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On Lisbon integrals

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Abstract

We introduce new complex analytic integral transforms, the Lisbon Integrals, which naturally arise in the study of the affine space $\mathbb{C}^k$ of unitary polynomials $P_s(z)$ where $s \in \mathbb{C}^k$ and $z \in \mathbb{C}$, s_i identified to the i-th symmetric function of the roots of $P_s(z)$. We completely determine the $\mathscr{D}$-modules (or systems of partial differential equations) the Lisbon Integrals satisfy and prove that they are their unique global solutions. If we specify a holomorphic function f in the z-variable, our construction induces an integral transform which associates a regular holonomic module quotient of the sub-holonomic module we computed. We illustrate this correspondence in the case of a 1-parameter family of exponentials $f_t(z) = exp(tz)$ with t a complex parameter.

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Contents

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1 Introduction

The main purpose of this paper is to understand the behaviour of functions obtained by integration of $\mathscr{C}^\infty$-forms on the fibers of a holomorphic proper fibration. This has been investigated by the first author in two extreme cases: when the basis is 1-dimensional (see [2]) and when the fibers are finite (see [3]), but also by many authors in more general settings (see for instance [4,12,13]).

In the present paper we look at a simple but very interesting case where the fibers are the roots of the universal monic equation of degree k. The general result proved in [3] says that the singularity of these functions are controlled by regular holonomic $\mathscr{D}$-modules.

Our purpose it to give a precise answer in this special context. For instance, giving two entire functions f and g in $\mathscr{O}(\mathbb{C})$ we want to compute the regular holonomic system whose solutions are the k-uples of continuous functions on $\mathbb{C}^k$ given by

$$\Psi_p(s_1, \dots, s_k) := \sum_{P_s(z_j)=0} {z_j}^p f(z_j)\bar{g}(z_j) \quad p \in [0, k-1]$$

where $P_s(z) = \sum_{h=0}^k (-1)^h s_h z^{k-h}$ is the universal monic polynomial of degree k.

As we are interested only in holomorphic derivatives in s, the function g is irrelevant for the $\mathscr{D}$-module we are interested in, but, more surprisingly, there exists a sub-holonomic $\mathscr{D}$-module of which all these k-uples of (continuous functions) are solutions (in the sense of distributions).

We determine precisely this $\mathscr{D}$-module, via formula (@@), for which it is useful to consider (in the simplest form, without g) the complex integral representation (4) of Ψ_P.

In order to make this computation, a main step is to introduce the trace of differential forms $f(z)ds_1\wedge\dots\wedge ds_{k-1}\wedge dz$ corresponding to the natural holomorphic volume forms on $H := \{(s,z) \in \mathbb{C}^k \times \mathbb{C} \ / \ P_s(z) = 0\}$ identified to $\mathbb{C}^k$ via the map $(s,z) \mapsto (s_1, \dots, s_{k-1}, z)$. These holomorphic traces[1] have a very simple integral representation via the "Lisbon integrals" (see integral representation (3) below).

Here we explicit the $\mathscr{D}$ -modules of which (3) are (the unique) solutions via formula (@) showing that they derive from a very simple one by the usual functorial operations on $\mathscr{D}$-modules (inverse image and direct image) as follows:

Note that the hypersurface H is also defined by the equation

$$s_k = (-1)^{k-1} \sum_{h=1}^{k} (-1)^h s_h z^{k-h}.$$

Let $j : H \subset \mathbb{C}^{k+1}$ denote the closed embedding and let $B_{H|\mathbb{C}^{k+1}}$ denote the regular holonomic $\mathscr{D}_{\mathbb{C}^{k+1}}$-module of holomorphic distributions supported by H. Since the restriction of the projection

$$\pi : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k, \ (s,z) \mapsto s$$

[1] despite the "denominators" in the formula

$$Trace(z^p f(z)ds_1 \wedge \dots \wedge ds_{k-1} \wedge dz) = \left(\sum_{P_s(z_j)=0} \frac{{z_j}^p f(z_j)}{P'_s(z_j)} \right) ds_1 \wedge \dots \wedge ds_k$$

theses forms have no singularity on the discriminant hypersurface $\{\Delta(s) = 0\}$ in $\mathbb{C}^k$.

to H is proper (with finite fibers), given a coherent $\mathscr{D}_{\mathbb{C}^{k+1}}$-module $\mathscr{L}$ non characteristic for H, following [11] we obtain a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{\mathbb{C}^k})$, the integral transformed of $\mathscr{L}$, given by the composition of the usual derived functors of direct image and inverse image for $\mathscr{D}$-modules,

$$D\pi_*(B_{H|\mathbb{C}^{k+1}} \overset{L}{\otimes}_{\mathscr{O}_{\mathbb{C}^{k+1}}} \mathscr{L}) \simeq D\pi_*(\mathscr{H}^1_{[H]}(\mathscr{L})) \simeq (\pi|_H)_* j^*\mathscr{L}$$

Note that $B_{H|\mathbb{C}^{k+1}} = \mathscr{H}^1_{[H]}(\mathscr{O}_{\mathbb{C}^{k+1}})$ and that for such a module $\mathscr{L}$, we have, thanks to Kashiwara's equivalence theorem (cf Theorem 4.1 and Proposition 4.2, [7]), $\mathscr{H}^1_{[H]}(\mathscr{L}) \simeq \mathscr{H}^0 Dj_* Dj^*\mathscr{L} \simeq j_*j^*\mathscr{L}$.

We show that the $\mathscr{D}_{\mathbb{C}^k}$-module determined by all vector functions Φ_f given by the integral transform (3) (f varying in the space of holomorphic functions in the z-variable) is obtained as an integral transform in the sense of Kashiwara and Schapira [11] of a coherent $\mathscr{D}_{\mathbb{C}^{k+1}}$-module $\mathscr{L}$.

In this note $\mathscr{L}$ is the quotient of $\mathscr{D}_{\mathbb{C}^{k+1}}$ by the ideal generated by the partial derivatives in s_i, $i = 1, \dots, k$, hence the sheaf of solutions of $\mathscr{L}$ is $p^{-1}\mathscr{O}_{\mathbb{C}}$ where $p(s, z) = z$.

To simplify we shall keep the notation π also for the restriction $\pi|_H$ of π to H.

As a consequence, we show that Lisbon Integrals (3) are exactly the global solutions of $\pi_* j^*\mathscr{L}$.

Moreover, once an entire function f is fixed, we can consider the regular holonomic $\mathscr{D}_{\mathbb{C}^{k+1}}$-module (denoted by $\mathscr{L}_f$) it defines:

$$\mathscr{L}_f = \mathscr{D}_{\mathbb{C}^{k+1}}/\mathscr{J}$$

where $\mathscr{J}$ is the coherent ideal of $\mathscr{D}_{\mathbb{C}^{k+1}}$ of operators P such that $Pf = 0$; hence, according to [9, Theorem 8.1], $\pi_* j^*\mathscr{L}_f$ is regular holonomic. We explicit this module in the case of the family $f_t(z) = e^{tz}$ where t is a complex parameter.

Since integrals (3) *and* (4) *are strongly related as explained below, for the sake of simplicity we call both Lisbon Integrals.*

We also prove that Lisbon integrals (4) are global solutions of another $\mathscr{D}_{\mathbb{C}^k}$-module $\widetilde{\mathscr{N}}$ which shares with the first this very simple relation:

Let $A(s)$ be the (k, k)-matrix companion of the unitary polynomial $P_s(z)$. If Φ is a solution of $\pi_* j^*\mathscr{L}$ then $\Psi := P'_s(A(s))\Phi$, where P'_s denotes the partial derivative of P_s with respect to z, is a solution of $\widetilde{\mathscr{N}}$. Furthermore, this correspondence $\Phi \leftrightarrow \Psi$ is a bijection when restricting to the complementary of the discriminant hypersurface $\{\Delta(s) = 0\}$.

