$ then X_t does not admit any SKT metric. Therefore by Corollary 3.3, for such values of t , X_t does not admit any Hermitian metric whose Bismut connection satisfies the first Bianchi identity and with parallel torsion. \square

4 Bismut Kähler-like almost abelian Lie groups

In this Section, we study the existence of Hermitian metrics with Kähler-like Bismut connection on simply-connected, almost abelian Lie groups in order to give new examples besides the ones provided in [4] on 6-dimensional solvmanifolds with invariant complex structures, trivial canonical bundle and invariant Hermitian metrics. Let G be a simply-connected, almost abelian Lie group namely, its Lie algebra \mathfrak{g} has a codimension-one abelian ideal \mathfrak{n} . In particular, notice that such a G is solvable. Let (J, g) be a left-invariant Hermitian structure on G , therefore there exists a basis $\{e_1, \dots, e_{2n}\}$ on \mathfrak{g} such that, setting $\mathfrak{n}_1 = \text{Span}_{\mathbb{R}} \langle e_2, \dots, e_{2n-1} \rangle$, one has

$$\mathfrak{n} = \text{Span}_{\mathbb{R}} \langle e_1, \dots, e_{2n-1} \rangle, \quad J e_1 = e_{2n}, \quad J(\mathfrak{n}_1) \subset \mathfrak{n}_1.$$

By [16] the complex structure J is integrable if and only if $\text{ad } e_{2n}$ leaves \mathfrak{n}_1 invariant, and $A := (\text{ad } e_{2n})|_{\mathfrak{n}_1}$ commutes with $J_1 := J|_{\mathfrak{n}_1}$. Hence one has

$$\text{ad } e_{2n} = \begin{pmatrix} a & 0 & 0 \\ v & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $a \in \mathbb{R}, v \in \mathfrak{n}_1, A \in \mathfrak{gl}(\mathfrak{n}_1)$ and $[A, J_1] = 0$. In particular, we have the following non-trivial Lie brackets

$$[e_{2n}, x] = Ax, \quad [e_{2n}, e_1] = ae_1 + v$$

for any $x \in \mathfrak{n}_1$.

We fix a real inner product g on \mathfrak{g} with an orthogonal decomposition

$$\mathfrak{g} = \mathbb{R}e_1 \oplus \mathfrak{n}_1 \oplus \mathbb{R}e_{2n}$$

and

$$\mathfrak{n} = \mathbb{R}e_1 \oplus \mathfrak{n}_1$$

and a compatible integrable complex structure J such that $J e_1 = e_{2n}$ and $J(\mathfrak{n}_1) \subset \mathfrak{n}_1$. We recall that by, [5, Lemma 4.2] the metric g is SKT if and only if

$$aA + A^2 + A^t A \in \mathfrak{so}(\mathfrak{n}_1).$$

In order to study the existence of Bismut Kähler-like metrics we need the expression of the Bismut connection. By using the formula (see [10])

$$2g \left(\nabla_x^B y, z \right) = g([x, y] - [Jx, Jy], z) - g([x, z] - [Jx, Jz], y) - g([y, z] + [Jy, Jz], x),$$

we get the following

Lemma 4.1 *Let G be an almost abelian Lie group. Then, with the previous notations, the only non-zero components of the Bismut connection are*

$$\begin{aligned} \nabla_{e_1}^B e_1 &= ae_{2n}, \\ \nabla_x^B e_1 &= \sum_{j=2}^{2n-1} g(S(A)Jx, e_j)e_j + g(v, x)e_{2n}, \\ \nabla_{e_1}^B e_{2n} &= -ae_1, \\ \nabla_x^B y &= g(x, S(A)Jy)e_1 + g(x, S(A)y)e_{2n}, \\ \nabla_{e_1}^B y &= -\sum_{j=2}^{2n-1} g(S(A)Jy, e_j)e_j, \\ \nabla_{e_{2n}}^B y &= \frac{1}{2} \sum_{j=2}^{2n-1} g((A - A^t)y, e_j)e_j, \\ \nabla_x^B e_{2n} &= -g(v, x)e_1 - \sum_{j=2}^{2n-1} g(S(A)x, e_j)e_j, \end{aligned}$$

with $x, y \in \mathfrak{n}_1$ and $S(A) := \frac{1}{2}(A + A^t)$.

