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The growth of dimension of cohomology of semipositive line bundles on Hermitian manifolds

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Abstract

In this paper, we study the dimension of cohomology of semipositive line bundles over Hermitian manifolds, and obtain an asymptotic estimate for the dimension of the space of harmonic $(0, q)$-forms with values in high tensor powers of a semipositive line bundle when the fundamental estimate holds. As applications, we estimate the dimension of cohomology of semipositive line bundles on q-convex manifolds, pseudo-convex domains, weakly 1-complete manifolds and complete manifolds. We also obtain the estimate of cohomology on compact manifolds with semipositive line bundles endowed with a Hermitian metric with analytic singularities and related results.

Contents

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1 Introduction

The purpose of this paper is to prove asymptotic estimates for the cohomology of semipositive line bundles over various non-compact manifolds. We generalize the asymptotics obtained by Berndtsson [4] in the compact case, which in turn refine the holomorphic Morse inequalities of Demailly [11].

Let X be a compact complex manifold, let L be a holomorphic line bundle and E be a holomorphic vector bundle on X. The Dolbeault cohomology $H^{0,q}(X, L^k \otimes E)$ plays a fundamental role in algebraic and complex geometry and is linked to the structure of the manifold, cf. [9,11,19]. If L is a positive line bundle, $H^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and k large enough, by the Kodaira-Serre vanishing theorem (see e.g. [19, Theorem 1.5.6]) and this can be used to prove that global holomorphic sections of $L^k \otimes E$ give a projective embedding of X for large k (Kodaira embedding theorem).

Assume now that L is semipositive. The solution of the Grauert-Riemenschneider conjecture [13] by Siu [28] and Demailly [11] shows that dim $H^{0,q}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. This can be used to show that X is a Moishezon manifold, if (L, h^L) is moreover positive at least at one point. Berndtsson [4] showed that we have actually dim $H^{0,q}(X, L^k \otimes E) = \mathcal{O}(k^{n-q})$ as $k \to \infty$ for $q \geq 1$.

We will consider first a very general situation where we can prove the decay of the cohomology groups as above. Let (X, ω) be a Hermitian manifold of dimension n. Let $dv_X := \omega^n/n!$ be the volume form on X. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X, where L is a line bundle. We denote by $(L^2_{0,q}(X, L^k \otimes E), \|\cdot\|)$ the space of square integrable $(0, q)$-forms with values in $L^k \otimes E$ with respect to the L^2 inner product induced by the above data. We denote by $\overline{\partial}^E_k$ the maximal extension of the Dolbeault operator on $L^2_{0,\bullet}(X, L^k \otimes E)$ and by $\overline{\partial}^{E*}_k$ its Hilbert space adjoint. Let $\mathcal{H}^{0,q}(X, L^k \otimes E)$ be the space of harmonic $(0, q)$-forms with values in $L^k \otimes E$ on X.

For a given $0 \leq q \leq n$, we say that **the concentration condition holds in bidegree** $(0, q)$ **for harmonic forms with values in $L^k \otimes E$ for large** k, if there exists a compact subset $K \subset X$ and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq C_0 \int_K |s|^2 dv_X, \tag{1.1}$$

for $s \in \mathrm{Ker}(\overline{\partial}^E_k) \cap \mathrm{Ker}(\overline{\partial}^{E*}_k) \cap L^2_{0,q}(X, L^k \otimes E)$. The set K is called the exceptional compact set of the concentration. We say that **the fundamental estimate holds in bidegree** $(0, q)$ **for forms with values in $L^k \otimes E$ for large** k, if there exists a compact subset $K \subset X$ and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq C_0(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E,*}_k s\|^2 + \int_K |s|^2 dv_X), \tag{1.2}$$

for $s \in \mathrm{Dom}(\overline{\partial}^E_k) \cap \mathrm{Dom}(\overline{\partial}^{E*}_k) \cap L^2_{0,q}(X, L^k \otimes E)$. The set K is called the exceptional compact set of the estimate.

The first observation of this paper is an asymptotic estimate for L^2-cohomology with semipositive line bundles over Hermitian manifolds as follows. It in some sense is a reformulation of Theorem 3.1, which was obtained in [31]. And the main results of this paper are how to use Theorem 1.1 as well as the local estimate of Theorem 3.1 to obtain global information such as the dimension of cohomology group over various non-compact complex manifolds.

Theorem 1.1 *Let (X, ω) be a Hermitian manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Assume that for some $1 \leq q \leq n$ the concentration condition holds in bidegree $(0, q)$ for harmonic forms with values in $L^k \otimes E$ for large k. Assume that (L, h^L) is semipositive on a neighbourhood of the exceptional compact set K. Then there exists $C > 0$ such that for sufficiently large k, we have*

$$\dim \mathscr{H}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.3}$$

The same estimate also holds for reduced L^2-Dolbeault cohomology groups,

$$\dim \overline{H}_{(2)}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.4}$$

In particular, if the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes E$ for large k, the same estimate holds for L^2-Dolbeault cohomology groups,

$$\dim H_{(2)}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.5}$$

Note that holomorphic Morse inequalities for the L^2-cohomology were obtained under the assumption that the fundamental estimate holds in [19, Theorem 3.2.13]. They can only deliver an estimate $\dim H_{(2)}^{0,q}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$.

A geometric situation when Theorem 1.1 can be applied is the case of a semipositive line bundle on a complete Kähler manifold which polarizes the Kähler metric at infinity.

Theorem 1.2 *Let (X, ω) be a complete Hermitian manifold of dimension n. Let (L, h^L) be a holomorphic Hermitian line bundle on X. Assume there exists a compact subset $K \subset X$ such that $\sqrt{-1}R^{(L,h^L)} = \omega$ on $X \backslash K$ and (L, h^L) is semipositive on K.*

Then there exists $C > 0$ such that for any $q \geq 1$ and sufficiently large k, we have

$$\dim H_{(2)}^{0,q}(X, L^k \otimes K_X) \leq Ck^{n-q}. \tag{1.6}$$

Note that by [19, Theorem 3.3.5] we have

$$\dim H_{(2)}^{0,0}(X, L^k \otimes K_X) \geq \frac{k^n}{n!} \int_X c_1(L, h^L)^n + o(k^n), \quad k \to \infty$$

in the situation of Theorem 1.2. Moreover, the Bergman kernel of $H_{(2)}^{0,0}(X, L^k \otimes K_X)$ has an asymptotic expansion in powers of k on the set where $c_1(L, h^L) > 0$ by [17, Theorem 1.7].

We consider further the case of q-convex manifolds as application of Theorem 1.1. For general holomorphic Morse inequalities on q-convex manifolds see [19, Theorem 3.5.8].

Theorem 1.3 *Let X be a q-convex manifold of dimension n, and let $(L, h^L), (E, h^E)$ be holomorphic Hermitian line bundles on X. Let ϱ be an exhaustion function of X and K the exceptional set. Let (L, h^L) be semipositive on a sublevel set $X_c := \{x \in X : \varrho(x) < c\}$ satisfying $K \subset X_c$, and let (L, h^L) be positive on $X_c \backslash K$.*

Then there exists $C > 0$ such that for any $j \geq q$ and $k \geq 1$, we have

$$\dim H^j(X, L^k \otimes E) \leq Ck^{n-j}. \tag{1.7}$$

We denote here by $H^j(X, L^k \otimes E)$ the cohomology groups of the sheaves $\mathscr{O}_X(L^k \otimes E)$ of holomoprhic sections of $L^k \otimes E$. They are isomorphic to the Dolbeault cohomology groups $H^{0,j}(X, L^k \otimes E)$.

In the case of 1-convex manifolds estimate (1.7) holds without additional hypothesis.

Theorem 1.4 *Let X be a 1-convex manifold of dimension n, and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let ϱ be an exhaustion function of X and K the exceptional set. Let (L, h^L) be semipositive on a sublevel set $X_c := \{x \in X : \varrho(x) < c\}$ satisfying $K \subset X_c$. Then there exists $C > 0$ such that for any $j \geq 1$ and $k \geq 1$ the estimate (1.7) holds.*

Similarly, we also have the estimates of cohomology over pseudoconvex domains and weakly 1-complete manifolds.

Theorem 1.5 *Let $M \Subset X$ be a smooth (weakly) pseudoconvex domain in a complex manifold X of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be semipositive on M. Moreover, assume (L, h^L) is positive in a neighbourhood of bM. Then there exists $C > 0$ such that for any $q \geq 1$ and sufficiently large k, we have*

$$\dim H^{0,q}_{(2)}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.8}$$

Theorem 1.6 *Let X be a weakly 1-complete manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be semipositive on X. Moreover, assume (L, h^L) is positive on $X \backslash K$ for a compact subset $K \subset X$. Then there exists $C > 0$ such that for any $q \geq 1$ and sufficiently large k, we have*

$$\dim H^q(X, L^k \otimes E) \leq Ck^{n-q}. \tag{1.9}$$

The next result is another generalization of [4] for line bundles endowed with a Hermitian metric with analytic singularities. Let us recall that the analogue of the Kodaira vanishing theorem in the case of singular metrics is the Nadel vanishing theorem [10,26]. If X is a compact Kähler manifold, L and E are holomorphic vector bundles with rank$(L) = 1$, and h^L is a singular Hermitian metric such that $c_1(L, h^L)$ is a Kähler current, then $H^q(X, E \otimes L^k \otimes \mathscr{J}(h^{L^k})) = 0$ for $q \geq 1$ and k suffiently large, where $\mathscr{J}(h^{L^k})$ is the Nadel multiplier ideal sheaf associated to h^{L^k}. Bonavero [5] obtained holomorphic Morse inequalities for singular Hermitian line bundles. These inequalities imply that for any $q \geq 1$ we have

$$\dim H^q(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) = o(k^n), \quad k \to \infty, \tag{1.10}$$

if the curvature $c_1(L, h)$ is semipositive on the set of points where it is smooth. We obtain the following refinement of (1.10).