Important features of the scalar components of Lisbon integrals are the following:

- They are common solutions of a particular sub-holonomic system. This aspect will be developed in another paper by the first author. Here we compute only the simplest case $k = 2$.
- Each entire function f determines a solution of $\widetilde{\mathscr{N}}$ which scalar component of order h is the trace with respect to π in the holomorphic sense of the function $f(z)z^h$ on H.

Last but not the least, these computations illustrate the fact that it is not so easy, even in a rather simple situation, to follow explicitly the computations hidden in the "yoga" of $\mathscr{D}$-module theory.

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2 Lisbon integrals and the differential system they satisfy

2.1 Lisbon integrals

For $(z_1, \dots, z_k) \in \mathbb{C}^k$ denote $s_1, \dots, s_k$ the elementary symmetric functions of $z_1, \dots, z_k$. Let $\mathfrak{S}_k$ denote the k-symmetric group, that is, the group of bijections of $\{1, \dots, k\}$. We shall consider in the sequel $s_1, \dots, s_k$ as coordinates on $\mathbb{C}^k \simeq \mathbb{C}^k \big/ \mathfrak{S}_k$, isomorphism given by the standard symmetric function theorem.

We shall denote $P_s(z) := \prod_{j=1}^k (z - z_j) = \sum_{h=0}^k (-1)^h s_h z^{k-h}$ with the convention $s_0 \equiv 1$.

We shall often write $P(s, z)$ instead of $P_s(z)$ with no risk of ambiguity.

The discriminant $\Delta(s)$ of P_s is the polynomial in s corresponding to the symmetric polynomial $\prod_{1 \le i < j \le k} (z_i - z_j)^2$ via the symmetric function theorem.

Lemma 2.1 *For $h \in \mathbb{N}$ and $f \in \mathscr{O}(\mathbb{C})$ any entire holomorphic function, let us define, for $R \gg ||s||$,*

$$\varphi_h(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)\zeta^h d\zeta}{P_s(\zeta)}. \tag{1}$$

Then $\varphi_h(s)$ is independent of the choice of R large enough and defines a holomorphic function on $\mathbb{C}^k$. For $\Delta(s) \neq 0$ we have

$$\varphi_h(s) = \sum_{j=1}^{k} \frac{z_j^h f(z_j)}{P_s'(z_j)} \tag{2}$$

where $z_1, \dots, z_k$ are the roots of $P_s(z)$.

Proof The independence on R large enough when s stays in a compact set of $\mathbb{C}^n$ is clear. For s in the interior of a compact set, $P_s(\zeta)$ does not vanish on $\{|\zeta| = R\}$ for R large enough, so we obtain the holomorphy of φ_h near any point in $\mathbb{C}^k$. The formula (2) is given by a direct application of the Residue's theorem. □

In fact, it will be convenient to consider the k functions $\varphi_0, \dots, \varphi_{k-1}$ as the component of a vector valued function $\Phi := \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \dots \\ \varphi_{k-1} \end{pmatrix}$. Defining $E(z) := \begin{pmatrix} 1 \\ z \\ \dots \\ z^{k-1} \end{pmatrix}$ we obtain

$$\Phi(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}. \tag{3}$$

Definition 2.2 We call Φ (sometimes also denoted by Φ_f when precision is required) the *Lisbon Integral associated to f*. The scalar components of Φ, denoted by $\varphi_h, h = 0, \dots, k-1$, are called the *scalar Lisbon Integrals*. One also denote by φ_h the functions constructed by the same formula, with $h \in \mathbb{N}$, still denominated by "scalar Lisbon Integrals".

It will be also interesting to introduce another type of integrals, still called Lisbon Integrals for the sake of simplicity:

$$\Psi(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta)P_s'(\zeta)d\zeta}{P_s(\zeta)} \tag{4}$$

Ψ will also be noted below by Ψ_f when precision is required.

It is easy to see that this is again a vector valued holomorphic function on $\mathbb{C}^k$ and the Residue's theorem entails that, for $\Delta(s) \neq 0$, the component ψ_h of Ψ is given by:

$$\psi_h(s) = \sum_{j=1}^{k} z_j^h f(z_j). \tag{5}$$

Proposition 2.3 *If f is not identically zero then Φ and Ψ are non zero vector-valued holomorphic functions on $\mathbb{C}^k$.*

Proof Suppose that f is non identically zero. Then the statement follows as an immediate consequence of the non vanishing of the Van der Monde determinant of $z_1, \ldots, z_k$ when these complex numbers are pairwise distinct. □

AN EXAMPLE Take $f \equiv 1$. Then formula (5) shows that $\psi_h(s)$ is the h-th Newton symmetric functions of the roots of the polynomial P_s. So it is a quasi-homogeneous polynomial in s of weight h (the weight of s_j is j by definition).

Let us show that we have $\varphi_h(s) \equiv 0$ for $h \in [0, k-2]$ and $\varphi_{k-1}(s) \equiv 1$ in this case. For $h \in [0, k-2]$ the formula (1) gives the estimate (with $f \equiv 1$)

$$|\varphi_h| \leq \frac{R^{h+1}}{(R-a)^{k-1}}$$

if each root of P_s is in the disc $\{|z| \leq a\}$ when $R > a > 0$. When $R \to +\infty$ this gives $\varphi_h(s) \equiv 0$ for $h \in [0, k-2]$. For $h = k-1$ write

$$kz^{k-1} = P_s'(z) - \sum_{h=1}^{k-1} (-1)^h (k-h) s_h z^{k-h-1}.$$

This gives, using the previous case and formula (2), that $\varphi_{k-1}(s) \equiv 1$.

2.2 The partial differential system

Let us introduce the (k, k) matrix A (the companion matrix) associated to the polynomial P_s:

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & \\ 0 & & \cdots & & 0 \\ & & & & \\ 0 & & \cdots & 0 & 1 \\ (-1)^{k-1}s_k & \cdots & (-1)^{h-1}s_h & \cdots \cdots & s_1 \end{pmatrix} \tag{6}$$

Theorem 2.b. 1 *The vector valued holomorphic function Φ on $\mathbb{C}^k$ satisfies the following differential system*

$$(-1)^{k+h}\frac{\partial \Phi}{\partial s_h}(s) = \frac{\partial (A^{k-h}\Phi)}{\partial s_k}(s) \qquad \forall s \in \mathbb{C}^k \quad \text{and} \quad \forall h \in [1, k-1]. \tag{@}$$

Moreover, this system is integrable[2] *and if Φ is a solution of this system, so is $A\Phi$.*

[2] We shall explain in the proof what we mean here.

The proof of this result will use several lemmas.

Lemma 2.4 *Let A be a (k, k) matrix with entries in $\mathbb{C}[x]$ and put $B := \lambda \frac{\partial A}{\partial x}$ where λ is a complex number. Let M be the $(2k, 2k)$ matrix given by*

$$M := \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

Then for each $p \in \mathbb{N}$ we have

$$M^p = \begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \tag{a}$$

where $B_p := \lambda \frac{\partial(A^p)}{\partial x}$.