In particular if we assume that $A \in \mathfrak{so}(\mathfrak{n}_1)$ the non-trivial components reduce to

$$\begin{aligned} \nabla_{e_1}^B e_1 &= ae_{2n}, \\ \nabla_x^B e_1 &= g(v, x)e_{2n}, \\ \nabla_{e_1}^B e_{2n} &= -ae_1, \\ \nabla_{e_{2n}}^B y &= Ay, \\ \nabla_x^B e_{2n} &= -g(v, x)e_1. \end{aligned}$$

If $y \in \mathfrak{n}_1$, by explicit computations one gets that the only non-trivial components of the Bismut curvature tensor are

$$\begin{aligned} R^B(e_{2n}, y)e_1 &= -g(v, Ay)e_{2n}, \\ R^B(e_{2n}, y)e_{2n} &= g(v, Ay)e_1, \\ R^B(e_1, e_{2n})e_1 &= (a^2 + |v|^2)e_{2n}, \\ R^B(e_1, e_{2n})e_{2n} &= (-a^2 - |v|^2)e_1 \end{aligned}$$

together with their symmetries. We can now prove

Theorem 4.2 *Let G be an almost abelian Lie group and assume that $A \in \mathfrak{so}(\mathfrak{n}_1)$. Then, the Bismut connection is Kähler-like if and only if Av is g -orthogonal to \mathfrak{n}_1 .*

Proof One can check that the first Bianchi identity holds if and only if $g(v, Ay) = 0$ for any $y \in \mathfrak{n}_1$. Indeed, for instance

$$R^B(y, e_{2n})e_1 + R^B(e_{2n}, e_1)y + R^B(e_1, y)e_{2n} = g(v, Ay)e_{2n}$$

and similarly for the other relations. For the J -invariance in the first two components of $R^B(\cdot, \cdot, \cdot, \cdot)$ one has the same conclusion, for example

$$R^B(e_{2n}, y, e_1, e_{2n}) - R^B(Je_{2n}, Jy, e_1, e_{2n}) = -g(v, Ay)$$

and so on. Hence, the Bismut connection is Kähler-like if and only if $g(v, Ay) = 0$ for every $y \in \mathfrak{n}_1$ if and only if Av is g -orthogonal to \mathfrak{n}_1 . □

Remark 4.3 Notice that in general these Bismut Kähler-like metrics are not Kähler. More precisely, the torsion of the Bismut connection is given by

$$\begin{aligned} T^B(e_1, y) &= -g(v, y)e_{2n}, \\ T^B(e_{2n}, y) &= g(v, y)e_1, \\ T^B(e_1, e_{2n}) &= v \end{aligned}$$

for $y \in \mathfrak{n}_1$.

And so the only non-zero components of the torsion 3-form are

$$T^B(e_1, y, e_{2n}) = -g(v, y).$$

In particular, one can check explicitly that $\nabla^B T^B = 0$. Hence, these exist explicit examples of metrics that are SKT, with $\nabla^B T^B = 0$ but they do not satisfy the condition $R^D(x, y, z, w) = R^D(z, y, x, w)$, for every tangent vectors x, y, z, w .

Remark 4.4 We notice that having $A \in \mathfrak{so}(\mathfrak{n}_1)$ is not necessary in order to have a Hermitian metric with Bismut Kähler-like connection on an almost abelian simply-connected Lie group. Indeed, by [4] the Lie algebra \mathfrak{h}_8 with structure equations $(0, 0, 0, 0, 12)$ is an almost abelian Lie algebra with Bismut Kähler-like connection but the corresponding matrix A is not antisymmetric.

