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Covering classes, strongly flat modules, and completions

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Abstract

We study some closely interrelated notions of Homological Algebra: (1) We define a topology on modules over a not-necessarily commutative ring R that coincides with the R-topology defined by Matlis when R is commutative. (2) We consider the class $\mathcal{SF}$ of strongly flat modules when R is a right Ore domain with classical right quotient ring Q. Strongly flat modules are flat. The completion of R in its R-topology is a strongly flat R-module. (3) We prove some results related to the question whether $\mathcal{SF}$ a covering class implies $\mathcal{SF}$ closed under direct limits. This is a particular case of the so-called Enochs' Conjecture (whether covering classes are closed under direct limits). Some of our results concern right chain domains. For instance, we show that if the class of strongly flat modules over a right chain domain R is covering, then R is right invariant. In this case, flat R-modules are strongly flat.

Keywords Covering class · Strongly flat module · Completion · Cotorsion module · R-topology

Mathematics Subject Classification Primary 16E30 · 16W80; Secondary 18G15

1 Introduction

The aim of this paper is to highlight some relations between completions, strongly flat modules and perfect rings in the non-commutative case. We explore the connections between some notions of Homological Algebra (cotorsion modules) and topological rings (completions in

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some natural topologies). These connections are well known for modules over commutative rings, thanks to Matlis, who proved that the completion in the R-toplogy for an integral domain R is closely related to the cotorsion completion functor $\mathrm{Ext}^1_R(K, -)$. Here Q is the field of fractions of R and $K := Q/R$ [20]. We investigate these connections in the non-commutative case, defining a suitable R-topology on any module over a not-necessarily commutative ring R. This leads us to the study of strongly flat modules, because the completion of R in its R-topology turns out to be a strongly flat R-module (Corollary 5.13).

We consider strongly flat modules over non-commutative rings as defined in [13, Sect. 3]. The class of strongly flat modules lies between the class of projective modules and the class of flat modules. In particular, we study when the class of strongly flat modules is covering, because this is related to an open problem posed by Enochs, that is, whether "every covering class is closed under direct limits" (see. for example [18, Open problem 5.4]). Since flat modules are direct limits of projective modules, the class of strongly flat modules is closed under direct limits if and only if flat modules are strongly flat. Bazzoni and Salce [4] gave a complete answer to this question for modules over commutative domains, completely determining when the class of strongly flat modules over a commutative domain is covering. Subsequently, Bazzoni and Positselski generalised this to arbitrary commutative rings [5]. They proved that, for a commutative ring R, the class $\mathcal{SF}$ of strongly flat modules is covering if and only if flat modules are strongly flat, if and only if R/aR is a perfect ring for every regular element $a \in R$ and the classical ring of quotients of R is perfect. In our Example 5.21, we will show that there exist non-invariant chain domains R for which $\mathrm{End}(R/I)$ is perfect for every non-zero principal right or left ideal I of R, but the class of strongly flat left R-modules is not covering. Very recent papers related to these topics are the articles [6] by Bazzoni and Positselski and [22] by Positselski.

For a commutative ring R, the set of regular elements is always an Ore set, and if Q denotes the classical quotient ring of R, the class of strongly flat modules is ${}^{\perp}\{Q^{\perp}\}$ [15]. The generalisation of strongly flat modules to non-commutative rings given in [13] depends on the choice of the overring Q of R. More precisely, if the inclusion $\varphi\colon R \to Q$ is a bimorphism in the category of rings, that is, φ is both a monomorphism and an epimorphism, we assume ${}_RQ$ is a flat left R-module. A left R-module ${}_RM$ is *Matlis-cotorsion* if $\mathrm{Ext}^1({}_RQ, {}_RM) = 0$ [13]. Let $\mathcal{MC}$ denote the class of Matlis-cotorsion left R-modules. For any class of left R-modules $\mathcal{A}$, set ${}^{\perp}\mathcal{A} := \{B \in R-\mathrm{Mod} \mid \mathrm{Ext}^1_R(B, A) = 0$ for every $A \in \mathcal{A}\}$ and $\mathcal{A}^{\perp} := \{B \in R{-}\mathrm{Mod} \mid \mathrm{Ext}^1_R(A, B) = 0$ for every $A \in \mathcal{A}\}$. A left R-module is *strongly flat* if it is in ${}^{\perp}\mathcal{MC}$. The class of strongly flat left R-modules will be denoted by $\mathcal{SF}$. By [18, Theorem 6.11], the cotorsion pair $(\mathcal{SF}, \mathcal{MC})$ is complete, that is, every left R-module has a special $\mathcal{MC}$-preenvelope (or, equivalently, every left R-module has a special $\mathcal{SF}$-precover). Thus, by [18, Corollary 6.13], the class $\mathcal{SF}$ consists of all direct summands of modules N such that N fits into an exact sequence of the form

$$0 \to F \to N \to G \to 0,$$

where F is a free R-module and G is $\{Q\}$-filtered. For the terminology, see [13].

Whenever R is a right Ore domain, i.e., the subset of non-zero elements is a right Ore set, the class of strongly flat left R-modules is the class ${}^{\perp}\{Q^{\perp}\}$, where Q is the classical right quotient ring of R.

Several of our results about strongly flat modules are for modules over a nearly simple chain domain. Recall that a chain domain R, that is, a not-necessarily commutative integral domain for which the modules R_R and ${}_RR$ are uniserial, is *nearly simple* if it has exactly three two-sided ideals, necessarily R, its Jacobson radical $J(R)$ and 0. The reason why we concentrate on chain domains R with classical quotient ring Q is due to the fact that for these rings the R-module ${}_RK := Q/R$ is uniserial, and thus, in the study of $\mathrm{End}({}_RK)$, we can take advantage of our knowledge of the endomorphism rings of uniserial modules [10–12,14,23,24]. In our Example 5.21, we also take advantage of our knowledge of the endomorphism rings of cyclically presented modules over local rings [1].

If R is a right chain domain and the class of strongly flat R-modules is covering, then R is right invariant, that is, $aR = Ra$ for every $a \in R$. In this case, flat modules are strongly flat (equivalently, the class $\mathcal{SF}$ of strongly flat modules is closed under direct limits). We began this paper in September 2017, when both of us were visiting the Department of Algebra of Charles University in Prague, and continued in March 2018 when the first named author was visiting the IPM (Institute for Research in Fundamental Sciences) in Tehran. We are very grateful to both institutions for their hospitality.

2 The *R*-topology

In Sects. 2, 3 and 4 of this paper, we suppose that *we have a ring R and a multiplicatively closed subset S of R satisfying:* (1) *If $a, b \in R$ and $ab \in S$, then $a \in S$.* (2) *S is a right Ore set in R.* (3) *The elements of S are regular elements of R.* (4) *The right ring of quotients $Q := R[S^{-1}]$ of R with respect to S is a directly finite ring*. That is, our setting is that of [13, Sect. 4].

Correspondingly, we have a Gabriel topology $\mathcal{G}$ on R consisting of all the right ideals I of R with $I \cap S \neq \emptyset$ (cf. [25, Sect. VI.6]). In particular, the Gabriel topology $\mathcal{G}$ consists of dense right ideals of R, the canonical embedding $\varphi\colon R \to Q := R[S^{-1}]$ is an epimorphism in the category of rings, we view R as a subring of Q and φ as the inclusion mapping, and ${}_RQ$ turns out to be a flat left R-module [25, Sect. XI.3]. There is a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ on Mod-R in which the torsion submodule of any right R-module M_R consists of all the elements $x \in M_R$ for which there exists an element $s \in S$ with $xs = 0$. If we indicate the torsion submodule of M by $t(M)$, then clearly $t(M) \otimes_R Q = 0$. A right R-module M_R is in $\mathcal{F}$, that is, is torsion-free, if and only if right multiplication $\rho_s\colon M_R \to M_R$ by s is an abelian group monomorphism for every $s \in S$. Dually, we will say that a right R-module M_R is *divisible* if right multiplication $\rho_s\colon M_R \to M_R$ by s is an abelian group epimorphism for every $s \in S$, that is, if $Ms = M$ for every $s \in S$. Every homomorphic image of a divisible right R-module is divisible. If A is a submodule of a right R-module B_R and both A_R and B/A are divisible, then B_R is divisible. Any sum of divisible submodules is a divisible submodule, so that every right R-module M_R contains a greatest divisible submodule, denoted by $d(M_R)$. A right R-module M_R is *reduced* if $d(M_R) = 0$. For every module M_R, $M_R/d(M_R)$ is reduced.

We have that $\mathcal{G} = \{\, I \mid I$ is a right ideal of R, and $\varphi(I)Q = Q \,\}$, and $\mathcal{G}$ has a basis consisting of the principal right ideals sR, $s \in S$. Let M_R be any right R-module. By [25, XI, Proposition 3.4], the kernel of the canonical right R-module morphism $M_R \to M \otimes_R Q$ is equal to $t(M)$. Note that if we set $K := Q/R$, then ${}_RK_R$ is an R-R-bimodule and $t(M_R) \cong \mathrm{Tor}_1^R(M_R, {}_RK)$ (see, (15) and (16) in [13, Sect. 3]).

We now define a topology on any right R-module in the attempt of generalising the R-topology studied by Matlis [20] for a commutative ring R. Our definition is as follows. Let R be any ring with identity, not necessarily commutative, and S be a subset of R with the properties written at the beginning of this section. Given any right R-module M_R, the *R-topology* on M_R has a neighbourhood base of 0 consisting, for every non-empty finite set of elements $s_1, \dots, s_n \in S$, of the submodules

$$U(s_1, \dots, s_n) := \{\, x \in M_R \mid xR \subseteq Ms_1 \cap \dots \cap Ms_n \,\}$$

of M_R. For the regular right module R_R, the R-topology on R has a neighbourhood base of 0 consisting, for every non-empty finite set of elements $s_1, \dots, s_n \in S$, of the right ideals

$$U(s_1, \dots, s_n) := \{ x \in R \mid xR \subseteq Rs_1 \cap \dots \cap Rs_n \}$$

of R.