Proof As the relation (a) is clear for $p = 0, 1$ let us assume that it has been proved for p and let us prove it for $p + 1$. We have:

$$\begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} A^{p+1} & A^p B + B_p A \\ 0 & A^{p+1} \end{pmatrix}$$

which allows to conclude. □

Corollary 2.b.2 *For each integer $p \in [0, k-1]$ the following equality holds in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]\big/(P^2)$ over the $\mathbb{C}$-algebra $\mathbb{C}[s_1, \dots, s_k, z]\big/(P^2)$*

$$z^p E(z) = A^p E(z) + (-1)^{k-1} P_s(z) \frac{\partial(A^p)}{\partial s_k} E(z) \tag{7}$$

In particular, for any entire function f (of the variable z), we have

$$\Phi_{zf} = A(s)\Phi_f$$

Moreover the following identity in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]\big/(P^2)$ holds

$$P_s'(z)E(z) = P_s'(A)E(z) + (-1)^{k-1} P_s(z) \frac{\partial(P_s'(A))}{\partial s_k} E(z). \tag{8}$$

Proof In the basis $1, z, \dots, z^{k-1}, P_s(z), zP_s(z), \dots, z^{k-1}P_s(z)$ of this algebra which is a free rank $2k$ module on $\mathbb{C}[s_1, \dots, s_k]$, the multiplication by z is given by the matrix M of the previous lemma with A as in (6) and with $B := (-1)^{k-1} \frac{\partial A}{\partial s_k}$. This proves equality (7).

As $P_s'(z) = \sum_{h=0}^{k-1} (-1)^h (k-h) z^{k-h-1}$ does not depend on s_k it is enough to sum up the previous equalities with $p = k - h - 1$ with the convenient coefficients to obtain the equality (8). □

Lemma 2.5 *For any $h \in [1, k]$ and any $p \in \mathbb{N}$ the matrix A in (6) satisfies the relation:*

$$(-1)^{k-h} \frac{\partial A^p}{\partial s_h} = \frac{\partial A^p}{\partial s_k} A^{k-h} \tag{9}$$

Proof The case $p = 1$ of (9) is an easy direct computation on the matrix A. Assume that the assertion is proved for $p \geq 1$. Then Leibnitz's rule gives:

$$\begin{aligned}(-1)^{k-h} \frac{\partial A^{p+1}}{\partial s_h} &= (-1)^{k-h} \frac{\partial A^p}{\partial s_h} A + A^p (-1)^{k-h} \frac{\partial A}{\partial s_h} \\ &= \frac{\partial A^p}{\partial s_k} A^{k-h+1} + A^p \frac{\partial A}{\partial s_k} A^{k-h} = \frac{\partial A^{p+1}}{\partial s_k} A^{k-h}\end{aligned}$$

concluding the proof of (9). □

Proof of the Theorem 2.b. 1 By derivation inside the integral in (3) we obtain:

$$\frac{\partial \Phi}{\partial s_h}(s) = \frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)E(\zeta)(-1)^{h+1}\zeta^{k-h}\frac{d\zeta}{P_s(\zeta)^2} \quad \text{and in particular}$$

$$\frac{\partial \Phi}{\partial s_k}(s) = \frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)E(\zeta)(-1)^{k+1}\frac{d\zeta}{P_s(\zeta)^2}$$

Now for $h \in [1, k-1]$ we use the formula of corollary 2.b.2 to obtain:

$$\frac{\partial \Phi}{\partial s_h} = (-1)^{h+1}A^{k-h}(-1)^{k-1}\frac{\partial \Phi}{\partial s_k} + (-1)^{h+1}(-1)^{k-1}\frac{\partial A^{k-h}}{\partial s_k}\Phi$$

that is to say, we obtain (@) as desired:

$$(-1)^{k+h}\frac{\partial \Phi}{\partial s_h} = \frac{\partial(A^{k-h}\Phi)}{\partial s_k} \qquad \forall h \in [1, k]$$

By the integrability of the system (@) we mean that for any Φ such that (@) holds, then the computation of the partial derivatives $\frac{\partial^2\Phi}{\partial s_h \partial s_j}$ using the system (@) gives a symmetric result in (h, j) for any pair (h, j) in $[1, k]$. Note that if h or j is equal to k the assertion is trivial.

So consider a couple $(h, j) \in [1, k-1]^2$. Thanks to Lemma 2.5 we have :

$$\begin{aligned}
(-1)^{h+j}\frac{\partial^2\Phi}{\partial s_j \partial s_h} &= (-1)^{k-j}\frac{\partial}{\partial s_k}\left[\frac{\partial(A^{k-h}\Phi)}{\partial s_j}\right] \\
&= (-1)^{k-j}\frac{\partial}{\partial s_k}\left[\frac{\partial A^{k-h}}{\partial s_j}\Phi + A^{k-h}\frac{\partial \Phi}{\partial s_j}\right] \\
&= \frac{\partial}{\partial s_k}\left[\frac{\partial A^{k-h}}{\partial s_k}A^{k-j}\Phi + A^{k-h}\frac{\partial(A^{k-j}\Phi)}{\partial s_k}\right] \\
&= \frac{\partial^2}{\partial s_k^2}\left[A^{2k-h-j}\Phi\right]
\end{aligned}$$

which is symmetric in (h, j).

To finish the proof of the theorem we have to show that $A\Phi$ is a solution of (@) when Φ is a solution of (@). This is given by the following computation

$$\begin{aligned}
(-1)^{k-h}\frac{\partial(A\Phi)}{\partial s_h} &= (-1)^{k-h}\frac{\partial A}{\partial s_h}\Phi + (-1)^{k-h}A\frac{\partial \Phi}{\partial s_h} \\
&= \frac{\partial A}{\partial s_k}.A^{k-h}\Phi + A\frac{\partial A^{k-h}\Phi}{\partial s_k} = \frac{\partial(A^{k-h}(A\Phi))}{\partial s_k}
\end{aligned}$$

which also uses Lemma 2.5. □

Remark 2.6 A consequence of our computation on the integrability of the system (@) is the fact that for any solution Φ and any pair $(h, j) \in [1, k]$ the second order partial derivative $\frac{\partial^2\Phi}{\partial s_j \partial s_h}$ only depends on $h + j$. This implies that any scalar Lisbon integral φ_h satisfies

$$\frac{\partial^2\varphi_h}{\partial s_p \partial s_{q+1}} = \frac{\partial^2\varphi_h}{\partial s_{p+1}\partial s_q} \qquad \forall\ p, q \text{ such that } 1 \le p < q \le k-1 \tag{10}$$

Let us denote by Δ the discriminant hypersurface $\Delta = \{\Delta(s) = 0\}$. An easy calculation shows that away of Δ the matrix $P_s'(A(s))$ is invertible.

The next corollary of theorem 2.b. 1 gives an analogous system to (@) for the vector function Ψ defined in (4) which is singular along Δ.

Corollary 2.b.3 (1) *The vector valued holomorphic function Ψ on $\mathbb{C}^k$ satisfies the following differential system:*

$$(-1)^{k+h}\frac{\partial\Psi}{\partial s_h}(s) = \frac{\partial(A^{k-h}\Psi)}{\partial s_k}(s) + (-1)^k(k-h)A^{k-h-1}P'_s(A)^{-1}\Psi(s) \qquad (@@)$$

$\forall h \in [1, k-1]$, *which is singular along the discriminant hyperdurface*

$$\Delta := \{s \in \mathbb{C}^k;\ \Delta(s) = 0\}.$$

(2) *If Ψ is any solution of (@@) then $A\Psi$ is also a solution of (@@).*
(3) *If Φ is any solution of (@) then $\Psi = P'_s(A)\Phi$ is a solution of (@@).*
(4) *If Ψ is any solution of (@@) on $\mathbb{C}^k \setminus \Delta$ then $\Phi := P'_s(A(s))^{-1}\Psi$ is a solution of (@) on $\mathbb{C}^k \setminus \Delta$.*

Statement (1) follows by (8) in Corollary 2.b.2.