We now discuss the existence of a compact quotient on an explicit example in dimension 6. First of all notice that if \mathfrak{g} is an almost abelian Lie algebra then it is unimodular (namely all the adjoint maps are traceless) if and only if $a + \text{tr } A = 0$.

If $A \in \mathfrak{so}(\mathfrak{n}_1)$, then \mathfrak{g} is unimodular if and only if $a = 0$.

Example 4.5 Let \mathfrak{g} be an almost abelian Lie algebra of dimension 6 with non trivial brackets

$$[e_6, e_1] = e_2, \quad [e_6, e_4] = e_5, \quad [e_6, e_5] = -e_4.$$

Hence we have

$$\tilde{A} := \text{ad } e_6 = \begin{pmatrix} a & 0 & 0 \\ v & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $v = e_2, a = 0$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We set $\varphi(t) = e^{t\tilde{A}}$, by [9] \mathfrak{g} admits a lattice if and only if there exists $t_0 \in \mathbb{R}$ such that $\varphi(t_0)$ is conjugate to an integral matrix. Set $t_0 := \pi$, then

$$\varphi(t_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \pi & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is conjugate to

$$\begin{pmatrix} 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, there exists a lattice Γ in G such that $\Gamma \backslash G$ is an almost abelian solvmanifold of dimension 6 admitting a Bismut Kähler-like Hermitian metric.

In relation to the condition

$$G^B(x, y, z, w) := R^B(x, y, z, w) - R^B(z, y, x, w) = 0, \tag{9}$$

for every tangent vector x, y, z, w . we can prove the following

Theorem 4.6 *An almost abelian Lie group G admits a left-invariant SKT metric satisfying (9) if and only if*

$$a = 0, \quad v = 0, \quad A \in \mathfrak{so}(\mathfrak{n}_1).$$

Proof We have

$$\begin{aligned} G^B(e_1, e_1, e_{2n}, e_{2n}) &= -R^B(e_{2n}, e_1, e_1, e_{2n}) = -g(R^B(e_{2n}, e_1)e_1, e_{2n}) \\ &= -g\left(\nabla_{e_{2n}}^B \nabla_{e_1}^B e_1 - \nabla_{e_1}^B \nabla_{e_{2n}}^B e_1 - \nabla_{[e_{2n}, e_1]}^B e_1, e_{2n}\right). \end{aligned}$$

Since

$$\nabla_{e_1}^B e_1 = ae_{2n}, \quad \nabla_{e_{2n}}^B e_1 = 0, \quad \nabla_{e_{2n}}^B e_{2n} = 0$$

and for every $x \in \mathfrak{n}_1$

$$\nabla_x^B e_1 = \sum_{j=2}^{2n-1} g(S(A)Jx, e_j)e_j + g(v, x)e_{2n},$$

where $S(A) := \frac{1}{2}(A + A^t)$, we have,

$$G^B(e_1, e_1, e_{2n}, e_{2n}) = a^2 + \sum_{j=1}^{2n-2} v_j^2,$$

hence $G^B(e_1, e_1, e_{2n}, e_{2n}) = 0$ if and only if $a = 0$ and $v = 0$. Similarly, assuming that $a = 0$ and $v = 0$ we get, for every $x \in \mathfrak{n}_1$

$$G^B(e_1, e_1, x, x) = \frac{1}{2} \sum_{j=2}^{2n-1} [g(S(A)Jx, e_j)]^2$$

hence $G^B(e_1, e_1, x, x) = 0$ for every $x \in \mathfrak{n}_1$ if and only if $g(S(A)Jx, e_j) = 0$ for every $x \in \mathfrak{n}_1$ and every $j = 1, \dots, 2n$ (notice that for $j = 1$ and $j = 2n$ it is obvious) if and only if $S(A) = 0$.