Lemma 2.1 *On the right R-module R_R, the right ideals $U(s)$ are two-sided ideals of R, $U(s)$ is the annihilator of the left R-module R/Rs, and the R-topology is a ring topology on R.*

Proof Clearly, $U(s) = \{ x \in R \mid xR \subseteq Rs \}$ is the annihilator of the cylic left R-module R/Rs, and hence $U(s)$ is a two-sided ideal. Moreover, R is a right linearly topological ring [25, p. 144], because every filter of two-sided ideals of a ring is a fundamental system of neighbourhoods of 0 for a right and left linear topology on the ring [25, p. 144]. □

More generally, notice that if $f : M_R \to N_R$ is a right R-module morphism and $x \in M_R$ is such that $xR \subseteq Ms_1 \cap \dots \cap Ms_n$, then $f(x)R \subseteq f(Ms_1 \cap \dots \cap Ms_n) \subseteq f(M)s_1 \cap \dots \cap f(M)s_n \subseteq Ns_1 \cap \dots \cap Ns_n$, so that f induces mappings $M_R/U_M(s_1, \dots, s_n) \to N_R/U_N(s_1, \dots, s_n)$, which form an inverse system of right R-module morphisms, hence they define a right R-module morphism

$$\widetilde{M_R} := \varprojlim M/U_M(s_1, \dots, s_n) \to \widetilde{N_R} := \varprojlim N/U_N(s_1, \dots, s_n).$$

Thus the completion in the R-topology is an additive functor Mod-$R \to$ Mod-R.

We will use $R_{R\text{-top}}$ to denote the topological ring R with the R-topology.

Lemma 2.2 *Every right R-module, with respect to its R-topology, is a linearly topological module over the topological ring $R_{R\text{-top}}$.*

Proof It suffices to check property $TM3$ in [25, p. 144]. That is, we must prove that $(U_M(s) : x) \supseteq U_R(s)$ for every $s \in S$, $x \in M_R$. Equivalently, that $xU_R(s) \subseteq U_M(s)$. Now if $r \in U_R(s)$, then $rR \subseteq Rs$, so that $xrR \subseteq xRs \subseteq Ms$, i.e., $rx \in U_M(s)$. □

Lemma 2.3 *If the ring R is commutative, the linear topology on any right R-module M defined by the submodules $U(s)$, $s \in S$, coincides with the R-topology defined by Matlis in* [20].

Proof $U(s) = \{ x \in M \mid xR \subseteq Ms \} = Ms$. □

In the next proposition, we consider the behaviour of continuity of right R-module morphisms when the modules involved are endowed with the R-topology. Recall that a submodule M of a right R-module N_R is an *RD-pure submodule* if $Mr = M \cap Nr$ for every $r \in R$ (equivalently, if the natural homomorphism $M \otimes R/Rr \to N \otimes R/Rr$ is injective for every $r \in R$, or if the natural homomorphism $\mathrm{Hom}(R/rR, N) \to \mathrm{Hom}(R/rR, N/M)$ is surjective for every $r \in R$.) See, [27, Proposition 2].

Proposition 2.4 (a) *Every right R-module morphism $f : M_R \to N_R$ between two right R-modules M_R and N_R endowed with their R-topologies is continuous.*

(b) *For every right R-module N_R and every $s \in S$, the R-submodule $U(s)$ of N_R is the largest R-submodule of N_R contained in Ns.*

(c) *A submodule M_R of a right R-module N_R endowed with the R-topology is an open submodule of N_R if and only if $M_R \supseteq U(s)$ for some $s \in S$.*

(d) *A right R-module morphism $f\colon M_R \to N_R$ between two right R-modules M_R and N_R with their R-topologies is an open map if and only if $f(M_R) \supseteq U(s)$ for some $s \in S$.*
(e) *Every right R-module epimorphism $f\colon M_R \to N_R$ between two right R-modules M_R and N_R is an open continuous map.*
(f) *Every right R-module isomorphism $f\colon M_R \to N_R$ is a homeomorphism when the two right R-modules M_R and N_R are endowed with their R-topologies.*
(g) *If M_R is an RD-pure submodule of a right R-module N_R, and M_R, N_R are endowed with their R-topologies, then the embedding $M_R \hookrightarrow N_R$ is a topological embedding.*

The proofs are easy and we omit them.

3 The right R-module $\mathrm{Hom}(K_R, M \otimes_R K)$

In this section, the hypotheses on R and S are the same as in the previous section. For any right R-module M_R, we will be interested in the right R-module

$$\mathrm{Hom}(K_R, M \otimes_R K).$$

Here the right R-module structure is given by the multiplication defined, for every $f \in \mathrm{Hom}(K_R, M \otimes_R K)$ and $r \in R$, by $(fr)(k) = f(rk)$ for all $k \in K$.

For any right R-module M_R, the right R-module

$$\mathrm{Hom}(K_R, M \otimes_R K)$$

can be endowed with the R-topology, defined by the submodules $U(s_1, \dots, s_n) := U(s_1) \cap \dots \cap U(s_n)$ as a neighbourhood base of 0. But we have that:

Lemma 3.1 *For the modules* $\mathrm{Hom}(K_R, M \otimes_R K)$*, one has that $U(s) = V(s)$, where, for every element $s \in S$,*

$$V(s) := \left\{ f \in \mathrm{Hom}(K_R, M \otimes_R K) \mid f(Rs^{-1}/R) = 0 \right\}.$$

Proof ($\subseteq$). Let f be an element of $U(s)$, so that $f \in \mathrm{Hom}(K_R, M \otimes_R K)$ and $fR \subseteq \mathrm{Hom}(K_R, M \otimes_R K)s$. In order to show that $f \in V(s)$ we have to prove that $f(Rs^{-1}/R) = 0$. Fix $r \in R$. Then $fr = gs$ for some $g \in \mathrm{Hom}(K_R, M \otimes_R K)$. Hence $f(rs^{-1} + R) = (fr)(s^{-1} + R) = (gs)(s^{-1} + R) = g(ss^{-1} + R) = 0$. Thus $f(Rs^{-1}/R) = 0$.

($\supseteq$). Suppose $f \in V(s)$, so that $f(Rs^{-1}/R) = 0$. In order to prove that $f \in U(s)$, we must show that, for every fixed element $r \in R$, there exists $g \in \mathrm{Hom}(K_R, M \otimes_R K)$ with $fr = gs$. Define $g\colon K_R \to M \otimes_R K_R$ by $g(q + R) = f(rs^{-1}q + R)$ for all $q \in Q$. Then g is a well defined right R-module morphism, because if $q \in R$, then $f(rs^{-1}q + R) = f(rs^{-1} + R)q \in f(Rs^{-1}/R)R = 0$, and $fr = gs$. □

We will denote by $V(s_1, \dots, s_n)$ the intersection $V(s_1) \cap \dots \cap V(s_n)$, but it is necessary to remark that:

Lemma 3.2 *For every $s, s' \in S$, there exists $t \in S$ such that $V(s) \cap V(s') \supseteq V(t)$.*

Proof Given $s, s' \in S$, there exist $t \in S$ and $r, r' \in R$ with $t = sr = s'r'$ [17, Lemma 4.21]. Then $s^{-1} = rt^{-1}$, so that $Rs^{-1} = Rrt^{-1} \subseteq Rt^{-1}$. Therefore $V(t) \subseteq V(s)$, because if $f \in \mathrm{Hom}(K_R, M \otimes_R K)$ and $f(Rt^{-1}/R) = 0$, then $f(Rs^{-1}/R) = 0$, that is, $f \in V(s)$. Similarly, $V(t) \subseteq V(s')$. □

A right (or left) R-module M_R is *h-divisible* if every homomorphism $R_R \to M_R$ extends to an R-module morphism $Q_R \to M_R$ [13, Sect. 2]. Any right (or left) R-module M contains a unique largest h-divisible submodule $h(M)$ that contains every h-divisible submodule of M. An R-module M_R is *h-reduced* if $h(M_R) = 0$, or, equivalently, if $\mathrm{Hom}(Q_R, M_R) = 0$ [13]. Obviously, h-divisible right R-modules are divisible.

Proposition 3.3 *Divisible torsion-free right R-modules are Q-modules. In particular, $h(M_R) = d(M_R)$ for any torsion-free right R-module M_R.*

Proof Suppose M_R torsion-free and divisible. Then right multiplication by s is an automorphism of the abelian group M for every $s \in S$. By the universal property of $Q = R[S^{-1}]$, the canonical ring antihomomorphism $R \to \mathrm{End}_{\mathbb{Z}}(M)$ extends to a ring antihomomorphism $Q \to \mathrm{End}_{\mathbb{Z}}(M)$ in a unique way. That is, there is a unique right Q-module structure on M that extends the right R-module structure of M_R. Thus M is a right Q-module. In particular, M is an h-divisible right R-module. □

Let M_R be a right R-module. For every element $x \in M_R$, there is a right R-module morphism $R_R \to M_R$, $1 \mapsto x$. Tensoring with ${}_RK$, we get a right R-module morphism

$$\lambda_x : K_R \to M \otimes_R K,$$

defined by $\lambda_x(k) = x \otimes k$. The canonical mapping $\lambda : M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$, defined by $\lambda(x) = \lambda_x$ for every $x \in M_R$, is a right R-module morphism, as is easily checked. In the rest of this section, all R-modules are endowed with their R-topologies.

Theorem 3.4 *Let M_R be an h-reduced torsion-free right R-module. Then the canonical mapping $\lambda : M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$ is an embedding of topological modules and $\mathrm{Hom}(K_R, M \otimes_R K)$ is complete.*

Proof The canonical mapping $\lambda : M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$ is injective by [13, Theorem 4.5]. In order to show that $\lambda : M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$ is an embedding of topological modules, it suffices to show that $\lambda^{-1}(V(s_1, \dots, s_n)) = U(s_1, \dots, s_n)$ for every $s_1, \dots, s_n \in S$. Now $x \in \lambda^{-1}(V(s_1, \dots, s_n))$ if and only if $\lambda_x \in V(s_1, \dots, s_n)$, that is, if and only if $x \otimes (Rs_1^{-1} + \dots + Rs_n^{-1}/R) = 0$ in $M \otimes_R K$. Equivalently, if and only if $x \otimes (rs_i^{-1} + R) = 0$ in $M \otimes_R K$ for every $r \in R$ and $i = 1, 2, \dots, n$. By [13, Step 3 of the proof of Theorem 4.5], this is equivalent to $xr \in Ms_i$ for every $r \in R$ and $i = 1, 2, \dots, n$, that is, if and only if $x \in U(s_1, \dots, s_n)$.