Statement (3) follows by direct computation:

For such a $\Psi = P'_s(A)\Phi$, for each $h \in [1, k-1]$

$$(-1)^{k-h}\frac{\partial\Psi}{\partial s_h}(s) = (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \sum_{p=0}^{k-1}(-1)^p(k-p)(-1)^{k-2h}s_p\frac{\partial(A^{k-p-1}\Phi)}{\partial s_h}(s)$$

and using now the fact that $A^{k-p-1}\Phi$ is solution of (@) we obtain

$$\begin{aligned}(-1)^{k-h}\frac{\partial\Psi}{\partial s_h}(s) &= (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \sum_{p=0}^{k-1}(-1)^p(k-p)s_p\frac{\partial(A^{2k-p-h-1}\Phi)}{\partial s_k}(s)\\ &= (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \frac{\partial(A^{k-h}P'_s(A)\Phi)}{\partial s_k}(s)\\ &= (-1)^{k-2h}(k-h)A^{k-h-1}P'_s(A)^{-1}\Psi(s) + \frac{\partial(A^{k-h}\Psi)}{\partial s_k}(s)\end{aligned}$$

which gives the formula (@@).

Since $P'_s(A)$ commutes with A, the assertion (2) is easy. □

The proof of (4) is a simple consequence of (9) in Lemma 2.5.

2.3 Example: the case $k = 2$

In this example we will explicit the system (@) and also a partial differential operators in the Weyl algebra $\mathbb{C}[s_1, s_2]\langle\partial_{s_1}, \partial_{s_2}\rangle$, which annihilates the scalar components of its solutions. The left ideals in $\mathscr{D}_{\mathbb{C}^k}$ of differential operators annihilating respectively the scalar components of the solutions of (@) and of (@@) for arbitrary k are described in [1].

Here we use the notations $s := s_1$ and $p := s_2$. In that case, with $\Phi = (\varphi_0, \varphi_1)$, the differential system (@) becomes:

$$\frac{\partial\varphi_0}{\partial s} = \frac{\partial\varphi_1}{\partial p} \qquad (11)$$

$$\frac{\partial \varphi_1}{\partial s} = -\varphi_0 - p\frac{\partial \varphi_0}{\partial p} + s\frac{\partial \varphi_1}{\partial p} \tag{12}$$

corresponding to the matrix $A := \begin{pmatrix} 0 & 1 \\ -p & s \end{pmatrix}$. Differentiating (12) with respect to p we obtain, after substituting via (11)

$$\frac{\partial^2 \varphi_0}{\partial s^2} = -2\frac{\partial \varphi_0}{\partial p} - p\frac{\partial^2 \varphi_0}{\partial p^2} - s\frac{\partial^2 \varphi_0}{\partial s \partial p} \tag{$\sharp$}$$

$$\text{and so} \quad \frac{\partial^2 \varphi_0}{\partial s^2} + s\frac{\partial^2 \varphi_0}{\partial s \partial p} + p\frac{\partial^2 \varphi_0}{\partial p^2} + 2\frac{\partial \varphi_0}{\partial p} = 0$$

Differentiating (12) with respect to s we obtain, after substituting via (11)

$$-\frac{\partial^2 \varphi_1}{\partial s^2} = \frac{\partial \varphi_1}{\partial p} + p\frac{\partial^2 \varphi_1}{\partial p^2} + \frac{\partial \varphi_1}{\partial p} + s\frac{\partial^2 \varphi_1}{\partial s \partial p} \tag{$\sharp\sharp$}$$

$$\text{and so} \quad \frac{\partial^2 \varphi_1}{\partial s^2} + s\frac{\partial^2 \varphi_1}{\partial s \partial p} + p\frac{\partial^2 \varphi_1}{\partial p^2} + 2\frac{\partial \varphi_1}{\partial p} = 0$$

Then the second order differential operator of weight -2

$$\Theta := \frac{\partial^2}{\partial s^2} + s\frac{\partial^2}{\partial s \partial p} + p\frac{\partial^2}{\partial p^2} + 2\frac{\partial}{\partial p} \tag{$\sharp\sharp\sharp$}$$

anihilates φ_0 and φ_1 for any solution Φ of the system ((11), (12)).

A DIRECT PROOF THAT Θ ANIHILATES SCALAR LISBON INTEGRALS FOR ALL $m \in \mathbb{N}$.

We have, for any entire function $f : \mathbb{C} \to \mathbb{C}$ and for $R \gg \max\{|s|, |p|\}$:

$$\varphi_m(s, p) = \frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^m d\zeta}{\zeta^2 - s\zeta + p} \tag{a}$$

This gives:

$$\frac{\partial \varphi_m}{\partial s}(s, p) = \frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^2} \tag{b}$$

$$\frac{\partial \varphi_m}{\partial p}(s, p) = -\frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{m} d\zeta}{(\zeta^2 - s\zeta + p)^2} \tag{c}$$

$$\frac{\partial^2 \varphi_m}{\partial s^2}(s, p) = 2\frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{m+2} d\zeta}{(\zeta^2 - s\zeta + p)^3} \tag{d}$$

$$\frac{\partial^2 \varphi_m}{\partial s \partial p}(s, p) = -2\frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^3} \tag{e}$$

$$\frac{\partial^2 \varphi_m}{\partial p^2}(s, p) = 2\frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{m} d\zeta}{(\zeta^2 - s\zeta + p)^3} \tag{f}$$

Now it is easy to check that $(d) + s(e) + p(f) + 2(c) = 0$.

3 The left action of $\Gamma(\mathbb{C}, \mathscr{D}_{\mathbb{C}})$ on Lisbon integrals

For each entire function f of the variable z we shall henceforward denote by Φ_f the associated Lisbon integral (previously generically denoted by Φ). Clearly the assignement

$$f \mapsto \Phi_f$$

is $\mathbb{C}$-linear and, according to Proposition 2.3, it is injective.

Lemma 3.1 *Let f be an entire function on $\mathbb{C}$ and let Φ_f the corresponding Lisbon integral.*

(1) *Let g be an entire function of z. Then $\Phi_{gf} = g(A(s))\Phi_f$. In particular $\Phi_f = f(A(s))\Phi_1$.*
(2) *We have the identity*

$$\Phi_{\partial_z(f)}(s) = -\nabla\Phi_f(s) + \left(\sum_{h=0}^{k-1}(k-h)s_h\partial_{s_{h+1}}\right)(\Phi_f)(s)$$

where ∇ is the constant (k, k) matrix $\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & & \cdots & 0 \\ 0 & \cdots & 0 & k-1 & 0 \end{pmatrix}$.

Proof When g is a polynomial on z, statement (1) follows easily in Corollary 2.b.2. For an arbitrary entire function g, it is a consequence of [5, Lem 3.1.8].

Let us prove (2): Consider the Lisbon integral

$$\Phi_{\partial_z f}(s) = \frac{1}{2i\pi}\int_{|\zeta|=R} f'(\zeta)E(\zeta)\frac{d\zeta}{P_s(\zeta)}$$

Integration by parts gives, as $\partial_z(E)(z) = \nabla E(z)$:

$$\Phi_{\partial_z(f)}(s) = -\nabla\Phi_f(s) + \frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)E(\zeta)\frac{P_s'(\zeta)}{P_s(\zeta)^2}d\zeta \qquad (*)$$

Now, using the equalities $P_s'(\zeta) = \sum_{h=0}^{k-1}(-1)^h(k-h)s_h\zeta^{k-h-1}$ and

$$\frac{\partial\Phi_f}{\partial s_h}(s) = -\frac{1}{2i\pi}\int_{|\zeta|=R} f(\zeta)E(\zeta)\frac{(-1)^h\zeta^{k-h}d\zeta}{P_s(\zeta)^2}$$

we obtain the formula of the lemma. □

Remark 3.2 From formula (*) and according to (8) we obtain also the formula

$$\Phi_{\partial_z(f)}(s) = -\nabla\Phi_f(s) + (-1)^{k-1}\frac{\partial(P_s'(A)\Phi_f)}{\partial s_k}(s) \qquad (**)$$

Note that the formula $\Phi_{(z.f)'} = \Phi_f + \Phi_{zf'}$ corresponding to the usual relation $\partial_z z - z\partial_z = 1$ follows from the linearity of the map $f \mapsto \Phi_f$ and the Leibniz rule $(zf)' = f + zf'$.