Thus, if g satisfies (9) then we have just proven that $a = 0, v = 0$, and $A \in \mathfrak{so}(\mathfrak{n}_1)$. Now we show the viceversa, suppose that $a = 0, v = 0$ and $A \in \mathfrak{so}(\mathfrak{n}_1)$ then by similar computations one can show that the only non trivial components of the Bismut connection are, for every $y \in \mathfrak{n}_1$

$$\nabla_{e_{2n}}^B y = \sum_{j=2}^{2n-1} g(Ay, e_j)e_j$$

hence, by definition

$$R^B(x, y)z = \nabla_x^B \nabla_y^B z - \nabla_y^B \nabla_x^B z - \nabla_{[x,y]}^B z = 0$$

for every $x, y, z \in \mathfrak{g}$, hence $R^B(x, y, z, w) = 0$ for every $x, y, z, w \in \mathfrak{g}$ concluding the proof. \square

Notice that, in particular, we have proven that an almost abelian Lie group G has $R^B(x, y)z = 0$ for every $x, y, z \in \mathfrak{g}$ if and only if

$$a = 0, \quad v = 0, \quad A \in \mathfrak{so}(\mathfrak{n}_1),$$

in particular almost abelian Lie groups satisfying (9) are Bismut flat. In fact, we prove the following

Theorem 4.7 *An almost abelian Lie group G with*

$$v = 0 \quad \text{and} \quad A \in \mathfrak{so}(\mathfrak{n}_1)$$

is Kähler. In particular, the only almost abelian examples satisfying (9) are Kähler and flat.

Proof If $v = 0$ and $A \in \mathfrak{so}(\mathfrak{n}_1)$ then the only non-zero commutators become

$$[e_{2n}, x] = Ax, \quad [e_{2n}, e_1] = ae_1$$

where $x \in \mathfrak{n}_1$. Hence, computing explicitly the Bismut connection for (g, J) as before one gets that the only non-trivial components are

$$\nabla_{e_1}^B e_1 = ae_{2n}, \quad \nabla_{e_1}^B e_{2n} = -ae_1, \quad \nabla_{e_{2n}}^B y = \sum_{j=2}^{2n-1} g(Ay, e_j)e_j$$

where $y \in \mathfrak{n}_1$.

Computing explicitly the torsion

$$T^B(x, y) = \nabla_x y - \nabla_y x - [x, y]$$

one gets $T^B = 0$. For instance we compute here, for $y \in \mathfrak{n}_1$,

$$T^B(e_{2n}, y) = \sum_{j=2}^{2n-1} g(Ay, e_j)e_j - [e_{2n}, y] = \sum_{j=2}^{2n-1} g(Ay, e_j)e_j - Ay = 0.$$

□

5 Bismut-Kähler like condition and pluriclosed flow

Let (X, J) be a complex manifold of complex dimension n . In [19] the authors introduce a family of flows called *Hermitian curvature flows*, among these a particular one is the so called *pluriclosed flow* which is defined by the equation

$$\frac{\partial}{\partial t} \omega(t) = - \left(\rho^B(\omega) \right)^{1,1}, \quad \omega(0) = \omega_0$$

where $(\rho^B(\omega))^{1,1}$ denotes the $(1, 1)$ -part of the Ricci form of the Bismut connection and ω_0 is a fixed Hermitian metric. It is easy to see that this flow preserves the SKT condition. In this section we study the behavior of the Bismut Kähler-like condition under the pluriclosed flow on complex surfaces and on almost abelian Lie groups considered in the previous section.

5.1 Pluriclosed flow on Vaisman surfaces

We recall that a Vaisman metric on a complex manifold (X, J) is a Hermitian metric ω such that

$$d\omega = \theta \wedge \omega \quad \text{and} \quad \nabla^{LC}\theta = 0,$$

where θ is a 1-form.

In fact, a Vaisman structure on a complex manifold is uniquely determined (up to a positive constant) by its Lee form θ via the following

$$\omega = \frac{1}{|\theta|^2}(\theta \wedge J\theta - dJ\theta).$$

On complex dimension 2, Belgun in [7] classified those compact complex surfaces admitting a Vaisman metric, and they are properly elliptic surfaces, primary and secondary Kodaira surfaces, elliptic Hopf surfaces and Hopf surfaces of class 1.