In order to prove that $\mathrm{Hom}(K_R, M \otimes_R K)$ is complete, we must show that every Cauchy net converges. Let A be a directed set with order relation $\leq$ and let $\{f_\alpha\}_{\alpha \in A}$ be a Cauchy net in $\mathrm{Hom}(K_R, M \otimes_R K)$. Define a morphism $f \in \mathrm{Hom}(K_R, M \otimes_R K)$ as follows. Since we are dealing with a Cauchy net, for every $s \in S$ there exists $\alpha \in A$ such that $f_\beta - f_\gamma \in V(s)$ for every $\beta, \gamma \in A$, $\beta, \gamma \geq \alpha$. Set $f(rs^{-1} + R) = f_\alpha(rs^{-1} + R)$ for every $r \in R$. We leave to the reader the easy verification that f is a well defined mapping. Let us check that $f(kr) = f(k)r$ for every $k \in K_R$ and $r \in R$. We have that $k = as^{-1} + R$ for some $a \in R$, $s \in S$. By the right Ore condition, there exist $r' \in R$ and $t \in S$ such that $as^{-1}r = r't^{-1}$. Since A is directed, there exists α such that $f(r't^{-1} + R) = f_\alpha(r't^{-1} + R)$ and $f(as^{-1} + R) = f_\alpha(as^{-1} + R)$. Therefore $f(kr) = f(k)r$. It is now easily seen that f is the limit of the Cauchy net. □

For any right R-module M_R endowed with its R-topology, the *(Hausdorff) completion* of M_R is $\widetilde{M_R} := \varprojlim M/U(s_1, \dots, s_n)$. Notice that the set of all the submodules $U(s_1, \dots, s_n)$ of M_R is downward directed under inclusion. Here $\{s_1, \dots, s_n\}$ ranges in the set of all non-empty finite subsets of S. There is a canonical mapping $\eta : M \to \widetilde{M_R}$, whose kernel is the closure $\overline{\{0\}}$ of 0 in the R-topology of M_R. Clearly, $\overline{\{0\}} = \bigcap_{s_1,\dots,s_n \in S} U(s_1, \dots, s_n) = \bigcap_{s \in S} U(s) = \{ x \in M_R \mid xR \subseteq \bigcap_{s \in S} Ms \}$.

From Lemma 2.2, we get that if M_R is a right R-module, the right R-module $\mathrm{Hom}(K_R, M \otimes_R K)$ with the topology defined by the submodules $V(s)$ is a topological module over the topological ring $R_{R\text{-top}}$.

Proposition 3.5 *The right R-submodules $V(s)$ of the ring* $\mathrm{End}(K_R)$ *are two-sided ideals of* $\mathrm{End}(K_R)$. *The topology they define on* $\mathrm{End}(K_R)$ *is a ring topology. If R is commutative, this topology on* $\mathrm{End}(K_R)$ *coincides with the topology on the completion H of R with respect to the R-topology* [20, p. 15].

Proof When we consider $M = R_R$, then, by [13, Step 2 of the proof of Theorem 4.5], the elements of K annihilated by right multiplications of an element $s \in S$ are those of Rs^{-1}/R. It follows that Rs^{-1}/R is a subgroup of K invariant under all endomorphisms of K_R. From this we get that every $V(s)$ is a two-sided ideal of the ring $\mathrm{End}(K_R)$.

Every filter of two-sided ideals of a ring is a fundamental system of neighbourhoods of 0 for a right and left linear topology on the ring [25, p. 144]. Thus the topology defined by the two-sided ideals $V(s)$ is a ring topology on $\mathrm{End}(K_R)$. Moreover, if R is commutative, the submodules $V(s)$ define the R-topology on the right R-module $\mathrm{Hom}(K_R, M \otimes_R K)$ for every module M (Lemma 2.2), which coincides with the R-topology defined by Matlis in [20] by Lemma 2.3. Furthermore, Matlis' R-topology on $\mathrm{End}(K_R)$ coincides with the topology on the completion H of R with respect to the R-topology, because the topology on the completion H coincides with the R-topology on H. □

4 Torsion-free modules

In this section, we keep the same hypotheses and notations as in the previous two sections.

As we have seen, for any right R-module M_R, there is a right R-module morphism

$$\lambda \colon M_R \to \mathrm{Hom}(K_R, M \otimes_R K),$$

defined by $\lambda(x) = \lambda_x$ for every $x \in M_R$, where $\lambda_x : k \to x \otimes k$, and there is a canonical mapping $\eta \colon M \to \widetilde{M_R}$ of M_R with its R-topology into its Hausdorff completion.

Proposition 4.1 *Let M_R be a torsion-free right R-module. Then:* (a) $\ker \lambda$ *is the closure of 0 in the R-topology;* (b) $\ker \lambda$ *is the kernel of the canonical mapping* $\eta \colon M \to \widetilde{M_R}$*; and* (c) $\ker \lambda$ *is equal to* $h(M_R)$.

Proof We have already remarked that the kernel of η is the closure $\overline{\{0\}}$ of 0. Hence (a) $\Leftrightarrow$ (b).

The right R-module $\mathrm{Hom}(K_R, N_R)$ is h-reduced for every right R-module N_R [13, Theorem 2.8]. Let M_R be a torsion-free right R-module. Since

$$\lambda \colon M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$$

is a homomorphism into an h-reduced R-module, it follows that $h(M) \subseteq \ker \lambda$.

Let us prove that $\ker \lambda \subseteq \overline{\{0\}}$. Suppose $x \in \ker \lambda$. Fix arbitrary $r \in R$ and $s \in S$. Then $x \otimes (rs^{-1} + R)$ is equal to zero in the tensor product $M \otimes K$. By [13, Theorem 3.1(1)], there exists an element $y_{r,s} \in M_R$ such that $x \otimes rs^{-1} = y_{r,s} \otimes 1$ in $M \otimes_R Q$. Thus $xr \otimes 1 = y_{r,s}s \otimes 1$ in $M \otimes_R Q$. Since M_R is torsion-free, it follows that $xr = y_{r,s}s$ in M_R by [13, Theorem 3.1(1)] again. This proves that $xR \subseteq \bigcap_{s \in S} Ms$, and so $\ker \lambda \subseteq \overline{\{0\}}$.

Conversely, $\overline{\{0\}} \subseteq \ker \lambda$, because if $x \in \overline{\{0\}}$, then $xR \subseteq Ms$ for every $s \in S$, that is, for every $s \in S$ and every $r \in R$ there exists $m_{r,s} \in M$ with $xr = m_{r,s}s$. Then, for every element $rs^{-1} + R \in K$, we have that $x \otimes (rs^{-1} + R) = xr \otimes (s^{-1} + R) = m_{r,s}s \otimes (s^{-1} + R) = m_{r,s} \otimes s(s^{-1} + R) = 0$ in $M \otimes_R K$. Thus $x \in \ker \lambda$. This proves that $\overline{\{0\}} = \ker \lambda$. Therefore (a) and (b) hold.

We now show that $\ker \lambda$ is divisible. For every $s \in S$, s is invertible in Q, hence $sQ = Q$, so $sK = K$. Now if $x \in \ker \lambda$ and $t \in S$, then $x \in \overline{\{0\}}$, hence $x = yt$ for some $y \in M_R$. We must prove that $y \in \ker \lambda$, that is, that $y \otimes K = 0$ in $M \otimes K$. But $y \otimes K = y \otimes sK = ys \otimes K = x \otimes K = 0$ in $M \otimes K$. This proves that $\ker \lambda = \overline{\{0\}}$ is divisible. Thus $\ker \lambda = h(M)$ by Proposition 3.3. □

Clearly, from Proposition 4.1 we have that:

Corollary 4.2 *If M_R is a torsion-free module, then $\widetilde{M_R} \cong \widetilde{M_R/h(M)}$.*

Lemma 4.3 *Let M be torsion-free right R-module. Then:*

(a) *Every element of $M \otimes_R K$ can be written in the form $x \otimes (s^{-1} + R)$ for suitable elements $x \in M_R$ and $s \in S$.*

(b) *Let s be an element of S. The elements y of $M \otimes_R K$ such that $ys = 0$ are those that can be written in the form $x \otimes (s^{-1} + R)$ for a suitable $x \in M_R$.*

(c) *If $x \in M_R$, $r \in R$ and $s \in S$, then $x \otimes (rs^{-1} + R) = 0$ in $M \otimes_R K$ if and only if $xr \in Ms$.*

(d) *The set $\{\, U(s) \mid s \in S \,\}$ is downward directed.*

Proof In the proof of Steps 1, 2 and 3 of [13, Theorem 4.5], we do not use the fact that M is h-reduced. So the proofs of (a), (b) and (c) are like those of Steps 1, 2 and 3 in [13, Theorem 4.5].

(d) Assume that $s, t \in S$. Then there exist $u \in S$ and $r_1, r_2 \in R$ such that $s^{-1} = r_1u^{-1}$ and $t^{-1} = r_2u^{-1}$. If $m \in U(u)$ and $r \in R$, then $m \otimes (rs^{-1} + R) = m \otimes (rr_1u^{-1} + R) = 0$. Part (c) implies that $m \in U(s)$, and so $U(u) \subseteq U(s)$. Similarly, $U(u) \subseteq U(t)$. □

Remark 4.4 By Lemma 4.3(d), for M torsion-free, we have that

$$\widetilde{M} = \varprojlim M/U(s).$$

Notice that the kernel of the canonical mapping $\eta \colon M \to \widetilde{M}$ is divisible by Theorem 4.1.

