



Counting tropical rational curves with cross-ratio constraints

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Abstract

We enumerate rational curves in toric surfaces passing through points and satisfying cross-ratio constraints using tropical and combinatorial methods. Our starting point is (Tyomkin in Adv Math 305:1356–1383, 2017), where a tropical-algebraic correspondence theorem was proved that relates counts of rational curves in toric varieties that satisfy point conditions and cross-ratio constraints to the analogous tropical counts. We proceed in two steps: based on tropical intersection theory we first study tropical cross-ratios and introduce degenerated cross-ratios. Second we provide a lattice path algorithm that produces all rational tropical curves satisfying such degenerated conditions explicitly. In a special case simpler combinatorial objects, so-called cross-ratio floor diagrams, are introduced which can be used to determine these enumerative numbers as well.

Keywords Tropical geometry · Enumerative geometry · Cross-ratios · Floor diagrams · Lattice path algorithm · Degenerations

Mathematics Subject Classification 14N10 · 14T05

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1 Introduction

Tropical geometry is a rather young field of mathematics that is intimately connected to algebraic geometry, non-Archimedean analytic geometry and combinatorics. In the past tropical geometry turned out to be a powerful tool to answer enumerative questions. To apply tropical geometry to enumerative questions, so-called correspondence theorems are needed. A correspondence theorem states that an enumerative number equals its tropical counterpart, where in tropical geometry we have to count each tropical object with a suitable multiplicity reflecting the number of classical objects in our counting problem that tropicalize to the given tropical object. Thus tropical geometry hands us a new approach to enumerative problems: first find a suitable correspondence theorem, then use combinatorics to enumerate the tropical objects in question. A famous example is the following: let $d \in \mathbb{N}_{>0}$ be a degree and assume that points in general position in \mathbb{P}^2 are given in such a way that only finitely many rational plane curves of degree d pass through these points. What is the number N_d of curves passing through these points? For small d , this question can be answered using methods from classical algebraic geometry. In the '90s, Kontsevich presented a recursive formula that computes N_d for arbitrary d [19]. Tropical geometry offers a new approach to compute the numbers N_d , and generalizations thereof: in [22], Mikhalkin pioneered the use of tropical methods in enumerative geometry by proving a correspondence theorem for counts of curves in toric surfaces satisfying point conditions.

Moduli spaces of (stable) curves resp. maps to toric surfaces are an important tool in enumerative geometry, both in algebraic and in tropical geometry. Often, an enumerative problem can be expressed as an intersection product on the moduli space parametrizing the objects to be counted. Gathmann and Markwig started to use tropical moduli space techniques in order to give a tropical proof of Kontsevich's formula in [15]. Both in the original proof of Kontsevich and in this tropical proof, the count of rational plane curves of degree d satisfying point, line and a cross-ratio condition is an essential ingredient.

A *cross-ratio* is a rational number associated to four collinear points. It encodes the relative position of these four points to each other. It is invariant under projective transformations and can therefore be used as a constraint that four points on \mathbb{P}^1 should satisfy. So a cross-ratio can be viewed as a condition on elements of the moduli space of n -pointed rational stable maps to a toric variety. Tropical cross-ratios were first introduced by Mikhalkin under the name "tropical double ratio" in [23] and can be thought of as paths of fixed lengths in a tropical curve. More precisely: A parameterized plane rational tropical curve (alternatively: tropical stable map) is a 1-dimensional polyhedral complex (mapped to \mathbb{R}^2 satisfying the balancing condition) whose first Betti number is zero and whose unbounded polyhedra (points on a tropical curve are contracted unbounded polyhedra) are uniquely labeled (see Definition 2.16). A *tropical cross-ratio* consists of two pieces of information: First a pair of pairs of labels of unbounded polyhedra such that all occurring labels are pairwise different and second a length. A parameterized plane rational tropical curve satisfies a tropical cross-ratio if forgetting the map to \mathbb{R}^2 and all unbounded polyhedra which are not given in the cross-ratio leaves an abstract tropical curve whose bounded parts' lengths sum up to the given length in the cross-ratio such that forgetting the bounded parts splits the four remaining labels into the two given pairs—see Fig. 1 for an example and Definition 3.1 for more details. It is natural to ask: Given point conditions p_1, \dots, p_n and cross-ratio constraints $\lambda_1, \dots, \lambda_l$ in such a way that there are only finitely many parametrized rational tropical curves of a given degree in a toric surface satisfying them, then

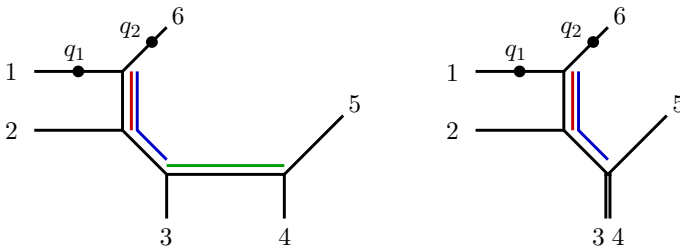


Fig. 1 Left: a degree two plane tropical curve that is fixed by two points q_1, q_2 and three cross-ratios: the red length associated to the four labels (16|23), the blue length associated to the four labels (1 q_2 |35) and the green length associated to the four labels (23|45) are fixed. Right: degenerating the cross-ratio associated to the green path means shrinking the green path, thus producing a 4-valent vertex (with two unbounded edges on top of each other) (color figure online)

- (1) How many of these curves are there and what are their multiplicities?
- (2) Can we construct them?

These questions motivated the study in this paper. Recall that applying tropical geometry to an enumerative problem happens in two steps: use a correspondence theorem, then use combinatorics. The correspondence theorem we are going to use is provided by Tyomkin in [31]. It also describes the multiplicities with which parameterized rational tropical curves have to be counted when they satisfy cross-ratio constraints.

Our approach to answer questions (1) and (2) can be subdivided into two steps. The first step is to develop a notion of *degenerated* tropical cross-ratios that helps us to simplify the combinatorics. Notice that Tyomkin’s correspondence theorem provides the multiplicities only in the non-degenerated case. The tradeoff when simplifying the combinatorics by considering degenerated tropical cross-ratios is that the multiplicities coming from the cross-ratio constraints get more involved. The second step to answer questions (1) and (2) is to explicitly construct all parameterized rational tropical curves that satisfy the given point and degenerated cross-ratio conditions using combinatorial methods. We want to explain these two steps and the methods used more precisely:

Degenerated cross-ratios

In Sect. 3 a generalization of Mikhalkin’s definition of tropical cross-ratios is introduced that allows us to use tropical intersection theory in order to degenerate tropical cross-ratios. If we think of a cross-ratio as a path of fixed length in a tropical curve, then a degenerated cross-ratio is a path of length zero—see Fig. 1. Obviously, the set of tropical curves satisfying given conditions becomes easier when degenerating the cross-ratios. The difficult part is to determine the multiplicities with which we have to count such curves. These multiplicities have a local description, which we present in Theorem 3.20 together with the fact that the number of parameterized tropical curves (counted with multiplicity) satisfying point and cross-ratio conditions stays invariant when degenerating the cross-ratios. The techniques used to prove Theorem 3.20 are *tropical moduli spaces* and *tropical intersection theory*.

Moduli spaces of abstract rational tropical curves were studied in [23]. They also show up in the study of the tropical Grassmannian as the space of trees [6,29]. It turns out that these tropical moduli spaces are tropicalizations of the corresponding moduli spaces in algebraic geometry in a suitable embedding [17,30]. Tropicalizations of moduli spaces of curves

of higher genus (in a toroidal and non-Archimedean setting) were studied by Abramovich, Caporaso and Payne [1]. The theory of rational tropical stable maps was introduced by Gathmann, Kerber and Markwig in [16]. Recently, Ranganathan [24] tropicalized the moduli space of stable rational maps to toric surfaces using logarithmic and non-Archimedean geometry. An excellent overview of the current development concerning compactifications of moduli spaces and tropical moduli spaces can be found in [12].

We use tropical intersection theory on moduli spaces of rational stable maps, building on Allermann and Rau [3,26]. Katz [18] related tropical intersection theory to intersection theory on toric varieties studied by Fulton and Sturmfels in [14]. For matroidal fans (i.e. tropicalizations of linear spaces) Shaw offers in [27] a framework of tropical intersection theory. Tropical intersection theory is still an active area of research.

All in all degenerating cross-ratios is a natural approach in the following sense: A parameterized rational tropical curve satisfying non-degenerated conditions can be *degenerated* to a parameterized rational tropical curve that satisfies degenerated conditions itself. This observation allows us to answer question (2) if we can construct parameterized rational tropical curves that satisfy degenerated conditions. We offer an algorithm for this construction in Sect. 4.

Combinatorial methods

Both the lattice path algorithm and floor diagrams are well-known combinatorial tools in tropical geometry. In Sect. 4 we generalize the *lattice path algorithm* to a *cross-ratio lattice path algorithm*. Lattice paths were used in [21] and [22] to construct curves satisfying point conditions. Since we want to find tropical curves that satisfy point and degenerated cross-ratio conditions, we need to generalize this approach. There are other generalizations (in particular [20]) of lattice paths that inspired our definition of cross-ratio lattice paths. The lattice path algorithm can also be extended to determine invariants connected to counts of real curves as well, see [28].

In Sect. 5 we prove Theorem 5.3, which states that the lattice path algorithm yields the number of rational parameterized tropical curves satisfying point and tropical cross-ratio conditions counted with multiplicity. Thus Theorem 5.3 answers question (1). Moreover, the cross-ratio lattice path algorithm we provide allows us to construct all tropical curves of a given degree that satisfy the given point conditions and the degenerated tropical cross-ratio constraints.

In Sect. 6 we restrict to curves in Hirzebruch surfaces and impose a restriction to our cross-ratios such that we can use simpler combinatorial objects than the ones we deal with when applying the cross-ratio lattice path algorithm. These simpler combinatorial objects are called *cross-ratio floor diagrams*. They are a generalization of *floor diagrams*. Floor diagrams are graphs that arise from so-called floor decomposed tropical curves by forgetting some information. Floor diagrams were introduced by Mikhalkin and Brugallé in [10] (and [11]) to give a combinatorial description of Gromov–Witten invariants of Hirzebruch surfaces. Floor diagrams have also been used to establish polynomiality of the node polynomials [13] and to give an algorithm to compute these polynomials in special cases—see [7]. Moreover, floor diagrams have been generalized, for example in case of Ψ -conditions, see [8], or for counts of curves relative to a conic [9].

Theorem 6.14 states that counting floor diagrams with multiplicities yields the same numbers as counting parameterized rational tropical curves with multiplicities that satisfy point and tropical cross-ratio conditions. Hence floor diagrams offer (besides the cross-ratio lattice path algorithm) another (simpler) way of answering question (1).

2 Preliminaries

In this preliminary section we give a short introduction to tropical intersection theory and tropical moduli spaces as needed in this paper. We fix the following conventions: polytopes are convex, and we work over a non-Archimedean closed field of characteristic zero.

Tropical intersection theory

This subsection summarizes intersection theoretic background from [2–4].

Definition 2.1 (*Normal vectors and balanced fans*) Let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space associated to a given lattice Γ and let X be a fan in V . The lattice generated by $\text{span}(\kappa) \cap \Gamma$, where κ is a cone of X , is denoted by Γ_{κ} . Let σ be a cone of X and τ be a face of σ of dimension $\dim(\tau) = \dim(\sigma) - 1$ (we write $\tau < \sigma$). A vector $u_{\sigma} \in \Gamma_{\sigma}$ that generates $\Gamma_{\sigma}/\Gamma_{\tau}$ such that $u_{\sigma} + \tau \subset \sigma$ defines a class $u_{\sigma/\tau} := [u_{\sigma}] \in \Gamma_{\sigma}/\Gamma_{\tau}$ that does not depend on the choice of u_{σ} . This class is called *normal vector of σ relative to τ* .

X is a *weighted fan of dimension k* if X is of pure dimension k and there are weights on its facets (i.e. its k -dimensional faces), that is there is a map $\omega_X : X^{(k)} \rightarrow \mathbb{Z}$. The number $\omega_X(\sigma)$ is called *weight* of the facet σ of X . To simplify notation, we write $\omega(\sigma)$ if X is clear. Moreover, a weighted fan (X, ω_X) of dimension k is called a *balanced fan of dimension k* if

$$\sum_{\sigma \in X^{(k)}, \tau < \sigma} \omega(\sigma) \cdot u_{\sigma/\tau} = 0$$

holds in $V/\langle \tau \rangle_{\mathbb{R}}$ for all faces τ of dimension $\dim(\tau) = \dim(\sigma) - 1$.

Definition 2.2 (*Affine cycles*) Let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space associated to a given lattice Γ . A *tropical fan X (of dimension k)* is a balanced fan of dimension k in V and $[(X, \omega_X)]$ denotes the refinement class of X with weights ω_X (see Definition 2.8 and Construction 2.10 of [3]). Such a class is also called an *affine (tropical) k -cycle* in V . Denote the set of all affine k -cycles in V by $Z_k^{\text{aff}}(V)$. For a fan X in V , we may also define an *affine k -cycle* in X as an element $[(Y, \omega_Y)]$ of $Z_k^{\text{aff}}(V)$ such that the support of Y with nonzero weights lies in the support of X (see Definition 2.15 of [3]). Define $[(X, \omega_X)] := X^*$, where X^* denotes the support of X with nonzero weights.

The set $Z_k^{\text{aff}}(V)$ (resp. $Z_k^{\text{aff}}([(X, \omega_X)])$) can be turned into an abelian group by taking unions while refining appropriately.

Definition 2.3 (*Rational functions*) Let $[(X, \omega_X)]$ be an affine k -cycle. A *(nonzero) rational function on $[(X, \omega_X)]$* is a continuous piecewise linear function $\varphi : [(X, \omega_X)] \rightarrow \mathbb{R}$, i.e. there exists a representative (X, ω_X) of $[(X, \omega_X)]$ such that on each cone $\sigma \in X$ the map φ is the restriction of an integer affine linear function. The set of (nonzero) rational functions of $[(X, \omega_X)]$ is denoted by $\mathcal{K}^*([(X, \omega_X)])$.

Define $\mathcal{K}([(X, \omega_X)]) := \mathcal{K}^*([(X, \omega_X)]) \cup \{-\infty\}$ such that $(\mathcal{K}([(X, \omega_X)]), \max, +)$ is a semifield, where the constant function $-\infty$ is the “zero” function.

Definition 2.4 (*Divisor associated to a rational function*) Let $[(X, \omega_X)]$ be an affine k -cycle in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $\varphi \in \mathcal{K}^*([(X, \omega_X)])$ a rational function on $[(X, \omega_X)]$. Let (X, ω) be a representative of $[(X, \omega_X)]$ on whose cones φ is affine linear and denote these linear pieces by φ_{σ} . We denote by $X^{(i)}$ the set of all i -dimensional cones of X . We define $\text{div}(\varphi) := \varphi \cdot [(X, \omega_X)] := [(\bigcup_{i=0}^{k-1} X^{(i)}, \omega_{\varphi})] \in Z_{k-1}^{\text{aff}}([(X, \omega_X)])$, where

$$\omega_\varphi : X^{(k-1)} \rightarrow \mathbb{Z}$$

$$\tau \mapsto \sum_{\sigma \in X^{(k)}, \tau < \sigma} \varphi_\sigma(\omega(\sigma)v_{\sigma/\tau}) - \varphi_\tau \left(\sum_{\sigma \in X^{(k)}, \tau < \sigma} \omega(\sigma)v_{\sigma/\tau} \right)$$

and the $v_{\sigma/\tau}$ are arbitrary representatives of the normal vectors $u_{\sigma/\tau}$. If $[(Y, \omega_Y)]$ is an affine k -cycle in $[(X, \omega_X)]$, we define $\varphi \cdot [(Y, \omega_Y)] := \varphi|_{[(Y, \omega_Y)]} \cdot [(Y, \omega_Y)]$.

Example 2.5 Let $[(X, \omega_X)]$ be the affine 1-cycle with representative (X, ω_X) whose weights are all 1 and whose 1-dimensional rays are given by $-e_x, -e_y, e_x + e_y$, where e_x, e_y are the vectors of the standard basis of \mathbb{R}^2 such that $X \subset \mathbb{R}^2$. Then

$$\varphi : X \rightarrow \mathbb{R}$$

$$(x, y) \mapsto \max(x, y, 0)$$

is a rational function on $[(X, \omega_X)]$ and (X, ω_X) is a representative such that φ is integer linear affine on each cone. The divisor associated to φ , namely $\varphi \cdot X$, is given by the 1-skeleton of X which is just one point (namely $0 \in \mathbb{R}^2$) and that point has weight 1. We calculate this weight as an example: Let $\tau = 0 \in \mathbb{R}^2, \sigma_1 = \text{cone}(-e_x), \sigma_2 = \text{cone}(-e_y)$ and $\sigma_3 = \text{cone}(e_x + e_y)$ be cones of X . Applying Definition 2.4, we get

$$\begin{aligned} \omega_\varphi(\tau) &= \varphi_{\sigma_1}(\omega(\sigma_1)v_{\sigma_1/\tau}) + \varphi_{\sigma_2}(\omega(\sigma_2)v_{\sigma_2/\tau}) + \varphi_{\sigma_3}(\omega(\sigma_3)v_{\sigma_3/\tau}) \\ &\quad - \varphi_\tau(\omega(\sigma_1)v_{\sigma_1/\tau} + \omega(\sigma_2)v_{\sigma_2/\tau} + \omega(\sigma_3)v_{\sigma_3/\tau}) \\ &= \varphi_{\sigma_3}(\omega(\sigma_3)v_{\sigma_3/\tau}) \\ &= \varphi_{\sigma_3}(1(e_x + e_y)) = 1 \end{aligned}$$

because $\varphi_{\sigma_1}, \varphi_{\sigma_2}, \varphi_\tau \equiv 0$ and $\varphi_{\sigma_3}(e_x + e_y) = \max(1, 1, 0)$.