Theorem 1.7 *Let X be a compact complex manifold of dimension n and let L be a holomorphic line bundle on X endowed with a Hermitian metric h^L with analytic singularities. Let (E, h^E) be a holomorphic Hermitian line bundle on X. Assume $c_1(L, h^L) \geq 0$ on the set $\{x \in X : h^L$ smooth on a neighborhood of $x\}$.*

Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$, we have

$$\dim H^q(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) \leq Ck^{n-q}. \tag{1.11}$$

In particular, this estimate still holds when $c_1(L, h^L)$ is a positive current on X.

Remark 1.8 Under the hypothesis of Theorem 1.7, if $c_1(L, h^L) \geq 0$ on the set $\{x \in X : h^L$ smooth on a neighborhood of $x\}$ and positive at least at one point, then there exist $C_1 > 0$ and $C_2 > 0$ such that for sufficiently large k, we have

$$C_1 k^n \leq \dim H^0(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) \leq C_2 k^n. \tag{1.12}$$

Thus L is big and X is Moishezon.

Remark 1.9 For a semipositive line bundle L on a compact complex manifold, if L is positive at some point, Demailly's Morse inequalities tells us that L is big. It is very interesting to know that when we can have many holomorphic sections if L is semipositive and degenerate everywhere. Marinescu and Savale [21] recently can produce many holomorphic sections for a semipositive line bundle L over a Riemann surface, where L is degenerate everywhere but of finite type. It would be desirable that if one can show that these asymptotic estimates of $\mathcal{O}(k^{n-q})$ type for semipositive line bundles can be used to produce holomorphic sections for some class of semipositive line bundles which is degenerate everywhere. Besides, for a compact Riemann surface Y with a semipositive line bundle L vanishing to finite order, $H^1(Y, L^k) = 0$ for sufficiently large k, see [21, Corollary 21], and this vanishing theorem for a semipositive line bundle works only in dimension one, see [21, Remark 22]. In general, without the assumption of the vanishing of R^L to finite order, $\dim H^1(Y, L^k) < C$ for all $k \in \mathbb{N}$ (see Theorem 3.17 for $n = q = 1$) and our estimates work in higher dimensional and various non-compact complex manifolds.

In this paper we consider the cohomology spaces on general (possibly non-compact) complex manifolds with semipositive line bundles. With the fundamental estimates fulfilled, Theorem 1.1 gives an estimate of L^2-Dolbeaut cohomology on arbitrary complex manifolds. On one hand, it generalizes [4] to general complex manifolds in the context of L^2-cohomology; on the other hand, it leads to the refinement of the estimates for complete manifolds, q-convex manifolds, pseudoconvex domains, weakly 1-convex manifolds, and semipositive line bundle endowed metric with analytic singularities, see Theorem 1.2, 1.3, 1.5, 1.6 and 1.7, respectively. Note also that the magnitude k^{n-q} in Theorems 1.1, 1.2, 1.3, 1.6, 1.7 and 3.22 cannot be improved in general, see [4, Proposition 4.2].

Our paper is organized in the following way. In Sect. 2 we introduce the notations and recall the necessary facts. In Sect. 3, we give an asymptotic estimate for L^2-cohomology with semipositive line bundles on Hermitian manifolds, which is a uniform approach to consider semipositive line bundles on complex manifolds. As main results, we obtain the estimate of growth of dimension in certain possibly non-compact complex manifolds. In additional, we revisited the compact and covering manifolds in this context. In Sect. 4, we prove the estimate of cohomology still holds when the Hermitian metric of the line bundle has analytic singularities. The techniques and formulations are mainly based on Berndtsson [4], Ma-Marinescu [19] and [31].

2 Preliminaries

2.1 L^2-cohomology

Let (X, ω) be a Hermitian manifold of dimension n and (F, h^F) be a holomorphic Hermitian vector bundle over X. Let $\Omega^{p,q}(X, F)$ be the space of smooth (p, q)-forms on X with values in F for $p, q \in \mathbb{N}$. If the $\mathrm{rank}(F) = 1$, the curvature of (F, h^F) is defined by $R^F = \overline{\partial}\partial \log |s|^2_{h^F}$ for any local holomorphic frame s, then the Chern-Weil form of the first Chern class of F is $c_1(F, h^F) = \frac{\sqrt{-1}}{2\pi} R^F$, which is a real $(1, 1)$-form on X. The volume form is given by $dv_X := \omega_n := \frac{\omega^n}{n!}$. We use the notion of positive (p, p)-form given by [9, Chapter III, §1, (1.1) (1.2) (1.5) (1.7)]. If a (p, p)-form T is positive, we write $T \geq 0$.

Definition 2.1 We say a holomorphic Hermitian line bundle (L, h^L) is semipositive on X, if $c_1(L, h^L)$ is positive semi-definite on X, equivalently $c_1(L, h^L) \geq 0$. For simplifying notations, we also denote $L \geq 0$.

Let $\Omega_0^{p,q}(X, F)$ be the subspace of $\Omega^{p,q}(X, F)$ consisting of elements with compact support. The L^2-scalar product on $\Omega_0^{p,q}(X, F)$ is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_h dv_X(x) \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_h := \langle \cdot, \cdot \rangle_{h^F, \omega}$ is the pointwise Hermitian inner product induced by ω and h^F. We denote by $L_{p,q}^2(X, F)$ the L^2 completion of $\Omega_0^{p,q}(X, F)$.

Let $\overline{\partial}^F : \Omega_0^{p,q}(X, F) \to L_{p,q+1}^2(X, F)$ be the Dolbeault operator and let $\overline{\partial}_{\max}^F$ be its maximal extension (see [19, Lemma 3.1.1]). From now on we still denote the maximal extension by $\overline{\partial}^F := \overline{\partial}_{\max}^F$ and the associated Hilbert space adjoint by $\overline{\partial}^{F*} := \overline{\partial}_H^{F*} := (\overline{\partial}_{\max}^F)_H^*$ for simplifying the notations. Consider the complex of closed, densely defined operators $L_{p,q-1}^2(X, F) \xrightarrow{\overline{\partial}^F} L_{p,q}^2(X, F) \xrightarrow{\overline{\partial}^F} L_{p,q+1}^2(X, F)$, then $(\overline{\partial}^F)^2 = 0$. By [19, Proposition 3.1.2], the operator defined by

$$\text{Dom}(\square^F) = \{s \in \text{Dom}(\overline{\partial}^F) \cap \text{Dom}(\overline{\partial}^{F*}) : \overline{\partial}^F s \in \text{Dom}(\overline{\partial}^{F*}), \ \overline{\partial}^{F*} s \in \text{Dom}(\overline{\partial}^F)\},$$
$$\square^F s = \overline{\partial}^F \overline{\partial}^{F*} s + \overline{\partial}^{F*} \overline{\partial}^F s \quad \text{for } s \in \text{Dom}(\square^F), \tag{2.2}$$

is a positive, self-adjoint extension of Kodaira Laplacian, called the Gaffney extension.

Definition 2.2 The space of harmonic forms $\mathcal{H}^{p,q}(X, F)$ is defined by

$$\mathcal{H}^{p,q}(X, F) := \text{Ker}(\square^F) = \{s \in \text{Dom}(\square^F) \cap L_{p,q}^2(X, F) : \square^F s = 0\}. \tag{2.3}$$

The q-th reduced L^2-Dolbeault cohomology is defined by

$$\overline{H}_{(2)}^{0,q}(X, F) := \frac{\text{Ker}(\overline{\partial}^F) \cap L_{0,q}^2(X, F)}{[\text{Im}(\overline{\partial}^F) \cap L_{0,q}^2(X, F)]}, \tag{2.4}$$

where $[V]$ denotes the closure of the space V. The q-th (non-reduced) L^2-Dolbeault cohomology is defined by

$$H_{(2)}^{0,q}(X, F) := \frac{\text{Ker}(\overline{\partial}^F) \cap L_{0,q}^2(X, F)}{\text{Im}(\overline{\partial}^F) \cap L_{0,q}^2(X, F)}. \tag{2.5}$$

According to the general regularity theorem of elliptic operators (cf. [19, Theorem A.3.4]), $s \in \mathcal{H}^{p,q}(X, F)$ implies $s \in \Omega^{p,q}(X, F)$. By weak Hodge decomposition (cf. [19, (3.1.21) (3.1.22)]), we have a canonical isomorphism

$$\overline{H}_{(2)}^{0,q}(X, F) \cong \mathcal{H}^{0,q}(X, F) \tag{2.6}$$

for any $q \in \mathbb{N}$, which associates to each cohomology class its unique harmonic representative. The q-th cohomology of the sheaf of holomorphic sections of F is isomorphic to the q-th Dolbeault cohomology, $H^q(X, F) \cong H^{0,q}(X, F)$.

For a given $0 \leq q \leq n$, we say the fundamental estimate holds in bidegree $(0, q)$ for forms with values in F, if there exists a compact subset $K \subset X$ and $C > 0$ such that

$$\|s\|^2 \leq C(\|\overline{\partial}^F s\|^2 + \|\overline{\partial}^{F*}\|^2 + \int_K |s|^2 dv_X), \tag{2.7}$$

for $s \in \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F,*}) \cap L^2_{0,q}(X, F)$. K is called the exceptional compact set of the estimate. If the fundamental estimate holds in bidegree $(0, q)$ for forms with values in F, the reduced and non-reduced L^2-Dolbeault cohomology coincide, see [19, Theorem 3.1.8].