Now let M_R be a torsion-free right R-module, so that

$$\lambda \colon M_R \to \operatorname{Hom}(K_R, M \otimes_R K)$$

is continuous with respect to the R-topologies (Proposition 2.4 (a)) and $\operatorname{Hom}(K_R, M \otimes_R K)$ is Hausdorff. Notice that $M \otimes_R K$ and $M/h(M) \otimes_R K$ are isomorphic, so that $\operatorname{Hom}(K_R, M \otimes_R K)$ is complete (Theorem 3.4). Thus λ extends in a unique way to a continuous morphism $\widetilde{\lambda} \colon \widetilde{M} \to \operatorname{Hom}(K_R, M \otimes_R K)$. In Theorem 4.5 and Example 4.6, we see that $\widetilde{\lambda}$ is a continuous monomorphism, but not necessary an isomorphism.

Theorem 4.5 *Let M_R be a torsion-free right R-module. Then there exists a right R-module monomorphism $\widetilde{\lambda} \colon \widetilde{M} \to \operatorname{Hom}(K_R, M \otimes_R K)$ such that $\lambda = \widetilde{\lambda}\eta$.*

Proof Define $\widetilde{\lambda}$ as follows. We know that

$$\widetilde{M} = \varprojlim M/U(s) \leq \prod_{s \in S} M/U(s),$$

so that every element of $\widetilde{M}$ is of the form $\widetilde{m} = (m_s + U(s))_{s\in S}$. Set $\widetilde{\lambda}(\widetilde{m})(rs^{-1} + R) = m_s \otimes (rs^{-1} + R)$ for every $r \in R$, $s \in S$. In order to prove that $\widetilde{\lambda}(\widetilde{m}) \colon K_R \to M \otimes_R K$ is a well defined mapping and is R-linear, note first of all that if $s, t \in S$ are such that $U(t) \subseteq U(s)$ and $r \in R$, then $m_s - m_t \in U(s)$ implies that $m_s \otimes rs^{-1} + R = m_t \otimes rs^{-1} + R$ by Lemma 4.3 (c). From this, it is easily shown that $\widetilde{\lambda}$ is a well defined R-module morphism. Also notice that $\lambda = \widetilde{\lambda}\eta$.

Now we prove that $\widetilde{\lambda}$ is a monomorphism. Suppose that $\widetilde{m} = (m_s + U(s))_{s\in S}$ is in $\ker \widetilde{\lambda}$. Then, for every $r \in R$ and $s \in S$, we have that $m_s \otimes rs^{-1} + R = 0$ in $M \otimes_R K$. By Lemma 4.3 (c), this means that $m_s r \in Ms$ for every r and s. Hence $m_s \in U(s)$ for every $s \in S$. This shows that $\widetilde{\lambda}$ is injective. □

Example 4.6 Let R be the nearly simple chain domain in [7, Example 6.5] (also see the example after Corollary 4.3 in [24]). In that example, the R-module $K = Q/R$ can be chosen to be countably generated, because the group G is countable, and so is its positive cone P. If the skew field F in that example is countable, then $F[P]$ is countable. In order to construct the ring R, we consider a right and left Ore subset S of $F[P]$, which is necessarily countable because $F[P]$ is countable, and then we set $R := F[P]S^{-1}$. Therefore if the skew field F is countable, then R is countable, and so K is a countably generated R-module. As R_R is torsion-free, its completion is $\varprojlim R/U(s)$ by Remark 4.4, and, for every non-zero element s of $J(R)$, $U(s) = 0$ because R is nearly simple. So $R = \varprojlim R/U(s)$. Let us prove that $R \ncong \mathrm{End}(K_R)$. The module K_R is a countably generated uniserial torsion locally coherent module (that is, every finitely generated submodule is coherent). By [24, Proposition 8.1], the module K_R is not quasi-small. Since uniserial modules with a local endomorphism ring are quasi-small [11], the ring $\mathrm{End}(K_R)$ cannot be isomorphic to R.

The same argument applies to any nearly simple chain domain R with $K = Q/R$ countably generated.

Proposition 4.7 *If R is a topological ring with a basis B of neighbourhoods of zero consisting of two-sided ideals, and R/I is a local ring for every proper ideal $I \in B$, then the Hausdorff completion of R is either 0 or a local ring.*

Remark 4.8 The case of completion of R equal to zero concernes only the trivial case of $B = \{R\}$. We will not consider this case in the proof.

Proof Let M_I be the maximal ideal of R such that M_I/I is the maximal ideal of R/I for every proper ideal $I \in B$. If $I, J \in B$, then considering the canonical projection $R/I \cap J \to R/I$, one sees that $M_{(I\cap J)} = M_I$. It follows that there exists a maximal ideal M of R such that $M_I = M$ for every proper ideal $I \in B$. The completion of R is the inverse limit of the rings R/I, which is a subring of the ring $\prod_{I\in B} R/I$, which has $\prod_{I\in B} M/I$ as a two-sided ideal, whose intersection N with the inverse limit is a two-sided ideal of the inverse limit. Let us prove that the inverse limit is a local ring with maximal ideal N. It suffices to show that every element of the inverse limit not in N is invertible. Let $(x_I + I)_{I\in B}$ be an element in the inverse limit, but not in N. Thus $x_I \in R$ and, for $I, J \in B$ with $I \subseteq J$, we have that $x_I - x_J \in J$, i.e., $x_I + I$ is mapped to $x_J + J$ via the canonical projection $R/I \to R/J$. Also, $x_I \notin M$ for some proper ideal I of B. Now if $J \in B$ is arbitrary, from $I \cap J \subseteq I$, we have that $x_{I\cap J} - x_I \in I \subseteq M$. But $I \cap J \subseteq J$, so $x_{I\cap J} - x_J \in J \subseteq M$. Therefore $x_I - x_J \in M$. It follows that $x_J \notin M$. Thus $x_I + I \notin M/I$ for every $I \in B$, hence $x_I + I$ is invertible in R/M. Let $y_I + I$ be the inverse of $x_I + I$ in R/I. Now the ring morphism $R/I \to R/J$ maps inverses to inverses. This shows that $(y_I + I)_{I\in B}$ is an element of the inverse limit, and concludes the proof. □

Therefore the completion of any local ring in the R-topology is a local ring.

5 Strongly flat modules

In all this section, we consider two rings R and Q, a bimorphism $\varphi\colon R \to Q$ in the category of rings, that is, φ is both a monomorphism and an epimorphism, and we assume that ${}_RQ$ is a flat left R-module. For simplicity, we will view R as a subring of Q and $\varphi\colon R \to Q$ as the inclusion. Then $\mathcal{F} = \{ I \mid I$ is a right ideal of R and $\varphi(I)Q = Q \}$ is a Gabriel topology consisting of dense right ideals [13, Sect. 3].

Let us recall some properties of such an inclusion $\varphi\colon R \hookrightarrow Q$. It is always possible to suppose $Q \subseteq Q_{\max}(R)$, the maximal right ring of quotients of R [25, proof of Theorem XI.4.1]. The inclusion $\varphi\colon R \to Q$ is an epimorphism in the category of rings if and only if the canonical R-R-bimodule morphism $Q \otimes_R Q \to Q$, induced by the multiplication $\cdot\colon Q \times Q \to Q$ of the ring Q, is an R-R-bimodule isomorphism [25, Proposition XI.1.2]. The family of all the subrings Q of $Q_{\max}(R)$ with $\varphi\colon R \hookrightarrow Q$ a bimorphism and ${}_RQ$ flat is directed under inclusion [25, Lemma XI.4.2]. Its direct limit is the "maximal flat epimorphic right ring of quotients" $Q_{\mathrm{tot}}(R)$ of R (see the paragraph after the proof of Corollary 5.7). By [13, (6) in Sect. 2], there is a torsion theory in Mod-R in which a module M_R is torsion if and only if $M \otimes_R Q = 0$. In this section, whenever we say "torsion" or "torsion-free", we refer to this torsion theory. For instance, the right R-module $K_R := Q/R$ is torsion [13, (10) in Sect. 2].

By [26, Theorem 4.8], $\mathrm{Ext}^1({}_RM, {}_RN) \cong \mathrm{Ext}^1({}_QM, {}_QN)$ for any pair M, N of left Q-modules, and similarly for right Q-modules. Recall that a left R-module ${}_RD$ is *divisible* if $D = ID$ for every $I \in \mathcal{F}$ (equivalently, if $M \otimes_R D = 0$ for every torsion right R-module M_R [25, Proposition VI.9.1]). For instance, $K \otimes_R D = 0$ for every divisible left R-module ${}_RD$. We use this fact in the proof of the following lemma.

Lemma 5.1 *Divisible strongly flat left R-modules are projective left Q-modules.*

Proof Let ${}_RD$ be a divisible strongly flat module. We have just seen that $K \otimes_R D = 0$. Since ${}_RD$ is flat, we have $D \cong Q \otimes D$. Thus D is a left Q-module. For any exact sequence $0 \to R^{(X)} \to D \oplus T \to Q^{(Y)} \to 0$, the corresponding exact sequence $0 \to Q^{(X)} \to D \oplus Q \otimes T \to Q^{(Y)} \to 0$ splits. Therefore D is a projective Q-module. □

We now pass to the study of strongly flat covers. Notice that strongly flat covers $f\colon {}_RS \to {}_RM$ are onto mappings, because we always have an onto morphism from a free module to the module ${}_RM$ and, since free modules are strongly flat, we have that f must be onto.

Recall that any left perfect ring is directly finite. The following result shows that when ${}_R\mathcal{SF}$ is covering, then Q is left perfect. Thus the results is the same as in the commutative case, but the proof is necessarily different.