Notice that in particular, on a compact complex surface, Vaisman metrics are SKT. We ask whether the Vaisman condition is preserved along the pluriclosed flow. We answer this question in the case the initial metric has constant scalar curvature.

In order to prove Theorem B we first need the following Lemma.

Lemma 5.1 *Let X be a compact complex surface and let ω be a Vaisman metric on X with Lee form θ . Then, the Ricci form of the Chern connection is*

$$\rho^{Ch} = h dJ\theta$$

for some $h \in C^\infty(X, \mathbb{R})$.

Moreover, the scalar curvature of ω is constant if and only if h is constant and, in particular, in such a case $c_1(X) = 0$.

Proof of Theorem B Let (ω_0, θ_0) be a Vaisman structure on X with constant scalar curvature; then, by Lemma 5.1, we have

$$c_1(X) = 0.$$

Moreover, since ω_0 is Vaisman, then

$$\omega_0 = \frac{1}{|\theta_0|_0^2}(\theta_0 \wedge J\theta_0 - dJ\theta_0)$$

where with $|\cdot|_0$ we denote the norm with respect to ω_0 and in particular $|\theta_0|_0$ is constant. We claim that the metrics

$$\omega_t = \frac{1}{|\theta_0|_0^2}(\theta_0 \wedge J\theta_0 - f(t)dJ\theta_0),$$

for a suitable positive function $f(t)$ depending only on t with $f(0) = 1$, are Vaisman and they are solutions of the pluriclosed flow.

First of all notice that the corresponding Lee form of ω_t is

$$\theta_t = \frac{1}{f(t)}\theta_0$$

indeed, $d\omega_t = \theta_t \wedge \omega_t$ and clearly θ_t is closed, hence they are locally conformally Kähler.

Moreover, the Lee field θ_t^{\sharp} with respect to ω_t is

$$\theta_t^{\sharp} = \frac{1}{f(t)}\theta_0^{\sharp_0}$$

hence by [18, Theorem 1] the metrics ω_t are Vaisman.

In fact, recall that on a compact complex manifold admitting Vaisman metrics, the Lee vector fields of all Vaisman structures are holomorphic, and coincide up to a positive multiplicative constant (cf. [23] and [17]).

Moreover, by a straightforward computation,

$$\omega_t^2 = f(t)^2\omega_0^2$$

therefore, the Ricci forms of the Chern connection of ω_t and ω_0 must coincide

$$\rho_{\omega_t}^{Ch} = \rho_{\omega_0}^{Ch}.$$

Hence, the Ricci forms of the Bismut connection of ω_t and ω_0 are related by

$$\rho_{\omega_t}^B = \rho_{\omega_t}^{Ch} - dJ\theta_t = \rho_{\omega_0}^{Ch} - \frac{1}{f(t)}dJ\theta_0.$$

Notice that, since ω_t is lck then $\rho_{\omega_t}^B$ is a closed $(1, 1)$ -form. For the pluriclosed flow we have

$$\frac{\partial}{\partial t}\omega_t = -\frac{1}{|\theta_0|_0^2}f'(t)dJ\theta_0 = d\left(-\frac{1}{|\theta_0|_0^2}f'(t)J\theta_0\right)$$

hence it is necessary to have $c_1(X) = 0$ to have ω_t solving

$$\frac{\partial}{\partial t}\omega_t = -(\rho_{\omega_t}^B)^{1,1}$$

because we have

$$d \left(-\frac{1}{|\theta_0|_0^2} f'(t) J\theta_0 \right) = \rho_{\omega_0}^{Ch} - d \left(\frac{1}{f(t)} J\theta_0 \right)$$

so $\rho_{\omega_0}^{Ch}$ is an exact form and, up to a constant, it represents the first Chern class of X .