For a given $0 \leq q \leq n$, we say that the concentration condition holds in bidegree $(0, q)$ for harmonic forms with values in F, if there exists a compact subset $K \subset X$ and $C > 0$ such that

$$\|s\|^2 \leq C \int_K |s|^2 dv_X, \tag{2.8}$$

for $s \in \mathrm{Ker}(\overline{\partial}^F) \cap \mathrm{Ker}(\overline{\partial}^{F*}) \cap L^2_{0,q}(X, F)$. The compact set K is called the exceptional compact set of the concentration. Note if the fundamental estimate holds in bidegree $(0, q)$ for forms with values in F, the concentration condition holds in bidegree $(0, q)$ for harmonic forms with values in F.

2.2 q-convex complex manifolds and Γ-coverings

Definition 2.3 [2] A complex manifold X of dimension n is called q-convex if there exists a smooth function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that the sublevel set $X_c := \{\varrho < c\} \Subset X$ for all $c \in \mathbb{R}$ and the complex Hessian $\partial\overline{\partial}\varrho$ has at least $n - q + 1$ positive eigenvalues outside a compact subset $K \subset X$. Here $X_c \Subset X$ means that the closure $\overline{X}_c$ is compact in X. We call ϱ an exhaustion function and K exceptional set. We say X is q-complete if $K = \emptyset$ in additional.

Definition 2.4 A complex manifold X of dimension n is called a q-convex manifold with a plurisubharmonic exhaustion function near the exceptional set, if there exists a compact subset $K \subset X$ and a smooth function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that the sublevel set $X_c := \{\varrho < c\} \Subset X$ for all $c \in \mathbb{R}$, and the complex Hessian $\partial\overline{\partial}\varrho$ has at least $n - q + 1$ positive eigenvalues on $X \backslash K$ and $\sqrt{-1}\partial\overline{\partial}\varrho \geq 0$ on $X_c \backslash K$ for some X_c with $K \subset X_c$.

Let M be a relatively compact domain with smooth boundary bM in a complex manifold X. Let $\rho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $M = \{x \in X : \rho(x) < 0\}$ and $d\rho \neq 0$ on $bM = \{x \in X : \rho(x) = 0\}$. We denote the closure of M by $\overline{M} = M \cup bM$. We say that ρ is a defining function of M. Let $T_x^{(1,0)}bM := \{v \in T_x^{(1,0)}X : \partial\rho(v) = 0\}$ be the analytic tangent bundle to bM at $x \in bM$. The Levi form of ρ is the 2-form $\mathscr{L}_\rho := \partial\overline{\partial}\rho \in \mathscr{C}^\infty(bM, T^{(1,0)*}bM \otimes T^{(0,1)*}bM)$.

Definition 2.5 A relatively compact domain M with smooth boundary bM in a complex manifold X is called strongly (resp. (weakly)) pseudoconvex if the Levi form $\mathscr{L}_\rho$ is positive definite (resp. semidefinite).

Note that any strongly pseudoconvex domain is 1-convex.

Definition 2.6 A complex manifold X is called weakly 1-complete if there exists a smooth plurisubharmonic function $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $\{x \in X : \varphi(x) < c\} \Subset X$ for any $c \in \mathbb{R}$. φ is called an exhaustion function.

Note that any 1-convex manifold is weakly 1-complete.

Definition 2.7 A Hermitian manifold (X, ω) is called complete, if all geodesics are defined for all time for the underlying Riemannian manifold.

If (X, ω) is complete, for arbitrary holomorphic Hermitian vector bundle (F, h^F) on X, $\Omega_0^{0,\bullet}(X, F)$ is dense in $\mathrm{Dom}(\overline{\partial}^F)$, $\mathrm{Dom}(\overline{\partial}_H^{F*})$ and $\mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}_H^{F*})$ in the graph-norms of $\overline{\partial}^F$, $\overline{\partial}_H^{F*}$ and $\overline{\partial}^E + \overline{\partial}_H^{E*}$ respectively, see [19, Lemma 3.3.1 (Andreotti-Vesentini), Corollary 3.3.3]. Here the graph-norm of R is defined by $\|s\| + \|Rs\|$ for $s \in \mathrm{Dom}(R)$.

Definition 2.8 Let (X, ω) be a Hermitian manifold of dimension n on which a discrete group Γ acts holomorphically, freely and properly such that ω is a Γ-invariant Hermitian metric and the quotient X/Γ is compact. We say (X, ω) is a Γ-covering manifold.

2.3 Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions

Let (X, ω) be a Hermitian manifold of dimension n and (F, h^F) be a holomorphic Hermitian vector bundle over X. Let M be a relatively compact domain in X. Let ρ be a defining function of M satisfying $M = \{x \in X : \rho(x) < 0\}$ and $|d\rho| = 1$ on bM, where the pointwise norm $|\cdot|$ is given by g^{TX} associated to ω.

Let $e_n \in TX$ be the inward pointing unit normal at bM and $e_n^{(0,1)}$ its projection on $T^{(0,1)}X$. In a local orthonormal frame $\{w_1, \ldots, w_n\}$ of $T^{(1,0)}X$, we have $e_n^{(0,1)} = -\sum_{j=1}^n w_j(\rho)\overline{w}_j$. Let $B^{0,q}(X, F) := \{s \in \Omega^{0,q}(\overline{M}, F) : i_{e_n^{(0,1)}}s = 0 \text{ on } bM\}$. We have $B^{0,q}(M, F) = \mathrm{Dom}(\overline{\partial}_H^{F*}) \cap \Omega^{0,q}(\overline{M}, F)$ and the Hilbert space adjoint $\overline{\partial}_H^{F*}$ of $\overline{\partial}^F$ coincides with the formal adjoint $\overline{\partial}^{F*}$ of $\overline{\partial}^F$ on $B^{0,q}(M, F)$, see [19, Proposition 1.4.19]. We consider the operator $\square_N s = \overline{\partial}^F \overline{\partial}^{F*} s + \overline{\partial}^{F*} \overline{\partial}^F s$ for $s \in \mathrm{Dom}(\square_N) := \{s \in B^{0,q}(M, F) : \overline{\partial}^F s \in B^{0,q+1}(M, F)\}$. The Friedrichs extension of $\square_N$ is a self-adjoint operator and is called the Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions, which coincides with the Gaffney extension of the Kodaira Laplacian, see [19, Proposition 3.5.2]. Note $\Omega^{0,\bullet}(\overline{M}, F)$ is dense in $\mathrm{Dom}(\overline{\partial}^F)$ in the graph-norms of $\overline{\partial}^F$, and $B^{0,\bullet}(M, F)$ is dense in $\mathrm{Dom}(\overline{\partial}_H^{F*})$ and in $\mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}_H^{F*})$ in the graph-norms of $\overline{\partial}_H^{F*}$ and $\overline{\partial}^E + \overline{\partial}_H^{E*}$, respectively, see [19, Lemma 3.5.1]. Here the graph-norm of R is defined by $\|s\| + \|Rs\|$ for $s \in \mathrm{Dom}(R)$.

2.4 Hermitian metric with analytic singularities on line bundles

Definition 2.9 Let X be a connected compact complex manifold of dimension n and L a holomorphic line bundle on X. On L we say h^L is a Hermitian metric with analytic singularities, if there exists a smooth Hermitian metric h_0^L and a function $\varphi \in L_{loc}^1(X, \mathbb{R})$ with locally

$$\varphi = \frac{c}{2} \log\left(\sum_{j \in J} |f_j|^2\right) + \psi, \tag{2.9}$$

where J is at most countable, c is a non-negative rational number, f_j are non-zero holomorphic functions and ψ is a smooth function, such that $h^L = h_0^L e^{-2\varphi}$.

Note that for a Hermitian metric with analytic singularities on line bundles over connected compact complex manifolds (see Definition 2.9), the non-negative rational number c does not depend on the local form (2.9) of the weight φ.

For any local holomorphic frame e_L of L, $h_0^L(e_L, e_L) = e^{-2\psi_0}$, where ψ_0 is smooth, thus the local weights of h^L is given by $\frac{c}{2}\log(\sum_J |f_j|^2) + (\psi + \psi_0)$. Because locally $\psi + \psi_0$ is smooth, we use (2.9) to represent the local weight of h^L for simplifying notations.

For $\varphi \in L_{loc}^1(X, \mathbb{R})$, the Nadel multiplier ideal sheaf $\mathscr{J}(\varphi)$ is the ideal subsheaf of germs of holomorphic functions $f \in \mathscr{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near x. We define $\mathscr{J}(h^L) := \mathscr{J}(\varphi)$, which does not depend on the choice of φ, see [10] or [19, Definition 2.3.13].

3 Asymptotic estimate for L^2-cohomology with semipositive line bundles

Let (X, ω) be a Hermitian manifold of dimension n and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles over X. Let $\mathscr{H}^{0,q}(X, L^k \otimes E)$ be the space of harmonic $(0, q)$-forms with values in $L^k \otimes E$. Let $\{s_j^k\}_{j \geq 1}$ be an orthonormal basis of $\mathscr{H}^{0,q}(X, L^k \otimes E)$ and let B_k^q be the Bergman density function defined by

$$B_k^q(x) = \sum_{j \geq 1} |s_j^k(x)|_{h_k,\omega}^2, \ x \in X, \tag{3.1}$$

where $|\cdot|_{h_k,\omega}$ is the pointwise norm of a form, see [31]. The function (3.1) is well-defined by an adaptation of [7, Lemma 3.1] to form case. By replacing $E \bigotimes \Lambda^n(T^{(1,0)}X)$ for E in $\mathscr{H}^{n,q}(X, L^k \otimes E)$, we can rephrase [31, Theorem 1.1] as follows.