It is not obvious that when Φ is solution of the system (@), then

$$(\Phi\partial_z)(s) := -\nabla\Phi(s) + (-1)^{k-1}\partial_{s_k}(P_s'(A)\Phi)(s)$$

(given by the formula (**)) is also a solution of the same system. A direct verification of this fact is consequence of the formula given in the lemma below.

Lemma 3.3 *We have the following identity:*

$$\nabla A^p - A^p\nabla + pA^{p-1} = (-1)^{k-1}\partial_{s_k}(A^p)P'_s(A), \quad \forall p \geq 1, \ \forall s \in \mathbb{C}^k.$$

Proof Let check the case $p = 1$ first. It is an easy computation to obtain that $\nabla A - A\nabla + Id$ is the matrix which have all lines equal to 0 excepted its last one which is given by $(x_1, \dots, x_k)$ with $x_h = (-1)^{k-h}hs_{k-h}$ for $h \in [1, k]$ with $s_0 \equiv 0$. On the other hand, the matrix $(-1)^{k-1}\partial_{s_k}A$ has only a non zero term at the place $(k, 1)$ which equal to 1, so it is quite easy to see that $(-1)^{k-1}(\partial_{s_k}A)A^p$ has only a non zero term at the place $(k, p+1)$ with value $(-1)^{k-1}$. According to the computation of $(-1)^{k-1}\partial_{s_k}(A)P'_s(A)$ we conclude the desired formula for $p = 1$

Now we shall make an induction on $p \geq 1$ to prove the general case. Then assume $p \geq 2$ and the formula proved for $p - 1$. Then write

$$\nabla A^p - A^p\nabla = (\nabla A^{p-1} - A^{p-1}\nabla)A + A^{p-1}(\nabla A - A\nabla).$$

Using the induction hypothesis and the case $p = 1$ gives

$$\begin{aligned}\nabla A^p - A^p\nabla &= -(p-1)A^{p-2}A - A^{p-1} + \\ &\quad + (-1)^{k-1}\partial_{s_k}(A^{p-1})P'_s(A)A + (-1)^{k-1}A^{p-1}\partial_{s_k}(A)P'_s(A) \\ &= -pA^{p-1} + (-1)^{k-1}(\partial_{s_k}(A^{p-1})A + A^{p-1}\partial_{s_k}(A))P'_s(A) \\ &= -pA^{p-1} + (-1)^{k-1}\partial_{s_k}(A^p)P'_s(A).\end{aligned}$$

Now we shall make the direct verification that Φ solution of (@) implies that

$$-\nabla\Phi + (-1)^{k-1}\partial_{s_k}(P'_s(A)\Phi)$$

is also solution of (@):

$$\begin{aligned}X &:= (-1)^{k+h}\partial_{s_h}(\nabla\Phi) - \partial_{s_k}(A^{k-h}\nabla\Phi) = \partial_{s_k}(\nabla A^{k-h}\Phi) - \partial_{s_k}(A^{k-h}\nabla\Phi) \\ &= \partial_{s_k}\big[-(k-h)A^{k-h-1}\Phi + (-1)^{k-1}\partial_{s_k}(A^{k-h})P'_s(A)\Phi\big]\end{aligned}$$

thanks to the previous lemma. Also, using the fact that $P'_s(A)\Phi$ is a simple linear combination of solutions of (@) (with non constant coefficients, but very simple), we obtain the formula:

$$(-1)^{k+h}\partial_{s_h}(P'_s(A)\Phi) = (-1)^k(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h}P'_s(A)\Phi).$$

Then:

$$\begin{aligned}Y &:= (-1)^{k+h}\partial_{s_h}(\partial_{s_k}(P'_s(A)\Phi)) - \partial_{s_k}(A^{k-h}\partial_{s_k}(P'_s(A)\Phi)) \\ &= \partial_{s_k}\big[(-1)^k(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h}P'_s(A))\Phi) - A^{k-h}\partial_{s_k}(P'_s(A)\Phi)\big] \\ &= \partial_{s_k}\big[(-1)^k(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h})P'_s(A)\Phi)\big]\end{aligned}$$

and $-X + (-1)^{k-1}Y = 0$, as desired. □

4 The $\mathscr{D}_{\mathbb{C}^k}$-module associated to Lisbon integrals

We shall begin by recalling some basic facts on $\mathscr{D}$-modules and by fixing notations.

For a morphism of manifolds $f : Y \to X$, we use the notation of [10]

$$f_d := {}^t f' : T^*X \times_X \times Y \longrightarrow T^*Y$$

and

$$f_\pi : T^*X \times_X \times Y \longrightarrow T^*X$$

the associated canonical morphisms of vector bundles.

We recall that a conic involutive submanifold V of the cotangent bundle T^*Z of a manifold Z (real or complex) is *regular* if the restriction $\omega|_V$ of the canonical 1-form ω on T^*Z never vanishes outside the 0-section. Recall also that if $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ are canonical symplectic coordinates on T^*Z, then $\omega(x, \xi) = \sum_{i=1}^n \xi_i dx_i$.

Let us fix some $k \in \mathbb{N}$, $k \geq 2$. In $\mathbb{C}^{k+1} = \mathbb{C}^k \times \mathbb{C}$ we consider the coordinates $(s_1, \dots, s_k, z)$. As in the previous sections we set

$$P(s, z) = z^k + \sum_{h=1}^{k-1} (-1)^h s_h z^{k-h}$$

Obviously

$$P(s_1, \dots, s_k, z) = 0 \Longleftrightarrow s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h},$$

where $s_0 = 1$. We note $s = (s_1, \dots, s_k)$ and $s' := (s_1, \dots, s_{k-1})$.

Let H be the smooth hypersurface of $\mathbb{C}^{k+1}$ given by the zeros of $P(s, z)$.

Let us denote by $\mathscr{L}$ the $\mathscr{D}_{\mathbb{C}^{k+1}}$-module with one generator u defined by the equations $\partial u/\partial s_1 = \cdots = \partial u/\partial s_k = 0$. Such module is an example of a so called *partial de Rham systems*, which have the feature, among others, that their characteristic varieties are non singular regular involutive. In our case we have

$$\text{Char}\, \mathscr{L} = \{(s, z); (\eta, \tau) \in \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \text{such that}\, \eta = 0\}.$$

Since $H \subset \mathbb{C}^{k+1}$ is defined by the equation

$$P(s, z) = (-1)^k s_k + \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h} = 0,$$

$T^*_H \mathbb{C}^{k+1}$ is the subbundle of $T^*\mathbb{C}^{k+1}|_H$ described by

$$\{(s, z; \eta, \tau), (s, z) \in H, \exists \lambda \in \mathbb{C} \quad \text{such that} \quad (\eta, \tau) = \lambda dP(s, z)\}.$$

This means that $P(s, z) = 0$ and that their exists $\lambda \in \mathbb{C}$ with $\eta_h = \lambda(-1)^h z^{k-h}$ for each $h \in [1, k]$ and that $\tau = \lambda P'(s, z)$. Hence as $\eta_k = \lambda(-1)^k$ and this implies:

$$\text{Char}\, \mathscr{L} \cap T^*_H \mathbb{C}^{k+1} \subset T^*_{\mathbb{C}^{k+1}} \mathbb{C}^{k+1}$$