Definition 2.6 (Affine intersection product) Let $[(X, w_X)]$ be an affine k -cycle. The subgroup of globally linear functions in $\mathcal{K}^*([(X, w_X)])$ with respect to $+$ is denoted by $\mathcal{O}^*([(X, w_X)])$. We define the group of affine Cartier divisors of $[(X, w_X)]$ to be the quotient group $\text{Div}([(X, w_X)]) := \mathcal{K}^*([(X, w_X)]) / \mathcal{O}^*([(X, w_X)])$. Let $[\varphi] \in \text{Div}([(X, w_X)])$ be a Cartier divisor. The divisor associated to this function is denoted by $\text{div}([\varphi]) := \text{div}(\varphi)$ and is well-defined. The following bilinear map is called *affine intersection product*

$$\cdot : \text{Div}([(X, w_X)]) \times Z_k^{\text{aff}}([(X, w_X)]) \rightarrow Z_{k-1}^{\text{aff}}([(X, w_X)])$$

$$([\varphi], [(Y, w_Y)]) \mapsto [\varphi] \cdot [(Y, w_Y)] := \varphi \cdot [(Y, w_Y)].$$

Definition 2.7 (Morphisms of fans) Let X be a fan in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and Y a fan in $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. A morphism $f : X \rightarrow Y$ is a \mathbb{Z} -linear map from $|X| \subseteq V$ to $|Y| \subseteq V'$ induced by a \mathbb{Z} -linear map on the lattices. A morphism of weighted fans is a morphism of fans. A morphism of affine cycles $f : [(X, \omega_X)] \rightarrow [(Y, \omega_Y)]$ is a morphism of weighted fans $f : X^* \rightarrow Y^*$ that is independent of the choice of representatives, where X^* (resp. Y^*) denotes the support of X (resp. Y) with nonzero weight.

Definition 2.8 (Push-forward of affine cycles) Let $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. Let $[(X, w_X)] \in Z_m^{\text{aff}}(V)$ and $[(Y, w_Y)] \in Z_n^{\text{aff}}(V')$ be cycles with representatives (X, ω_X) and (Y, ω_Y) . Let $f : X \rightarrow Y$ be a morphism. Choosing a refinement of (X, ω_X) , the set of cones

$$f_*X := \{f(\sigma) \mid \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a tropical fan in V' of dimension m with weights

$$\omega_{f_*X}(\sigma') := \sum_{\sigma \in X^{(m)}: f(\sigma)=\sigma'} \omega_X(\sigma) \cdot |\Gamma'_{\sigma'} / f(\Gamma_\sigma)|$$

for all $\sigma' \in f_*X^{(m)}$. The equivalence class of (f_*X, ω_{f_*X}) is uniquely determined by the equivalence class of (X, ω_X) . For $[(Z, \omega_Z)] \in Z_k^{\text{aff}}([(X, \omega_X)])$ we define

$$f_*[(Z, \omega_Z)] := [(f_*(Z^*), \omega_{f_*(Z^*)})] \in Z_k^{\text{aff}}([(Y, \omega_Y)])$$

The map

$$Z_k^{\text{aff}}([(X, w_X)]) \rightarrow Z_k^{\text{aff}}([(Y, w_Y)]), [(Z, w_Z)] \mapsto f_*[(Z, w_Z)]$$

is well-defined, \mathbb{Z} -linear and $f_*[(Z, w_Z)]$ is called *push-forward of $[(Z, w_Z)]$ along f* .

Definition 2.9 (*Pull-back of Cartier divisors*) Let $[(X, w_X)] \in Z_m^{\text{aff}}(V)$ and $[(Y, w_Y)] \in Z_n^{\text{aff}}(V')$ be cycles in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. Let $f : [(X, w_X)] \rightarrow [(Y, w_Y)]$ be a morphism. The map

$$\begin{aligned} \text{Div}([(Y, w_Y)]) &\rightarrow \text{Div}([(X, w_X)]) \\ [h] &\mapsto f^*[h] := [h \circ f] \end{aligned}$$

is well-defined, \mathbb{Z} -linear and $f^*[h]$ is called *pull-back of $[h]$ along f* .

So far, we introduced affine cycles only. Affine cycles are building blocks of *abstract* cycles. Since the whole “affine-to-abstract”-procedure is quite technical, we omit it here and refer to section 5 of [3] instead. For our purposes the following definition of abstract cycles is sufficient:

Definition 2.10 (*Abstract cycles*) An *abstract k -cycle C* is a class under a refinement relation of a balanced polyhedral complex of pure dimension k which is locally isomorphic to tropical fans.

Remark 2.11 (*Rational functions on abstract cycles*) In the same way rational functions on affine cycles led to an affine intersection product, one can also consider *rational functions on abstract cycles* to obtain an intersection product. Again, we want to omit technicalities and refer to Definition 6.1 of [3] instead. The main point of considering rational functions on abstract cycles is that they are no longer piecewise linear but piecewise affine linear.

As we see below, it happens that we start with an affine cycle $[(X, \omega_X)]$ and want to intersect it with a rational function f that is piecewise affine linear. In order to do so, we need to refine $[(X, \omega_X)]$ in such a way that f is linear on faces. Hence $[(X, \omega_X)]$ becomes a polyhedral complex which is a representative of an abstract cycle C . Then we can intersect f with C .

In the following we want to restrict to tropical intersection theory on \mathbb{R}^n .

Definition 2.12 (*Degree map*) Let $A_0(\mathbb{R}^n)$ denote the set of abstract 0-cycles in \mathbb{R}^n up to rational equivalence. The map

$$\begin{aligned} \text{deg} : A_0(\mathbb{R}^n) &\rightarrow \mathbb{Z} \\ [\omega_1 P_1 + \dots + \omega_r P_r] &\mapsto \sum_{i=1}^r \omega_i \end{aligned}$$

is a well-defined morphism and for $D \in A_0(\mathbb{R}^n)$ the number $\text{deg}(D)$ is called the *degree of D* .

Remark 2.13 (Rational equivalence) Concepts like pull-backs and push-forwards carry over to abstract cycles. Moreover, there is a concept of rational equivalence of abstract cycles (section 8 of [3]). When we consider abstract cycles, we usually consider them up to this equivalence relation. The most important facts about rational equivalence that we use are the following:

- (a) Pull-backs of rationally equivalent cycles are rationally equivalent.
- (b) If two 0-dimensional cycles are rationally equivalent, then their degrees are the same.
- (c) Two cycles in \mathbb{R}^n that only differ by a translation are rationally equivalent.

Tropical moduli spaces

This subsection collects background on tropical moduli spaces following [16].

Definition 2.14 (*Moduli space of abstract tropical curves of genus zero*) An abstract rational tropical curve is a metric tree Γ with unbounded edges called ends and with $\text{val}(v) \geq 3$ for all vertices $v \in \Gamma$. It is called rational n -marked tropical curve of genus zero $(\Gamma, x_1, \dots, x_n)$ if Γ has exactly n ends that are labeled with pairwise different $x_1, \dots, x_n \in \mathbb{N}$. Two n -marked tropical curves of genus zero $(\Gamma, x_1, \dots, x_n)$ and $(\tilde{\Gamma}, \tilde{x}_1, \dots, \tilde{x}_n)$ are isomorphic if there is a homeomorphism $\Gamma \rightarrow \tilde{\Gamma}$ mapping x_i to \tilde{x}_i for all i and each edge of Γ is mapped onto an edge of $\tilde{\Gamma}$ by an affine linear map of slope ± 1 . The set $\mathcal{M}_{0,n}$ of all n -marked tropical curves of genus zero up to isomorphism is called *moduli space of n -marked tropical curves of genus zero*. Forgetting all lengths of an n -marked tropical curve gives us its *combinatorial type*.

Remark 2.15 ($\mathcal{M}_{0,n}$ is a tropical fan) We have the distance map

$$\text{dist} : \mathcal{M}_{0,n} \rightarrow \mathbb{R}^{\binom{n}{2}}$$

$$\Gamma \mapsto (\text{length of the path from end } i \text{ to end } j)_{ij}$$

and define v_I ($I \subset \{1, \dots, n\}, |I| \geq 2, |I^C| \geq 2$) to be the image under dist of the n -marked tropical curve that has only one bounded edge of length one with markings I on one and markings I^C on the other side. Moreover, the map

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}^{\binom{n}{2}}$$

$$a \mapsto (a_i + a_j)_{ij}$$

induces (by abuse of notation) an injective map

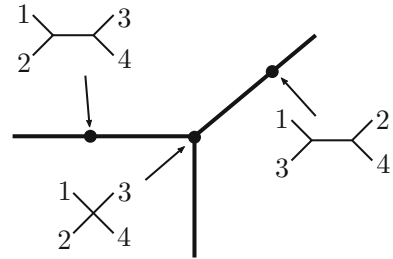
$$\text{dist} : \mathcal{M}_{0,n} \rightarrow \mathbb{R}^{\binom{n}{2}} / \text{Im}(\phi).$$

If we choose

$$\Lambda_n := \sum_{\substack{I \subset \{1, \dots, n\} \\ |I| \geq 2}} v_I \mathbb{Z}$$

to be the lattice of $\mathbb{R}^{\binom{n}{2}} / \text{Im}(\phi)$, then $\mathcal{M}_{0,n} \subseteq \mathbb{R}^{\binom{n}{2}} / \text{Im}(\phi)$ is a tropical fan of pure dimension $n - 3$ with its fan structure given by combinatorial types, and with all weights equal one, i.e. $\mathcal{M}_{0,n}$ represents an affine cycle in some \mathbb{R}^t , see Fig. 2. This allows us to use tropical intersection theory on $\mathcal{M}_{0,n}$.

Fig. 2 One way of embedding the moduli space $\mathcal{M}_{0,4}$ into \mathbb{R}^2 centered at the origin of \mathbb{R}^2 . The length of a bounded edge of a tropical curve depicted above is given by the distance between the point in $\mathcal{M}_{0,4}$ corresponding to this curve and the origin of \mathbb{R}^2



Definition 2.16 (*Degree*) Let $\#\Delta \in \mathbb{N}_{>0}$. A set $\Delta := \{(v_i, x_i)\}_{i=1, \dots, \#\Delta}$ of tuples is called *degree* if

- (1) $0 \neq v_i \in \mathbb{R}^2$ for all $i = 1, \dots, \#\Delta$, and $\langle v_1, \dots, v_{\#\Delta} \rangle = \mathbb{R}^2$, and $\sum_i v_i = 0$.
- (2) $x_i \in \mathbb{N}_{>0}$ for all $i = 1, \dots, \#\Delta$, and $x_i \neq x_j$ for all $i \neq j$. An x_i is called *label*.

Let Σ be a 2-dimensional lattice polytope in \mathbb{R}^2 with facets E_1, \dots, E_m whose lattice lengths are denoted by $|E_1|, \dots, |E_m|$ and let e_1, \dots, e_m be unordered partitions of E_1, \dots, E_m , that is e_i is a partition of E_i of some length denoted by $l(e_i)$ for $i = 1, \dots, m$. If

$$\{v_i\}_{i=1, \dots, \#\Delta} = \bigcup_{i=1}^m \bigcup_{j=1}^{l(e_i)} \{e_{ij} \cdot \text{pnv}(E_i)\},$$

where $\text{pnv}(E_i)$ is the primitive normal vector of E_i^\perp for $i = 1, \dots, m$, then Δ is said to be associated to a polytope Σ with partitions e_1, \dots, e_m and is referred to as $\Delta(\Sigma(e_1, \dots, e_m))$.

Important special cases that we use later are the following:

- If each entry of each partition e_i is one, then the associated degree is denoted by $\Delta(\Sigma)$.
- In case of degree d curves in \mathbb{P}^2 , the degree Δ is defined as follows: Let Σ_d be the convex hull of $\{(0, 0), (d, 0), (d, 0)\} \in \mathbb{R}^2$ for some $d \in \mathbb{N}_{>0}$, then Δ_d is the degree associated to Σ_d , where the labels are given by: vectors parallel to (and with the same direction as) $(-1, 0) \in \mathbb{R}^2$ have labels $1, \dots, d$, vectors parallel to (and with the same direction as) $(0, -1)$ have labels $d + 1, \dots, 2d$ and vectors parallel to (and with the same direction as) $(1, 1)$ have labels $2d + 1, \dots, 3d$.
- In case of degree $(|\alpha|, |\beta|)$ curves of contact orders α, β in the first Hirzebruch surface, the degree Δ is defined as follows: Let $s \in \mathbb{N}_{>0}$ and $b \in \mathbb{N}$. Let $\alpha = (\alpha_1, \dots)$ be an unordered partition of $b + s$, let $\beta = (\beta_1, \dots)$ be an unordered partition of b and let $\Sigma(\alpha, \beta)$ be the convex hull of $\{(0, 0), (s, 0), (s, b), (0, b + s)\} \in \mathbb{R}^2$. We associate the degree $\Delta(\alpha, \beta)$ to the polytope $\Sigma(\alpha, \beta)$, where the partition of the left facet is given by α and the partition of the right facet is given by β . Moreover, vectors parallel to (and with the same direction as) $(-1, 0) \in \mathbb{R}^2$ have labels $1, \dots, l(\alpha)$, vectors parallel to (and with the same direction as) $(1, 0)$ have labels $l(\alpha) + 1, \dots, l(\alpha) + l(\beta)$.

Definition 2.17 (*Moduli space of rational tropical stable maps to \mathbb{R}^2*) An n -pointed rational tropical stable map of degree Δ to \mathbb{R}^2 (alternatively: rational tropical curve with n points) is a tuple $(\Gamma, x_1, \dots, x_N, h)$, where $(\Gamma, x_1, \dots, x_N)$ is an N -marked rational tropical curve (with $N = \#\Delta + n$ and x_{n+1}, \dots, x_N the labels given by Δ) and $h : \Gamma \rightarrow \mathbb{R}^2$ such that:

- (a) Let $e \in \Gamma$ be an edge with length $l(e) \in [0, \infty]$, identify e with $[0, l(e)]$ and denote the vertex of e that is identified with $0 \in [0, l(e)] = e$ by V . The map h is integer affine linear when restricted to e , i.e. $h|_e : t \mapsto tv + a$ with $a \in \mathbb{R}^2$ and $v(e, V) := v \in \mathbb{Z}^2$,

where $v(e, V)$ is called *direction vector of e at V* and the *weight* of an edge (denoted by $\omega(e)$) is the gcd of the entries of $v(e, V)$. If $e = x_i \in \Gamma$ is an end, then $v(x_i)$ denotes the direction vector of x_i pointing away from its one vertex it is adjacent to.

(b) If $i > n$, then the direction vector $v(x_i)$ of an end labeled with x_i is given by

$$v(x_i) := v_{i-n},$$

where v_{i-n} is defined by Δ . If $i \leq n$, then the direction vector of the end labeled with x_i is zero. Ends with direction vector zero are called *contracted ends* or *points*.

(c) The *balancing condition*

$$\sum_{\substack{e \in \Gamma \text{ an edge,} \\ V \text{ vertex of } e}} v(e, V) = 0$$

holds for every vertex $V \in \Gamma$.

Two n -pointed rational tropical stable maps of degree Δ , namely $(\Gamma, x_1, \dots, x_N, h)$ and $(\Gamma', x'_1, \dots, x'_N, h')$, are isomorphic if there is an isomorphism φ of their underlying N -marked tropical curves of genus zero such that $h' \circ \varphi = h$.

The set $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ of all n -pointed rational tropical stable maps of degree Δ up to isomorphism is called *moduli space of n -pointed rational tropical stable maps of degree Δ* .

Remark 2.18 ($\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is a fan) The map

$$\begin{aligned} \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) &\rightarrow \mathcal{M}_{0,N} \times \mathbb{R}^2 \\ (\Gamma, x_1, \dots, x_N, h) &\mapsto ((\Gamma, x_1, \dots, x_N), h(x_1)) \end{aligned}$$

with $N = \#\Delta + n$ is bijective and $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is a tropical fan of dimension $\#\Delta - 1$, see Proposition 4.7 of [16]. Hence $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ represents an affine cycle in some \mathbb{R}^t . This allows us to use tropical intersection theory on $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$.

Definition 2.19 (*Evaluation maps*) For $i = 1, \dots, n$, the map

$$\begin{aligned} \text{ev}_i : \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) &\rightarrow \mathbb{R}^2 \\ (\Gamma, x_1, \dots, x_N, h) &\mapsto h(x_i) \end{aligned}$$

is called *i -th evaluation map*. Under the identification from Remark 2.18 the i -th evaluation map is a morphism of fans $\text{ev}_i : \mathcal{M}_{0,N} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, see Proposition 4.8 of [16]. This allows us to pull-back cycles via the evaluation map.

Example 2.20 (*Pull-back of a point*) A point $p = (p_1, p_2) \in \mathbb{R}^2$ is an intersection product of two rational functions, e.g.

$$p = \max\{p_1, x\} \cdot \max\{p_2, y\} \cdot \mathbb{R}^2,$$

where x, y are the coordinates in \mathbb{R}^2 . The pull-back of the point p under ev_i is defined to be

$$\text{ev}_i^*(p) := \text{ev}_i^*(\max\{p_1, x\}) \cdot \text{ev}_i^*(\max\{p_2, y\}) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta).$$

Definition 2.21 [*Forgetful maps*] For $n \geq 4$ the map

$$\begin{aligned} \text{ft} : \mathcal{M}_{0,n} &\rightarrow \mathcal{M}_{0,n-1} \\ (\Gamma, x_1, \dots, x_n) &\mapsto (\Gamma', x_1, \dots, x_{n-1}) \end{aligned}$$

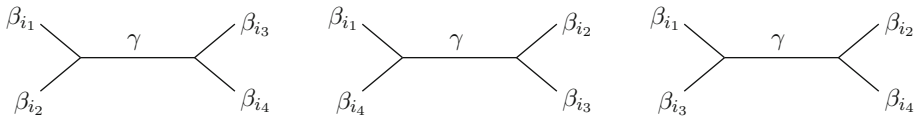


Fig. 3 Schematic picture of the three cases of $\epsilon(\gamma, i)$ for λ'_i as in Definition 2.22 for a 4-marked curve. From left to right: $\epsilon(\gamma, i) = 1, -1, 0$

where Γ' is the stabilization (straighten 2-valent vertices) of Γ after removing its end marked by x_n is called the n -th forgetful map. Applied recursively, it can be used to forget several ends with markings in $I^C \subset \{x_1, \dots, x_n\}$, denoted by ft_I , where I^C is the complement of $I \subset \{x_1, \dots, x_n\}$. With the identification from Remark 2.18, and additionally forgetting the map to the plane, we can also consider

$$\begin{aligned} \text{ft}_I &: \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \rightarrow \mathcal{M}_{0,|I|} \\ (\Gamma, x_1, \dots, x_n, h) &\mapsto \text{ft}_I(\Gamma, x_i | i \in I). \end{aligned}$$

Any forgetful map is a morphism of fans. This allows us to pull-back cycles via the forgetful map.

Correspondence theorem

The tropical counterpart to classical cross-ratios was first introduced by Mikhalkin under the name *tropical double ratio* in [23]. The correspondence theorem of [31] we use states that the number of classical curves satisfying point and cross-ratio conditions and the number of tropical curves satisfying point and tropical double ratio conditions are equal. Since different classical curves may tropicalize to the same tropical curve, each tropical curve has to be counted with a *multiplicity*. We recall the definition of these multiplicities. For that we stick to the notation used in [31], for more details see (4.1) of [31].