Theorem 3.1 *Let (X, ω) be a Hermitian manifold of dimension n and let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles over X. Let $K \subset X$ be a compact subset and assume that (L, h^L) is semipositive on a neighborhood of K.*

Then there exists $C > 0$ depending on the compact set K, the metric ω and the bundles (L, h^L) and (E, h^E), such that for any $x \in K$, $k \geq 1$ and $q \geq 1$,

$$B_k^q(x) \leq Ck^{n-q}, \tag{3.2}$$

where $B_k^q(x)$ is the Bergman density function (3.1) of harmonic $(0, q)$-forms with values in $L^k \otimes E$.

A general result on asymptotic estimate for L^2-cohomology with semipositive line bundles over Hermitian manifolds follows immediately.

Theorem 3.2 (Theorem 1.1) *Let (X, ω) be a Hermitian manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let $1 \leq q \leq n$. Assume the concentration condition holds in bidegree $(0, q)$ for harmonic forms with values in $L^k \otimes E$ for large k. Moreover, assume (L, h^L) is semipositive on a neighbourhood of the exceptional set K. Then there exists $C > 0$ such that for sufficiently large k we have*

$$\dim \mathscr{H}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.3}$$

The same estimate also holds for reduced L^2-Dolbeault cohomology groups,

$$\dim \overline{H}_{(2)}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.4}$$

In particular, if the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes E$ for large k, the same estimate holds for L^2-Dolbeault cohomology groups

$$\dim H_{(2)}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.5}$$

Proof By Theorem 3.1 and the concentration condition, we have

$$\dim \overline{H}_{(2)}^{(0,q)} = \dim \mathscr{H}^{0,q}(X, L^k \otimes E) \tag{3.6}$$

$$= \sum_{j \geq 1} \|s_j^k\|^2 \leq C_0 \int_K B_k^q(x) dv_X \leq C_0 C \, \mathrm{vol}(K) k^{n-q} \tag{3.7}$$

for sufficiently large k. Note that $H_{(2)}^{0,q}(X, F) = \overline{H}_{(2)}^{0,q}(X, F)$ and the dimension is finite, when the fundamental estimate holds in bidegree $(0, q)$ for forms with values in a holomorphic Hermitian vector bundle (F, h^F) by [19, Theorem 3.1.8]. □

3.1 q-Convex manifolds

3.1.1 Exhaustion functions with the plurisubharmonic near the exceptional set

In this section we prove the following general result about the growth of the cohomology of q-convex manifolds.

Theorem 3.3 *Let X be a q-convex manifold of dimension n, and let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let ϱ be an exhaustion function of X and K the exceptional set. Let (L, h^L) be semipositive on a sublevel set $X_c := \{x \in X : \varrho(x) < c\}$ satisfying $K \subset X_c$, and let $\sqrt{-1}\partial\overline{\partial}\varrho \geq 0$ on $X_c \backslash K$. Then there exists $C > 0$ such that for any $j \geq q$ and $k \geq 1$, we have*

$$\dim H^j(X, L^k \otimes E) \leq C k^{n-j}. \tag{3.8}$$

Let X be a q-convex manifold of dimension n, let ϱ be a plurisubharmonic exhaustion function of X and K the exceptional set. By the definition, $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies $X_c := \{\varrho < c\} \Subset X$ for all $c \in \mathbb{R}$, $\sqrt{-1}\partial\overline{\partial}\varrho$ has $n - q + 1$ positive eigenvalues on $X \backslash K$. In this section, we fix real numbers u_0, u and v satisfying $u_0 < u < c < v$ and $K \subset X_{u_0}$.

Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. We have that the fundamental estimate holds in bidegree $(0, j)$ for forms with values in $L^k \otimes E$ for large k and each $q \leq j \leq n$ on X_c when X is a q-convex manifold, see Proposition 3.8, which was obtained in [19, Theorem 3.5.8] for the proof of Morse inequalities on q-convex manifold. For the sake of completeness, we prove it here.

Firstly, we choose now a Hermitian metric on a q-convex manifold X.

Lemma 3.4 ([19, Lemma 3.5.3]) *For any $C_1 > 0$ there exists a metric g^{TX} (with Hermitian form ω) on X such that for any $j \geq q$ and any holomorphic Hermitian vector bundle (F, h^F) on X,*

$$\langle (\partial\overline{\partial}\varrho)(w_l, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_l}s, s\rangle_h \geq C_1 |s|^2, \quad s \in \Omega_0^{0,j}(X_v \backslash \overline{X}_u, F), \tag{3.9}$$

where $\{w_l\}_{l=1}^n$ is a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w^l\}_{l=1}^n$ of $T^{(1,0)*}X$.

Now we consider the q-convex manifold X associated with the metric ω obtained above as a Hermitian manifold (X, ω). Note for arbitrary holomorphic vector bundle F on a relatively compact domain M in X, the Hilbert space adjoint $\overline{\partial}_H^{F*}$ of $\overline{\partial}^F$ coincides with the formal adjoint $\overline{\partial}^{F*}$ of $\overline{\partial}^F$ on $B^{0,j}(M, F) = \mathrm{Dom}(\overline{\partial}_H^{F*}) \cap \Omega^{0,j}(\overline{M}, F)$, $1 \leq j \leq n$. So we simply use the notion $\overline{\partial}^{F*}$ on $B^{0,j}(M, F)$, $1 \leq j \leq n$.

Secondly, we modify Hermitian metric h_χ^L on L and show the fundamental estimate fulfilled. Let $\chi(t) \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi'(t) \geq 0$, $\chi''(t) \geq 0$. We define a Hermitian metric $h_\chi^{L^k} := h^{L^k} e^{-k\chi(\varrho)}$ on L^k for each $k \geq 1$ and set $L_\chi^k := (L^k, h_\chi^{L^k})$. Thus

$$R^{L_\chi^k} = kR^{L_\chi} = kR^L + k\chi'(\varrho)\partial\overline{\partial}\varrho + k\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho. \tag{3.10}$$

Lemma 3.5 ([19, (3.5.19)]) *There exists $C_2 > 0$ and $C_3 > 0$ such that, if $\chi'(\varrho) \geq C_3$ on $X_v \backslash \overline{X}_u$, then*

$$\|s\|^2 \leq \frac{C_2}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) \tag{3.11}$$

for $s \in B^{0,j}(X_c, L^k \otimes E)$ with $\mathrm{supp}(s) \subset X_v \backslash \overline{X}_u$, $j \geq q$ and $k \geq 1$, where the L^2-norm $\| \cdot \|$ is given by ω, $h_\chi^{L^k}$ and h^E on X_c.

Lemma 3.6 *Let $\epsilon > 0$ satisfying $X_{c+\epsilon} \backslash \overline{X}_{c-\epsilon} := \{c - \epsilon < \varrho < c + \epsilon\} \Subset X_v \backslash \overline{X}_u$. Let $\phi \in \mathscr{C}_0^\infty(X_v, \mathbb{R})$ with $\mathrm{supp}(\phi) \subset X_v \backslash \overline{X}_u$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $X_{c+\epsilon} \backslash \overline{X}_{c-\epsilon}$. Let $K' := \overline{X}_{c-\epsilon} := \{\varrho \leq c - \epsilon\}$. Then, for any $s \in B^{0,p}(X_c, L^k \otimes E)$, $1 \leq p \leq n$, we have*

$$\|\phi s\|^2 \geq \|s\|^2 - \int_{K'} |s| dv_X, \tag{3.12}$$

where the Hermitian norm $| \cdot |$ and the L^2-norm $\| \cdot \|$ are given by ω, $h_\chi^{L^k}$ and h^E on X_c.

Proof For $s \in B^{0,p}(X_c, L^k \otimes E) = \mathrm{Dom}(\overline{\partial}_k^{E*}) \cap \Omega^{0,p}(\overline{X}_c, L^k \otimes E)$, $\phi s \in \Omega^{0,p}(\overline{X}_c, L^k \otimes E)$ and $i_{e_n^{(0,1)}}(\phi s) = i_{e_n^{(0,1)}}(s) = 0$ on bX_c by $\phi s = s$ on the neighbourhood $X_{c+\epsilon} \backslash \overline{X}_{c-\epsilon}$ of bX_c. Thus $\phi s \in B^{0,p}(X_c, L^k \otimes E)$ with $\mathrm{supp}(\phi s) \subset X_v \backslash \overline{X}_u$,

$$\|\phi s\|^2 = \int_{X_c} |\phi s|^2 dv_X = \int_{X_c \backslash \overline{X}_u} |\phi s|^2 dv_X = \int_{\{c-\epsilon < \varrho < c\}} |\phi s|^2 dv_X + \int_{u < \varrho \leq c-\epsilon} |\phi s|^2 dv_X$$

$$= \int_{\{c-\epsilon < \varrho < c\}} |s|^2 dv_X + \int_{\{u < \varrho \leq c-\epsilon\}} |\phi s|^2 dv_X$$

$$\geq \int_{X_c \backslash \overline{X}_{c-\epsilon}} |s|^2 dv_X = \|s\|^2 - \int_{K'} |s|^2 dv_X. \tag{3.13}$$

$\square$

Lemma 3.7 *Let ϕ be in Lemma 3.6, and let $\xi := 1 - \phi$ and $C_1 := \sup_{x \in X_c} |d\xi(x)|_{g^{T*X}}^2 > 0$. Then, for any $s \in B^{0,p}(X_c, L^k \otimes E)$, $1 \leq p \leq n$, and $k \geq 1$, we have*

$$\|\overline{\partial}_k^E(\xi s)\|^2 + \|\overline{\partial}_k^{E*}(\xi s)\|^2 \leq \frac{3}{2}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) + 6C_1\|s\|^2, \tag{3.14}$$

$$\frac{1}{k}(\|\overline{\partial}_k^E(\phi s)\|^2 + \|\overline{\partial}_k^{E*}(\phi s)\|^2) \leq \frac{5}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) + \frac{12C_1}{k}\|s\|^2, \tag{3.15}$$

where the L^2-norm $\| \cdot \|$ is given by ω, $h_\chi^{L^k}$ and h^E on X_c.