(as usual, $T^*_{\mathbb{C}^{k+1}} \mathbb{C}^{k+1}$ denotes the zero section of $T^*\mathbb{C}^{k+1}$), in other words H is non characteristic for $\mathscr{L}$. Let us denote by $j : H \hookrightarrow \mathbb{C}^{k+1}$ the closed embedding. By Kashiwara's classical results (which can be found in [8]) it follows that the induced system $Dj^*\mathscr{L}$ by $\mathscr{L}$

on H is concentrated in degree zero and $\mathscr{N} := \mathscr{H}^0 Dj^*\mathscr{L}$ is a $\mathscr{D}_{\mathbb{C}_H}$-coherent module whose characteristic variety is exactly

$$j_d j_\pi^{-1} \operatorname{Char}(\mathscr{L}).$$

Recall that

$$Dj^*\mathscr{L} := \mathscr{O}_H \overset{L}{\otimes}_{j^{-1}\mathscr{O}_{\mathbb{C}^{k+1}}} j^{-1}\mathscr{L}$$

and in this non-characteristic case

$$\simeq j^{-1}\frac{\mathscr{O}_{\mathbb{C}^{k+1}}}{P(s', s_k, z)\mathscr{O}_{\mathbb{C}^{k+1}}} \otimes_{j^{-1}\mathscr{O}_{\mathbb{C}^{k+1}}} j^{-1}\mathscr{L}$$

We have

$$\mathscr{N} := j^{-1}(\frac{\mathscr{D}_{\mathbb{C}^{k+1}}}{P\mathscr{D}_{\mathbb{C}^{k+1}} + \mathscr{D}_{\mathbb{C}^{k+1}}\partial_{s_1} + \cdots + \mathscr{D}_{\mathbb{C}^{k+1}}\partial_{s_k}}) \simeq j^{-1}(\mathscr{O}_{\mathbb{C}^{k+1}}/P\mathscr{O}_{\mathbb{C}^{k+1}}) < \partial_z >$$

which is isomorphic as a $\mathscr{D}_H$-module to

$$\mathscr{O}_H < \partial_z > \simeq \frac{\mathscr{D}_H}{\mathscr{D}_H\partial_{s_1} + \cdots + \mathscr{D}_H\partial s_{k-1}}$$

where ∂_{s_i} stands for the derivation $\partial/\partial s_i$ on $\mathscr{O}_H$ and ∂_z as a derivation on $\mathscr{O}_H$ is the class of ∂_z in the quotient above.

In particular $\mathscr{N}$ is sub-holonomic and it is a partial de Rham system similarly to $\mathscr{L}$.

Let us now determine the image of $\mathscr{N}$ under the morphism $\pi : \mathbb{C}^k \simeq H \to \mathbb{C}^k$ given by $(s', z) \mapsto (s', s_k)$. Clearly π is proper surjective with finite fibers.

Recall that one denotes by $\mathscr{D}_{\mathbb{C}^k \leftarrow H}$ the transfer $(\pi^{-1}\mathscr{D}_{\mathbb{C}^k}, \mathscr{D}_H)$-bimodule

$$(\pi^{-1}\mathscr{D}_{\mathbb{C}^k} \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{C}^k}} \pi^{-1}\Omega_{\mathbb{C}^k}^{\otimes^{-1}}) \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{C}^k}} \Omega_H$$

Recall also that, according to the properness and the fiber finiteness of π, we have

$$D\pi_*\mathscr{N} \simeq \mathscr{H}^0 D\pi_*\mathscr{N} = \pi_*(\mathscr{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathscr{D}_H} \mathscr{N})$$

where we abusively use the notation π_* for the direct image functor in the categoy of $\mathscr{D}$-modules in the two left terms and for the direct image functor for sheaves in the right term. According to [8, Theorems 4.25 and 4.27] (see also the comments in loc.cit. before Theorem 4.27), one knows that $D\pi_*\mathscr{N}$ is concentrated in degree zero and that

$$\operatorname{Char} \mathscr{H}^0 D\pi_*\mathscr{N} = \pi_\pi \pi_d^{-1} \operatorname{Char} \mathscr{N}.$$

So we may henceforward denote for short $\pi_*\mathscr{N} := D\pi_*\mathscr{N}$ without ambiguity.

Let Δ as above be the zero set of the discriminant of P, which can also be defined as the image by π of the subset of H defined by $\{P'(s, z) = 0\}$.

Since π_d is given by the $k \times k$ matrix

$$\begin{pmatrix} Id & 0 \\ \partial s_k/\partial s' & \partial s_k/\partial z \end{pmatrix}^T$$

we conclude that $\operatorname{Char} \pi_*\mathscr{N}$ is the image by π_π of the set

$$\{(s', z); (\eta', \tau) \in \mathbb{C}^k \times \mathbb{C}^k \;/\; \eta_j = -(-z)^{k-j-1}\tau, \quad \forall j \in [1, k-1]\}$$

so it is given by the set

$$\{(s,\eta) \in \mathbb{C}^k \times \mathbb{C}^k \ / \ \exists z \in \mathbb{C} \text{ such that } P_s(z) = 0 \text{ and with } \eta_j = (-z)^{k-j-1}\eta_k \quad \forall j \in [1, k-1]\}$$

Then $\mathrm{Char}\, \pi_*\mathscr{N}$ is an involutif analytic subset of $T^*\mathbb{C}^k$ with codimension $k-1$ which proves the following:

Lemma 4.1 *$\pi_*\mathscr{N}$ is a subholonomic $\mathscr{D}_{\mathbb{C}^{k+1}}$-module.*

Remark 4.2 Let $\widetilde{\mathscr{N}}$ denote the $\mathscr{D}_{\mathbb{C}^k}$-module associated to (@@). Then $\widetilde{\mathscr{N}}$ is clearly not subholonomic.

Proposition 4.3 *The $\mathscr{D}_{\mathbb{C}^k}$-module $\pi_*\mathscr{N}$ is the quotient of $\mathscr{D}^k_{\mathbb{C}^k} \simeq \mathscr{D}_{\mathbb{C}^k} \otimes_{\mathbb{C}} \mathbb{C}^k$ by the action of*

$$\mathscr{A}_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1}\partial_{s_k} \otimes A(s)^{k-h} \quad \text{for } j \in [1, k-1].$$

Moreover the action of z and ∂_z on $\pi_*\mathscr{N}$ deduced from the action of $\mathscr{D}_H$ on $\mathscr{N}$[3] *are given respectively by*

$$\mathscr{A}_0 := 1 \otimes A(s) \quad \text{and} \quad \mathscr{B} := 1 \otimes \nabla + (-1)^{k-1}\partial_{s_k} \otimes P'_s(A(s))$$

where we put $P'_s(z) := (\partial_z(P_s(z))$ and $\nabla := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots \\ \cdots\cdots & & \cdots & \cdots \\ 0 & \cdots & k-1 & 0 \end{pmatrix}$

Proof Our goal is to explicit $\pi_*\mathscr{N}$ and to check that it coincides with the $\mathscr{D}_{\mathbb{C}^k}$-module associated to the system (@) in Theorem 2.b. 1.

In a first step we explicit the transfer-module

$$\mathscr{D}_{\mathbb{C}^k \leftarrow H} := \pi^{-1}\mathscr{D}_{\mathbb{C}^k} \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{C}^k}} (\pi^{-1}\Omega_{\mathbb{C}^k}^{\otimes^{-1}} \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{C}^k}} \Omega_H)$$

as a $(\pi^{-1}\mathscr{D}_{\mathbb{C}^k}, \mathscr{D}_H)$-bimodule. The next step is to determine the cokernel of

$$\alpha : (\mathscr{D}_{\mathbb{C}^k \leftarrow H})^{k-1} \longrightarrow \mathscr{D}_{\mathbb{C}^k \leftarrow H}$$
$$(u_1, \dots, u_{k-1}) \mapsto \sum_{i=1}^{k-1} u_i \partial_{s_i}$$

The last step is to apply π_*.