Definition 2.22 (*Tropical double ratios defined by [23,31]*) Let $(\Gamma, x_1, \dots, x_N, h) \in \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. Let $\{\beta_{i_1}, \beta_{i_3}\}$ and $\{\beta_{i_2}, \beta_{i_4}\}$ be two sets of labels of ends of Γ such that $\beta_{i_1}, \dots, \beta_{i_4}$ are pairwise different. A bounded edge γ of Γ separates β_{i_1}, β_{i_2} from β_{i_3}, β_{i_4} if β_{i_1}, β_{i_2} belong to one of the two connected components of $\Gamma \setminus \{\gamma\}$ and β_{i_3}, β_{i_4} to another.

The *tropical double ratio* λ'_i of $\{\beta_{i_1}, \beta_{i_2}\}$ and $\{\beta_{i_3}, \beta_{i_4}\}$ is given by

$$\lambda'_i := \sum_{\gamma} \epsilon(\gamma, i) |\gamma|,$$

where the sum goes over all bounded edges of Γ and $|\gamma|$ is the length of a bounded edge and

$$\epsilon(\gamma, i) := \begin{cases} 1, & \text{if } \gamma \text{ separates the ends } \beta_{i_1}, \beta_{i_2} \text{ from } \beta_{i_3}, \beta_{i_4}, \\ -1, & \text{if } \gamma \text{ separates the ends } \beta_{i_1}, \beta_{i_4} \text{ from } \beta_{i_2}, \beta_{i_3}, \\ 0, & \text{otherwise.} \end{cases}$$

See Fig. 3 for an example. Notice that by abuse of notation we do not incorporate the β_i 's into the notation of a tropical double ratio λ'_i .

Remark 2.23 (*Tropical double ratios and tropicalizations*) Note that tropical double ratios are indeed tropicalizations of classical cross-ratios (see Lemma 3.1 of [31]), i.e. given a classical

curve that satisfies a classical cross-ratio, then its tropicalization satisfies a tropical double ratio which is given by applying the valuation map to the classical cross-ratio.

Definition 2.24 (*Multiplicities*) Let $C = (\Gamma, x_1, \dots, x_N, h)$ be a tropical curve that satisfies given point conditions p_1, \dots, p_n and tropical double ratios $\lambda'_1, \dots, \lambda'_l$.

Let x_1 be the end of Γ that is contracted to p_1 under h . We refer to the vertex adjacent to x_1 in Γ as *root vertex* and orient all edges of Γ away from the root vertex. The *head* of a bounded edge γ is denoted by $h(\gamma)$ and its *tail* by $t(\gamma)$. Let $V(\Gamma)$ be the set of vertices of Γ and let $E^b(\Gamma)$ be the set of bounded edges of Γ . We refer to a vertex of Γ as v and to a bounded edge of Γ as γ for now. The vertices adjacent to ends x_1, \dots, x_N are denoted by v_1, \dots, v_N and do not need to be different. Define the complex

$$\theta : \underbrace{\bigoplus_{v \in V(\Gamma)} \mathbb{Z}^2 \oplus \bigoplus_{\gamma \in E^b(\Gamma)} \mathbb{Z}}_{M_1} \xrightarrow{B} \underbrace{\bigoplus_{\gamma \in E^b(\Gamma)} \mathbb{Z}^2 \oplus \bigoplus_{i=1}^n \mathbb{Z}^2 \oplus \bigoplus_{j=1}^l \mathbb{Z}}_{M_2} \tag{1}$$

given by the maps (that are defined copywise)

$$1_\gamma \mapsto n_\gamma + \sum_{i=1}^l \epsilon(\gamma, i) \quad \text{and} \quad a_v \mapsto \sum_\gamma \tilde{\epsilon}(\gamma, v) a_v + \sum_{i=1}^n \delta(v, v_i) a_{v_i},$$

where a_v is the coordinate vector of $h(v)$ and where (see Definition 2.17 for the notation of $v(\gamma, t(\gamma))$)

$$\tilde{\epsilon}(\gamma, v) := \begin{cases} 1, & \text{if } v = t(\gamma) \\ -1, & \text{if } v = h(\gamma) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad n_\gamma := v(\gamma, t(\gamma))$$

and

$$\delta(v, v_i) := \begin{cases} 1, & \text{if } v = v_i \\ 0, & \text{otherwise.} \end{cases}$$

Let $\theta_{\mathbb{Z}}$ be the map from above in the complex (1) $\otimes_{\mathbb{Z}} \mathbb{Z}$. Finally, we can define the *multiplicity of C*

$$m_{\mathbb{C}}(\Gamma, h) := \# \text{coker } \theta_{\mathbb{Z}},$$

which is equal to $|\det(B)|$.

Theorem 2.25 (Correspondence Theorem 5.1 of [31]) *Let Σ be a 2-dimensional lattice polytope and let X_Σ be its toric variety. Let q_1, \dots, q_n be points in X_Σ and let μ_1, \dots, μ_l be classical cross-ratio constraints. Let these conditions be in general position such that there is only a finite number of rational curves in X_Σ that fulfill them. Denote this number by $N_{0,n}^{\text{class}}(\mu_1, \dots, \mu_l)$. Let $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_l$ be the tropicalizations (see Remark 2.23) of the conditions above. Then*

$$N_{0,n}^{\text{class}}(\mu_1, \dots, \mu_l) = N_{0,n}(\lambda'_1, \dots, \lambda'_l)$$

holds, where $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ is the number of rational tropical curves of degree $\Delta(\Sigma)$ that satisfy the point conditions p_1, \dots, p_n and the tropical double ratio constraints $\lambda'_1, \dots, \lambda'_l$.

Example 2.26 When going through the (tropical) proof of Kontsevich’s formula [15], we can see that it allows us to determine the number of unlabeled tropical curves of degree Δ_d satisfying point conditions and exactly one tropical double ratio constraint which involves two point conditions and two line conditions. In this case *unlabeled* means that non-contracted edges not involved in any tropical double ratio condition are not equipped with a label.

In case of $d = 3$, Kontsevich’s formula yields 40 unlabeled curves (counted with multiplicity). Moreover, the proof of Kontsevich’s formula allows us to actually draw these tropical curves. Hence we can determine the number of labeled curves by putting labels on ends, which yields 1440 labeled curves.

3 Tropical cross-ratios

In this section we introduce tropical cross-ratios and their degenerations from an intersection theoretic point of view. Given a tropical curve that satisfies degenerated cross-ratios, we express its multiplicity locally.

Definition 3.1 (*Cross-ratios*) A (tropical) cross-ratio λ' is an unordered pair of pairs of unordered numbers $(\beta_1\beta_2|\beta_3\beta_4)$ together with an element in $\mathbb{R}_{>0}$ denoted by $|\lambda'|$, where β_1, \dots, β_4 are pairwise distinct ends of a tropical curve of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. We say that $C \in \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ satisfies the cross-ratio constraint λ' if $C \in \text{ft}_{\lambda'}^*(|\lambda'|) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$, where $|\lambda'|$ is the canonical local coordinate of the ray $(\beta_1\beta_2|\beta_3\beta_4)$ in $\mathcal{M}_{0,4}$.

Remark 3.2 Definition 3.1 generalizes Definition 2.22 of tropical double ratios used by Mikhalkin and Tyomkin since we can find a suitable projection $\pi : \mathcal{M}_{0,4} \rightarrow \mathbb{R}$ shrinking on ray to zero, sending another one to $\mathbb{R}_{>0}$ and the last one to $\mathbb{R}_{<0}$ such that $\pi \circ \text{ft}_{\lambda'}$ coincides with Definition 2.22. In particular, Theorem 2.25 holds for our notion of tropical cross-ratios.

Definition 3.3 (*General position I*) Let p_1, \dots, p_n be points in \mathbb{R}^2 and $\lambda'_1, \dots, \lambda'_l$ be cross-ratios that have pairwise distinct pairs of unordered numbers. These conditions are in *general position* if $\prod_{j=1}^l \text{ft}_{\lambda'_j}^*(|\lambda'_j|) \cdot \prod_{i=1}^n \text{ev}_i^*(p_i) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is a nonempty finite set that is contained in the union of the interiors of top-dimensional polyhedra of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ and $n + l = \#\Delta - 1$. We say that $p_1, \dots, p_n, \lambda_1, \dots, \lambda_{l'}$ with $n + l' < \#\Delta - 1$ are in general position if there are $p_{n+1}, \dots, p_n, \lambda'_{l'+1}, \dots, \lambda'_l$ such that $n + l = \#\Delta - 1$ and $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_l$ are in general position. If $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_l$ with $n + l = \#\Delta - 1$ are in general position, we define

$$N_{0,n}(\lambda'_1, \dots, \lambda'_l) := \text{deg} \left(\prod_{j=1}^l \text{ft}_{\lambda'_j}^*(|\lambda'_j|) \cdot \prod_{i=1}^n \text{ev}_i^*(p_i) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \right), \tag{2}$$

the number of rational tropical curves of degree Δ satisfying the point conditions p_i and the cross-ratio conditions λ'_i . Denote by $\mathcal{C}_{0,n}(\lambda'_1, \dots, \lambda'_l)$ the set of tropical curves contributing to $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$.

Remark 3.4 The numbers $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ are independent of the exact positions of the points since two sets of n points are rationally equivalent and so their pull-backs are rationally equivalent leading to the same degree (see Remark 2.13). Notice also that all points in $\mathcal{M}_{0,4}$ are rationally equivalent using Remark 2.13 since $\mathcal{M}_{0,4}$ can be embedded (cf. Fig. 2)

by a morphism into \mathbb{R}^2 and all points of \mathbb{R}^2 are rationally equivalent. Hence the numbers $N_{0,n}(\lambda'_1, \dots, \lambda'_j)$ are independent of the lengths $|\lambda'_i|$ of the cross-ratios. In particular, the lengths can be zero. This observation is crucial and is used extensively later. Moreover, $N_{0,n}(\lambda'_1, \dots, \lambda'_j)$ does not depend on the partition of the four entries of each cross-ratio into pairs.

Note that the intersection theoretic definition of tropical cross-ratios automatically assigns a multiplicity to each tropical curve satisfying given point conditions and cross-ratio constraints. In our case, Lemma 1.2.9 of [25] states that the intersection theoretic multiplicity of a tropical curve C is the absolute value of the determinant of the so called ev-ft-matrix which is given by the locally (around C) linear maps $ev : \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \rightarrow \mathbb{R}^{2n}$ and $ft : \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \rightarrow \mathcal{M}_{0,4}$, where the coordinates on $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ and $\mathcal{M}_{0,4}$ are the bounded edges' lengths.

Often, tropical intersection theory yields multiplicities needed for correspondence theorems, which enables us to count tropical curves by means of tropical intersection theory on tropical moduli spaces. The same holds true for the counts of curves satisfying cross-ratio conditions we consider here. We prove this in the following proposition, using methods well-known to the experts in the area.

Proposition 3.5 *Let C be a tropical curve contributing to (2). The intersection theoretic multiplicity of C coincides with $m_C(\Gamma, h)$ defined in Definition 2.24.*

Proof Let $C = (\Gamma, x_1, \dots, x_N, h)$ be a tropical curve that contributes to $N_{0,n}(\lambda'_1, \dots, \lambda'_j)$. In terms of tropical intersection theory the multiplicity of C is given by $|\det(A)|$, where A is the ev-ft-matrix that is given by the (around C) linear maps ev, ft and the lengths of the edges as coordinates on the moduli space. We want to sketch how to prove that $|\det(A)|$ and $|\det(B)|$ (from Definition 2.24) are equal. For that, we start with the following complex

$$\mathbb{Z}^2 \oplus \underbrace{\bigoplus_{\gamma \in E^b(\Gamma)} \mathbb{Z}}_{N_1} \xrightarrow{A} \underbrace{\bigoplus_{i=1}^n \mathbb{Z}^2 \oplus \bigoplus_{j=1}^l \mathbb{Z}}_{N_2}$$

where the first summand on the left belongs to the root vertex defined in Definition 2.24. There are maps between the complex above and the complex (1) in the following way: Let $\alpha_2 : N_2 \rightarrow M_2$ be the canonical embedding and let

$$\alpha_1 : N_1 \rightarrow M_1, (a, \underline{e}) \mapsto \left(a, a + \sum \pm e_i u_{e_i}, \underline{e} \right)$$

be a map where a is the coordinate of the root vertex, e_i is the length of the edge γ_i and u_{e_i} is the primitive direction vector of γ_i . Moreover, we choose $a + \sum \pm e_i u_{e_i}$ in such a way that it is the shortest path between the root vertex and the vertex associated to the j -th contracted end depending on which entry of the vector in the image we are considering (the choice of \pm should be consistent with the orientation on Γ). Note that α_1, α_2 are both injective and that the diagram given by the maps A, B, α_1, α_2 commutes. This commutative diagram extends to the commutative diagram shown below. By definition

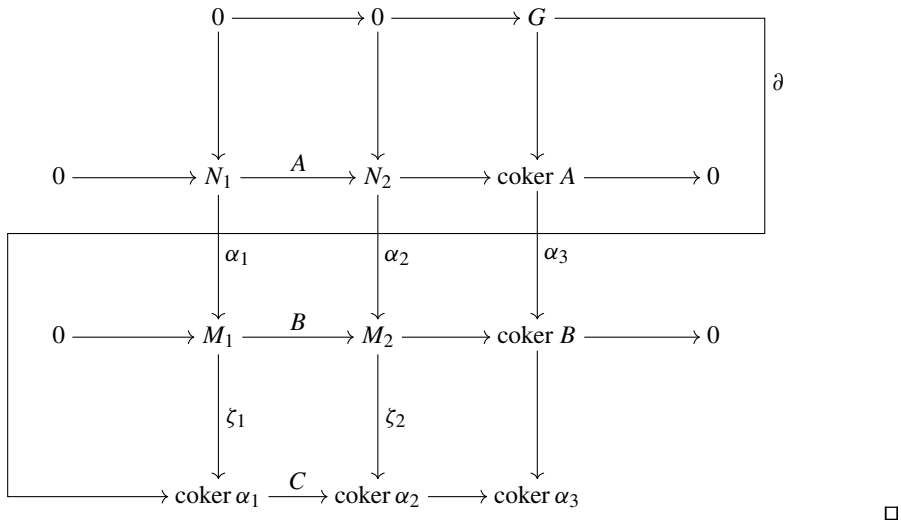
$$\text{coker } \alpha_1 \cong (\mathbb{Z}^2)^{\#V(\Gamma)-1} \quad \text{and} \quad \text{coker } \alpha_2 = (\mathbb{Z}^2)^{\#E^b(\Gamma)}.$$

Considering the definitions of B, ζ_2 , we can see that $\zeta_2 \circ B$ is surjective. Hence C is surjective. Since C is a surjective morphism of free module of the same rank it is an isomorphism.

Therefore $\text{coker } \alpha_3$ vanishes which guarantees that α_3 is surjective. The map ∂ which we obtain from applying the snake lemma yields that G vanishes. Therefore α_3 is an isomorphism. Thus

$$|\det(A)| = |\det(B)|$$

follows.



The strength of our intersection theoretic definition of tropical cross-ratios is that it allows us to *degenerate* tropical cross-ratios easily. For that note that from an intersection theoretic point of view it does not matter if we pull-back $0 \in \mathcal{M}_{0,4}$ instead of a nonzero point.

Definition 3.6 (*Cross-ratios with $|\lambda| = 0$*) A (tropical) cross-ratio λ with $|\lambda| = 0$ (or *degenerated* cross-ratio) is defined as a set $\{\beta_1, \dots, \beta_4\}$, where β_1, \dots, β_4 are pairwise distinct ends of a tropical curve $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. We say that $C \in \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ satisfies the cross-ratio constraint λ (with $|\lambda| = 0$) if $C \in \text{ft}_\lambda^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. Notice that $|\lambda|$ does not denote the number of elements in the set λ here. We use the symbol $\#$ to indicate that we mean the number of elements in a set.

Another way to think about a cross-ratio λ with $|\lambda| = 0$ is that λ is the *degeneration* of cross-ratios $\lambda'_j, j \in \mathbb{N}$ which have the same pairs of unordered numbers and $|\lambda'_j| \rightarrow 0$ for $j \rightarrow \infty$, where the pairs become a set in the limit. Because of Remark 3.4 it makes sense to refer to λ as the *degeneration* of λ'_j for some $j \in \mathbb{N}$.

Definition 3.7 (*General position II*) Let $\lambda_1, \dots, \lambda_{l'}$ be cross-ratios with $|\lambda_j| = 0$ for $j = 1, \dots, l'$. These cross-ratios are in *general position* if there are general positioned cross-ratios $\lambda'_1, \dots, \lambda'_{l'}$ such that λ_j is the degeneration of λ'_j for $j = 1, \dots, l'$. More precisely, points p_1, \dots, p_n in \mathbb{R}^2 , cross-ratios $\lambda_1, \dots, \lambda_{l'}, \lambda_{l'+1}, \dots, \lambda_{l'}$ with $|\lambda_j| = 0$ for $j = 1, \dots, l'$ and $|\lambda_j| > 0$ otherwise are in general position if $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_{l'}, \lambda_{l'+1}, \dots, \lambda_{l'}$ are in general position, where λ_j is the degeneration of λ'_j for $j = 1, \dots, l'$.

Notation 3.8 We want to fix the following conventions. If we mention a set of conditions, then we assume that these conditions are in general position and that the cross-ratio constraints are totally ordered by their lengths, i.e. $|\lambda_1| > |\lambda_2| > \dots$. Point conditions are always denoted

by p_1, \dots, p_n . Cross-ratios are denoted by λ'_i , where we have l' of these cross-ratios if the intersection defined by the conditions $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_{l'}$ is not a 0-dimensional cycle, and we have l cross-ratios if the intersection defined by the conditions $p_1, \dots, p_n, \lambda_1, \dots, \lambda_l$ is 0-dimensional. If we write λ_i , then λ_i is the degeneration of λ'_i . It may also happen that we need classical (i.e. non-tropical) cross-ratios. A classical cross-ratio is denoted by μ_i and its tropical counterpart obtained from applying the valuation map of the ground field is denoted by λ'_i .

Our next aim is to describe the multiplicity of a curve that satisfies point conditions and degenerated cross-ratio conditions. For that we observe that degenerating a cross-ratio means to shrink an edge, i.e. degenerating the tropical curve satisfying it as well. Therefore the multiplicity of such a degenerated tropical curve C can be described in terms of the number of tropical curves degenerating to C .