Proof The first inequality follows from [19, (3.2.8)]. For simplifying notations, we use $\overline{\partial}$ and $\overline{\partial}^*$ instead of $\overline{\partial}_k^E$ and $\overline{\partial}_k^{E*}$ respectively. From $\frac{1}{2}\|\overline{\partial}(\phi s)\|^2 - \|\overline{\partial}s\|^2 \leq \|\overline{\partial}s - \overline{\partial}(\phi s)\|^2$, $\frac{1}{2}\|\overline{\partial}^*(\phi s)\|^2 - \|\overline{\partial}^*s\|^2 \leq \|\overline{\partial}^*s - \overline{\partial}^*(\phi s)\|^2$ and the first inequality, we have

$$\frac{1}{2}(\|\overline{\partial}(\phi s)\|^2 + \|\overline{\partial}^*(\phi s)\|^2) \leq \frac{5}{2}(\|\overline{\partial}s\|^2 + \|\overline{\partial}^*s\|^2) + 6C_1\|s\|^2, \tag{3.16}$$

thus the second inequality follows.   □

Proposition 3.8 *Let X be a q-convex manifold of dimension n with the exceptional set $K \subset X_c$. Then there exists a compact subset $K' \subset X_c$ and $C_0 > 0$ such that for sufficiently large k, we have*

$$\|s\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_{k,H}^{E*} s\|^2) + C_0 \int_{K'} |s|^2 dv_X \qquad (3.17)$$

for any $s \in \mathrm{Dom}(\overline{\partial}_k^E) \cap \mathrm{Dom}(\overline{\partial}_{k,H}^{E*}) \cap L_{0,j}^2(X_c, L^k \otimes E)$ and $q \leq j \leq n$, where $\chi'(\varrho) \geq C_3$ on $X_v \setminus \overline{X}_u$ in Lemma 3.5 and the L^2-norm is given by ω, $h_\chi^{L^k}$ and h^E on X_c.

Proof We follow [19, Theorem 3.5.8]. Since $B^{0,j}(X_c, L^k \otimes E)$ is dense in $\mathrm{Dom}(\overline{\partial}_k^E) \cap \mathrm{Dom}(\overline{\partial}_{k,H}^{E*}) \cap L_{0,j}^2(X_c, L^k \otimes E)$ with respect to the graph norm of $\overline{\partial}_k^E + \overline{\partial}_{k,H}^{E*}$, we only to show this inequality holds for $s \in B^{0,j}(X_c, L^k \otimes E)$ with $j \geq q$ and large k.

Suppose now $s \in B^{0,j}(X_c, L^k \otimes E)$. Let ϕ be in Lemma 3.6. Thus $\phi s \in B^{0,j}(X_c, L^k \otimes E)$ with $\mathrm{supp}(\phi s) \subset X_v \setminus \overline{X}_u$. By Lemma 3.5, there exists $C_2 > 0$ and $C_3 > 0$ such that for $j \geq q$ and $k \geq 1$, we have

$$\|\phi s\|^2 \leq \frac{C_2}{k}(\|\overline{\partial}_k^E(\phi s)\|^2 + \|\overline{\partial}_k^{E*}(\phi s)\|^2) \qquad (3.18)$$

where $\chi'(\varrho) \geq C_3$ on $X_v \setminus \overline{X}_u$ and the L^2-norm $\| \cdot \|$ is given by ω, $h_\chi^{L^k}$ and h^E on X_c. Next applying (3.15) and Lemma 3.6, we obtain

$$\|s\|^2 - \int_{K'} |s|^2 dv_X \leq \frac{5C_2}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) + \frac{12C_1 C_2}{k}\|s\|^2. \qquad (3.19)$$

For $k \geq 24C_1 C_2$, it follows that $1 - \frac{12C_1 C_2}{k} \geq \frac{1}{2}$ and

$$\|s\|^2 \leq \frac{10C_2}{k}(\|\overline{\partial}_k^E s\|^2 + \|\overline{\partial}_k^{E*} s\|^2) + 2\int_{K'} |s|^2 dv_X. \qquad (3.20)$$

The proof is complete by choosing $C_0 := \max\{10C_2, 2\}$ and $k \geq 24C_1 C_2$.   □

Thirdly, we will show that (L_χ, h^{L_χ}) is semipositive if (L, h^L) is semipositive by choosing a appropriate χ. Let $C_2 > 0$ and $C_3 > 0$ be in Lemma 3.5. We choose $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi''(t) \geq 0$, $\chi'(t) \geq C_3$ on (u, v) and $\chi(t) = 0$ on $(-\infty, u_0)$. Therefore, $\chi'(\varrho(x)) \geq C_3 > 0$ on $X_v \setminus \overline{X}_u$ and $\chi(\varrho(x)) = \chi'(\varrho(x)) = 0$ on X_{u_0}. Note that $K \subset X_{u_0}$ and $u_0 < u < c < v$. Now we have a fixed χ which leads to the following proposition.

Proposition 3.9 *Let X be a q-convex manifold with $\sqrt{-1}\partial\overline{\partial}\varrho \geq 0$ on $X_c \setminus K$. If (L, h^L) is semipositive on X_c, then (L, h_χ^L) is semipositive on X_c for χ defined above.*

Proof From the above definition of χ, we have $\chi'(\varrho) \geq 0$ on X, $\chi'(\varrho) = 0$ on K. Since ϱ is plurisubharmonic on $X_c \setminus K$, i.e., $\sqrt{-1}\partial\overline{\partial}\varrho \geq 0$ on $X_c \setminus K$, we have $\sqrt{-1}\chi'(\varrho)\partial\overline{\partial}\varrho \geq 0$ on X_c. Since $\chi''(\varrho) \geq 0$ and $\sqrt{-1}\partial\varrho \wedge \overline{\partial}\varrho \geq 0$ on X_c, we have $\sqrt{-1}\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho \geq 0$ on X_c. Finally $\sqrt{-1}R^{L_\chi} = \sqrt{-1}R^L + \sqrt{-1}\chi'(\varrho)\partial\overline{\partial}\varrho + \sqrt{-1}\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho \geq 0$ on X_c.   □

Now we can prove the main result of this section as follows.

Proof of Theorem 3.3 By Proposition 3.9 with the fixed χ, Proposition 3.8 and using Theorem 1.1 for X_c endowed with Hermitian metric ω obtained in Lemma 3.4, there exists $C > 0$ such that for any $j \geq q$ and sufficiently large k,

$$\dim H^{0,j}_{(2)}(X_c, L^k \otimes E) = \dim \mathscr{H}^{0,j}(X_c, L^k \otimes E) \leq Ck^{n-j} \tag{3.21}$$

holds with respect to the metrics ω, h^L_χ and h^E on X_c. From [19, Theorem 3.5.6 (Hörmander), Theorem 3.5.7 (Andreotti-Grauert)(i), Theorem B.4.4 (The Dolbeault isomorphism)], we have for $j \geq q$,

$$H^j(X, L^k \otimes E) \cong H^j(X_v, L^k \otimes E) \cong H^{0,j}(X_v, L^k \otimes E) \cong H^{0,j}_{(2)}(X_c, L^k \otimes E).$$

Thus the conclusion holds for sufficiently large k. Also we know that for any holomorphic vector bundle F, $\dim H^j(X, F) < \infty$ for $j \geq q$ by [19, Theorem B.4.8 (Andreotti-Grauert)]. So the conclusion holds for all $k \geq 1$. □

Proof of Theorem 1.4 Apply Theorem 3.3 for 1-convex manifolds. □

By adapting the duality formula [15, 20.7 Theorem] for cohomology groups to Theorem 3.3, we have the analogue result to [31, Remark 4.4] for seminegative line bundles.

Corollary 3.10 *Let X be a q-convex manifold of dimension n with a plurisubharmonic exhaustion function ϱ near the exceptional set K, and let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be seminegative on a sublevel set $X_c = \{x \in X : \varrho(x) < c\}$ satisfying $K \subset X_c$. Then there exists $C > 0$ such that for any $0 \leq j \leq n - q$ and $k \geq 1$, the j-th cohomology with compact supports*

$$\dim[H^{0,j}(X, L^k \otimes E)]_0 \leq Ck^j. \tag{3.22}$$

In particular, $\dim[H^{0,0}(X, L^k \otimes E)]_0 \leq C$.

Proof Combine [15, 20.7 Theorem] and Theorem 3.3. □

3.1.2 Line bundles with positivity near the exceptional set

A natural question in Theorem 3.3 is whether the hypothesis on plurisubharmonic exhaustion function is necessary. In this section, we show that such hypothesis can be replaced by the positivity assumption of line bundle near the exceptional set.

Proof of Theorem 1.3 Let X be a q-convex manifold of dimension n, let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ be the exhaustion function satisfying $\sqrt{-1}\partial\overline{\partial}\varrho$ has $n - q + 1$ positive eigenvalues on $X \backslash K$.

Since (L, h^L) be semipositive on a sublevel set $X_c = \{x \in X : \varrho(x) < c\}$ satisfying the exceptional set $K \subset X_c$ and positive on $X_c \backslash K$, we fix real numbers u_0, u, c' and v satisfying $u_0 < u < c' < v < c$ and $K \subset X_{u_0}$. Thus for $X_{c'} := \{x \in X : \varrho(x) < c'\}$, we see $K \subset X_{c'}$, $L \geq 0$ on $X_{c'}$ and $L > 0$ on $X_v \backslash K$. For simplifying notions, we still denote c' by c in this proof, that is, there exists real numbers u_0, u, c and v satisfying $u_0 < u < c < v$ and $K \subset X_{u_0}$ and $L \geq 0$ on X_c and $L > 0$ on $X_v \backslash K$.