Theorem 5.2 *If all left Q-modules have a strongly flat cover as left R-modules, then Q is left perfect.*

Proof Assume that ${}_QM$ is a left Q-module and $f\colon {}_RS \to {}_RM$ is a strongly flat cover of ${}_RM$. Then we have an epimorphism $1 \otimes f\colon Q \otimes S \to M$, $1 \otimes f\colon q \otimes s \mapsto qf(s)$. Since ${}_RS$ is strongly flat, ${}_QQ \otimes_R S$ is a direct summand of a direct sum of copies of Q, i.e., it is a projective left Q-module. Since projective left Q-modules are strongly flat left R-modules, the left R-module ${}_RQ \otimes S$ is strongly flat. But f is a strongly flat precover of M, so that there exists

$g\colon Q\otimes S\to S$ with $fg=1\otimes f$. Note that ${}_RS$ is flat, and so S can be embedded in $Q\otimes S$, that is, there is a left R-module monomorphism $h\colon {}_RS\to {}_RQ\otimes_R S$, defined by $h\colon s\mapsto 1\otimes s$. Then $f(gh)=f$, and thus gh is an automorphism of ${}_RS$ because $f\colon {}_RS\to {}_RM$ is a cover. Thus $(gh)^{-1}gh=1$, so that $e:=h(gh)^{-1}g$ is an idempotent endomorphism of the left R-module ${}_RQ\otimes S$. Hence e is an idempotent endomorphism of the left Q-module ${}_QQ\otimes S$. This shows that ${}_QQ\otimes S$ is the direct sum of the image and the kernel of e, which are Q-modules. But the image of e is the image of h. Hence the splitting monomorphism $h\colon s\mapsto 1\otimes s$ induces by corestriction a right R-module isomorphism of ${}_RS$ onto the Q-module ${}_Qh(S)$. By [13, Sect. 2(7)], if a left R-module ${}_RA$ is a left Q-module ${}_QA$, then its unique left Q-module structure is given by the canonical isomorphism $\mathrm{Hom}({}_RQ,{}_RA)\to {}_RA$. Therefore S has a unique left Q-module structure that extends its left R-module structure, and as such ${}_QS$ is a projective Q-module. Thus $f\colon {}_QS\to {}_QM$ is a left Q-module morphism. Note that projective Q-modules are strongly flat, and so $f\colon {}_QS\to {}_QM$ is a projective cover of ${}_QM$. Therefore Q is left perfect. □

The following result has a proof similar to that of [4, Proposition 2.4 ((1) and (2))]. We give a complete proof for convenience of the reader.

Lemma 5.3 *Let A be a module with a strongly flat cover and let*

$$0\to C\to M\to A\to 0 \tag{1}$$

be a special strongly flat precover of A. Then the exact sequence (1) is a strongly flat cover if and only if C is $\mathcal{MC}$-small (i.e., for every submodule H of M, $C+H=M$ and $C\cap H$ Matlis-cotorsion imply $H=M$).

Proof Assume that (1) is a strongly flat cover. Let $H\le M$ be such that $C+H=M$ and $C\cap H$ is Matlis-cotorsion. Let f denote the map $M\to A$ in (1), and let $f|_H$ be the restriction of f to H. We have an exact sequence $0\to C\cap H\to H\to A\to 0$. Since M is strongly flat and $C\cap H$ is Matlis-cotorsion, we know that $\mathrm{Ext}^1_R(M,C\cap H)=0$. Apply the functor $\mathrm{Hom}(M,-)$ to the exact sequence $0\to C\cap H\to H\to A\to 0$, getting the short exact sequence $\mathrm{Hom}(M,H)\to\mathrm{Hom}(M,A)\to\mathrm{Ext}^1_R(M,C\cap H)=0$. It follows that there exists $\varphi\colon M\to H$ such that $f|_H\varphi=f$. If $\iota\colon H\to M$ is the inclusion, we obtain that $f\iota\varphi=f$. But f is a strongly flat cover, hence $\iota\varphi$ is an isomorphism. Therefore $H=M$.

Conversely, suppose C is $\mathcal{MC}$-small in M. As we are assuming that A admits a strongly flat cover, we can use [29, Corollary 1.2.8]. Hence it is enough to show that the module C in the sequence (1) does not contain any non-zero summand of M. Suppose $M=X\oplus Y$ with $X\le C$. Then $M=C+Y$ and $C=X\oplus(Y\cap C)$. Since the precover (1) is special, C is Matlis-cotorsion, so its direct summand $C\cap Y$ is Matlis-cotorsion. But C is $\mathcal{MC}$-small, therefore $X=0$. □

Theorem 5.4 *Let I be a two-sided ideal of R such that $IQ=Q$. If all left R/I-modules have a strongly flat cover as left R-modules, then R/I is left perfect.*

Proof It is enough to show that every left R/I-module has a projective R/I-cover. Let M be an R/I-module and $f\colon {}_RA\to {}_RM$ be a strongly flat cover of ${}_RM$. Since $IM=0$, we have that $IA\subseteq\ker(f)$. But ${}_RA$ is strongly flat, so that there exists an exact sequence $0\to {}_RR^{(X)}\to {}_RA\oplus {}_RT\to {}_RQ^{(Y)}\to 0$, where X and Y are sets. Since $IQ=Q$, we have $R/I\otimes Q=0$. Thus we see that A/IA is a projective left R/I-module. So f induces a map $h\colon A/IA\to M$, $h\colon a+IA\mapsto f(a)$, and $\ker(h)=\ker(f)/IA$. We now show that h is a projective cover for M or, equivalently, that $\ker(f)/IA$ is small in A/IA. Assume that

$T + \ker(f) = A$, where T is an R-submodule of A such that $IA \subseteq T$. From $IQ = Q$, we get that $\mathrm{Hom}(Q, \ker(f)/\ker(f) \cap T) = 0$. On the other hand, since $f \colon {}_RA \to {}_RM$ is a strongly flat cover of ${}_RM$, the module $\ker(f)$ is Matlis-cotorsion by Wakamatsu Lemma (see [18, Lemma 5.13]), and thus $\ker(f) \cap T$ is Matlis-cotorsion. Therefore $T = A$ by Lemma 5.3. □

Lemma 5.5 *Assume that R is a local ring with Jacobson radical J. Let $0 \to C \to S \to M \to 0$ be a strongly flat cover for M. Then $C \leq JS$.*

Proof Assume that $C \nleq JS$. Then $JS \neq S$. Since R/J is a division ring, there exists a proper submodule T/JS of S/JS such that $T/JS + (C + JS)/JS = S/JS$. Consequently $T + C = S$. Consider the exact sequence $0 \to T \cap C \to C \to S/T \to 0$. Let us show that $\mathrm{Hom}(Q, S/T) = 0$. Note that R_R is essential in Q_R (because Q is a subring of $Q_{\max}(R)$). Thus if $x \in Q \setminus R$, then the right ideal $I := \{r \mid xr \in R\}$ is proper ideal of R, and so $I \leq J$. By [16, Theorem 3.9 (b)], $IQ = Q$ and so $JQ = Q$. If $f \in \mathrm{Hom}(Q, S/T) \neq 0$, then $f(Q) = f(JQ) \subseteq Jf(Q) \subseteq J(S/T) = 0$. Therefore $\mathrm{Hom}(Q, S/T) = 0$, and so $T \cap C \in Q^{\perp}$. Since C is $\mathcal{MC}$-small, we have $T = S$, which is a contradiction. □

It is known that if R is commutative, Q is the total ring of fractions of R, that is, the localization with respect to the set of all regular elements of R, and ${}_R\mathcal{SF}$ is covering, then $\mathrm{p.dim}({}_RQ) \leq 1$ [5, Propositions 7.9 and 8.7]. We do not know what occurs in the non-commutative case. Therefore we now study the projective dimension of ${}_RQ$.

Proposition 5.6 *Suppose ${}_RQ$ is a projective left R-module. Then ${}_RQ$ is a finitely generated left R-module.*

Proof Since ${}_RQ$ is projective, it has a dual basis [2, Exercise 11, p. 202–203], that is, there are elements $x_\alpha \in Q$ and morphisms $f_\alpha \colon {}_RQ \to {}_RR$ ($\alpha \in A$) such that, for all $x \in Q$, $f_\alpha(x) \neq 0$ for only finitely many $\alpha \in A$ and $x = \sum_{\alpha \in A} f_\alpha(x)x_\alpha$. Applying the functor

$$ {}_QQ \otimes_R - \colon R\text{–Mod} \to Q\text{–Mod}, $$

we get left Q-module morphisms $1 \otimes f_\alpha \colon {}_QQ \otimes_R Q \to {}_QQ \otimes_R R$. Now there are left Q-module isomorphisms $Q \to {}_QQ \otimes_R Q$, $q \mapsto 1 \otimes q$, and ${}_QQ \otimes_R R \to {}_QQ$, $q \otimes r \mapsto qr$. Composing the mappings $Q \to {}_QQ \otimes_R Q$, $1 \otimes f_\alpha \colon {}_QQ \otimes_R Q \to {}_QQ \otimes_R R$ and ${}_QQ \otimes_R R \to {}_QQ$, we get left Q-module endomorphisms ${}_QQ \to {}_QQ$, which are necessarily right multiplications ρ_{y_α} by elements $y_\alpha \in Q$. That is, we have commutative diagrams

$$ \begin{array}{ccc} {}_QQ \otimes_R Q & \xrightarrow{1 \otimes f_\alpha} & {}_QQ \otimes_R R \\ \cong \uparrow & & \downarrow \cong \\ {}_QQ & \xrightarrow[\rho_{y_\alpha}]{} & {}_QQ. \end{array} $$

Now, for all $x \in Q$, $f_\alpha(x) \neq 0$ for only finitely many $\alpha \in A$. For $x = 1$, we get that there is a finite subset F of A such that $f_\alpha(1) = 0$ for every $\alpha \in A \setminus F$. Thus $(1 \otimes f_\alpha)(1 \otimes 1) = 0$ for every $\alpha \in A \setminus F$. It follows that right multiplication by y_α maps 1 to 0, that is, $y_\alpha = 0$ for every $\alpha \in A \setminus F$. Hence $1 \otimes f_\alpha \colon {}_QQ \otimes_R Q \to {}_QQ \otimes_R R$ is the zero mapping for every $\alpha \in A \setminus F$. Thus $(1 \otimes f_\alpha)(q \otimes q')$ is the zero element of ${}_QQ \otimes_R R$ for every $q, q' \in Q$. Hence $1 \otimes f_\alpha(q')$ is the zero element of ${}_QQ \otimes_R R$. It remains to show that the mapping ${}_RR \to {}_QQ \otimes_R R$, $r \to 1 \otimes r$, is injective, which is easily seen because $\mathrm{Tor}_1^R(K, R) = 0$. This proves that $f_\alpha = 0$ for every $\alpha \in A \setminus F$. As a consequence, ${}_RQ$ is isomorphic to a direct summand of ${}_RR^F$, so that ${}_RQ$ is a finitely generated left R-module. □