Definition 3.9 (*Resolving vertices w.r.t. a cross-ratio with $|\lambda| = 0$*) The combinatorial type of a polyhedron $\tau \subset \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ (resp. $\mathcal{M}_{0,m}$) is denoted by $c(\tau)$. Let $\lambda_1, \dots, \lambda_{l'}$ be degenerated cross-ratios and let $\tau \subset \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ be some polyhedron. The set λ_v of cross-ratios associated to a vertex v of $c(\tau)$ consists of the cross-ratios λ_j such that the image of v under ft_{λ_j} is 4-valent. If

$$\text{val}(v) = 3 + \#\lambda_v$$

holds, then we say that v is *resolved according to λ'_i* (we use Notation 3.8) if we replace v by two vertices v_1, v_2 that are connected by a new edge such that

$$\lambda_v = \{\lambda_i\} \cup \lambda_{v_1} \cup \lambda_{v_2}$$

is a union of pairwise disjoint sets and

$$\text{val}(v_k) = 3 + \#\lambda_{v_k}$$

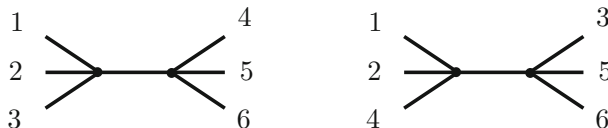
holds for $k = 1, 2$.

Example 3.10 In this example we want to point out that resolving a vertex according to a cross-ratio is not unique. It is neither unique in the sense (A) that the edges adjacent to v_1, v_2 are uniquely determined nor in the (weaker) sense (B) that the λ_{v_i} are uniquely determined.

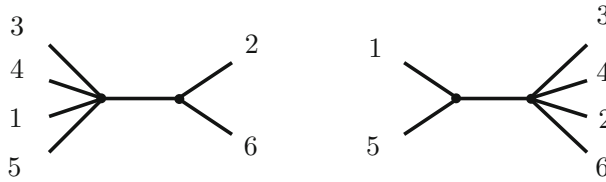
Let τ be the 0-dimensional cell of $\mathcal{M}_{0,6}$, that is $c(\tau)$ has only one vertex v to which all ends are adjacent to. We choose the following cross-ratios:

$$\begin{aligned} \lambda_1 &= \{1, 2, 3, 4\}, & \lambda'_1 &= (12|34) \\ \lambda_2 &= \{3, 4, 5, 6\}, & \lambda'_2 &= (34|56) \\ \lambda_3 &= \{1, 2, 5, 6\}, & \lambda'_3 &= (12|56) \end{aligned}$$

(A) If we resolve v according to λ'_3 , we have at least two choices shown in the figure below.



(B) If we choose another λ'_3 , namely $\lambda'_3 = (15|26)$, we also have at least two choices shown in the figure below.



Lemma 3.11 For notation, see Notation 3.8. The intersection product $X := \prod_{j=1}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ lies in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)^{(l')}$ and its top-dimensional polyhedra are top-dimensional polyhedra τ of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)^{(l')}$ such that for all vertices v of $c(\tau)$

$$\text{val}(v) = 3 + \#\lambda_v$$

holds and the weight of a top-dimensional polyhedron τ of X is given recursively by

$$\omega(\tau) = \begin{cases} 1, & \text{if } l' = 1 \\ \sum_{\sigma} \omega(\sigma), & \text{otherwise} \end{cases}$$

where the sum runs over all top-dimensional polyhedra of $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ such that $c(\sigma)$ is given by resolving the vertex $v \in c(\tau)$, that is defined by $\lambda_1 \in \lambda_v$, according to λ'_1 . In particular, all weights of X are non-negative.

Note that the intersection product X above does not depend on the non-degenerated cross-ratios $\lambda'_1, \dots, \lambda'_{l'}$ that degenerate to $\lambda_1, \dots, \lambda_{l'}$. We consider X up to rational equivalence. We use $\lambda'_1, \dots, \lambda'_{l'}$ to describe a representative of X under this equivalence relation.

Proof Let $\lambda'_1, \dots, \lambda'_{l'}$ be cross-ratios such that λ_j is the degeneration of λ'_j for $j = 1, \dots, l'$. The pull-back of 0 along ft_{λ_j} is given by a Cartier divisor $\max(\text{ft}_{\lambda_j}(\star), 0)$ (see Example 2.5), where $\max(\star, \star, 0) : (x, y) \mapsto \max(x, y, 0)$ is a Cartier divisor on $\mathcal{M}_{0,4} \subset \mathbb{R}^2$ (see Fig. 2). Note that $\max(\text{ft}_{\lambda_j}(\star), 0)$ is a linear function on every cell of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ for $j = 1, \dots, l'$. Therefore no refinement of $\prod_{j \neq i} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is necessary when intersecting with some $\text{ft}_{\lambda_j}^*(0)$. Hence X lies in the codimension- l' -skeleton of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. Moreover, every intersection with a Cartier divisor lowers the dimension by one, so the dimension of X is exactly the dimension of top-dimensional cells of the codimension- l' -skeleton of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$.

To prove the last part of the lemma, we set $m = n + \#\Delta$ and identify

$$\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \cong \mathcal{M}_{0,m} \times \mathbb{R}^2$$

as in Remark 2.18 such that it is sufficient to prove the statements for $\mathcal{M}_{0,m}$ because cross-ratio constraints only fix a tropical curve up to translation in \mathbb{R}^2 . To do so, we use induction on the number of cross-ratio constraints. Let $m \in \mathbb{N}_{>3}$.

We start with one cross-ratio $\lambda_1 = \{\beta_1, \dots, \beta_4\}$ with $|\lambda_1| = 0$. Obviously, a top-dimensional polyhedron τ of $\text{ft}_{\lambda_1}^*(0) \cdot \mathcal{M}_{0,m}$ is a top-dimensional polyhedron $\mathcal{M}_{0,m}^{(1)}$ such that $\text{val}(v) = 3 + \#\lambda_v$ holds for the only 4-valent vertex v of $c(\tau)$ since $\#\lambda_v = \#\{\lambda_1\} = 1$. Note that the three resolutions of v correspond to three top-dimensional polyhedra $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}, \sigma_{(\beta_1\beta_3|\beta_2\beta_4)}, \sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$ of $\mathcal{M}_{0,m}$ that arise from inserting a new edge e , where $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$ denotes the polyhedron where e separates $\{\beta_1, \beta_2\}$ from $\{\beta_3, \beta_4\}$. On two of

the polyhedra $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$, $\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}$, $\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$ the map $\max(\text{ft}_{\lambda_j}(\star), 0)$ is the zero function and on one of that polyhedra it maps each point to the length of the edge that was obtained from resolving the vertex v . Which of the $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$, $\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}$ or $\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$ are mapped to zero depends on the choice of coordinates of $\mathcal{M}_{0,4} \subset \mathbb{R}^2$. Let $v_{(\beta_1\beta_2|\beta_3\beta_4)}$ denote the direction vector in $\mathcal{M}_{0,m}$ associated to a tropical curve that has only one edge of length one that separates the ends β_1, β_2 from β_3, β_4 (see the following figure), and define $v_{(\beta_1\beta_3|\beta_2\beta_4)}$, $v_{(\beta_1\beta_4|\beta_2\beta_3)}$, respectively. See Fig. 4 for an example of the notation used. We assume without loss of generality that $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$ is not mapped to zero under $\max(\text{ft}_{\lambda_j}(\star), 0)$. Therefore $v_{(\beta_1\beta_2|\beta_3\beta_4)}$ is mapped to 1 under $\max(\text{ft}_{\lambda_j}(\star), 0)$ and $v_{(\beta_1\beta_3|\beta_2\beta_4)}$, $v_{(\beta_1\beta_4|\beta_2\beta_3)}$ are mapped to zero. We write $\varphi := \max(\text{ft}_{\lambda_j}(\star), 0)$. The weight $\omega_\varphi(\tau)$ is

$$\omega_\varphi(\tau) = \sum_{\substack{\sigma=\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}, \\ \sigma_{(\beta_1\beta_3|\beta_2\beta_4)}, \sigma_{(\beta_1\beta_4|\beta_2\beta_3)}}} \varphi_\sigma (\omega(\sigma) \cdot v_{\sigma/\tau}) - \varphi_\tau \left(\sum_{\substack{\sigma=\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}, \\ \sigma_{(\beta_1\beta_3|\beta_2\beta_4)}, \sigma_{(\beta_1\beta_4|\beta_2\beta_3)}}} \omega(\sigma) \cdot v_{\sigma/\tau} \right),$$

where $\varphi_\sigma, \varphi_\tau$ denote the linear parts of φ on σ, τ , $\omega(\sigma) = 1$ denotes the weight of σ in $\mathcal{M}_{0,m}$ and $v_{\sigma/\tau}$ denotes an arbitrary representative of the normal vector $u_{\sigma/\tau}$. Moreover, $v_{\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}/\tau} = v_{(\beta_1\beta_2|\beta_3\beta_4)}$ and $v_{\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}/\tau}, v_{\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}/\tau}$, respectively. Note that the second sum is in τ as $\mathcal{M}_{0,m}$ is balanced and because of $\tau \subset \text{ft}_{\lambda_1}^{-1}(0)$ this second sum vanishes under φ_τ . As discussed above only one summand of the first sum is nonzero, namely $\sigma = \sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$. Hence $\omega_\varphi(\tau) = 1$.

Next, we perform the induction step from $l' - 1$ to l' . We denote the elements of λ_1 as above, that is $\lambda_1 = \{\beta_1, \dots, \beta_4\}$ with $|\lambda_1| = 0$. We use the fact that

$$\prod_{j=1}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m} = \text{ft}_{\lambda_1}^*(0) \cdot \left(\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m} \right)$$

and then use the induction hypothesis for $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m}$. A top-dimensional polyhedron τ of $\text{ft}_{\lambda_1}^*(0) \cdot \left(\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m} \right)$ is a top-dimensional polyhedron of $\mathcal{M}_{0,m}^{(l')}$ such that there is a vertex v of τ with $\lambda_1 \in \lambda_v$. Since the interior of τ is in the codimension-1-boundary of $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m}$ and the cross-ratio lengths are without loss of generality small, the vertex v is obtained by shrinking an edge connecting two vertices v_1, v_2 in the combinatorial type of a top-dimensional polyhedron of $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,m}$ such that

$$\begin{aligned} \text{val}(v) &= 3 + \#\lambda_{v_1} + 3 + \#\lambda_{v_2} - 2 \\ &= 4 + \#(\lambda_{v_1} \cup \lambda_{v_2}) \\ &= 3 + \#(\lambda_{v_1} \cup \lambda_{v_2} \cup \{\lambda_1\}) \\ &= 3 + \#\lambda_v. \end{aligned}$$

Again there are three resolutions of v and we choose the coordinates on $\mathcal{M}_{0,4}$ such that the top-dimensional polyhedra of $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ given by resolving the vertex v according to the pairs of unordered numbers of λ'_1 are not mapped to zero. The weight $\omega_\varphi(\tau)$ is

$$\omega_\varphi(\tau) = \sum_{\sigma} \varphi_\sigma (\omega(\sigma) \cdot v_{\sigma/\tau}) - \varphi_\tau \left(\sum_{\sigma} \omega(\sigma) \cdot v_{\sigma/\tau} \right),$$

where the sums run over all top-dimensional polyhedra of $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ that have τ in their boundaries. Since $\prod_{j=2}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is balanced, the second sum is in τ and vanishes. Moreover, the arguments above yield that $\varphi_\sigma(v_{\sigma/\tau})$ is zero if and only if v is not resolved according to λ'_1 . By definition $\varphi_\sigma(v_{\sigma/\tau}) = 1$ otherwise.

Definition 3.12 (*Local description of the weights of X*) Let τ be a top-dimensional polyhedron of X (for notation, see Lemma 3.11) of weight $\omega(\tau)$. Let $\mathfrak{c}(\tau)$ be the combinatorial type of τ such that $\mathfrak{c}(\tau)$ satisfies all given degenerated cross-ratios $\lambda_1, \dots, \lambda_l$. That is, the disjoint union over all λ_v of $\mathfrak{c}(\tau)$ is exactly $\lambda_1, \dots, \lambda_l$ and each vertex v of $\mathfrak{c}(\tau)$ satisfies $\text{val}(v) = 3 + \#\lambda_v$. If $v \in \mathfrak{c}(\tau)$ is a vertex with $\text{val}(v) > 3$, then cut all adjacent bounded edges of v , stretch the remaining edges to infinity and denote the component that contains v by C_v . If $\lambda = \{\beta_1, \dots, \beta_4\} \in \lambda_v$ is a given cross-ratio and β_i is not adjacent to v after cutting some bounded edges, then replace β_i by the label of the edge adjacent to v that is contained in the shortest path from v to β_i in $\mathfrak{c}(\tau)$. Let $\tilde{\lambda}_1, \dots, \tilde{\lambda}_r$ be the cross-ratios obtained this way such that $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_r\} = \lambda_v$ in C_v and let Δ' be the degree associated to C_v . The component of v is by definition the 0-dimensional cell of $\prod_{j=1}^r \text{ft}_{\tilde{\lambda}_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta')$. We call its weight the *local weight of v* and denote it by $\omega_v(\tau)$.

Using the proof of Lemma 3.11, we can deduce the following corollary:

Corollary 3.13 *Under the same assumption as Lemma 3.11, we have that*

$$\omega(\tau) = \prod_v \omega_v(\tau),$$

where the product runs over all vertices of $\mathfrak{c}(\tau)$ and $\omega_v(\tau)$ is the local weight of v .

Corollary 3.13 allows us to deduce the following:

Lemma 3.14 *For notation, see Notation 3.8. Let C be a point in the interior of a top-dimensional polyhedron τ of $X := \prod_{j=1}^{l'} \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ such that its multiplicity $\omega(\tau)$ is nonzero. Let $v \in C$ be a vertex of C such that $\text{val}(v) > 3$. Then for every edge e adjacent to v in C there is a β_i in some $\lambda_j \in \lambda_v$ such that e is in the shortest path from v to β_i .*

Proof We use the notation from Definition 3.12: Let C_v be the component of v in C and let μ_1, \dots, μ_r be the cross-ratios associated to v in C_v . Then $\text{val}(v) = 3 + r$ by Lemma 3.11. Denote the ends adjacent to v by e_1, \dots, e_{3+r} suppose that there is an end e_i adjacent to v in C_v such that there is no μ_j with $e_i \in \mu_j$. Since the multiplicity of τ is nonzero, Corollary 3.13 guarantees that there is a *total resolution* of v , that is there is a tropical curve C'_v and cross-ratios μ'_1, \dots, μ'_r such that C'_v is 3-valent and C'_v arises from resolving μ_1, \dots, μ_r in C_v according to μ'_1, \dots, μ'_r . The end e_i does not appear in any μ_j and therefore it does not appear in any μ'_j for $j = 1, \dots, r$. Let v_i be the vertex of C'_v to which e_i is adjacent to. Note that there is a bounded edge b adjacent to v_i that is shrunk first when degenerating μ'_1, \dots, μ'_r step by step. Therefore there is a cross-ratio μ'_j shrinking exactly b . Hence e_i appears in μ'_j as v_i is 3-valent. This is a contradiction.

Remark 3.15 Let Δ be a degree. Let p_1, \dots, p_n be points in \mathbb{R}^2 and let $\lambda'_1, \dots, \lambda'_{l'}, \lambda'_{l'+1}, \dots, \lambda_l$ be cross-ratios such that $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_{l'}, \lambda'_{l'+1}, \dots, \lambda_l$ are in general position and $n + l = \#\Delta - 1$ holds. Let

$$X := \prod_{k=1}^{l'} \text{ft}_{\lambda_k}^*(0) \cdot \prod_{j=l'+1}^l \text{ft}_{\lambda_j}^*(|\lambda_j|) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$$

be an intersection product, where $\lambda_1, \dots, \lambda_{l'}$ are the degenerations of $\lambda'_1, \dots, \lambda'_{l'}$. Then, using general position, the curves $\prod_{i=1}^n \text{ev}_i^*(p_i) \cdot X$ are in the interior of top-dimensional cells of X .

Proposition 3.16 *Let Δ be a degree, let $p_1, \dots, p_n, \lambda_1, \dots, \lambda_{l'}, \lambda'_{l'+1}, \dots, \lambda'_l$ be conditions as in Notation 3.8 such that*

$$n + l = \#\Delta - 1$$

and let

$$X := \prod_{k=1}^{l'} \text{ft}_{\lambda_k}^*(0) \cdot \prod_{j=l'+1}^l \text{ft}_{\lambda'_j}^*(|\lambda'_j|) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta).$$

Then the multiplicity $\text{mult}(C)$ with which a curve C in $\prod_{i=1}^n \text{ev}_i^*(p_i) \cdot X$ contributes to the degree of this 0-dimensional cycle is

$$\text{mult}(C) = \text{mult}_{\text{ev}}(C) \cdot \omega(\sigma_C),$$

where $\omega(\sigma_C)$ is the weight of the top-dimensional cell σ_C of X that contains C and $\text{mult}_{\text{ev}}(C)$ is the absolute value of the determinant of the locally (around C) linear map $\text{ev} : X \rightarrow \mathbb{R}^{2n}$.

Proof This follows from Lemma 3.11, Remark 3.15 and Lemma 1.2.9 of [25].

Having expressed $\omega(\sigma_C)$ locally already (see Corollary 3.13), our next goal is to express $\text{mult}_{\text{ev}}(C)$ locally.

Definition 3.17 (*Free and fixed components*) Let C be a rational tropical curve (possibly with vertices of higher valence) that is fixed by general positioned points p_1, \dots, p_n . Let v be an m -valent vertex of C such that there is no point lying on v and denote adjacent edges of v by e_1, \dots, e_m . Fix $i \in \{1, \dots, m\}$, cut the edge e_i and stretch it to infinity. Now there are two tropical curves, namely one that contains v and one that does not. The tropical curve C_i that does not contain v is called a *component of v* . A component of v is called a *fixed component of v* if it is fixed by the points on it (if this component is only a line, then this line is considered fixed if there is a point on it). Otherwise it is called a *free component of v* .

Note that there are exactly two fixed components of v : It is clear that every vertex has at least two fixed components, otherwise it could be moved. On the other hand general positioned points do not allow the number of fixed components to be greater than two. Hence the following multiplicities that generalize the well-know local ev -multiplicities for 3-valent vertices are well-defined.

Definition 3.18 (*Local multiplicities*) Let C be a rational tropical curve (possibly with vertices of higher valence) that is fixed by general positioned points p_1, \dots, p_n . Let v be a vertex of C . If there is a point on v , then define $\text{mult}(v) = 1$. Otherwise, let v be a vertex of C with fixed components C_1, C_2 associated to the edges e_1, e_2 adjacent to v . Let v_i denote the weighted primitive vector of e_i for $i = 1, 2$. The multiplicity of v is defined as

$$\text{mult}_{\text{ev}}(v) := |\det(v_1, v_2)|.$$

Another way to think about the multiplicity of a higher-valent vertex is to add up edges of free components, to be more precise, consider the following example:



On the left there is a 4-valent vertex whose black edges belong to fixed components and its blue edges belong to free components. The multiplicity of this vertex is completely determined by its black edges. If we “add” these blue edges (add their direction vectors), we obtain the 3-valent vertex on the right whose multiplicity is again completely determined by its black edges.