Now, by Lemma 5.1

$$\rho_0^{Ch} = h dJ\theta_0$$

for some constant h . Hence,

$$\rho_{\omega_t}^B = \left(h - \frac{1}{f(t)} \right) dJ\theta_0.$$

and so

$$-(\rho_{\omega_t}^B)^{1,1} = -\rho_{\omega_t}^B = -\left(h - \frac{1}{f(t)} \right) dJ\theta_0.$$

The equation of the pluriclosed flow reduces to find a solution $f(t)$ of

$$\frac{1}{|\theta_0|_0^2} f'(t) dJ\theta_0 = \left(h - \frac{1}{f(t)} \right) dJ\theta_0$$

or equivalently

$$\left(\frac{1}{|\theta_0|_0^2} f'(t) - h + \frac{1}{f(t)} \right) dJ\theta_0 = 0.$$

In fact, the equation

$$\frac{1}{|\theta_0|_0^2} f'(t) - h + \frac{1}{f(t)} = 0$$

admits a unique solution $f(t) > 0$ and with $f(0) = 1$. Therefore, (ω_t, θ_t) are Vaisman metrics and solutions of the pluriclosed flow. □

Remark 5.2 We notice that, if X is a compact complex surface admitting a Vaisman metric ω_0 with constant scalar curvature, then the pluriclosed flow starting with ω_0 preserves such a condition. Indeed, by Lemma 5.1 one has for every $t \neq 0$

$$\rho_{\omega_t}^{Ch} = h_t dJ\theta_t$$

for some smooth function h_t on X , and for $t = 0$

$$\rho_{\omega_0}^{Ch} = h_0 dJ\theta_0$$

with h_0 constant by hypothesis.

Then, by the proof of Theorem B we have that

$$\rho_{\omega_t}^{Ch} = \rho_{\omega_0}^{Ch} \quad \text{and} \quad \theta_t = \frac{1}{f(t)} \theta_0$$

therefore $h_t = f(t)h_0$ for every t , giving h_t constant and so ω_t has constant scalar curvature.

We now give a proof of Lemma 5.1

Proof of Lemma 5.1 Let ω be a Vaisman metric on X . Then, by [2, Lemma 4.4] the Ricci forms of the Bismut and Weyl connections coincide

$$\rho^B = \rho^W,$$

where the Weyl connection determined by the Hermitian structure g of X is the unique torsion-free connection ∇^W such that $\nabla^W g = \theta \otimes g$. In fact, the Weyl connection is related with the Levi-Civita connection of g via

$$\nabla_x^W y = \nabla_x^{LC} y - \frac{1}{2}\theta(x)y - \frac{1}{2}\theta(y)x + \frac{1}{2}g(x, y)\theta^\sharp.$$

Since $\rho^{Ch} = \rho^B + dJ\theta = \rho^W + dJ\theta$ we just need to prove that

$$\rho^W = h dJ\theta$$

for some smooth function h on X .

Since the metric is Vaisman then the Ricci tensor Ric^W is symmetric and one has (cf. [3])

$$\text{Ric}^W(x, x) = \text{Ric}^{LC}(x, x) - \frac{1}{2}(|\theta|^2|x|^2 - \theta(x)^2)$$

where Ric^{LC} is the Ricci tensor of the Levi-Civita connection.

Therefore, for the Ricci form of the Weyl connection we can compute

$$\rho^W(\theta^\sharp, J\theta^\sharp) = -\text{Ric}^W(\theta^\sharp, \theta^\sharp) = -\text{Ric}^{LC}(\theta^\sharp, \theta^\sharp) + \frac{1}{2}(|\theta|^2|\theta|^2 - \theta(\theta^\sharp)^2) = -\text{Ric}^{LC}(\theta^\sharp, \theta^\sharp).$$