Firstly, we choose the metric ω on X from Lemma 3.4.

Secondly, we show the fundamental estimate holds. Note $L > 0$ on $X_v \backslash K$, by the same argument in Lemma 3.5 without modification of h^L by h^L_χ, we have that there exists $C_2 > 0$ and $k_0 > 0$ such that

$$\|s\|^2 \leq \frac{C_2}{k}(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E*}_k s\|^2) \tag{3.23}$$

for $s \in B^{0,j}(X_c, L^k \otimes E)$ with $\mathrm{supp}(s) \subset X_v \backslash \overline{X}_u$, $j \geq q$ and $k \geq k_0 > 0$, where the L^2-norm $\| \cdot \|$ is given by ω, h^{L^k} and h^E on X_c. As in Proposition 3.8 without the modification of h^L, we conclude that there exist a compact subset $K' \subset X_c$ (In fact, let $\epsilon > 0$ such that $\{c - \epsilon < \varrho < c + \epsilon\} \Subset X_v \backslash \overline{X}_u$, $K' := \overline{\{\varrho < c - \epsilon\}}$ as in Lemma 3.6) and $C_0 > 0$ such that for sufficiently large k, we have

$$\|s\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}^E_k s\|^2 + \|\overline{\partial}^{E*}_{k,H} s\|^2) + C_0 \int_{K'} |s|^2 dv_X \tag{3.24}$$

for any $s \in \mathrm{Dom}(\overline{\partial}^E_k) \cap \mathrm{Dom}(\overline{\partial}^{E*}_{k,H}) \cap L^2_{0,j}(X_c, L^k \otimes E)$ and each $q \leq j \leq n$, where the L^2-norm is given by ω, h^{L^k} and h^E on X_c.

Finally, we can apply Theorem 1.1 on X_c. Therefore, there exists $C > 0$ such that for any $j \geq q$ and sufficiently large k,

$$\dim H^{0,j}_{(2)}(X_c, L^k \otimes E) \leq Ck^{n-j} \tag{3.25}$$

holds with respect to the metrics ω, h^L and h^E on X_c. As in the proof of Theorem 3.3, the conclusion holds for all $k \geq 1$.   □

By adapting the duality formula [15, 20.7 Theorem] for cohomology groups to Theorem 1.3, we have the analogue result to [31, Remark 4.4] for seminegative line bundles.

Corollary 3.11 *Let X be a q-convex manifold of dimension n, and let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let ϱ be an exhaustion function of X and K the exceptional set. Let (L, h^L) be seminegative on a sublevel set $X_c = \{x \in X : \varrho(x) < c\}$ satisfying $K \subset X_c$, and let (L, h^L) be negative on $X_c \backslash K$. Then there exists $C > 0$ such that for any $0 \leq j \leq n - q$ and $k \geq 1$, the j-th cohomology with compact supports*

$$\dim[H^{0,j}(X, L^k \otimes E)]_0 \leq Ck^j. \tag{3.26}$$

In particular, $\dim[H^{0,0}(X, L^k \otimes E)]_0 \leq C$.

Proof Combining [15, 20.7 Theorem] and Theorem 1.3.   □

Remark 3.12 (Compatibility to the vanishing theorem on q-convex manifolds) Let (E, h^E) be a holomorphic vector bundle on X. If $(L, h^L) > 0$ on X_c with $K \subset X_c$ instead of the hypothesis $(L, h^L) \geq 0$ on $X_c \backslash K$ in Theorem 1.3, then for $j \geq q$ and sufficiently large k,

$$\dim H^j(X, L^k \otimes E) = 0. \tag{3.27}$$

In particular, if X is q-complete, then $H^p(X, E) = 0$ for $p \geq q$ and arbitrary holomorphic vector bundle E. In fact, the fundamental estimates hold with $K = \emptyset$ in these cases, thus the space of harmonic forms is trivial, see [19, Theorem 3.5.9].

3.2 Pseudo-convex domains

Theorem 3.13 *Let $M \Subset X$ be a smooth strongly pseudoconvex domain in a complex manifold X of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be semipositive on M. Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$, we have*

$$\dim H^q(M, L^k \otimes E) \leq Ck^{n-q}. \tag{3.28}$$

Proof Note a strongly pseudocovex domain is 1-convex and applying Theorem 1.4. $\qquad\square$

Proof of Theorem 1.5 We follow [19, Theorem 3.5.10]. Let ω be a Hermitian metric on X. Note $L > 0$ around bM, by the same argument in Proposition 3.8 without the modification of h^L, we conclude that there exist a compact subset $K' \subset M$ and $C_0 > 0$ such that for sufficiently large k,

$$\|s\|^2 \leq \frac{C_0}{k}(\|\bar{\partial}_k^E s\|^2 + \|\bar{\partial}_{k,H}^{E*} s\|^2) + C_0 \int_{K'} |s|^2 dv_X \tag{3.29}$$

for any $s \in \mathrm{Dom}(\bar{\partial}_k^E) \cap \mathrm{Dom}(\bar{\partial}_{k,H}^{E*}) \cap L^2_{0,q}(M, L^k \otimes E)$ and each $1 \leq q \leq n$, where the L^2-norm is given by ω, h^{L^k} and h^E on M. Finally, we apply Theorem 1.1 on M. $\qquad\square$

3.3 Weakly 1-complete manifolds

Proof of Theorem 1.6 We follow [19, Theorem 3.5.12]. Let $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$ be an exhaustion function of X such that $\sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ on X and $X_c := \{\varphi < c\} \Subset X$ for all $c \in \mathbb{R}$. We choose a regular value $c \in \mathbb{R}$ of φ such that $K \subset X_c$ by Sard's theorem. Thus X_c is a smooth pseudoconvex domain and $L > 0$ on a neighbourhood of bX_c. We apply Theorem 1.5, for any $q \geq 1$ and sufficiently large k

$$\dim H^{0,q}_{(2)}(X_c, L^k \otimes E) = \dim \mathscr{H}^{0,q}(X_c, L^k \otimes E) \leq Ck^{n-q}. \tag{3.30}$$

Finally, by [29, Theorem 6.2] (see [19, Theorem 3.5.11]) and Dolbeault isomorphism, it follows that $H^q(X, L^k \otimes E) \cong H^q(X_c, L^k \otimes E) \cong H^{0,q}(X_c, L^k \otimes E) \cong \mathscr{H}^{0,q}(X_c, L^k \otimes E)$ for $q \geq 1$ and sufficiently large k. $\qquad\square$

Remark 3.14 Marinescu [22] positively answered a question of Ohsawa [27, §1. Remark 2] by proving Morse inequalities on weakly 1-complete manifolds. If L is q-positive outside a compact subset $K \subset X_c$, $\dim H^p(X_c, L^k)$ are at most of polynomial growth of degree n with respect to k for $p \geq q$. Theorem 1.6 says that if L is 1-positive outside a compact subset $K \subset X_c$ and $L \geq 0$ on X_c additionally, then $\dim H^p(X_c, L^k)$ are at most of polynomial growth of degree $n - p$ with respect to k for $p \geq 1$.

Generally we have the following result when (L, h^L) might be not semipositive.

Corollary 3.15 *Let X be a weakly 1-complete manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Suppose there exists $f \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $\sqrt{-1}R^{(L,h^L)} + \sqrt{-1}\partial\bar{\partial}f \geq 0$ on X, and $\sqrt{-1}R^{(L,h^L)} + \sqrt{-1}\partial\bar{\partial}f > 0$ on $X \backslash K$ for a compact subset $K \subset X$. Then there exists $C > 0$ such that for any $q \geq 1$ and sufficiently large k, we have*

$$\dim H^q(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.31}$$

Proof Apply Theorem 1.6 for the line bundle $(L, h^L e^{-f})$. □

Note that, by definition, there always exists a smooth function f on a weakly 1-convex manifold X such that $\sqrt{-1}\partial\overline{\partial}f \geq 0$, thus Corollary 3.15 implies the estimate may still hold when (L, h^L) is not semipositive everywhere.

Remark 3.16 (Compatibility to Nakano vanishing theorem on weakly 1-convex manifolds [19, Theorem 3.5.15]) If $(L, h^L) > 0$ on X instead of $(L, h^L) \geq 0$ in Theorem 1.6, then for $q \geq 1$ and sufficiently large k, $H^q(X, L^k \otimes E) = 0$, which implies the vanishing theorem on 1-convex (in particular, compact) manifold in Remark 3.12.

3.4 Complete manifolds

Proof of Theorem 1.2 We follow [19, Theorem 3.3.5]. Since (X, ω) is complete, $\overline{\partial}_{k,H}^{E*} = \overline{\partial}_k^{E*}$, that is, the Hilbert space adjoint and the maximal extension of the formal adjoint of $\overline{\partial}_k^E$ coincide for arbitrary holomorphic Hermitian vector bundle (E, h^E). Let $\Lambda = i(\omega)$ be the adjoint of the operator $\omega \wedge \cdot$ with respect to the Hermitian inner product induced by ω and h^L. In a local orthonormal frame $\{w_j\}_{j=1}^n$ of $T^{(1,0)}X$ with dual frame $\{w^j\}_{j=1}^n$ of $T^{(1,0)*}X$, $\omega = \sqrt{-1}\sum_{j=1}^n w^j \wedge \overline{w}^j$ and $\Lambda = -\sqrt{-1}i_{\overline{w}_j}i_{w_j}$. Thus $\sqrt{-1}R^{(L,h^L)} = \sqrt{-1}\sum_{j=1}^n w^j \wedge \overline{w}^j$ outside K.