Corollary 5.7 *Let R be a ring, S a multiplicatively closed subset of regular elements of R, and suppose that S is a right denominator set, so that the right ring of fractions $Q := R[S^{-1}]$ exists. If ${}_RQ$ is a projective left R-module, then $Q = R$, that is, all the elements of S are invertible in R.*

Proof By Proposition 5.6, there are finitely many elements $r_1 s_1^{-1}, \dots, r_n s_n^{-1}$ that generate Q as a left R-module. Reducing to the same denominator [17, Lemma 4.21], we find elements $r_i' \in R$ and $s \in S$ such that $s_i r_i' = s$ for every i. Multiplying by s^{-1} on the right and by s_i^{-1} on the left, we get that $r_i' s^{-1} = s_i^{-1}$. Thus $Q = \sum_{i=1}^n R r_i s_i^{-1} \subseteq Rs^{-1}$. This proves that $Q = Rs^{-1}$. In particular, $s^{-2} \in Rs^{-1}$, from which $1 \in Rs$. Let $t \in R$ be such that $1 = ts$. Then $t = s^{-1}$ in Q. Thus $Q = Rs^{-1} = Rt \subseteq R$, hence $Q = R$. □

On p. 235 of [25], Stenström asks for necessary and sufficient conditions for $Q_{\max}(R)$ to be equal to the maximal flat epimorphic right ring of quotients $Q_{\text{tot}}(R)$. He shows that if $Q_{\max}(R)$ is a right Kasch ring (i.e., a ring that contains a copy of its simple right modules), then $Q_{\max}(R) = Q_{\text{tot}}(R)$. If R is right hereditary right noetherian [25, Example 3, p. 235] or commutative noetherian [25, Example 4, p. 237] or a right Goldie ring [25, Theorem XII 2.5], then $Q_{\max}(R)$ is known to be Kasch.

Example 5.8 Here is an example of a ring R for which $Q_{\text{tot}}(R) = Q_{\max}(R)$ is a projective right and left R-module, but $R \neq Q_{\max}(R)$. Let R be the ring of all lower triangular 2×2 matrices over a field F. The ring R is right nonsingular and $E(R_R) = S^0 R = Q_{\max}(R)$ is a projective right and left R-module [16, Exercise 14 on P. 78, and Corollary 2.31] (in Goodearl's notation, $S^0 A := E(A/Z(A_A))$, the injective envelope of $A/Z(A_A)$ for any ring A). More precisely, $Q_{\max}(R)$ is the 2×2 matrix ring over the field F, which is a semisimple artinian ring, hence a right and left Kasch ring, and so $Q_{\max}(R) = Q_{\text{tot}}(R)$ as we have seen above.

We are now ready to consider the case of $\text{p.}\dim({}_RQ) \leq 1$. Recall that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *hereditary* if $\text{Ext}_R^i(A, B) = 0$ for all $i \geq 1$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Note that if $\mathcal{F}$ is the class of flat modules and $\mathcal{EC}$ the class of Enochs cotorstion modules, the cotorsion pair $(\mathcal{F}, \mathcal{EC})$ is always hereditary. Similarly to [18, Lemma 7.53], we can show that:

Lemma 5.9 *The following conditions are equivalent for the pair of rings $R \subseteq Q$:*
(a) $\text{p.}\dim({}_RQ) \leq 1$.
(b) *The cotorsion pair $(\mathcal{SF}, \mathcal{MC})$ is hereditary.*

Proof (a) $\Rightarrow$ (b). Assume $\text{p.}\dim({}_RQ) \leq 1$. Then strongly flat modules, which are summands of extensions of a direct sum of copies of Q by a free module, are of p. dim at most 1. Thus the cotorsion pair $(\mathcal{SF}, \mathcal{MC})$ is hereditary.

(b) $\Rightarrow$ (a). By [3, Theorem 3.5], it is enough to show that $\text{Ext}^1(K, M)$ is h-reduced Matlis-cotorsion for any left R-module M. From the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$, we have the exact sequence

$$0 \to A \to B \to \text{Ext}^1(K, M) \to 0,$$

where $A = \text{Hom}(K, E(M))/\text{Hom}(K, M)$ and $B = \text{Hom}(K, E(M)/M)$. Notice that, for every left R-module N, $\text{Hom}(K, N)$ is Matlis-cotorsion and h-reduced [13, Theorem 2.8]. So $\text{Ext}^1(K, M)$ is h-reduced if and only if $A \in Q^\perp$. Now $A \in Q^\perp$ follows from the exact sequence $0 \to \text{Hom}(K, M) \to \text{Hom}(K, E(M)) \to A \to 0$ and the fact that $(\mathcal{SF}, \mathcal{MC})$ is hereditary. As the module A is Matlis-cotorsion, from the exact sequence $0 \to A \to B \to \text{Ext}^1(K, M) \to 0$ and the fact that $(\mathcal{SF}, \mathcal{MC})$ is hereditary, we get that $\text{Ext}^1(K, M)$ is Matlis-cotorsion. □

As a consequence, ${}_R\mathcal{SF} = {}_R\mathcal{F}$ implies $\mathrm{p.\,dim}({}_RQ) \leq 1$.

When R is a right Ore domain and Q is its classical right quotient ring, then Q_R is injective and so, by Proposition 3.3, every torsion-free right R-module M is the direct sum of its h-divisible part and an h-reduced submodule. In Example 5.10, we will see that in general ${}_RQ$ can be not injective, but nevertheless strongly flat left R-modules still have a decomposition into an h-divisible and an h-reduced submodule (Lemma 5.11).

Example 5.10 Let R be a right noetherian right chain domain of type $> \omega$, i.e., such that the noetherian linearly ordered set of non-zero right ideals of R is order antiisomorphic to an ordinal $\alpha > \omega$). Then R is a right hereditary ring and R is not left Ore [7, Proposition 3.7]. Thus ${}_RR$ is not uniform, so that there exist non-zero left ideals I and J of R with $I \cap J = 0$. We can assume I maximal with respect to the property that $I \cap J = 0$. Thus I is an essentially closed left ideal of R, that is, if ${}_RI$ is essential in B, where B is a left ideal of R, then $I = B$. This ideal I cannot be an annihilator ideal, so that the left and the right maximal quotient ring of R do not coincide [16, Theorem 2.38]. Also, if Q is the maximal right quotient ring of R, which is equal to the classical right quotient ring of R, then ${}_RQ$ is flat, but Q_R is not flat [9, Proposition 0.8.6]. We claim that ${}_RQ$ is not injective. To prove the claim, notice that ${}_RR$ is not uniform. Hence there exist non-zero elements $x, y \in R$ with $Rx \cap Ry = 0$. The left ideal $Rx \oplus Ry$ of R is a free left R-module of rank 2. There is a unique left R-module morphism $f : Rx \oplus Ry \to {}_RQ$ such that $f(x) = x$ and $f(y) = 0$. If ${}_RQ$ is injective, f extends to a left R-module morphism $g : {}_RR \to {}_RQ$, which is necessarily right multiplication by an element $q \in Q$. Thus $f(x) = g(x)$, that is, $x = xq$, and $f(y) = g(y)$, so that $0 = yq$. Since Q is a division ring, we get that $q = 1$ and $q = 0$, a contradiction. This concludes the proof of our claim.

Lemma 5.11 *Let R be a right Ore domain and Q the classical right quotient ring of R. If S is a strongly flat left R-module, then S is the direct sum of the Q-module $h(S)$ and a strongly flat reduced R-submodule of S isomorphic to $S/h(S)$.*

Proof Assume that R is not a division ring. There exists an exact sequence

$$0 \to R^{(X)} \to S \oplus C \to Q^{(Y)} \to 0.$$

We claim that $\mathrm{Hom}(Q, R) = 0$. Otherwise, i.e., if ${}_RQ$ can be embeded in ${}_RR$, there exists a monomorphism $\varepsilon : {}_RQ \to {}_RR$. Then ε can be viewed as a monomorphism ${}_RQ \to {}_RQ$. This monomorphism ε is right multiplication by an element q of Q. Now ε a monomorphism implies $q \neq 0$, and R right Ore domain implies Q division ring. Hence q is invertible in Q, so that $R = Q$, which is a contradiction. This proves our claim. Now we have the embedding $\mathrm{Hom}(Q, S \oplus C) \to \mathrm{Hom}(Q, Q^{(Y)})$. So we have an exact sequence $0 \to R^{(X)} \to (S \oplus C)/h(S \oplus C) \to Q^{(Y)}/h(S \oplus C) \to 0$. Since $h(S \oplus C)$ is a torsion-free divisible module, it is a Q-module. But Q is division ring, so $h(S \oplus C)$ is a direct summand of $Q^{(Y)}$. It follows that $S/h(S)$ is strongly flat.

Finally, consider the exact sequence $0 \to h(S) \to S \to S/h(S) \to 0$. The module $h(S)$ is isomorphic to a direct sum of copies of Q, hence $h(S)$ is a Matlis-cotorsion module. Therefore the exact sequence splits. □

We now consider the relation between strong flatness and completions in the R-topology. Notice that, by Theorem 4.5 and [13, Proposition 2.6], if a module M_R is torsion-free, then $\widetilde{M_R}$ is torsion-free.