Lemma 3.19 *Let $p_1, \dots, p_n, \lambda_1, \dots, \lambda_l$ be in general position, where $|\lambda_j| = 0$ for $j = 1, \dots, l$ and let C be a rational tropical curve of some degree such that C is fixed by $p_1, \dots, p_n, \lambda_1, \dots, \lambda_l$, then*

$$\text{mult}_{\text{ev}}(C) = \prod_{v|v \text{ vertex of } C} \text{mult}_{\text{ev}}(v)$$

Proof We prove this by induction on the number of vertices of C which is denoted by k . Let $k = 1$ and denote the vertex of C by v . There are two choices of general positioned conditions that fix this curve:

1. If there is no point on v and v is at least 3-valent, then we have $n + 2$ parameters of C that need to be fixed. On the other hand each point p_i for $i = 1, \dots, n$ is in \mathbb{R}^2 and therefore $2n = n + 2$ for a natural number $n > 0$. Hence $n = 2$, so there are two ends e_1, e_2 that are equipped with points. Denote the weighted primitive vector of e_1 (pointing away from v) by $u = (u_1, u_2)$ and the vector of e_2 by w , respectively. If we choose p_1 as the base point of the ev-matrix $M(C)$ of C , then

$$M(C) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -u_1 & w_1 \\ 0 & 1 & -u_2 & w_2 \end{pmatrix} \tag{3}$$

has determinant $\text{mult}(v)$.

2. If there is a point on v and this point fixes the position of C , then $\text{mult}_{\text{ev}}(C) = 1$ since it is the determinant of the 2×2 identity matrix.

Let $k > 1$. In order to use induction and lower the number of vertices, we have to split off components. This has been done in the case where all vertices are 3-valent, see Proposition 3.8 of [15]. Let v be a vertex of C and let C_1 be a component of v that contains at least on vertex. Denote by C' the tropical curve after cutting e_1 that belongs to v . Introduce a new point p on $e'_1 \in C'$, where e'_1 denotes the cut and stretched edge e_1 in C' and denote C' with its new point by C'' . The proof of Proposition 3.8 in [15] given by Gathmann and Markwig can easily be adapted to our situation, such that

$$\text{mult}_{\text{ev}}(C) = \text{mult}_{\text{ev}}(C_1) \cdot \text{mult}_{\text{ev}}(C'')$$

holds and the induction hypothesis can be applied. □

We finish this section by summing up the most important results of this section in a theorem.

Theorem 3.20 *Let Δ be a degree and let $p_1, \dots, p_n, \lambda'_1, \dots, \lambda'_l$ be conditions as defined in Notation 3.8. Let $\lambda_1, \dots, \lambda_l$ denote the degenerations of $\lambda'_1, \dots, \lambda'_l$ and define*

$$N_{0,n}(\lambda_1, \dots, \lambda_l) := \deg \left(\prod_{j=1}^l \text{ft}_{\lambda_j}^*(0) \cdot \prod_{i=1}^n \text{ev}_i^*(p_i) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \right).$$

Then

$$N_{0,n}(\lambda'_1, \dots, \lambda'_l) = N_{0,n}(\lambda_1, \dots, \lambda_l)$$

holds, where $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ is defined in Definition 3.3. Moreover, the multiplicity of a tropical curve contributing to the right side can be expressed locally as

$$\text{mult}(C) = \prod_{v|v \text{ vertex of } C} \text{mult}_{\text{ev}}(v) \cdot \omega_v(\sigma_C),$$

where $\omega_v(\sigma_C)$ is the local weight of the top-dimensional cell σ_C of X that contains C (see Definition 3.12) and $\text{mult}_{\text{ev}}(v)$ is defined in Definition 3.18.

Proof The first part is a consequence of Remark 2.13. For the second part, note that if C is a tropical curve corresponding to a point in $\prod_{i=1}^n \text{ev}_i^*(p_i) \cdot X$ such that

$$X = \prod_{j=1}^l \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta),$$

then the contribution of C to $N_{0,n}(\lambda_1, \dots, \lambda_l)$ is

$$\text{mult}(C) = \prod_{v|v \text{ vertex of } C} \text{mult}_{\text{ev}}(v) \cdot \omega_v(\sigma_C)$$

due to Proposition 3.16, Lemma 3.19 and Corollary 3.13. □

Combining the Correspondence Theorem 2.25 and Theorem 3.20 enables us to enumerate classical curves satisfying point and classical cross-ratio conditions using degenerated tropical cross-ratios. We state this in the following corollary, which is used to obtain a cross-ratio lattice path algorithm in the next section.

Corollary 3.21 *Use the same notations/assumptions as in the Correspondence Theorem 2.25 and denote the degenerations of $\lambda'_1, \dots, \lambda'_l$ by $\lambda_1, \dots, \lambda_l$. Then*

$$N_{0,n}^{\text{class}}(\mu_1, \dots, \mu_l) = N_{0,n}(\lambda_1, \dots, \lambda_l)$$

holds.

The results of this section can be generalized to counts of curves satisfying tangency conditions to the toric boundary, point conditions and cross-ratio conditions in a straightforward way. We make use of this in Sect. 6 when dealing with floor diagrams. Here, we sum up the relevant notations.

Lemma 3.22 (Evaluation of horizontal ends) *Let $\Delta(\alpha, \beta)$ be a degree associated to curves in the first Hirzebruch surface with given contact orders as in Definition 2.16. The pull-backs of the maps*

$$\begin{aligned} \partial \text{ev}_k : \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\alpha, \beta)) &\rightarrow \mathbb{R} \\ (\Gamma, x_1, \dots, x_N, h) &\mapsto (h(x_k))_y \end{aligned}$$

are well-defined for $k = 1, \dots, l(\alpha) + l(\beta)$.

Proof This follows immediately from

$$\partial \text{ev}_k = \pi_y \circ \text{ev}_h \tag{4}$$

for some label h of an ending, where π_y is the projection on the y -coordinate of \mathbb{R}^2 and Proposition 1.12 of [26]. □

Remark 3.23 The pull-back of a map ∂ev_k for some k imposes a condition on the height of a horizontal end, corresponding to tangency conditions with the toric boundary. General position for point-, end- and cross-ratio conditions can be defined analogously to Definitions 3.3 and 3.7. The multiplicity of a curve in a 0-dimensional cycle in the moduli space of rational tropical stable maps corresponding to point-, end- and cross-ratio conditions can be computed similarly to Lemma 3.19 (i.e. locally on the vertices): If there is no end with an end condition adjacent to a vertex v , then its evaluation multiplicity equals $\text{mult}_{\text{ev}}(v)$. Otherwise (note that in this case there cannot be a point on v since all conditions are in general position), its local evaluation multiplicity equals $\frac{1}{\omega} \text{mult}_{\text{ev}}(v)$, where ω is the weight of the end adjacent to v that fulfills a end condition. This can be seen from an easy Laplace expansion argument that leads to the matrix (3) occurring in the proof of Lemma 3.19. The matrix M used in the ∂ev_k case is obtained from the one in (3) by erasing the third row and the fourth column, or in other words (using notations as in the proof of Lemma 3.19), we can pick $w_1 = \omega$ and $w_2 = 0$ such that

$$\det(M) = \frac{1}{\omega} \det(M(C))$$

follows.

4 Cross-ratio lattice path algorithm

In this section we present a generalized lattice path algorithm to determine the number of rational tropical curves passing through prescribed points and satisfying given degenerated cross-ratio constraints. Before diving into technical details, we want to shortly recall the “usual” lattice path algorithm introduced by Mikhalkin in [21,22].

The lattice path algorithm determines the number of rational tropical of degree Δ_d (see Definition 2.16) in \mathbb{R}^2 that satisfy $n = 3d - 1$ general positioned point conditions. It does so by explicitly constructing these curves for a specific configuration of points (the points are still in general position).

To obtain a suitable point configuration, pick points p_1, \dots, p_n in general position linearly ordered on a line L with a small negative slope such that distances of consecutive points grow, i.e.

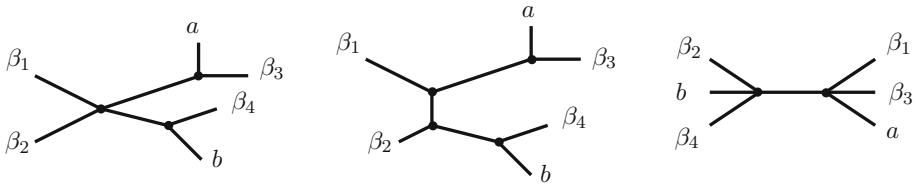


Fig. 4 From left to right: an arbitrary τ with its $\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}$ and the curve associated to $v_{(\beta_1\beta_3|\beta_2\beta_4)}$

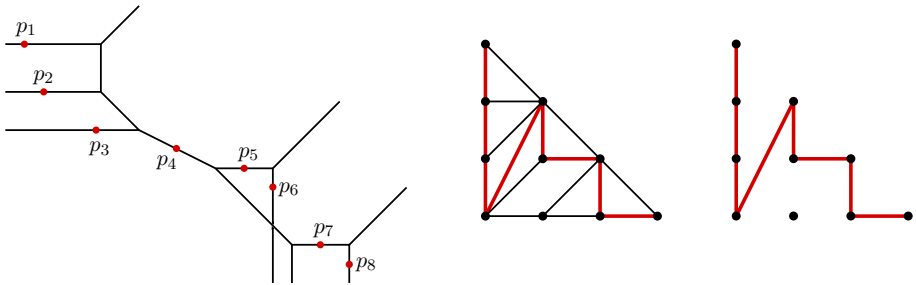


Fig. 5 From left to right: a rational degree Δ_3 curve in \mathbb{R}^2 satisfying the point conditions p_1, \dots, p_8 , its associated dual subdivision and its associated lattice path in bold red. Although p_1, \dots, p_8 are not lying on a line with small negative slope, they can be moved into this position without effecting the combinatorial type (resp. the dual subdivision) of the curve drawn. We just draw the points this way to get a better picture (color figure online)

$$|p_i - p_{i-1}| \ll |p_{i+1} - p_i|.$$

Let C be a curve satisfying these point conditions and let \mathcal{S}_C be its dual subdivision of Δ_d . It can be achieved that \mathcal{S}_C consists of triangles and parallelograms only since C is 3-valent. Hence each contracted end x_i of C that satisfies a point condition p_i is dual to an edge a_i of \mathcal{S}_C . A crucial observation is that the set $\mathcal{A} := \{a_i \mid i = 1, \dots, n\}$ forms a path in \mathcal{S}_C , a so-called *lattice path* with respect to the chosen line L on which p_1, \dots, p_n lie. Figure 5 provides an example of a rational degree Δ_3 curve satisfying eight point conditions, its dual subdivision and its associated lattice path.

The idea of the lattice path algorithm is to go the other way round: start with a lattice path \mathcal{A} and reconstruct all tropical curves C that satisfy p_1, \dots, p_n and yield the given lattice path \mathcal{A} . To do so, construct all possible dual subdivisions by recursively filling in the missing polytopes Δ_d such that these polytopes are compatible with the given lattice path \mathcal{A} . The lattice path algorithm provides the necessary rules which govern how triangles and parallelograms can be filled in. For more details about the “usual” lattice path algorithm, we refer to [21,22] (Fig. 7).

We now want to generalize the lattice path algorithm to curves satisfying point conditions and degenerated cross-ratio constraints. Notice that degenerated cross-ratios lead to vertices with valency > 3 , which means that nor there are only triangles and parallelograms in our subdivisions, neither \mathcal{A} needs to be a path (i.e. a collection of connected edges). Moreover, edges may be mapped onto vertices when our curves are embedded into \mathbb{R}^2 , see Fig. 8. We overcome these technical problems by equipping polytopes with additional information and carefully adapting the rules of filling in missing polytopes.

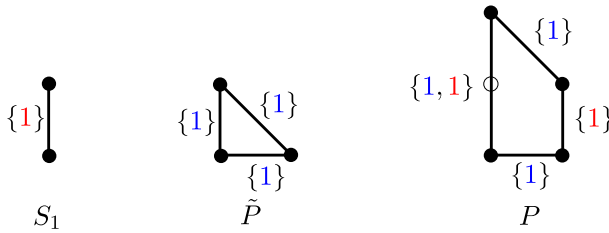


Fig. 6 From left to right: a segment S_1 and a 2-dimensional labeled polytope \tilde{P} whose Minkowski sum forms the labeled polytope P on the right. The colors indicate the matching of labelings of P_1, P_2 to their Minkowski summands (color figure online)

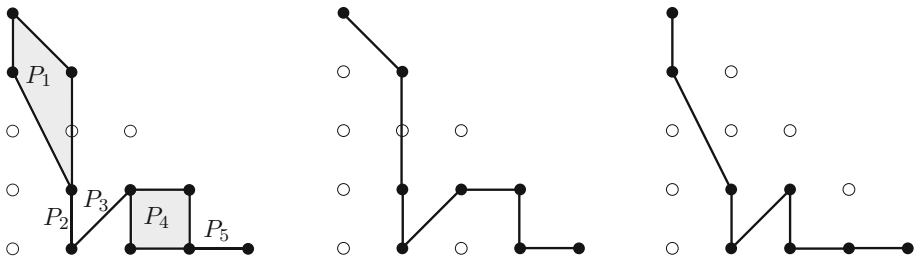
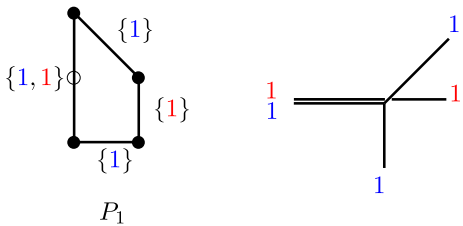


Fig. 7 Let $\Sigma = \text{conv}((0, d), (0, 0), (d, 0))$. From left to right: $\mathcal{A} = \{P_1, \dots, P_5\}, \gamma_+, \gamma_-$

Fig. 8 On the left is the Minkowski labeled polytope P introduced in Fig. 6 and on the right is its dual tropical curve



Definition 4.1

- An edge E is a 1-dimensional lattice polytope in \mathbb{R}^2 consisting of one 1-dimensional face and two 0-dimensional faces. A labeled edge is a tuple (E, τ^E) , where τ^E is a multiset of $m > 0$ elements denoted by $\tau_1^E, \dots, \tau_m^E$ in $\mathbb{N}_{>0}$ such that $\sum_i \tau_i^E = |E|$, where $|E|$ denotes the lattice length of E . We refer to τ^E as labeling of E and to $\tau_1^E, \dots, \tau_m^E$ as labels of E .
- In particular, we call a labeled edge (E, τ^E) where $\tau^E = \{n\}$ for some $n \in \mathbb{N}_{>0}$ a segment.
- Let P be a lattice polytope in \mathbb{R}^2 where each of its e facets is a labeled edge. Denote the labeling of an edge E^j of P by τ^j . Then (P, τ) with $\tau = (\tau^1, \dots, \tau^e)$ is called a labeled polytope.

Definition 4.2 (Minkowski labeled polytopes) Let P be the Minkowski sum of a labeled polytope $\tilde{P} \subset \mathbb{R}^2$ that is either 0-dimensional or 2-dimensional and segments S_1, \dots, S_r such that each segment is parallel to an edge of \tilde{P} and P is 2-dimensional. Note that if \tilde{P} is a point, then every segment is by definition parallel to it. Moreover, we require that if \tilde{P} is 0-dimensional, then there are two segments $S_{i_1}, S_{i_2} \in \{S_1, \dots, S_r\}$ such that all other Minkowski summands of P are parallel to one of them. Let E be an edge of P and denote

by F_1, \dots, F_k edges of the Minkowski summands $\tilde{P}, S_1, \dots, S_r$ that contribute to E . If τ^{F_i} is the labeling of F_i , then we define \ll^E to be the multiset

$$\tau^E := \tau^{F_1} \cup \dots \cup \tau^{F_k}.$$

A pair (P, τ) of such a polytope P with e edges E^1, \dots, E^e and a tuple of multisets $\tau = (\tau^{E^1}, \dots, \tau^{E^e})$, where τ^{E^i} is defined above, together with maps that *match* labels to the summands they come from

$$f_P|_E: \tau^E \rightarrow \{\tilde{P}, S_1, \dots, S_r\}$$

such that if $f_P|_E(t) = A \in \{\tilde{P}, S_1, \dots, S_r\}$, then $t \in \tau^{F_i}$ for $F_i \subset A$, is called a *Minkowski labeled polytope*. See Fig. 6 for an example.

We always denote the non-segment Minkowski summand of a Minkowski labeled polytope P by \tilde{P} .

Definition 4.3

- A Minkowski labeled polytope P is called *k-marked* if \tilde{P} has e edges E^j with labelings τ^j such that $\sum_{j=1}^e \#\tau^j = 3 + k$ holds, where $\#\tau^j \in \mathbb{N}_{>0}$ is the number of entries of τ^j . If $k = 0$ or \tilde{P} is 0-dimensional, then P is called *unmarked*.
- A Minkowski labeled polytope is called *valid polytope* if it is either unmarked or *k*-marked. Two valid polytopes that share an edge E are *compatible* if their labelings of E coincide.
- Let \tilde{P} be a 1-dimensional polytope where each side of its edge E is equipped with a labeling. The Minkowski sum of \tilde{P} with segments S_1, \dots, S_r parallel to it, where each summand contributes a label to the two labelings of E as in Definition 4.2 is called a *pointed segment*. If \tilde{P} is 0-dimensional, then it is called a *non-pointed segment* (all S_i are then parallel). The notion of compatibility extends to (non-)pointed segments as well: If a valid polytope and a (non-)pointed segment share an edge, then they are compatible if their labelings on this (side of the) edge coincide. We can refer to a (non-)pointed segment as *k*-marked as above.

Definition 4.4 (Coloring) A *coloring* of a labeled polytope P is a 2-coloring of all of its labels on each of its edges. The two colors are called *fixed* and *free*. A colored polytope is called *free* (or *fixed*) if it is monochromatic of the color free (or fixed). Given a colored Minkowski labeled polytope P , we say that exactly \tilde{P} is fixed if all labels associated to \tilde{P} are colored fixed and the rest is colored free.