Let $\{e_k\}$ be a local frame of L^k. For $s \in \Omega_0^{n,q}(X\backslash K, L^k)$, we can write $s = \sum_{|J|=q} s_J w^1 \wedge \cdots \wedge w^n \wedge \overline{w}^J \otimes e_k$ locally, thus

$$[\sqrt{-1}R^L, \Lambda]s = \sum_{|J|=q}(qs_J w^1 \wedge \cdots \wedge w^n \wedge \overline{w}^J) \otimes e_k = qs. \tag{3.32}$$

Since $(X\backslash K, \sqrt{-1}R^{(L,h^L)})$ is Kähler, we apply Nakano's inequality [19, (1.4.52)],

$$\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2 = \langle \Box^{L^k} s, s\rangle \geq k\langle[\sqrt{-1}R^L, \Lambda]s, s\rangle. \tag{3.33}$$

Thus for any $1 \leq q \leq n$,

$$\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2 \geq qk\|s\|^2 \geq k\|s\|^2. \tag{3.34}$$

Therefore, we have

$$\|s\|^2 \leq \frac{1}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2), \tag{3.35}$$

for $s \in \Omega_0^{n,q}(X\backslash K, L^k)$ with $1 \leq q \leq n$.

Next we follow the analogue argument in Proposition 3.8 to obtain the fundamental estimates as follows. Let V and U be open subsets of X such that $K \subset V \Subset U \Subset X$. We choose a function $\xi \in \mathscr{C}_0^\infty(U, \mathbb{R})$ such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $\overline{V}$. We set $\phi := 1 - \xi$, thus $\phi \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfying $0 \leq \phi \leq 1$ and $\phi \equiv 0$ on $\overline{V}$.

Now let $s \in \Omega_0^{n,q}(X, L^k)$, thus $\phi s \in \Omega_0^{n,q}(X\backslash K, L^k)$. We set $K' := \overline{U}$, then

$$\|\phi s\|^2 \geq \|s\|^2 - \int_{K'} |s|^2 dv_X, \tag{3.36}$$

and similarly there exists a constant $C_1 > 0$ such that

$$\frac{1}{k}(\|\overline{\partial}_k(\phi s)\|^2 + \|\overline{\partial}_k^*(\phi s)\|^2) \leq \frac{5}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2) + \frac{12C_1}{k}\|s\|^2. \tag{3.37}$$

By combining the above three inequalities, there exists $C_0 > 0$ such that for any $s \in \Omega_0^{n,q}(X, L^k) = \Omega_0^{0,q}(X, L^k \otimes K_X)$ with $1 \leq q \leq n$ and k large enough

$$\|s\|^2 \leq \frac{C_0}{k}(\|\overline{\partial}_k s\|^2 + \|\overline{\partial}_k^* s\|^2) + C_0 \int_{K'} |s|^2 dv_X. \tag{3.38}$$

Finally, since $\Omega_0^{0,\bullet}(X, L^k \otimes K_X)$ is dense in $\mathrm{Dom}(\overline{\partial}_k^{K_X}) \cap \mathrm{Dom}(\overline{\partial}_k^{K_X*})$ in the graph-norm, for each $1 \leq q \leq n$ the fundamental estimate holds in bidegree $(0, q)$ for forms with values in $L^k \otimes K_X$ for k large. So the conclusion follows from Theorem 1.1. $\qquad \square$

3.5 Compact manifolds and coverings revisited

Theorem 3.17 ([4]) *Let X be a compact complex manifold of dimension n. Let (L, h^L) and (E, h^E) be holomorphic Hermitian line bundles on X. Assume (L, h^L) is semipositive on X. Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$ we have*

$$\dim H^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.39}$$

Note that this theorem is a special case of Theorem 1.1, 1.2, 1.3, 1.4, 1.6, 1.7 and the following Theorem 3.22.

Corollary 3.18 *Let X be a compact manifold of dimension n, and let (L, h^L), (E, h^E) be holomorphic Hermitian line bundles on X. Let (L, h^L) be seminegative on X. Then there exists $C > 0$ such that $0 \leq q \leq n - 1$,*

$$\dim H^{0,q}(X, L^k \otimes E) \leq Ck^q. \tag{3.40}$$

In particular, $\dim H^{0,0}(X, L^k \otimes E) \leq C$.

Proof It follows from Serre duality and Theorem 3.17. $\qquad \square$

For the case of nef line bundles, the following observation refines the estimates in Theorem 3.17 as well as Corollary 3.18, and reflects that the magnitude k^{n-q} are precise.

Lemma 3.19 [12] *Let L be a nef holomorphic line bundle on a compact complex manifold X. Then every non-trivial section in $H^0(X, L^{-1})$ has no zero at all.*

Corollary 3.20 *Let X be a compact manifold of dimension n, and let L be holomorphic Hermitian line bundles on X. Let L be nef. Then, for any $k \in \mathbb{N}$,*

$$0 \leq \dim H^{0,n}(X, L^k \otimes K_X) = \dim H^{0,0}(X, L^{-k}) \leq 1. \tag{3.41}$$

Proof Suppose there exists $k \geq 1$ such that $H^0(X, L^{-k}) \neq 0$, then there exists $s_0 \in H^0(X, L^{-k})$ such that $s_0(x) \neq 0$ for all $x \in X$ by Lemma 3.19. Let $s \in H^0(X, L^{-k})$. It follows that $s \otimes s_0^{-1} \in \mathscr{O}(X) = \mathbb{C}$, thus $H^0(X, L^{-k}) = \mathbb{C}s_0$. And the case of $k = 0$ is trivial. $\qquad \square$

Remark 3.21 Let X be a compact complex manifold of dimension n and (L, h^L) a holomorphic Hermitian line bundle on X. Let $M := \{v \in L^* : |v|_{h^{L^*}} = 1\}$. It is known that the $\overline{\partial}_b$ (Kohn-Rossi) cohomology $H_{b,k}^q(M) \cong H^q(X, L^k)$, see [20, Section 1.5] and [16, (2.8)]. Thus if $L \geq 0$ on X, $\dim H_{b,k}^q(M) \leq Ck^{n-q}$ for all $q \geq 1$ and $k \geq 1$.

The study of L^2 cohomology spaces on coverings of compact manifolds has also interesting applications, cf. [14,18]. The results are similar to the case of compact manifolds, but we have to use the reduced L^2 cohomology groups and von Neumann dimension instead of the usual dimension, see [31, Theorem 1.2] or Theorem 3.22. For example, in the situation of Theorem 3.22, if the line bundle (L, h^L) is positive, the Andreotti-Vesentini vanishing theorem [1] shows that $\overline{H}_{(2)}^{0,q}(X, L^k \otimes E) \cong \mathscr{H}^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and k large enough. The holomorphic Morse inequalities of Demailly [11] were generalized to coverings by Chiose-Marinescu-Todor [23,30] (cf. also [19, (3.6.24)]) and yield in the conditions of Theorem 3.22 that $\dim_\Gamma \overline{H}_{(2)}^{0,q}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. Hence Theorem 3.22 generalizes [4] to covering manifolds and refines the estimates obtained in [23,30].

Theorem 3.22 [31] *Let (X, ω) be a Γ-covering manifold of dimension n. Let (L, h^L) and (E, h^E) be two Γ-invariant holomorphic Hermitian line bundles on X. Assume (L, h^L) is semipositive on X. Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$ we have*

$$\dim_\Gamma \overline{H}_{(2)}^{0,q}(X, L^k \otimes E) = \dim_\Gamma \mathscr{H}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}. \tag{3.42}$$

See [31, Theorem 1.2] for the complete proof. As a remark, note that for a fundamental domain $U \Subset X$ with respect to Γ, we used

$$\dim_\Gamma \mathscr{H}^{0,q}(X, L^k \otimes E) = \int_U B_k^q(x) dv_X, \tag{3.43}$$

which is similar to the formula

$$\dim \mathscr{H}^{0,q}(X, L^k \otimes E) \leq C_0 \int_K B_k^q(x) dv_X \tag{3.44}$$

used in the proof of Theorem 1.1.

The following estimate was firstly obtained by Morse inequalities for covering manifolds [23,30]. We do not assume the holomorphic Morse inequalities for coverings in our proof.

Corollary 3.23 *Under the hypothesis of Theorem 3.22, if (L, h^L) is positive at least at one point additionally, then there exists $C_1 > 0$ and $C_2 > 0$ such that for k large enough*

$$C_1 k^n \leq \dim_\Gamma H^0(X, L^k \otimes E) \leq C_2 k^n. \tag{3.45}$$

Proof It follows from Riemann-Roch-Hirzebruch formula for coverings [3]

$$\sum_{q=0}^n (-1)^{n-q} \dim_\Gamma \overline{H}_{(2)}^{0,q}(X, L^k \otimes E) = \frac{k^n}{n!} \int_{X/\Gamma} (-1)^n c_1(L/\Gamma, h^{L/\Gamma})^n + o(k^n) \tag{3.46}$$

(see also [19, Theorem 3.6.7]) and Theorem 3.22. $\square$

4 Semipositive line bundles endowed metric with analytic singularities

Let X be a connected compact complex manifold of dimension n and L a holomorphic line bundle on X. Let h^L be a Hermitian metric with analytic singularities with local weight

$$\varphi = \frac{c}{2} \log \left(\sum_{j \in J} |f_j|^2 \right) + \psi, \tag{4.1}$$

where J is at most countable, c is a non-negative rational number, f_j are non-zero holomorphic functions and ψ is a smooth function, such that $h^L(e_L, e_L) = e^{-2\varphi}$ for a local holomorphic frame e_L of L. We denote by $\mathcal{J}(h^L) := \mathcal{J}(\varphi)$ the Nadel multiplier ideal sheaf of h^L. We define the regular part of X with respect to h^L by $R(h^L) := \{x \in X : h^L \text{ smooth on a neighborhood of } x\}$ and the singular part by $S(h^L) := X \backslash R(h^L)$.