Theorem 5.12 *Let R be an Ore domain. Then the Hausdorff completion $\widetilde{S_R}$ of any strongly flat right R-module S_R in the R-topology is strongly flat.*

Proof Let Q be the classical quotient ring of R and $K := Q/R$. If S is a strongly flat right R-module, then $S/h(S)$ is strongly flat by Lemma 5.11, and $\widetilde{S_R} \cong \widetilde{S_R/h(S)}$ by Corollary 4.2. Hence in the proof of the theorem we can suppose S h-reduced, and prove that if S_R is an h-reduced strongly flat right R-module, then its completion $\widetilde{S_R}$ is strongly flat. We have the short exact sequence

$$0 \longrightarrow S_R \longrightarrow \mathrm{Hom}(K_R, S \otimes_R K) \longrightarrow \mathrm{Ext}^1_R({}_RQ_R, S_R) \longrightarrow 0 \qquad (2)$$

[13, Theorem 4.5]. We know that $\widetilde{S_R}$ is a submodule of $\mathrm{Hom}(K_R, S \otimes_R K)$ that contains S_R. Hence $\widetilde{S_R}/S_R$ is isomorphic to a submodule of $\mathrm{Ext}^1_R({}_RQ_R, S_R)$. In particular, $\widetilde{S_R}/S_R$ is torsion-free, because $\mathrm{Ext}^1_R({}_QQ_R, S_R)$ is a Q-module, hence torsion-free. Let us prove that $\widetilde{S_R}/S_R$ is divisible, i.e., that $(\widetilde{S_R}/S_R)r = \widetilde{S_R}/S_R$ for every non-zero $r \in R$. Equivalently, we must prove that $\widetilde{S_R} \subseteq \widetilde{S_R}r + S_R$. Now S_R is dense in $\widetilde{S_R}$, so that, for every $\widetilde{s} \in \widetilde{S_R}$ and every non-zero element t of R, we have that $(\widetilde{s} + U(t)) \cap S_R \neq \emptyset$. In particular, $(\widetilde{s} + U(r)) \cap S_R \neq \emptyset$. Notice that $U(r) \subseteq \widetilde{S_R}r$, because, for every $x \in U(r)$, we have that $xR \subseteq \widetilde{S_R}r$, hence $x \in \widetilde{S_R}r$. It follows that $(\widetilde{s} + \widetilde{S_R}r) \cap S_R \neq \emptyset$. Thus there exist $\widetilde{s'} \in \widetilde{S_R}$ and $s'' \in S_R$ with $\widetilde{s} + \widetilde{s'}r = s''$. Therefore $\widetilde{s} = -\widetilde{s'}r + s'' \in \widetilde{S_R}r + S_R$. This proves that $\widetilde{S_R}/S_R$ is divisible and torsion-free, hence a module over the division ring Q. Thus $\widetilde{S_R}/S_R \cong Q^{(X)}$ for some set X. From the short exact sequence

$$0 \longrightarrow S_R \longrightarrow \widetilde{S_R} \longrightarrow Q^{(X)} \longrightarrow 0,$$

we have that S_R strongly flat and $Q^{(X)}$ strongly flat imply $\widetilde{S_R}$ strongly flat. □

Corollary 5.13 *Let R be an Ore domain. Then the completion $\widetilde{P_R}$ of any projective right R-module in the R-topology is a strongly flat right R-module. In particular, the completion $\widetilde{R_R}$ of R_R is a strongly flat R-module.*

Recall that a *left coherent* ring is a ring over which every finitely generated left ideal is finitely presented or, equivalently, any intersection of two finitely generated left ideals is finitely generated.

Theorem 5.14 *Assume that R is a left coherent Ore domain with classical quotient ring Q. A left ideal ${}_RI$ of R is a strongly flat left module if and only if ${}_RI$ is finitely generated projective.*

Proof The result is clearly true for R a division ring, so that we can suppose $R \neq Q$. Assume ${}_RI$ a non-zero strongly flat module. We have the exact sequence of R-R-bimodules $0 \to R \to Q \to Q/R = K \to 0$. Since ${}_RI$ is flat, we get the exact sequence of left R-modules $0 \to R \otimes I \to Q \otimes I \to K \otimes I \to 0$. Therefore $K \otimes I \cong (Q \otimes I)/(R \otimes I)$. We want to show that R/I embeds in $K \otimes I$ as a left R-module. Consider the sequence of left R-modules $0 \to {}_RI \to {}_RQ \to {}_RQ/I \to 0$ and apply to it the functor $Q \otimes_R -$. Since Q_R is flat, we get to an exact sequence $0 \to Q \otimes_R I \to Q \otimes_R Q \to Q \otimes_R Q/I \to 0$. Under the natural isomorphism $f : Q \otimes_R Q \to Q$, the image of $Q \otimes I$ is $QI = Q$, because I is non-zero, and the image of $R \otimes I$ is I, and so $K \otimes I \cong (Q \otimes I)/(R \otimes I) \cong Q/I$ as a left R-module. Now $R/I \leq Q/I$ implies that R/I embeds in $K \otimes I$ as an R-module. There exists an exact sequence $0 \to R^{(X)} \to I \oplus T \to Q^{(Y)} \to 0$ with $X \neq \emptyset$. Since $K \otimes_R Q = 0$, we conclude that $K \otimes I$, and so R/I, embed in $K^{(X)}$ as left R-modules. Consequently, there exists an element $x \in {}_RK^{(X)}$ whose annihilator is equal to I. But the annihilator of an element of $K^{(X)}$ is equal to the intersection of finitely many annihilators of elements of K. If $ab^{-1} + R \in {}_RK$, then $\mathrm{ann}(ab^{-1} + R) = R \cap Rba^{-1}$. Note that $R \cap Rba^{-1} \cong Ra \cap Rb$, which is a finitely generated left ideal of R because R is left coherent. Thus I is a finitely generated left ideal of R and, since it is flat, I is projective [19, Theorem 4.30]. □

Lemma 5.15 *Let R be a right Ore domain with classical right quotient ring Q. Then the strongly flat cover of any h-reduced flat left R-module is h-reduced.*

Proof Assume that M is a flat h-reduced module and $0 \to C \to S \to M \to 0$ is a strongly flat cover of M. Since M is h-reduced, we can assume $D := h(C) = h(S)$ and that C is Matlis-cotorsion. So we have an exact sequence

$$0 \to C/D \to S/D \to M \to 0. \tag{3}$$

By Lemma 5.11, S/D is strongly flat. Since C is torsion-free, D is a left Q-module. Thus $\mathrm{Ext}^2_R(Q, D) = 0$, and so $\mathrm{Ext}^1_R(Q, C/D) = 0$. Hence the sequence (3) is a special strongly flat precover for M. Let us see that C/D is $\mathcal{MC}$-small in S/D. Let T be a submodule of S that contains D and suppose that $C/D + T/D = S/D$ and $(C \cap T)/D$ is Matlis-cotorsion. Therefore $C + T = S$. On the other hand, D is a Q-module, so $\mathrm{Ext}^1_R(Q, D) = 0$. From the sequence $0 \to D \to C \cap T \to (C \cap T)/D \to 0$, we conclude that $C \cap T$ must be Matlis-cotorsion. Since $0 \to C \to S \to M \to 0$ is a special strongly flat cover of M, by Lemma 5.3 we get that C is $\mathcal{MC}$-small in S, and $T = S$. It follows that $0 \to C/D \to S/D \to M \to 0$ is a strongly flat cover of M by Lemma 5.3, and so $S \cong S/D$. Therefore S is h-reduced. □

Proposition 5.16 *Assume that R is an Ore local domain with classical quotient ring Q. Suppose that $K \otimes_R S$ is direct sum of copies of K for every strongly flat module ${}_RS$. If ${}_R\mathcal{SF}$ is a covering class, then ${}_R\mathcal{SF} = {}_R\mathcal{F}$.*

Proof Firstly, notice that left Q-modules are injective as R-modules because R is both a right and a left Ore domain. If M is flat, then $h(M)$ is a direct summand of M, and therefore $M \cong h(M) \oplus M/h(M)$. Clearly, Q-modules are strongly flat, and thus it is enough to show that any flat h-reduced module is strongly flat. Let M be an h-reduced flat left module and $0 \to C \to S \to M \to 0$ be a strongly flat cover of M. By Lemma 5.15, S is also h-reduced, and thus C is an h-reduced flat left R-module. Assume that $C \neq 0$, and let $0 \to C' \to S' \to C \to 0$ be a strongly flat cover of C. Then S' is Matlis-cotorsion h-reduced strongly flat. By the left version of [13, Theorem 4.5], we have an exact sequence $0 \to S' \to \mathrm{Hom}({}_RK, {}_RK \otimes_R S') \to \mathrm{Ext}^1({}_RQ, {}_RS') \to 0$. Thus ${}_RS' \cong \mathrm{Hom}({}_RK_R, {}_RK \otimes_R S')$. Since S' is strongly flat, there exists an index set Z such that ${}_RK \otimes_R S' \cong {}_RK^{(Z)}$. Thus ${}_RS' \cong \mathrm{Hom}(K_R, K \otimes S') \cong \mathrm{Hom}(K_R, K^{(Z)}) \cong \mathrm{Hom}(K_R, K \otimes R^{(Z)})$. By [13, Theorem 4.5], we have an exact sequence $0 \to R^{(Z)} \to \mathrm{Hom}(K_R, K \otimes R^{(Z)}) \to \mathrm{Ext}^1({}_RQ_R, {}_RR^{(Z)}) \to 0$. Since $\mathrm{Ext}^1({}_RQ_R, {}_RR^{(Z)}) = \mathrm{Ext}^1({}_RQ_Q, {}_RR^{(Z)})$ is a left Q-module, it follows that $R/J \otimes \mathrm{Ext}^1({}_RQ_R, {}_RR^{(Z)}) = 0$, where J denotes the Jacobson radical of R. Moreover, the left Q-module $\mathrm{Ext}^1({}_RQ_Q, {}_RR^{(Z)})$ is isomorphic to a direct sum of copies of ${}_QQ$, so $\mathrm{Ext}^1({}_RQ_R, {}_RR^{(Z)})$, isomorphic to a direct sum of copies of ${}_RQ$, is a flat left R-module. Therefore $R/J \otimes R^{(Z)} \cong R/J \otimes \mathrm{Hom}(K_R, K \otimes R^{(Z)})$. Since $R/J \otimes R^{(Z)} \cong (R/J)^{(Z)}$ is non-zero, the module $R/J \otimes \mathrm{Hom}(K_R, K \otimes R^{(Z)}) \cong R/J \otimes S' \cong S'/JS'$ is non-zero. This proves that $JS' \neq S'$.