Since θ is parallel with respect to the Levi-Civita connection we have that $R^{LC}(x, y, \theta^\sharp) = 0$ for any tangent vectors x, y . Hence, $\text{Ric}^{LC}(\theta^\sharp, \theta^\sharp) = 0$ and so

$$\rho^W(\theta^\sharp, J\theta^\sharp) = 0.$$

Now assume, without loss of generality, that $|\theta| = 1$ and take a orthonormal basis $\theta, J\theta, \xi, J\xi$. Then, we have

$$\omega = \xi \wedge J\xi + \theta \wedge J\theta, \quad dJ\theta = \xi \wedge J\xi,$$

and

$$\rho^W = h\xi \wedge J\xi + k\theta \wedge J\theta$$

for some smooth functions h, k . Since, $\rho^W(\theta^\sharp, J\theta^\sharp) = 0$, then $k = 0$ and so

$$\rho^W = h\xi \wedge J\xi = h dJ\theta.$$

Now, notice that the scalar curvature s of g is constant if and only if the same holds for the trace b of ρ^B . This follows by [2, Formula (2.12)],

$$b = s - 2|\theta|^2 + \frac{1}{2}|d\omega|^2$$

having $|d\omega|^2 = |\theta|^2$ constant for a Vaisman metric on a complex surface. Therefore, b is constant if and only if the same holds for the function h , indeed

$$b = \rho^B(J\xi^\sharp, \xi^\sharp) + \rho^B(J(J\xi^\sharp), J\xi^\sharp) + \rho^B(J\theta^\sharp, \theta^\sharp) + \rho^B(J(J\theta^\sharp), J\theta^\sharp) = -2h,$$

where we have used that $\rho^B = \rho^W = h\xi \wedge J\xi$. □

5.2 Pluriclosed Flow on almost abelian Lie groups

In [5] it is shown that the pluriclosed flow on almost abelian Lie algebras reduces to

$$\begin{cases} a' := ca \\ v' := cv + Sv - \frac{1}{2}|v|^2v ; \\ A' := cA \end{cases}$$

where

$$c = \left(\frac{k}{4} - \frac{1}{2}\right)a^2 - \frac{1}{2}|v|^2 \in \mathbb{R}, \quad 2k = \text{rk}(A + A')$$

and

$$S = \left(\frac{k}{4} - \frac{1}{2}\right)a^2\text{Id}_{n_1} - \frac{1}{2}AA^t + \frac{a}{4}(A + A').$$

We prove the following

Theorem 5.3 *Let G be a simply-connected almost abelian Lie group. If $A_0 \in \mathfrak{so}(n_1)$, then the pluriclosed flow preserves the Bismut Kähler-like condition.*

Proof Notice that if $A_0 \in \mathfrak{so}(n_1)$ then by uniqueness of the solution $A(t) \in \mathfrak{so}(n_1)$. Hence, if we assume $A_0 \in \mathfrak{so}(n_1)$ one gets

$$c = -\frac{1}{2}a^2 - \frac{1}{2}|v|^2, \quad k = 0$$

and

$$S = -\frac{1}{2}a^2\text{Id}_{n_1} + \frac{1}{2}A^2.$$

We show that if $A_0 \in \mathfrak{so}(n_1)$ and at $t = 0$ the metric is Bismut Kähler-like then it remains Bismut Kähler-like along the flow. Indeed, if $A_0 \in \mathfrak{so}(n_1)$ and at $t = 0$ the metric is Bismut Kähler-like one has that A_0v_0 is orthogonal to n_1 and by [5] n_1 is preserved along the flow. We compute, for any $y \in n_1$

$$\begin{aligned} \frac{d}{dt}(g(v, Ay)) &= g(v', Ay) + g(v, A'y) = g(cv + Sv - \frac{1}{2}|v|^2v, Ay) + cg(v, Ay) \\ &= 2cg(v, Ay) - \frac{1}{2}a^2g(v, Ay) + \frac{1}{2}g(A^2v, Ay) - \frac{1}{2}|v|^2g(v, Ay) \\ &= 3cg(v, Ay) + \frac{1}{2}g(A^2v, Ay) = g\left(v, A(3c\text{Id}_{n_1} + \frac{1}{2}A^2)y\right), \end{aligned}$$

hence

$$\frac{d}{dt}(g(Av, y)) = g\left(Av, (3c\text{Id}_{n_1} + \frac{1}{2}A^2)y\right).$$

Then, if A_0v_0 is orthogonal to n_1 by uniqueness of the solution it remains orthogonal along the flow and so the Bismut Kähler-like condition is preserved. □

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