Algorithm 4.5 (Adjusting colors of two compatible polytopes) Let P_1, P_2 be two colored polytopes that are compatible and denote their shared edge by E with labelings $\tau_{P_1}^E, \tau_{P_2}^E$. Let $f_{P_1}|_E, f_{P_2}|_E$ be maps as in Definition 4.2 and let $g: \tau_{P_1}^E \rightarrow \tau_{P_2}^E$ be a bijective map such that $g(t) = t$ for all $t \in \tau^E \cap \mathbb{N}_{>0}$. Let $t \in \tau_{P_1}^E$ be a colored label of E in P_1 and let $g(t)$ be its image under g in $\tau_{P_2}^E$. When comparing and adjusting the colors of t and $g(t)$, we follow the slogan “fixed wins”:

- (1) If t is colored fixed and $g(t)$ is colored fixed, we leave the colors the way they are.
- (2) If t is colored fixed and $g(t)$ is colored free, we change $g(t)$ to fixed. When changing $g(t)$ to fixed, we check whether all other labels coming from $f_{P_2}|_E(g(t))$ are fixed. If this is not the case, then change them to fixed if $f_{P_2}|_E(g(t))$ is a segment. If $f_{P_2}|_E$ associates $g(t)$ to \tilde{P}_2 , then change the labels associated to \tilde{P}_2 to fixed if exactly two of the labels associated to \tilde{P}_2 are fixed (where $g(t)$ is one of them).

- (3) If t is colored free and $g(t)$ is colored fixed, then do the same as in (2) but with the roles of $t, g(t)$ and P_1, P_2 exchanged.
- (4) If t is colored free and so is $g(t)$, then do nothing.

We repeat this procedure using different labels in $\tau_{P_1}^E$ until no color of labels of P_1, P_2 can be changed according to the rules above. Note that this algorithm terminates since colors can only be changed from free to fixed.

Algorithm 4.6 (*Adjusting colors of a set of polytopes*) Let P_1, \dots, P_z be a finite set of colored polytopes, where two polytopes are compatible if they share an edge. Go through all pairs of compatible polytopes of P_1, \dots, P_z and adjust their colors according to Algorithm 4.5. Repeat this procedure until no colors can be changed. This algorithm terminates because we only allow changing a color from free to fixed, following the slogan that fixed wins.

Note that the notion of coloring and adjusting colors extends to (non-)pointed segments. The following definitions can be found in [21] and [20].

Definition 4.7 (*Lattice path*) Fix θ to be a linear map of the form

$$\theta : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x - \epsilon y,$$

where ϵ is a small irrational number. A path $\gamma : [0, n] \rightarrow \mathbb{R}^2$ is called a *lattice path* if $\gamma|_{[j-1, j]}$ for $j = 1, \dots, n$ is an affine-linear map and $\gamma(j) \in \mathbb{Z}^2$ for all $j = 0, \dots, n$. For $j = 1, \dots, n$, we call $\gamma|_{[j-1, j]}$ ($[j - 1, j]$) a *step* (the j -th step) of the lattice path γ . A lattice path is called θ -*increasing* if $\theta \circ \gamma$ is strictly increasing. If every step in a lattice path is a labeled edge, the lattice path is called *labeled lattice path*.

Definition 4.8 (*Cross-ratio lattice path*) Let Σ be a polytope in \mathbb{R}^2 and let $n \in \mathbb{N}_{>0}$. Let \mathcal{A} be a set $\{P_1, \dots, P_{n+z}\}$ of colored polytopes in Σ such that there are polytopes $\{P_{i_1}, \dots, P_{i_n}\} \subset \mathcal{A}$ such that P_{i_j} is a pointed segment or a valid polytope such that P_{i_j} is fixed and not 0-dimensional for $j = 1, \dots, n$. The other polytopes in $\mathcal{A} \setminus \{P_{i_1}, \dots, P_{i_n}\}$ are non-pointed segments that are colored free. The set \mathcal{A} is called a *cross-ratio lattice path* if the following conditions are satisfied:

- (1) two polytopes P_i, P_j intersect in at most one point,
- (2) if an edge E of a polytope P_i lies in the boundary $\partial\Sigma$ of Σ it is labeled by $\tau^E = (1, \dots, 1)$,
- (3) there are sets γ_+, γ_- of edges of P_1, \dots, P_{n+z} such that γ_+, γ_- form θ -increasing labeled lattice paths, $\gamma_+ \cup \gamma_-$ is the set of all edges of P_1, \dots, P_{n+z} and for all $x \in \pi_x(\Sigma)$ (where π_x is the projection of \mathbb{R}^2 to the x -axis) and all $E_+ \in \gamma_+, E_- \in \gamma_-$ such that there are points $(x, y_+) \in E_+ \subset \mathbb{R}^2, (x, y_-) \in E_- \subset \mathbb{R}^2$ the inequality $y_+ \geq y_-$ holds (see Fig. 7),
- (4) the order of the polytopes P_1, \dots, P_{n+z} agrees with the obvious order given by γ_+ and γ_- , respectively,
- (5) let p and q be the points in Σ where $\theta|_\Sigma$ reaches its minimum (resp. maximum), then $p = \gamma_+(0) = \gamma_-(0)$ and $q = \gamma_+(n_+) = \gamma_-(n_-)$, where $\gamma_+ : [0, n_+] \rightarrow \mathbb{R}^2$ and $\gamma_- : [0, n_-] \rightarrow \mathbb{R}^2$ are defined as above.

Figure 7 provides an example of a cross-ratio lattice path. Throughout the following, we fix a degree $\Delta(\Sigma)$ from a polytope $\Sigma \subset \mathbb{R}^2$, see Definition 2.16, point conditions p_1, \dots, p_n and degenerated cross-ratio constraints $\lambda_1, \dots, \lambda_l$ in general position.

Construction 4.9 (*Constructing subdivisions of Σ from a cross-ratio lattice path \mathcal{A}*) Let \mathcal{A} be a cross-ratio lattice path in the polytope Σ with $\#\mathcal{A} = n + z$ for some $z \in \mathbb{N}$ such that $z \leq \#(\Sigma \cap \mathbb{Z}^2)$. Let γ_+ be the associated labeled lattice path as before. Recall that in the “standard” lattice path algorithm left (resp. right) turns of a given lattice path are filled up with triangles and parallelograms. In our case we must allow more polytopes than only triangles and parallelograms.

Let $\gamma_+(j)$ and $\gamma_+(j + 1)$ be the j -th and the $(j + 1)$ -th labeled edge of γ_+ that form the first left turn. Fill up this left turn with a valid polytope $P \subset \Sigma$ that is colored free, whose edges that equal $\gamma_+(j)$ and $\gamma_+(j + 1)$ are compatible with $\gamma_+(j)$ and $\gamma_+(j + 1)$ and if P shares other edges with our polytopes, it should there be compatible, too. Whenever two compatible labeled edges with labelings τ^E come together, we choose a bijective map $g : \tau^E \rightarrow \tau^E$ such that $g(t) = t$ for all $t \in \tau^E \cap \mathbb{N}_{>0}$. Moreover, we use Algorithm 4.6 to adjust the colors of the set of polytopes we have so far. If P shares an edge E with $\partial\Sigma$, then we require $\tau^E = (1, \dots, 1)$ and we choose a bijective map $g' : \tau^E \rightarrow M$, where M is a submultiset of the labels of the degree $\Delta(\Sigma)$ that are associated to vectors orthogonal (and pointing away from Σ) to E (see Definition 2.16). When another polytope P' shares an edge with $\partial\Sigma$, then we choose M' in the set of labels of $\Delta(\Sigma)$ minus M and so on. In the same way the right turns of γ_- can be filled up.

Repeating these steps, we obtain subdivisions of Σ if and only if $\Sigma = \mathcal{A} \cup \bigcup\{P\}$, where the union runs over all valid polytopes P used to fill up turns during the process described above. The cells of such a subdivision are valid polytopes which are compatible and connected via maps called g above. Such a subdivision is called a *lattice path subdivision* of \mathcal{A} if all polytopes are fixed. The set of all lattice path subdivisions of \mathcal{A} is denoted by $\mathcal{S}_0(\mathcal{A})$.

Construction 4.10 (*Dual tropical curve*) Let $S \in \mathcal{S}_0(\mathcal{A})$ be a lattice path subdivision. We want to construct the dual tropical curve C_S associated to a point in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\Sigma))$ to S . For that draw a k -valent vertex v for every k -marked ($k > 0$) polytope P in S and an edge passing through this vertex for every segment of P . An edge e adjacent to v is dual to an edge E of \tilde{P} , that is the weight of e is given by an entry of the labeling τ^E of E . The weight of an edge passing through v is given by the label of its associated segment that is dual to this edge. If two polytopes $P, Q \in \mathcal{S}_0(\mathcal{A})$ share an edge E with labeling τ^E , we connect the edge associated to τ_i^E in P with the edge associated to $g(\tau_i^E)$ in Q for all i , where g is a map as in Construction 4.9. Moreover, if $P \in \mathcal{A}$ and P is neither a pointed segment nor a non-pointed segment, then add a point (a contracted end) to the vertex dual to \tilde{P} . If $P \in \mathcal{A}$ and P is a pointed segment, then the edges dual to the labelings associated to \tilde{P} meet in one vertex which is in addition adjacent to a point. In this way, we obtain the combinatorial type of C_S . From the general construction of tropical curves dual to lattice paths (see [22]) and the fact that all polytopes are fixed, it follows that for given points p_1, \dots, p_n in general position linearly ordered on a line with a small negative slope such that distances grow ($|p_i - p_{i-1}| \ll |p_{i+1} - p_i|$) there is exactly one curve of type C_S that satisfies the point conditions.

Since we are only interested in genus zero curves, we need to remove subdivisions whose dual tropical curves are reducible. We denote the set of lattice path subdivisions for a given cross-ratio lattice path \mathcal{A} which are dual to irreducible tropical curves by $\mathcal{S}_1(\mathcal{A})$.

Definition 4.11 Let $\Lambda := \bigcup_{j=1}^l \lambda_j$ the union of all given degenerated cross-ratio constraints. Let S be a lattice path subdivision in $\mathcal{S}_1(\mathcal{A})$ and let P be a valid polytope or a pointed segment in S . Consider the summand \tilde{P} of P and define for all entries τ_1, \dots, τ_m of labelings of edges of P associated to \tilde{P} the sets $\Lambda(P, i) \subset \Lambda$ of points and ends appearing in the cross-ratios

$\lambda_1, \dots, \lambda_l$ that can be reached from P via τ_i . That is, we obtain the elements of $\Lambda(P, i)$ with the following procedure:

- If the edge E of P where τ_i appears is contained in $\partial\Sigma$, then its dual edge is a labeled end determined by $g(\tau_i)$ (Construction 4.9), and we add it to $\Lambda(P, i)$.
- Else there is a valid polytope (or a pointed segment) Q in \mathcal{S} such that $Q \neq P$ and P, Q share an edge E such that τ_i appears in τ^E . Then either:
 - τ_i is mapped to \tilde{Q} (via the map $f_{\tilde{Q}}|_E$ from Definition 4.2) and $Q \notin \mathcal{A}$, then continue with all other labels mapped to \tilde{Q} instead of τ_i .
 - τ_i is mapped to \tilde{Q} and $Q = P_j \in \mathcal{A}$, then add the marked point x_j to $\Lambda(P, i)$ and continue with all other labels mapped to \tilde{Q} instead of τ_i
 - τ_i is mapped to a segment of Q , then there is exactly one τ'_i in another edge E' of Q that is mapped to the same segment. We continue with this.

In each case, we follow all appearing edges until we reach edges in $\partial\Sigma$ for which we add the labels of the dual ends to $\Lambda(P, i)$.

Furthermore, if P is a polytope appearing in the lattice path \mathcal{A} itself as j -th step, then we set $\Lambda(P, 0) := \{x_j\}$, the j -th marked point. Otherwise, we set $\Lambda(P, 0) := \emptyset$.

Moreover, we define

$$\Lambda(P) := \{\lambda_j = \{\beta_{j_1}, \dots, \beta_{j_4}\} \mid \beta_{j_i} \in \Lambda(P, k_i) \text{ for } i = 1, \dots, 4 \text{ and } k_i \neq k_{i'} \text{ if } i \neq i'\},$$

and we say that the lattice path subdivision \mathcal{S} fits the cross-ratios $\lambda_1, \dots, \lambda_l$ if

$$\sum_P \#\Lambda(P) = l,$$

where the sum goes over all valid polytopes and pointed segments in \mathcal{S} and

$$\#\Lambda(P) = \begin{cases} k, & \text{if } P \text{ is } k\text{-marked} \\ 0, & \text{otherwise.} \end{cases}$$

For a cross-ratio lattice path \mathcal{A} , the subset of $\mathcal{S}_1(\mathcal{A})$ of subdivisions which fit the given cross-ratios is denoted by $\mathcal{S}_2(\mathcal{A})$.

Definition 4.12 (*Multiplicity of a subdivision*) In order to associate a multiplicity to a lattice path subdivision \mathcal{S} in $\mathcal{S}_2(\mathcal{A})$, define

$$\text{mult}_{\text{ev}}(\mathcal{S}) := \prod_P \text{mult}_{\text{ev}}(P),$$

where the product goes over all valid polytopes and pointed segments in \mathcal{S} , and $\text{mult}(P)$ is defined as follows: If \tilde{P} is 0-dimensional or $P \in \mathcal{A}$, then $\text{mult}(P) := 1$. Otherwise let τ_1, \dots, τ_m denote the entries of labelings of edges of P associated to \tilde{P} , let \mathcal{E}_i be the number of ends that can be reached from P via τ_i and let \mathcal{C}_i be the number of constraints that can be reached from P via τ_i (using the procedure from Definition 4.11), that is

$$\begin{aligned} \mathcal{C}_i &:= \mathcal{C}_i^{(\text{points})} + \mathcal{C}_i^{(\text{cross-ratios})}, \\ \mathcal{C}_i^{(\text{cross-ratios})} &:= \sum_{P'} \#\Lambda(P'), \end{aligned}$$

where the sum goes over all valid polytopes and pointed segments in \mathcal{S} that can be reached from P via τ_i , $\Lambda(P')$ is defined in Definition 4.11 and $\mathcal{C}_i^{(\text{points})}$ is the number of points that

can be reached from P via τ_i . We have either $\mathcal{E}_i - 1 = C_i$ or $\mathcal{E}_i - 2 = C_i$: in the first case, the edge dual to τ_i in the tropical curve leads to a fixed component, in the second to a free component (see Definition 3.17). Every vertex of the dual tropical curve has exactly two fixed components, we use the indices i_0 and i_1 for those labels corresponding to edges in the dual tropical curve that lead to a fixed component. Then we set

$$\text{mult}_{\text{ev}}(P) := |\det(\tau_{i_0} \cdot v_0, \tau_{i_1} \cdot v_1)|,$$

where v_0 is the primitive vector of the edge E_0 of P that belongs to τ_{i_0} and v_1 , respectively.

Furthermore let C_S be the dual tropical curve of \mathcal{S} (see Construction 4.10). Let $X := \prod_{j=1}^l \text{ft}_{\lambda_j}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\Sigma))$. Note that $C_S \in X$ since the lattice path subdivision \mathcal{S} fits the cross-ratios $\lambda_1, \dots, \lambda_l$. Moreover, C_S passes through the points p_1, \dots, p_n by Construction 4.10. Using Remark 3.15, we know that C_S lies in the interior of a top-dimensional cell of X . Denote this top-dimensional cell by σ_S and define $\omega(\sigma_S)$ to be its weight. Recall that this weight has a local structure, see Corollary 3.13.

We define the *multiplicity* $\text{mult}(\mathcal{S})$ of \mathcal{S} as

$$\text{mult}(\mathcal{S}) := \text{mult}_{\text{ev}}(S) \cdot \omega(\sigma_S).$$

By definition, we have $\text{mult}(\mathcal{S}) = \text{mult}(C_S)$ for all $\mathcal{S} \in \mathcal{S}_2(\mathcal{A})$.

Definition 4.13 Given cross-ratio constraints $\lambda_1, \dots, \lambda_l$, we denote the sum over all $\mathcal{S} \in \mathcal{S}_2(\mathcal{A})$ (counted with multiplicity) for all cross-ratio lattice paths \mathcal{A} with $n + z$ steps for all z by $N_{0,n}^{\text{lpa}}(\lambda_1, \dots, \lambda_l)$.

Remark 4.14 (Arbitrary degree) Note that we do not need to restrict to a degree Δ coming from a polytope where all entries of all partitions are one (see Definition 2.16). We restrict to $\Delta(\Sigma)$ here to keep notation as simple as possible. The cross-ratio lattice path algorithm can be extended to arbitrary degrees.

Example 4.15 We want to give an example of the lattice path algorithm. Fix the degree Δ_d for $d = 3$ (cf. Definition 2.16). We choose points p_1, \dots, p_7 and a degenerated cross-ratio $\lambda = \{x_1, x_2, 7, 8\}$. It turns out that all cross-ratio lattice paths we need to consider have 7 steps. The top row of Fig. 9 shows these cross-ratio lattice paths. There are no labels on polytopes and colors in Fig. 9 because all labels are 1 and all labels are colored fixed. The column under each of these cross-ratio lattice paths shows the subdivisions arising from these lattice paths. The maps that glue together the polytopes in a subdivision (maps like g from Construction 4.9) are not mentioned in Fig. 9 since they are the obvious ones. However, the glueing maps that connect the polytopes in the subdivision to the boundary of Δ_3 are not unique since we labeled ends of tropical curves (we come back to this later). The grey polytopes are 1-marked, that is λ sits at these polytopes. Note that all subdivisions fit the cross-ratio λ for an appropriate choice of glueing the polytopes to the boundary.

The numbers in the rightmost column correspond to subdivisions shown on the left. Each of these numbers is a product, where the first factor is the multiplicity $\text{mult}(\mathcal{S})$ of its associated subdivision \mathcal{S} . Note that $\omega(\sigma_S) = 1$ for all subdivisions since there is only one way of resolving the 4-valent vertex dual to each 1-marked polytope according to some λ' degenerating to λ . The second factor comes from different glueings of polytopes to the boundary of Δ_3 and can easily be seen from an example, see Fig. 10.