Let us denote for short

$$\sigma := \omega_H \otimes \omega_{\mathbb{C}^k}^{\otimes^{-1}} := ds_1 \wedge \cdots \wedge ds_{k-1} \wedge d_z \otimes (ds_1 \wedge \cdots \wedge ds_{k-1} \wedge ds_k)^{\otimes^{-1}}$$

the generator of the line bundle $\pi^{-1}\Omega_{\mathbb{C}^k}^{\otimes^{-1}} \otimes_{\pi^{-1}\mathscr{O}_{\mathbb{C}^k}} \Omega_H$.

Recall that $\mathscr{O}_H = \mathscr{O}_{\mathbb{C}^{k+1}}/\mathscr{J}$, where $\mathscr{J}$ is the ideal generated by $P(s,z)$. Hence $\mathscr{O}_H$ is a $\pi^{-1}\mathscr{O}_{\mathbb{C}^k}$-free module with rank k since each section $a(s', z)$ of $\mathscr{O}_H$ is equivalent, by Weierstrass Division Theorem, to a unique polynomial $\sum_{j=0}^{k-1} a_j(s', s_k)z^j$ modulo $P(s,z)$, for some sections a_j of $\mathscr{O}_{\mathbb{C}^k}$.

Hence $\mathscr{D}_{\mathbb{C}^k \leftarrow H}$ is a left $\pi^{-1}\mathscr{D}_{\mathbb{C}^k}$-free module of rank k generated by the k-sections

[3] Note that z and ∂_z commute with ∂_{s_h} for $h \in [1, k-1]$ in $\mathscr{D}_H$.

$(1 \otimes z^j \sigma)_{j=0,\dots,k-1}$. Since the right action of each operator in $\mathscr{D}_H$ is $\pi^{-1}\mathscr{D}_{\mathbb{C}^k}$-linear, it is sufficient to calculate each $(1 \otimes z^j \sigma)\partial_{s_i}$, $i = 1, \dots, k-1$, $j = 0, \dots, k-1$.

Now recall that H is defined in $\mathbb{C}^{k+1}$ by the equation $s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h}$ with the convention $s_0 = 1$ and so $s_1, \dots, s_{k-1}, z$ are global coordinates on H. Then we have in H

$$\frac{\partial s_k}{\partial s_h} = (-1)^{k-h-1} z^{k-h} \quad \text{and} \quad \frac{\partial s_k}{\partial z} = (-1)^{k-1} P_s'(z)$$

where $P_s'(z)$ does not depend on s_k.

Let $F := 1 \otimes E(z)\sigma$ denote the basis $(1 \otimes z^j \sigma)$, $j \in [0, k-1]$ of the free rank k left $\pi^{-1}(\mathscr{D}_{\mathbb{C}^k})$-module $\mathscr{D}_{\mathbb{C}^k \leftarrow H}$. Recall that, according to [8, Remark 4.18], in view of the generators described above, the action of $\mathscr{D}_H$ in $\mathscr{D}_{\mathbb{C}^k \leftarrow H}$ is defined by the following formulas, where we consider F as a k-vector and use the usual matrix product

$$F\theta(s') = \theta(s')F \quad \text{where} \quad \theta \in \mathscr{O}_H \quad \text{does not depend on } z \tag{0*}$$

$$Fz = A(s)F \tag{1*}$$

$$-F\partial_{s_h} = \partial_{s_h} F + (-1)^{k-h-1} \partial_{s_k} (A(s)^{k-h} F) \quad \forall h \in [1, k-1] \tag{2*}$$

$$-F\partial_z = \nabla F + (-1)^{k-1} \partial_{s_k} \big(P_s'(A(s))F\big) \tag{3*}$$

where we have used the equalities $zE(z) = A(s)E(z)$ and $\partial_z(E(z)) = \nabla E(z)$.

Summing up:

- For $g \in \mathscr{O}_H$ represented by $\sum_{r=0}^{k-1} g_r(s) z^r$, the $(k \times k)$ matrix G of the $\pi^{-1}\mathscr{D}_{\mathbb{C}^k}$-linear morphism defined by g on $\mathscr{D}_{\mathbb{C}^k \leftarrow H}$ is given by $G := \sum_{r=0}^{k-1} g_r(s) A(s)^r$
- Let us consider the $\mathscr{D}_{\mathbb{C}^k}$-linear morphism $\alpha : (\mathscr{D}^k_{\mathbb{C}^k})^{(k-1)} \to \mathscr{D}^k_{\mathbb{C}^k}$ defined by the following $k-1$ (k,k)-matrices

$$\mathscr{A}_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1} \partial_{s_k} \otimes A(s)^{k-h}$$

Let Φ be in $\mathscr{O}^k_{\mathbb{C}^k}$. In view of the relation (2*), the map $(1 \otimes z^h \sigma) \mapsto \Phi_h$, for $h \in [0, k-1]$ will induce an element of $\mathscr{H}om_{D_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$, that is to say a solution of $\pi_*\mathscr{N}$, if and only if we have $\partial_{s_h}(\Phi) = (-1)^{k-h} \partial_{s_k}(A^{k-h}\Phi)$, that is, in and only if Φ satisfies (@), since the generator of $\pi_*\mathscr{N}$ is anihilated by the action of ∂_{s_h} for each $h \in [1, k-1]$. In conclusion:

$$\pi_*\mathscr{N} \simeq \operatorname{coker} \alpha$$

by the finitness of the fibers of π.

Remark 4.4 $\mathscr{N}$ is naturally endowed with a structure of right $\Gamma(\mathbb{C}; \mathscr{D}_{\mathbb{C}})$-module. By functoriality $\pi_*\mathscr{N}$ is also a right $\Gamma(\mathbb{C}; \mathscr{D}_{\mathbb{C}})$-module and its structure coincides with the induced by the right $\mathscr{D}_{\mathbb{C}}$ action defined by (1*) and (3*) on $\mathscr{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathscr{D}_H} \mathscr{N}$, since it commutes with each ∂_{s_i}, for $i = 1, \dots, k-1$. Therefore we obtain a natural left action of $\Gamma(\mathbb{C}, \mathscr{D}_{\mathbb{C}})$ on $\mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$.

We also conclude, according to Lemma 3.1:

Proposition 4.5 *The left action of $\Gamma(\mathbb{C}; \mathscr{D}_{\mathbb{C}})$ defined by the above Remark 4.4 on $\mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$ coincides with the left action of $\Gamma(\mathbb{C}; \mathscr{D}_{\mathbb{C}})$ on Lisbon integrals.*

If Φ is a solution of $\pi_*\mathscr{N}$, replacing in the formula (3*) the second term thanks to the equality obtained for Φ after applying (2*) or equivalently (@), we also derive a right action of ∂_z which is given by the formula

$$-\Phi\partial_z = \nabla\Phi - \sum_{h=0}^{k-1}(k-h)s_h\partial_{s_{h+1}}\Phi$$

Our next goal is to conclude in Proposition 4.7 below that there are no global holomorphic solutions of $\pi_*\mathscr{N}$ other than those of the form Φ_f, for some holomorphic function f only depending on z. Since j is non-characteristic we have an isomorphism

$$j^{-1}R\mathscr{H}om_{\mathscr{D}_{\mathbb{C}^{k+1}}}(\mathscr{L}, \mathscr{O}_{\mathbb{C}^{k+1}}) \simeq R\mathscr{H}om_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H)$$

According to Theorem 4.33 (2) of [8], making $X = H, Y = \mathbb{C}^k, f = \pi, \mathscr{N} = \mathscr{O}_{\mathbb{C}^k}$ in loc.cit, we obtain

Theorem 4.6 *For any open subset Ω of $\mathbb{C}^k$ we have an isomorphism functorial on $\mathscr{N}$ compatible with restrictions to open subsets*

$$R\Gamma(\pi^{-1}(\Omega); R\mathscr{H}om_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H)) \simeq R\Gamma(\Omega; R\mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k}))$$

Recall that this isomorphism uses as a tool the "trace morphism": $\pi_*\mathscr{O}_H \to \mathscr{O}_{\mathbb{C}^k}$ constructed in [8, Proposition 4.34].