In this section, we follow the argument of Bonavero's singular holomorphic Morse inequalities [5] closely and provide three lemmas, see [19, 2.3.2] for details. In the end, we combine them to prove a result which is analogue to Theorem 3.17.

Firstly, we blow up the singularities of h^L as below, see [19, Lemma 2.3.19].

Lemma 4.1 *There exists a proper modification $\widetilde{\pi} : \widetilde{X} \longrightarrow X$ such that the local weight $\widetilde{\varphi}$ of the metric $h^{\widetilde{L}} = \widetilde{\pi}^* h^L$ on $\widetilde{L} = \widetilde{\pi}^* L$ has the form $\varphi \circ \widetilde{\pi} = \frac{c}{2} \log |g|^2 + \widetilde{\psi}$, where g is holomorphic and $\widetilde{\psi}$ is smooth.*

Secondly, we construct a smooth Hermitian metric $h^{\widehat{L}}$ on a modified line bundle $\widehat{L}$ on $\widetilde{X}$, see [19, Lemma 2.3.20, Lemma 2.3.21]. For a holomorphic vector bundle F over X, we denote by $\widetilde{F} := \pi^* F$ the pull-back on $\widetilde{X}$.

Lemma 4.2 *With the proper modification in Lemma 4.1, there exists a holomorphic line bundel $\widehat{L}$ on $\widetilde{X}$ and a smooth Hermtian metric $h^{\widehat{L}}$ satisfying the following conditions:*

(1) There exists $m \in \mathbb{N}\backslash\{0\}$ such that the curvature is locally given by

$$R^{(\widehat{L}, h^{\widehat{L}})} = 2m\partial\overline{\partial}\widetilde{\psi}; \tag{4.2}$$

(2) For any $k \in \mathbb{N}\backslash\{0\}$ and arbitrary holomorphic vector bundle $\widetilde{E}$ on $\widetilde{X}$, there exists $k' \in \mathbb{N}$, $m' \in [0, m)$ with $k = mk' + m'$, and a holomorphic vector bundle $\widetilde{E}_{m'}$ over $\widetilde{X}$ with $\mathrm{rank}(\widetilde{E}_{m'}) = \mathrm{rank}(\widetilde{E})$, such that

$$H^q(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E} \otimes \mathcal{J}(h^{\widetilde{L}^k})) = H^q(\widetilde{X}, \widehat{L}^{k'} \otimes \widetilde{E}_{m'}). \tag{4.3}$$

Here note that there exists $C_1 > 0$ and $C_2 > 0$ such that for any integer $p \in [0, n]$ and k large enough, $C_1 k^p \leq k'^p \leq C_2 k^p$.

Thirdly, relation to the cohomology on X, see [19, (2.3.45)].

Lemma 4.3 *Let E be an arbitrary holomorphic vector bundle over X. With the proper modification in Lemma 4.1, for all $q \geq 0$ and k large enough, there exists an isomorphism*

$$H^q(X, L^k \otimes E \otimes \mathcal{J}(h^{L^k})) \cong H^q(\widetilde{X}, \widetilde{L}^k \otimes \widetilde{E} \otimes K_{\widetilde{X}} \otimes \widetilde{K_X^*} \otimes \mathcal{J}(h^{\widetilde{L}^k})). \tag{4.4}$$

Finally we substitute these into our setting of semi-positivity of (L, h^L) on X, and obtain another generalization of Theorem 3.17.

Proof of Theorem 1.7 Consider a local weight φ like in (4.1) defined on an open connected subset $U \subset X$. Thus

$$S(h^L) \cap U = \{x \in U : \varphi \text{ not smooth at } x\} = \cap_{j \in J} Z(f_j), \tag{4.5}$$

where $Z(f_j) := \{x \in U : f_j(x) = 0\}$.

Let $\widetilde{\pi} : \widetilde{X} \to X$ be a proper modification of Lemma 4.1. The local weight of $(\widetilde{L}, h^{\widetilde{L}})$ on $\widetilde{U} := \widetilde{\pi}^{-1}(U)$ has the form

$$\varphi \circ \widetilde{\pi} = \frac{c}{2}\log\left(\sum_{j \in J} |f_j \circ \widetilde{\pi}|^2\right) + \psi \circ \widetilde{\pi} = \frac{c}{2}\log|g|^2 + \widetilde{\psi}, \tag{4.6}$$

where $\widetilde{\psi}$ is smooth on $\widetilde{U}$. Thus $Z(g) := \{z \in \widetilde{U} : g(y) = 0\}$ satisfies

$$Z(g) = \{y \in \widetilde{U} : \varphi \circ \widetilde{\pi} \text{ not smooth at } y\} = \widetilde{\pi}^{-1}(S(h^L) \cap U). \tag{4.7}$$

Since $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi} R^{(L, h^L)} \geq 0$ on $X \backslash S(h^L)$, φ is smooth plurisubharmonic on $U \backslash S(h^L)$, i.e., $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ on $U \backslash S(h^L)$. Therefore, $\varphi \circ \widetilde{\pi}$ is smooth plurisubharmonic on $\widetilde{\pi}^{-1}(U \backslash S(h^L)) = \widetilde{U} \backslash Z(g)$, and since $\partial\overline{\partial} \log |g|^2 = 0$ on $\widetilde{U} \backslash Z(g)$, we have on $\widetilde{U} \backslash Z(g)$

$$\sqrt{-1}\partial\overline{\partial}\widetilde{\psi} = \sqrt{-1}\partial\overline{\partial}(\varphi \circ \widetilde{\pi}) \geq 0. \tag{4.8}$$

Next we show $\sqrt{-1}\partial\overline{\partial}\widetilde{\psi} \geq 0$ on $Z(g)$. In fact, suppose there exists $y_0 \in Z(g)$ such that $\sqrt{-1}\partial\overline{\partial}\widetilde{\psi}(y_0)$ has at least one negative eigenvalue. By the smoothness of $\widetilde{\psi}$, there exists an open neighbourhood $V_0 \subset \widetilde{U}$ of y_0 satisfying $\sqrt{-1}\partial\overline{\partial}\widetilde{\psi}$ has at least one negative eigenvalue on V_0. Since $Z(g)$ is nowhere dense subset in $\widetilde{U}$, there exists $y_1 \in V_0$ such that $y_1 \in \widetilde{U} \backslash Z(g)$. So we obtain a contradiction.

Finally, by using Lemma 4.2 (1), we obtain $c_1(\widehat{L}, h^{\widehat{L}}) = \frac{\sqrt{-1}}{2\pi} R^{(\widehat{L}, h^{\widehat{L}})} = \frac{m}{\pi}\sqrt{-1}\partial\overline{\partial}\widetilde{\psi} \geq 0$ on $\widetilde{U}$, i.e., $(\widehat{L}, h^{\widehat{L}})$ is a semipositive line bundle on $\widetilde{X}$. By (4.4), (4.3) and Theorem 3.17 applied to $\widetilde{X}$ with $(\widehat{L}, h^{\widehat{L}})$, we have

$$\dim H^q(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) = H^q(\widetilde{X}, \widehat{L}^{k'} \otimes (\widetilde{E} \otimes K_{\widetilde{X}} \otimes \widetilde{K_X^*})_{m'}) \leq Ck'^{n-q} \leq CC_2 k^{n-q}.$$

Since a positive current $c_1(L, h^L)$ on X is semipositive on $R(h^L)$, the last assertion follows. $\qquad\square$

In analogy to the covering manifolds case, we have the following estimate of the space of holomorphic sections obtained firstly by Bonavero, see [6] and [19, Corollary 2.3.46].

Corollary 4.4 *Under the hypothesis of Theorem 1.7, if (L, h^L) is positive at least at one point additionally, then for k large enough*

$$C_1 k^n \leq \dim H^0(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) \leq C_2 k^n. \tag{4.9}$$

In particular, if X is additionally Kähler, then X is projective.

Proof It follows from Theorem 1.7 and asymptotic Riemann-Roch-Hirzebruch formula for h^L with analytic singularities (see [19, (2.3.45), (2.3.31), (1.7.1)])

$$\sum_{q=0}^{n} (-1)^{n-q} \dim H^q(X, L^k \otimes E \otimes \mathscr{J}(h^{L^k})) = \frac{k^n}{n!} \int_{R(h^L)} (-1)^n c_1(L, h^L)^n + o(k^n), \tag{4.10}$$

where $R(h^L) := \{x \in X : h^L \text{ smooth on a neighborhood of } x\}$ is the regular part of X with respect to h^L. Thus L is big, X is Moishezon. So X is projective when it is Kähler. $\qquad\square$

Corollary 4.5 *Let X be a compact manifold of dimension n, let L and E be holomorphic line bundles on X. Let h^L be the Hermitian metric on L with analytic singularities as in (4.1). Assume $c_1(L, h^L) \geq 0$ on $R(h^L)$. Then, as $k \to \infty$, we have*

$$\dim H^q(X, E \otimes L^k \otimes \mathscr{J}(h^{L^k})) = \begin{cases} \frac{k^n}{n!} \int_{X(0)} c_1(L, h^L)^n + o(k^n), & \text{for } q = 0, \\ \mathcal{O}(k^{n-q}), & \text{for } 1 \leq q \leq n. \end{cases} \tag{4.11}$$

Proof The second equality is from Theorem 1.7 and the first is given by combining the second equality and (4.10). □

Remark 4.6 The smooth version (i.e., h^L is smooth everywhere) of Corollary 4.5 are given by [4,11]. Matsumura [24, Problem 3.4] asked the question if one can remove the assumption that h^L has analytic singularities. He proved this in the case where X is projective in [25, Theorem 4.1]. Note that on projective manifolds, Demailly had similar estimates $\mathcal{O}(k^{n-q})$ for nef line bundles, see [8].

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