The total sum of the numbers in the right column is 40, which is the number of unlabeled tropical curves satisfying the given point conditions and the cross-ratio constraint. Since the

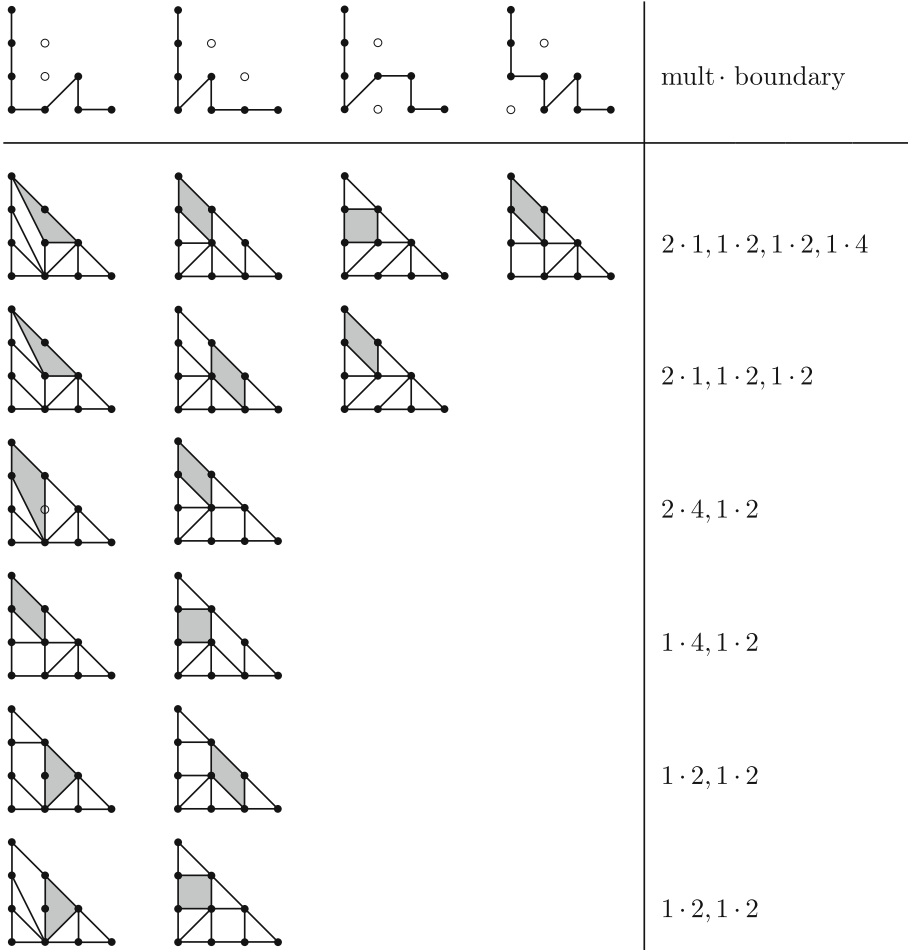


Fig. 9 A complete example of lattice paths, subdivisions and their multiplicities

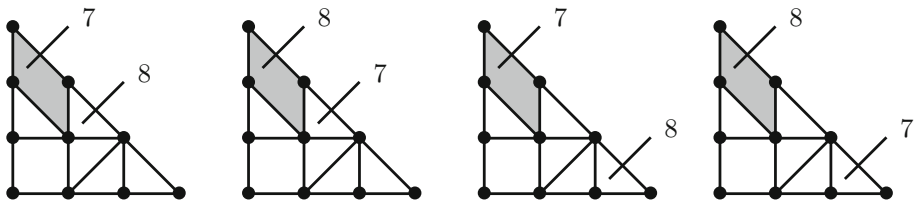


Fig. 10 The subdivision in the right top corner of Fig. 9 and the 4 different choices of labels of ends in λ such that the subdivision still fits λ

second factor of each product in the rightmost column equals the number of ways to label ends parallel to the vector $(1, 1) \in \mathbb{R}^2$, we obtain the number of labeled tropical curves satisfying our given conditions by multiplying 40 with $(3!)^2$, which is 1440 as we would expect considering Example 2.26. Thus we checked that we are not missing any subdivisions.

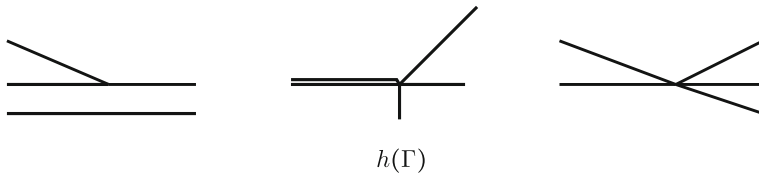


Fig. 11 Left: a local picture of Γ and its graph structure. Middle: a local picture of a vertex of $h(\Gamma)$, where the two edges on the left are mapped on top of each other, we shifted them slightly to get a better picture. Right: the graph structure of $h(\Gamma)$ induced from intersections of edges

5 Duality: tropical curves and subdivisions

In this section we want to prove Theorem 5.3 that relates the numbers obtained from the cross-ratio lattice path algorithm to the enumerative numbers $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ of tropical curves satisfying point conditions and cross-ratio constraints. Moreover, it makes $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ computable using the cross-ratio lattice path algorithm. As a consequence the numbers $N_{0,n}^{\text{class}}(\mu_1, \dots, \mu_l)$ (we use Notation 3.8) become computable too.

Definition 5.1 (*Simple tropical curves*) An element $(\Gamma, x_1, \dots, x_n, h)$ in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is called *simple* if it satisfies:

- the map h that embeds Γ in \mathbb{R}^2 is injective on vertices,
- if $h(v) \in h(e)$ for a vertex v and an edge e , then there is an edge e' adjacent to v such that $h(e)$ and $h(e')$ intersect in infinitely many points and then there are a vertex v' and finite sequences $(e_i)_i^r, (e'_j)_j^l$ of edges (with $e_0 = e, e'_0 = e'$) that lie in $\text{span}(e)$ such that two consecutive elements in a sequence meet in a vertex and such that $h(e_r)$ and $h(e'_l)$ are adjacent to $h(v')$,
- assume $p \in \mathbb{R}^2$ is a point through which more than two edges pass. Divide these edges into equivalence classes depending on the slope of the line they are mapped to. Then there are at most two equivalence classes.

Remark 5.2 Given a rational tropical stable map $(\Gamma, x_1, \dots, x_n, h)$ to \mathbb{R}^2 , we can associate two different graph structures to the image of Γ in \mathbb{R}^2 . The first graph structure is the one coming from Γ . The second graph structure is the one of $h(\Gamma) \subset \mathbb{R}^2$, i.e. whenever edges of $h(\Gamma)$ in \mathbb{R}^2 intersect, this intersection is considered a vertex, see Fig. 11. Notice that Minkowski labeled polytopes as in Fig. 8 help us to keep track of the graph structures.

Theorem 5.3 For notation, see Notation 3.8. The number $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ of rational tropical curves satisfying point and cross-ratio conditions (see Definition 3.3) equals the number $N_{0,n}^{\text{lpa}}(\lambda_1, \dots, \lambda_l)$ obtained from the cross-ratio lattice path algorithm (see Definition 4.13) if the input data of the algorithm are the number of point conditions and the degenerated cross-ratios. More precisely, the equality

$$N_{0,n}(\lambda'_1, \dots, \lambda'_l) = N_{0,n}^{\text{lpa}}(\lambda_1, \dots, \lambda_l)$$

holds.

Proof Using Theorem 3.20, we deduce that $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ equals the number of tropical curves satisfying the degenerated cross-ratio conditions $\lambda_1, \dots, \lambda_l$.

Let $\mathcal{S}_{0,n}(\lambda_1, \dots, \lambda_l)$ denote the set of elements that contribute to $N_{0,n}^{\text{lpa}}(\lambda_1, \dots, \lambda_l)$. As before, we pick points p_1, \dots, p_n in general position linearly ordered on a line with a small negative slope such that distances grow ($|p_i - p_{i-1}| \ll |p_{i+1} - p_i|$), and we let $\mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l)$ denote the set of degenerated tropical curves satisfying degenerated cross-ratio constraints, that is $\mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l)$ denotes the set of elements that contribute to $N_{0,n}(\lambda_1, \dots, \lambda_l)$. Consider the map

$$\begin{aligned} \phi : \mathcal{S}_{0,n}(\lambda_1, \dots, \lambda_l) &\rightarrow \mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l) \\ \mathcal{S} &\mapsto C_{\mathcal{S}} \end{aligned}$$

that maps a lattice path subdivision \mathcal{S} to its dual tropical curve $C_{\mathcal{S}}$ given by Construction 4.10. This map is obviously well-defined because we only have subdivisions where all polytopes are fixed and the map is injective because curves with different combinatorial types are different. To see that ϕ is surjective, we need to construct a preimage for a given curve $C = (\Gamma, x_1, \dots, x_n, h)$ in $\mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l)$. Note that C carries two different graph structures, namely one induced by Γ and one induced by $h(\Gamma)$, see Remark 5.2. If we refer to a vertex in $h(\Gamma)$, we mean the graph structure induced by h and if we refer to a vertex in Γ , we mean the graph structure of Γ .

First of all, associate a valid polytope (resp. a pointed segment) to every vertex $v \in h(\Gamma)$: Let v be a vertex of $h(\Gamma)$ and consider its dual polytope P_v . The polytope P_v can be turned into a labeled polytope (resp. a pointed segment) if we label its edges E_i with weights of its dual edges $e_{i_1}, \dots, e_{i_m} \in \Gamma$. Moreover, denote by \tilde{P}_v the dual polytope of $v \in \Gamma$ and label its edges as before. Note that P_v is a Minkowski sum of \tilde{P}_v and segments S_1, \dots, S_r that correspond to edges of $v \in h(\Gamma)$ that are no edges of $v \in \Gamma$. We can choose the points p_1, \dots, p_n in such a way that C is a simple tropical curve. Then, edges of $v \in h(\Gamma)$ that are no edges of $v \in \Gamma$ can only be parallel to edges of $v \in \Gamma$. Furthermore, if \tilde{P} is 0-dimensional, then there are two segments $S_{i_1}, S_{i_2} \in \{S_1, \dots, S_r\}$ such that all other Minkowski summands of P are parallel to one of them. Note also that there are mappings of entries of labeled edges of P_v to its Minkowski summands. In addition, P_v is unique because permuting parallel edges of $v \in h(\Gamma)$ leads to the same dual polytope. In this way, we can assign a valid polytope (resp. pointed segment) to every vertex $v \in h(\Gamma)$.

The second step is to associate a subdivision $\mathcal{S}_C \in \mathcal{S}_{0,n}(\lambda_1, \dots, \lambda_l)$ to C : The tropical curve $h(\Gamma)$ determines how to glue the polytopes P_v (via maps called g in Construction 4.9) for all vertices $v \in h(\Gamma)$ together. Note that if two vertices $v, v' \in h(\Gamma)$ are adjacent, then their dual valid polytopes $P_v, P_{v'}$ are compatible. Denote the subdivision obtained this way by \mathcal{S}_C . The dual polytopes resp. segments associated to the vertices and edges of $h(\Gamma)$ meeting the points p_1, \dots, p_n and non-pointed segment we associate in the obvious way to the edges of C intersecting the line the points p_1, \dots, p_n lie on form a cross-ratio lattice path \mathcal{A} . Hence \mathcal{S}_C is a lattice path subdivision whose dual tropical curve is C , the genus of C is zero, all polytopes of \mathcal{S}_C are fixed and \mathcal{S}_C fits to the given cross-ratios by definition. Therefore $\mathcal{S}_C \in \mathcal{S}_2(\mathcal{A})$ for a cross-ratio lattice path \mathcal{A} . Thus ϕ is bijective and preserves weights. □

Now that we established Theorem 5.3, we can apply Corollary 3.21 and in particular the correspondence theorem shown by Tyomkin in [31] such that the next corollary follows immediately.

Corollary 5.4 *We use the notation from Notation 3.8. Under the same assumptions as in Theorem 2.25 the equality*

$$N_{0,n}^{class}(\mu_1, \dots, \mu_l) = N_{0,n}^{lpa}(\lambda_1, \dots, \lambda_l)$$

holds.

6 Floor diagrams for cross-ratio counts

In this section, we want to impose some restrictions on the degree Δ and the cross-ratios such that we can work with simple combinatorial objects called floor diagrams. Let Σ_d be the convex hull of $\{(0, 0), (d, 0), (d, 0)\} \in \mathbb{R}^2$ for some $d \in \mathbb{N}_{>0}$ and $\Delta_d := \Delta(\Sigma_d)$ (see Definition 2.16). As in Sect. 4, we first want to recall “usual” floor diagrams introduced by Mikhalkin and Brugallé in [10,11]. Floor diagrams are degenerations of tropical curves. These diagrams are combinatorial objects that reflect the combinatorial properties of tropical curves well. If each floor diagram is counted with the weighted number of tropical curves degenerating to it, we obtain in total the count of tropical curves we are looking for. The idea of floor diagrams is to choose points in a specific configuration, namely:

Definition 6.1 (*Stretched point configuration*) General positioned points p_1, \dots, p_n in \mathbb{R}^2 are in a *stretched configuration* if the y -coordinates of the points p_1, \dots, p_n are contained in a small interval $I \subset \mathbb{R}$ while the distances between the x -coordinates are large compared to I .

Points in a stretched point configuration yield curves that are of a particularly nice form, namely:

Definition 6.2 (*Floors and elevators*) An *elevator* of a tropical curve of degree Δ_d is an edge that is parallel to $(-1, 0) \in \Delta_d$. A connected component of a tropical curve that remains if the interiors of the elevators are removed is called *floor of size s* if there are exactly s ends that are in this connected component and that are parallel to $(1, 1) \in \Delta_d$. The case $s = 0$ is possible for floors consisting of a single contracted marked point. A tropical curve that is fixed by points and cross-ratios is called *floor decomposed* if each point lies on its own floor.

Figure 12 provides an example of a floor decomposed tropical rational curve C of degree Δ_3 satisfying eight point conditions. This curve can be degenerated the following way: Forget ends and shrink floors to points. What remains is a tree without ends on n vertices (there are two types of vertices), where n is the number of given point conditions. This tree is called a *floor diagram*. The floor diagram associated to C is shown in Fig. 12. In case of point conditions only, each floor is either of size 0 or of size 1 and no floor of size 1 is connected to another floor of size 1. Therefore enumerating floor diagrams is straightforward in this case. Notice that floor diagrams allow us to boil the counting problem of tropical rational plane curves of degree Δ_d through $3d - 1$ general positioned points down to a purely combinatorial counting problem.

To obtain floor decomposed tropical curves from a stretched point configuration while allowing degenerated cross-ratios conditions, we need to restrict the cross-ratios we consider.

Definition 6.3 A degenerated cross-ratio $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is said to have t points if the number of β_i that are labels of contracted ends that satisfy point conditions is t . A set of degenerated cross-ratios $\lambda_1, \dots, \lambda_l$ has t points if each cross-ratio in the set does. This definition also applies to non-degenerated cross-ratios.

Assume in the following that all cross-ratios have 4 points. Now we can show that our curves also decompose into floors.

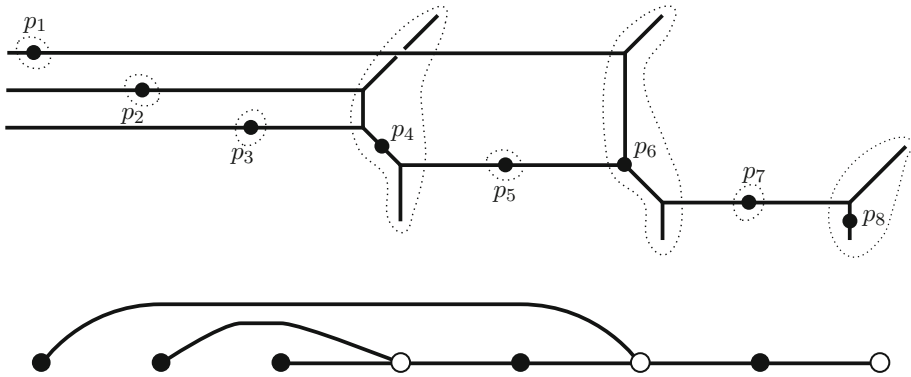


Fig. 12 A floor decomposed rational tropical degree Δ_3 curve satisfying eight point conditions p_1, \dots, p_8 . Its floors are indicated by dotted lines. Under the curve, its associated floor diagram is shown. White vertices indicate floors of size 1 and black ones indicate floors of size 0

Lemma 6.4 *A tropical curve C of degree Δ_d (see Definition 2.16) that is fixed by general positioned points p_1, \dots, p_n in a stretched configuration (Definition 6.1) and degenerated cross-ratio constraints $\lambda_1, \dots, \lambda_l$ that have 4 is floor decomposed.*

Proof A string is a path in a tropical curve connecting two non-contracted ends such that no point lies on that path. A string gives rise to a 1-dimensional family of tropical curves. Let $I \subset \mathbb{R}$ be a compact interval such that p_1, \dots, p_n lie in the stripe $\mathbb{R} \times I$ of \mathbb{R}^2 . Assume there is a vertex v of C whose y -coordinate (among all vertices of C) is (without loss of generality) maximal and v lies above the stripe. There are two cases.

- (1) Assume v has valency greater 3, that is there are cross-ratios such that $\text{val}(v) = 3 + \#\lambda_v$ (see Definition 3.9). By the balancing condition there is an edge adjacent to v whose direction vector has y -coordinate greater zero. But this edge cannot lead to a point since all points lie beneath v and v has maximal y -coordinate. This contradicts Lemma 3.14 since all cross-ratios have 4 points.
- (2) Assume v is 3-valent. We follow the proof of Proposition 5.3 of [11]: Since the y -coordinate of v is maximal there is an edge e_1 that is an end with direction vector u_1 adjacent to v . The given degree Δ_d guarantees that $u_1 = (\alpha, 1)$ for some α . Denote the two other direction vectors by u_2, u_3 . Using the balancing condition, we can (without loss of generality) write $u_2 = (\gamma, \beta)$ and $u_3 = (\epsilon, \delta)$ for some integers $\beta \geq 0, \delta < 0$. Note that the edge e_2 associated to u_2 is an end if $\beta > 0$ and this leads to a string from e_1 to e_2 which is a contradiction. Therefore $\beta = 0$ and e_2 is no end. Let v' be the vertex to which v is connected to via e_2 . By case (1) v' is also 3-valent, and v' is (by the balancing condition) adjacent to an end denoted by e'_1 . Thus there is a string from e_1 to e'_1 which is a contradiction.

Since no vertex of C lies outside the stripe $\mathbb{R} \times I$, Corollary 5.4 of [11] can be applied, which yields that C is floor decomposed. □

The next step is to define suitable cross-ratio floor diagrams.

Definition 6.5 (*Cross-ratio floor diagrams*) Let $d \in \mathbb{N}_{>0}$ and let \mathcal{F} be a tree on a totally ordered set of vertices v_1, \dots, v_n , then \mathcal{F} is called a *cross-ratio floor diagram of degree Δ_d* if:

- (1) Each edge of \mathcal{F} consists of two half-edges. There are two types of half-edges, *thin* and *thick* ones. A thin half-edge can only be completed to an edge with a thick half-edge and vice versa.
- (2) Each vertex v is labeled with $s_v, \#\lambda_v \in \mathbb{N}$ and a set δ_v of labels that appear in Δ_d , where $\#\lambda_v$ is called the *number of cross-ratios of v* and s_v is called the *size of v* such that

$$s_v = \{x \in \delta_v \mid d + 1 \leq x \leq 2d\} = \{x \in \delta_v \mid 2d + 1 \leq x \leq 3d\}$$

and $\emptyset = \delta_v \cap \delta_{v'}$ for all $v \neq v'$ and $\bigcup_v \delta_v$ is the set of all labels appearing in Δ_d .