Since for any open subset Ω and any $\mathscr{D}_H$-module $\mathscr{P}$, $\Gamma(\Omega;\cdot)$ and $\mathscr{H}om_{\mathscr{D}_H}(\mathscr{P},\cdot)$ are left exact functors, since if Ω is a Stein open set and if $\mathscr{P}$ admits a global resolution by free $\mathscr{D}_H$-modules of finite rank, then $R\mathscr{H}om_{\mathscr{D}_H}(\mathscr{P}, \mathscr{O})$ is represented by a complex in degrees ≥ 0 with $\Gamma(\Omega,\cdot)$-acyclic entries, we conclude that

$$H^0(R\Gamma(H; R\mathscr{H}om_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H))) = \Gamma(H; \mathscr{H}om_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H)) \quad \text{and}$$
$$H^0(R\Gamma(\mathbb{C}^k; R\mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k}))) = \Gamma(\mathbb{C}^k; \mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k}))$$

therefore Theorem 4.6 entails a $\mathbb{C}$-linear isomorphism

$$T : \mathrm{Hom}_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H) \simeq \mathrm{Hom}_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$$

Proposition 4.7 *The correspondence*

$$f \mapsto \Phi(f) := \Phi_f$$

defines a $\mathbb{C}$-linear isomorphism

$$\Phi : \Gamma(\mathbb{C}; \mathscr{O}_{\mathbb{C}}) \longrightarrow \Gamma(\mathbb{C}^k; \mathscr{H}om_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})) = \mathrm{Hom}_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$$

Moreover, this isomorphism is also $\Gamma(\mathbb{C}, \mathscr{D}_{\mathbb{C}})$-left linear.

Proof The last statement is clear from Proposition 4.5 and Lemma 3.1.

The remaining of the statement is equivalent to prove that $\Phi : f \mapsto \Phi_f$ defines an isomorphism $\mathrm{Hom}_{\mathscr{D}_H}(\mathscr{N}, \mathscr{O}_H) \to \mathrm{Hom}_{\mathscr{D}_{\mathbb{C}^k}}(\pi_*\mathscr{N}, \mathscr{O}_{\mathbb{C}^k})$.

We already know that Φ is injective. It remains to prove that Φ is surjective. For each $f \in \Gamma(\mathbb{C}, \mathscr{O}_{\mathbb{C}})$, we introduce the regular holonomic $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{M}_f$ (a regular flat holomorphic connection on $\mathbb{C}$) of which the constant sheaf $\mathbb{C}f$ in degree zero is the complex of holomorphic solutions.

Note that $\mathcal{N} \simeq \mathcal{O}_{\mathbb{C}^{k-1}} \boxtimes \mathcal{D}_{\mathbb{C}}$ where we consider $\mathbb{C}^{k-1}$ endowed with the coordinates $(s_1, \dots, s_{k-1})$ and $\mathbb{C}$ with the coordinate z. We denote by $\mathcal{N}_f$ the regular holonomic $\mathcal{D}_H$-module (a regular flat holomorphic connection on H)

$$\mathcal{N}_f := \mathcal{O}_{\mathbb{C}^{k-1}} \boxtimes \mathcal{M}_f.$$

It is clear that $\mathcal{N}_f$ is a quotient of $\mathcal{N}$, and, by the left exactness of π_*, $\pi_*\mathcal{N}_f$ is a quotient of $\pi_*\mathcal{N}$. Moreover, according to Proposition 4.5 and Lemma 3.1, Φ_f belongs to $\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$. According to Theorem 4.6, for each f we have a $\mathbb{C}$-linear isomorphism $T_f : \mathrm{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ which makes this last one a one dimensional $\mathbb{C}$-vector space. Moreover, by left exactness of Hom and the exactness of π_*, we have monomorphisms $\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k}) \subset \mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ and, by functoriality, we have $T(f) = T_f(f)$.

We shall use the following result:

Lemma 4.8 *Suppose that* $f \neq 0$. *Then* $\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ *is a one dimensional* $\mathbb{C}$*-vector space generated by* Φ_f.

Proof The result follows by Proposition 2.3 since Φ_f is a non zero element of $\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ hence it is a generator as a $\mathbb{C}$-vector space. □

Let us now end the proof of Proposition 4.7.

Clearly $\mathrm{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) = \sum_f \mathrm{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H)$ and, according to Lemma 4.8, for each f, $\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ is the $\mathbb{C}$-vector space spanned by Φ_f; hence $T(f) = \lambda\Phi_f$ for some $\lambda \in \mathbb{C}^*$. Since $\Phi_{\lambda f} = \lambda\Phi_f$ we conclude that Φ is surjective which gives the desired result. □

As a consequence, isomorphism Φ explicits isomorphism of Theorem 4.6 since they coincide up to the multiplication by a constant $\lambda \neq 0$.

4.1 An example

To conclude this article, let us give an interesting example of choice of the entire function f on $\mathbb{C}$ for which we explicit the regular holonomic system on $\mathbb{C}^k$ associated to the corresponding Lisbon integrals.

THE CASE $f_t(z) := e^{tz}$ Let us fix a parameter $t \in \mathbb{C}^*$ and consider the entire function $f_t(z) := e^{tz}$.

First remark that according to Lemma 3.1, we have $\frac{\partial E(z)}{\partial z} = \nabla E(z)$ where ∇ is the (k, k) matrix given by

$$\nabla := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots \\ \cdots\cdots & & \cdots & \cdots \\ 0 & \cdots & k-1 & 0 \end{pmatrix}$$

We have[4]

$$\Phi_{f_t}(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{t\zeta} E(\zeta) d\zeta}{P_s(\zeta)}$$

[4] Remember that t is a fixed complex parameter.

and, according to the linearity of $\Phi_{(\cdot)}$, we also have

$$t\Phi_{f_t}(s) = \Phi_{\partial_z(f_t)}(s)$$

which, applying (**) in Remark 3.2, entails

$$t\Phi_{f_t}(s) = -\nabla\Phi_{f_t}(s) + (-1)^{k-1}\frac{\partial(P'_s(A)\Phi_{f_t})}{\partial s_k}(s) \tag{13}$$

Hence

$$(tId + \nabla)\Phi_{f_t}(s) = (-1)^{k-1}\frac{\partial(P'_s(A)\Phi_{f_t})}{\partial s_k}(s). \tag{14}$$

This also implies the following equation for Ψ_{f_t} away of the discriminant hypersurface Δ:

$$(tId + \nabla)P'_s(A)^{-1}\Psi_{f_t}(s) = (-1)^{k-1}\frac{\partial(\Psi_{f_t})}{\partial s_k}(s) \tag{15}$$

for

$$\Psi_{f_t}(s) = P'_s(A)\Phi_{f_t}(s) = \frac{1}{2i\pi}\int_{|\zeta|=R}\frac{e^{t\zeta}P'_s(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.$$

Combining (14) with the system (@) it is easy to see that we obtain a meromorphic integrable connexion on the trivial vector bundle of rank k on $\mathbb{C}^k$ with a pole along the discriminant hypersurface.

The regularity of this meromorphic connexion is then consequence of the regularity of the $\mathscr{D}_H$-module $\mathscr{N}_{e^{tz}} = \mathscr{D}_H u$ which is given by the equations

$$\partial_{s_h}u = 0, \quad \forall h \in [1, k-1] \quad \text{and} \quad (\partial_z - t)u = 0$$

which is clearly regular holonomic on H. So its direct image by π (as a $\mathscr{D}_{\mathbb{C}^k}$-module) is regular holonomic on $\mathbb{C}^k$ (see [9, Theorem 8.1]).

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