- (3) The number of thick edges adjacent to a vertex v is $2 - 2s_v + \#\lambda_v$.
- (4) The total ordering on the vertices induces directions on the edges in the following way: we order the vertices on a line starting with the smallest vertex v_1 on the left and direct the edges from smaller to larger vertices. Each edge e of the graph is equipped with a weight $\omega(e) \in \mathbb{N}$ such that the *balancing condition*

$$s_v - (\#\delta_v - 2s_v) + \sum \pm \omega(e) = 0$$

holds for all vertices v , where the sign is $+$ for outgoing edges and $-$ for incoming edges of v . □

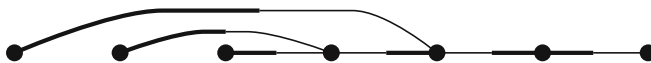
Definition 6.6 Let $\lambda = \{\beta_1, \dots, \beta_4\}$ be a degenerated cross-ratio on $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta_d)$. Let \mathcal{F} be a floor diagram of degree Δ_d . Each element β_i of λ is associated to a vertex of \mathcal{F} the following way:

- (1) If β_i is the label $t \in \{1, \dots, 3d\}$ of an end, then β_i is associated to the unique vertex $v \in \mathcal{F}$ such that $t \in \delta_v$.
- (2) If β_i is the label of a contracted end $x_j \in \{x_1, \dots, x_n\}$, then β_i is associated to v_j .

Hence a pair $\{\beta_i, \beta_j\}$ induces a unique path in \mathcal{F} . If the paths associated to $\{\beta_{i_1}, \beta_{i_2}\}$ and $\{\beta_{i_3}, \beta_{i_4}\}$ intersect in exactly one vertex v of \mathcal{F} for all pairwise different choices of i_1, \dots, i_4 such that $\{i_1, \dots, i_4\} = \{1, \dots, 4\}$, then the cross-ratio λ is *satisfied at v* . A cross-ratio floor diagram *satisfies* the degenerated cross-ratios $\lambda_1, \dots, \lambda_l$ if for each cross-ratio there is a vertex of \mathcal{F} satisfying it and $\#\lambda_v$ is exactly the total number of cross-ratios that are satisfied at a vertex v for each vertex.

Remark 6.7 Note that the condition ‘all choices of i_1, \dots, i_4 lead to exactly one vertex in the intersection of the paths’ is equivalent to ‘one choice of i_1, \dots, i_4 leads to exactly one vertex in the intersection of the paths’. This makes it easier to check if \mathcal{F} satisfies a degenerated cross-ratio.

Example 6.8 The figure below shows a cross-ratio floor diagram, where all weights on the edges are 1 and where thick edges are drawn thick. Note that we have $d = 3$ and this cross-ratio floor diagram satisfies the degenerated cross-ratio $\lambda = \{x_1x_4x_5x_6\}$.



i	1	2	3	4	5	6	7
s_{v_i}	0	0	0	1	1	0	1
$ \lambda_{v_i} $	0	0	0	0	1	0	0
δ_{v_i}	{1}	{2}	{3}	{4,9}	{5,8}	\emptyset	{6,7}

Definition 6.9 (*i*-th piece of \mathcal{F}) Let \mathcal{F} be a cross-ratio floor diagram of degree Δ_d on the ordered set of vertices v_1, \dots, v_n corresponding to given point conditions p_1, \dots, p_n such that \mathcal{F} satisfies the degenerated cross-ratios $\lambda_1, \dots, \lambda_l$. The *i*-th piece $(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \tilde{\lambda}_{i_1}, \dots, \tilde{\lambda}_{i_{\#\lambda_{v_i}}})$ (for $i = 1, \dots, n$) of \mathcal{F} is obtained from \mathcal{F} in the following way: Cut all edges that connect the vertex v_i to other vertices of \mathcal{F} into (thick or thin) half-edges, and call the connected component containing v_i now \mathcal{F}_i , equip the cut edges with the labels indicating the vertices that they used to be connected to. Moreover, we want to adapt the cross-ratios that are satisfied at v_i : If $\lambda = \{\beta_1, \dots, \beta_4\}$ is a degenerated cross-ratio which is satisfied at v_i , the paths associated to λ in \mathcal{F} (see Definition 6.6) might have been cut by cutting the edges connecting v_i to the rest of \mathcal{F} . Let $\beta_j \in \lambda$ be such that the path from the vertex associated to β_j to v_i is cut. Replace β_j by the label of the edge in the path that has been cut and denote the cross-ratio obtained that way by $\tilde{\lambda}$. We shorten the notation to \mathcal{F}_i if the additional data $(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \tilde{\lambda}_{i_1}, \dots, \tilde{\lambda}_{i_{\#\lambda_{v_i}}})$ is obvious from the context.

Definition 6.10 (*Multiplicities of cross-ratio floor diagrams*) Let \mathcal{F} be a cross-ratio floor diagram of degree Δ_d on the ordered set of vertices v_1, \dots, v_n that satisfies the degenerated cross-ratios $\lambda_1, \dots, \lambda_l$ and let p_1, \dots, p_n be points in a stretched configuration. Let \mathcal{F}_i be a piece of a floor diagram \mathcal{F} like above. The weighted incoming edges of \mathcal{F}_i induce a partition α of the sum of all weights of incoming edges of \mathcal{F}_i in a natural way and the weighted outgoing edges induce a partition β , respectively. Let κ be the set of labels of thin edges adjacent to $v_i \in \mathcal{F}_i$. The *multiplicity* of the piece \mathcal{F}_i is defined as

$$\text{mult}(\mathcal{F}_i) := \deg \left(\text{ev}_i^*(p_i) \cdot \prod_{k \in \kappa} \partial \text{ev}_k^*(y_k) \cdot \prod_{j=1}^{\#\lambda_{v_i}} \text{ft}_{\tilde{\lambda}_{i_j}}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\alpha, \beta)) \right),$$

where \deg is the degree of a cycle (Definition 2.12) and $p_i, \lambda_{i_1}, \dots, \lambda_{i_{\#\lambda_{v_i}}}, \{y_k \mid k \in \kappa\}$ are in general position (cf. Lemma 3.22). The *multiplicity* of \mathcal{F} is defined as

$$\text{mult}(\mathcal{F}) := \prod_e \omega(e) \prod_{i=1}^n \text{mult}(\mathcal{F}_i),$$

where the first product goes over all edges of \mathcal{F} and $\omega(e)$ is the weight of an edge e .

Construction 6.11 (*Floor decomposed curve \mapsto cross-ratio floor diagram*) Let Δ_d be a degree, let $p_1, \dots, p_n, \lambda_1, \dots, \lambda_l$ be in general position, where $p_1, \dots, p_n \in \mathbb{R}^2$ are points in a stretched configuration (Definition 6.1), $\lambda_1, \dots, \lambda_l$ are degenerated cross-ratios with 4 points such that $3d - 1 = n + l$ holds. Curves satisfying these conditions are floor decomposed by Lemma 6.4. We obtain a cross-ratio floor diagram \mathcal{F}_C the following way: Cut all elevators of C , that is cut all edges parallel to $(1, 0) \in \mathbb{R}^2$ such that each remaining component contains exactly one point. Shrinking these components to points v_i , we get the vertices of \mathcal{F}_C . We connect $v_i, v_j \in \mathcal{F}_C$ if and only if the components obtained from p_i, p_j are connected by an elevator. Distribute the conditions $\lambda_1, \dots, \lambda_l$ to the components analogous to Definition 6.9. We draw half-edges thin if they lead to a fixed component, and thick if they lead to a free component (see Definition 3.17). We set

$$\#\lambda_{v_i} := \sum_u \#\lambda_u,$$

where the sum runs over all vertices u in the component of p_i where λ_u is introduced in Definition 3.9, s_{v_i} is the size of the component associated to p_i and δ_{v_i} is the set of labels

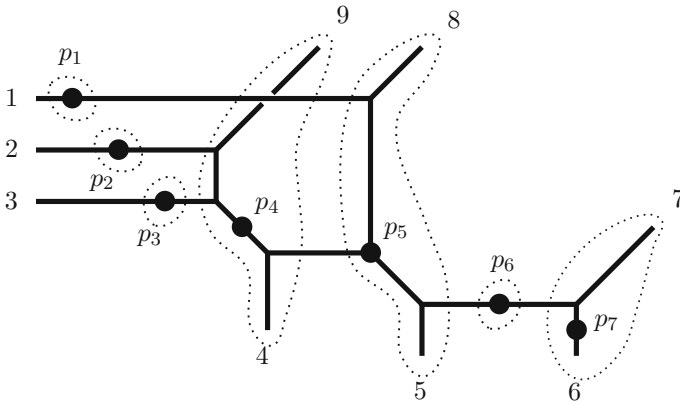


Fig. 13 A floor decomposed curve

of ends in Δ_d that are adjacent to the component associated to p_i by cutting. Finally, the balancing condition of C turns \mathcal{F}_C into a cross-ratio floor diagram.

Example 6.12 In order to illustrate Construction 6.11, a tropical curve (see Fig. 13) of degree Δ_3 through points p_1, \dots, p_7 in a stretched configuration satisfying the cross-ratio $\lambda = \{x_1x_4x_5x_6\}$ is given such that this curve is by Construction 6.11 associated to the cross-ratio floor diagram of Example 6.8. The floors of the curve are indicated by dotted lines.

Lemma 6.13 Let G be a tree without ends such that each edge of G consists of two half-edges and there are two types of half-edges, thin and thick ones. A thin half-edge can only be completed to an edge with a thick half-edge and vice versa. Then there is a vertex of G that is only adjacent to thick half-edges.

Proof This can be shown by induction over the number n of vertices of G . For $n = 2$ it is obviously true. If $n > 2$, there is a 1-valent vertex v of G since G is a tree. There are two cases: either v is adjacent to a thick half-edge, then we are done or v is adjacent to a thin half-edge. If v is adjacent to a thin half-edge, then remove this edge and v from G . The graph G' obtained this way has one vertex less than G such that there is a vertex $v' \in G'$ that is only adjacent to thick half-edges. Again there are two cases: if v' is not connected to v in G , then we are done. Otherwise, the edge connecting v' to v in G is thick at v' since it is thin at v . □

Theorem 6.14 For notation, see Notation 3.8. Let $d \in \mathbb{N}_{>0}$ and let Δ_d be its associated degree (see Definition 2.16). The number $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ of rational tropical curves satisfying point and cross-ratio conditions with 4 points (see Definitions 3.3, 6.3) equals the number obtained from counting floor diagrams. More precisely, the equality

$$N_{0,n}(\lambda'_1, \dots, \lambda'_l) = \sum_{\mathcal{F}} \text{mult}(\mathcal{F}) \tag{5}$$

holds, where the sum goes over all cross-ratio floor diagrams of degree Δ_d on an ordered set of vertices v_1, \dots, v_n that satisfy $\lambda_1, \dots, \lambda_l$.

Proof We use Theorem 3.20 to show that $N_{0,n}(\lambda'_1, \dots, \lambda'_l)$ equals the number of tropical curves satisfying the degenerated cross-ratio conditions $\lambda_1, \dots, \lambda_l$.

Let $p_1, \dots, p_n \in \mathbb{R}^2$ be points in a stretched configuration as in Definition 6.1. Let $\mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l)$ denote the set of tropical curves that contribute to $N_{0,n}(\lambda_1, \dots, \lambda_l)$. Then all curves in $\mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l)$ are floor decomposed by Lemma 6.4. Let C be such a curve. By Construction 6.11 there is a cross-ratio floor diagram \mathcal{F}_C associated to C . Recall that all weights are local (see Theorem 3.20), hence \mathcal{F}_C contributes to the right-hand side of (5) since cutting C along its elevators yields $\text{mult}(\mathcal{F}_{C_i}) \neq 0$ for all pieces of \mathcal{F}_C .

Let $\mathcal{F}_{0,n}(\lambda_1, \dots, \lambda_l)$ denote the set of elements that contribute to the right-hand side of (5). The arguments above show that

$$\begin{aligned} \phi : \mathcal{R}_{0,n}(\lambda_1, \dots, \lambda_l) &\rightarrow \mathcal{F}_{0,n}(\lambda_1, \dots, \lambda_l) \\ C &\mapsto \mathcal{F}_C \end{aligned}$$

is a well-defined map. We want to show that ϕ is onto by constructing preimages. Let $\mathcal{F} \in \mathcal{F}_{0,n}(\lambda_1, \dots, \lambda_l)$. Using Lemma 6.13, there is a vertex v_i of \mathcal{F} such that v_i is only adjacent to thick half-edges. Let $(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \tilde{\lambda}_{i_1}, \dots, \tilde{\lambda}_{i_{\#\lambda_{v_i}}})$ be the piece of \mathcal{F} that includes v_i . The weighted incoming elevators and ends of \mathcal{F}_i induce an unordered partition $\alpha^{(i)}$ and the weighted outgoing elevators and ends of \mathcal{F}_i induce $\beta^{(i)}$, respectively. Since $\text{mult}(\mathcal{F}_i) \neq 0$ there is a curve C_i corresponding to a point in $\text{ev}_i^*(p_i) \cdot \prod_{j=1}^{\#\lambda_{v_i}} \text{ft}_{\tilde{\lambda}_{i_j}}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\alpha^{(i)}, \beta^{(i)}))$ (see Definition 2.16) that is fixed by $p_i, \tilde{\lambda}_{i_1}, \dots, \tilde{\lambda}_{i_{\#\lambda_{v_i}}}$. Remove v_i and its adjacent edges from \mathcal{F} . The resulting graph might be disconnected. Let K be a component of this graph. Using Lemma 6.13, there is a vertex v_j of K such that v_j is only adjacent to thick half-edges. There are two cases:

- (1) If $v_j \in \mathcal{F}$ is only adjacent to thick half-edges, then associate a curve C_j to v_j like we did before for v_i .
- (2) There is an edge e in \mathcal{F} that connects v_i and v_j such that the thick half-edge of e is adjacent to v_i . Let $y_e \in \mathbb{R}$ be the height of the horizontal end associated to e in C_i . Now that we fixed that height, we can argue like before: Let $(\mathcal{F}_j, \delta_{v_j}, s_{v_j}, \#\lambda_{v_j}, \tilde{\lambda}_{j_1}, \dots, \tilde{\lambda}_{j_{\#\lambda_{v_j}}})$ be the piece of \mathcal{F} that includes v_j . The weighted incoming elevators and ends of \mathcal{F}_j induce $\alpha^{(j)}$ and $\beta^{(j)}$ as before. Since $\text{mult}(\mathcal{F}_j) \neq 0$ there is a curve C_j corresponding to a point in $\text{ev}_j^*(p_j) \cdot \partial \text{ev}_e^*(y_e) \cdot \prod_{z=1}^{\#\lambda_{v_j}} \text{ft}_{\tilde{\lambda}_{z_j}}^*(0) \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta(\alpha^{(j)}, \beta^{(j)}))$ that is fixed by $p_i, \tilde{\lambda}_{i_1}, \dots, \tilde{\lambda}_{i_{\#\lambda_{v_i}}}$.

Iterating this procedure gives us a curve C_t for each piece \mathcal{F}_t of \mathcal{F} such that C_1, \dots, C_n can be glued together by construction. Denote the curve obtained from this gluing by C . The multiplicity of C is given by

$$\text{mult}(C) = \prod_{t=1}^n \text{mult}(C_t)$$

because of Theorem 3.20. Therefore $C \in \phi^{-1}(\mathcal{F})$.

Note that the procedure above does not depend on the choice of C_t we associated to each \mathcal{F}_t . Hence (consider Remark 3.23)

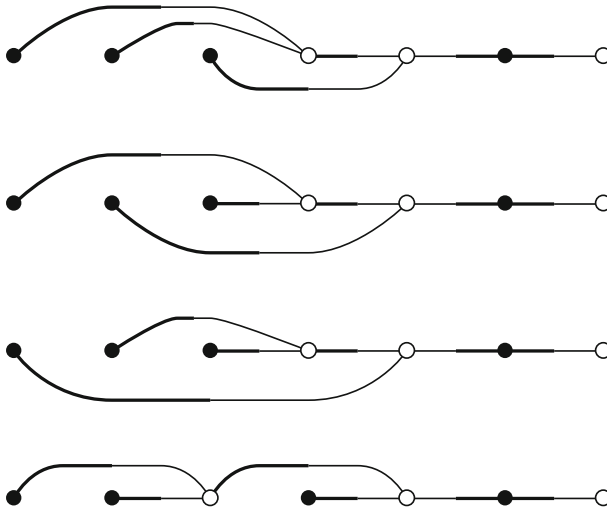


Fig. 14 Cross-ratio floor diagrams with floors of size 0 (black) and 1 (white)

$$\text{mult}(\mathcal{F}) = \sum_{C \in \phi^{-1}(\mathcal{F})} \text{mult}(C)$$

holds.

□

We can now apply Corollary 3.21 and the Correspondence Theorem 2.25 such that the next corollary follows immediately.

Corollary 6.15 *We use the notation from Notation 3.8. Under the same assumptions as in Theorem 6.14 the equality*

$$N_{0,n}^{\text{class}}(\mu_1, \dots, \mu_l) = \sum_{\mathcal{F}} \text{mult}(\mathcal{F})$$

holds, where the sum goes over all cross-ratio floor diagrams of degree Δ_d on an ordered set of vertices v_1, \dots, v_n that satisfy $\lambda_1, \dots, \lambda_l$.

Remark 6.16 The results of this section are not restricted to degree Δ_d curves and can be generalized to Hirzebruch surfaces or other surfaces with h -transverse polytopes (see [5]) since the cross-ratio floor diagram techniques can be extended to these degrees in a straightforward way.

Example 6.17 Fix the degree Δ_3 , let p_1, \dots, p_7 be points and let $\lambda = \{x_1, \dots, x_4\}$ be a degenerated cross-ratio. We want to determine the number $N_{0,7}(\lambda)$ using floor diagrams. For that draw all floor diagrams of degree Δ_3 on 7 vertices that satisfy the degenerated cross-ratio λ . Since we have 7 points, there are no floors of size 3 or 2. Figure 14 shows all possible floor diagrams. Note that in this example we do not need all discrete data a floor diagram is equipped with, i.e. floors of size 1 are drawn white and floors of size 0 are drawn black (instead of specifying s_{v_i} for each floor), the number of degenerated cross-ratios satisfied at each floor is obvious (we only have one cross-ratio) and the labels of ends adjacent to each floor are dropped here, so we need to add a factor of $(d!)^3$ to the final count. By considering

the multiplicities of each piece \mathcal{F}_i of a floor diagram \mathcal{F} in Fig. 14, we end up with multiplicity 1 for all floor diagrams shown in Fig. 14. Hence

$$N_{0,7}(\lambda) = 4 * (3!)^3 = 864.$$

Note that this number is not the same as the one in Example 4.15 because we considered a cross-ratio with 4 points here, whereas we considered a cross-ratio with 2 points in Example 4.15.

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