Now the module ${}_RM$ is flat, so its strongly flat cover $0 \to {}_RC \to {}_RS \to {}_RM \to 0$ is a pure exact sequence. Hence the sequence $0 \to R/J \otimes_R C \to R/J \otimes_R S \to R/J \otimes_R M \to 0$ is exact. That is, the sequence $0 \to C/JC \to S/JS \to M/JM \to 0$ is exact. From Lemma 5.5, we have that $C \leq JS$, so the first monomorphism in the last short exact sequence is the zero morphism. It follows that $C/JC = 0$.

The proof that $JC = C$ follows from the fact that M is an h-reduced flat left module and $0 \to C \to S \to M \to 0$ is a strongly flat cover of M. In the first paragraph of proof of this proposition, we showed that C is also an h-reduced flat left module, and $0 \to C' \to S' \to$

$C \to 0$ is a strongly flat cover of C. Hence the argument also holds for C, and shows that $C'/JC' = 0$.

The sequence $0 \to C' \to S' \to C \to 0$ is pure exact, so the sequence $0 \to R/J \otimes_R C' \to R/J \otimes_R S' \to R/J \otimes_R C \to 0$ is exact. Equivalently, the sequence $0 \to C'/JC' \to S'/JS' \to C/JC \to 0$ is exact. But C'/JC' and C/JC are both zero, so $S'/JS' = 0$. It follows that $JS' = S'$, which is a contradiction. This proves that $C = 0$, so M is strongly flat. □

For any left module ${}_RM$, let $\mathrm{Add}({}_RM)$ denote the class of all left R-modules isomorphic to direct summands of direct sums of copies of ${}_RM$. We will say that $\mathrm{Add}({}_RM)$ is *trivial* if every direct summand of a direct sum of copies of ${}_RM$ is isomorphic to a direct sum of copies of ${}_RM$.

Lemma 5.17 *Let R be a nearly simple chain domain and let ${}_RK$ be the uniserial left R-module Q/R. Suppose* $\mathrm{Add}({}_RK)$ *is not trivial. Then there exists a submodule V of ${}_RK$ that is not quasismall. Moreover, all the elements of* $\mathrm{Add}({}_RK)$ *are isomorphic to R-modules of the form ${}_RK^{(X)} \oplus {}_RV^{(Y)}$.*

Proof See [23, Theorem 1.1(ii)]. □

In the next proposition, we describe uniserial strongly flat modules over Ore domains.

Proposition 5.18 *If R is an Ore domain with classical quotient ring Q and with a non-zero uniserial strongly flat left R-module ${}_RU$, then R is a left chain domain and ${}_RU$ is isomorphic to ${}_RQ$ or ${}_RR$.*

Proof Let ${}_RU$ be a non-zero uniserial strongly flat left module over an Ore domain R. Since ${}_RU$ is flat, considering the exact sequence $0 \to R \to Q$, we have an embedding $U \to Q \otimes_R U$. Hence the annihilator of every non-zero element of ${}_RU$ is zero, and so cyclic submodules of ${}_RU$ are isomorphic to ${}_RR$. In particular, the ring R is a left chain ring. Moreover, U is the union of cyclic submodules isomorphic to ${}_RR$, that is, a direct limit of copies of ${}_RR$, where the connecting homomorphisms are right multiplications by non-zero elements of R. Applying the functor ${}_RQ \otimes_R -$, since tensor product commutes with direct limits, we get that ${}_RQ \otimes_R U$ is a direct limit of a direct system of copies of ${}_RQ$, in which the connecting isomorphisms are right multiplications by non-zero elements of R, that is, the connecting isomorphisms are all left R-module automorphisms of ${}_RQ$. Thus ${}_RQ \otimes_R U \cong {}_RQ$. Hence there is an embedding $\varepsilon\colon {}_RU \to {}_RQ$. If this embedding ε is onto, then ${}_RU \cong {}_RQ$.

Assume that ε is not onto. We claim that ${}_RU$ is isomorphic to a proper left ideal of R. By Theorem 5.14, ${}_RU$ is cyclic, and so isomorphic to ${}_RR$, which concludes the proof of the proposition.

To prove the claim, we first show that ${}_RQ$ is uniserial. In fact, suppose $x, x' \in {}_RQ$. Then $x = rs^{-1}$ and $x' = r's'^{-1}$ for suitable $r, r', s, s' \in R$, s, s' non-zero. Reducing to the same denominator [17, Lemma 4.21], we have that there exist elements $t, t' \in R$ such that $st = s't' \neq 0$. As ${}_RR$ is uniserial, without loss of generality we can suppose $Rrt \subseteq Rr't'$. Multiplying by the inverse of $st = s't'$, we get that $Rrt(st)^{-1} \subseteq Rr't'(s't')^{-1}$, that is, $Rx \subseteq Rx'$. This proves that ${}_RQ$ is uniserial. Since the embedding ε is not onto, if y is an element of ${}_RQ$ not in the image of ε, then ${}_RU$ is isomorphic to a proper submodule of $Ry \cong {}_RR$, hence ${}_RU$ is isomorphic to a proper left ideal of R. This concludes the proof of the claim. □

Lemma 5.19 *Let R be a nearly simple chain domain with Jacobson radical J. If $\mathrm{Add}({}_RK)$ is not trivial, ${}_RV$ is as in Lemma 5.17 and ${}_RM := \mathrm{Hom}(K, V^{(X)})$, where X is a non-empty set, then $JM \neq M$. That is, ${}_RM$ has a maximal submodule.*

Proof The module ${}_RM = \mathrm{Hom}({}_RK, {}_RV)$ is a left R-module because ${}_RK_R$ is a bimodule. Notice that ${}_RM$ always has a direct summand isomorphic to $\mathrm{Hom}({}_RK, {}_RV)$, so that we can suppose that X has exactly one element. By [23, Theorem 1.1(ii)], K has an endomorphism whose image is contained in V, say $\varphi\colon {}_RK \to {}_RV$, that is injective but not surjective. Let us show that φ is in M but not in JM. For every $j \in J$ and $\psi \in \mathrm{Hom}({}_RK, {}_RV)$, the left R-module morphism $j\psi$ is not injective. In fact, $j\psi$ is right multiplication by j viewed as a morphism ${}_RK \to {}_RK$ composed with $\psi\colon {}_RK \to {}_RV$. Thus the first morphism annihilates the element $j^{-1} + R$, so that the kernel of $j\psi$ is non-zero. (This proves that $j\psi$ is not injective for $j \neq 0$. But also when $j = 0$, $j\psi$ is not injective.) Now every element of JM is a finite sum of elements of the form $j\psi$, i.e., of non-injective homomorphisms, hence is not injective because ${}_RK$ is uniserial, thus uniform. Therefore $\varphi\colon {}_RK \to {}_RV$ is not an element of JM. □

Recall that a two-sided ideal I of R is *completely prime* if $xy \in I$ implies $x \in I$ or $y \in I$ for every $x, y \in R$. A right chain domain is *exceptional* if it contains a prime ideal that is not completely prime [8].

Theorem 5.20 *If R is a right chain domain with classical right quotient ring Q such that ${}_R\mathcal{SF}$ is a covering class, then R is invariant and ${}_R\mathcal{SF} = {}_R\mathcal{F}$.*

Proof Let J be the Jacobson radical of R. If I is a non-zero completely prime two-sided ideal of R, R/I is a left perfect domain by Theorem 5.4, and so it is a division ring. Since J/I is an ideal of R/I, we conclude that the only proper non-zero completely prime ideal of R is J. A chain domain R is said to be of rank one if J is its only non-zero completely prime ideal. By [8], such a ring is either invariant, i.e., $aR = Ra$ for all $a \in R$, or it is nearly simple, in which case 0 and J are the only proper two-sided ideals, or R is exceptional and there exists a non-zero prime ideal P properly contained in J. In this last case, $\bigcap_n P^n = 0$ and there are no further ideals between P and J. In the second and the third case, J is neither right nor left finitely generated and $J^2 = J$. Now we break the proof in three steps.

Step 1: The ring R cannot be exceptional.

Suppose that R is an exceptional ring with a prime ideal P, $0 \subset P \subset J$. The Jacobson radical of $\overline{R} := R/P$ is $\overline{J} := J/P$, which is not nilpotent because $J^2 = J$. Let us show that $\overline{J}$ is not right T-nilpotent (the proof is similar to that of [21, Lemma 3.33]). Construct by induction a sequence $\overline{a_1}, \overline{a_2}, \overline{a_3}, \dots$ of elements of $\overline{J}$ such that $\overline{J}\,\overline{a_n} \cdots \overline{a_1} \neq 0$ for every $n \geq 1$ as follows. Since $\overline{J}^2 = \overline{J} \neq 0$, there exists an element $\overline{a_1} \in \overline{J}$ such that $\overline{J}\,\overline{a_1} \neq 0$. Suppose $\overline{a_1}, \overline{a_2}, \dots, \overline{a_n} \in \overline{J}$ with $\overline{J}\,\overline{a_n} \cdots \overline{a_1} \neq 0$ have been constructed. Then $\overline{J}^2\overline{a_n} \cdots \overline{a_1} \neq 0$, so that there exists $\overline{a_{n+1}} \in \overline{J}$ with $\overline{J}\,\overline{a_{n+1}}\,\overline{a_n} \cdots \overline{a_1} \neq 0$. This completes the construction of the sequence by induction, and shows that $\overline{J}$ is not right T-nilpotent. Similarly, $\overline{J}$ is not left T-nilpotent. So $\overline{R} := R/P$ is neither a right nor a left perfect ring, and therefore the class of strongly flat left R-modules is not covering by Theorem 5.4.

Step 2: The ring R cannot be a nearly simple chain domain.

Suppose R a nearly simple chain domain. For every strongly flat module ${}_RS$, $K \otimes S$ is direct summand of a direct sum of copies of K, so that $K \otimes S$ belongs to $\mathrm{Add}(K)$. We have two cases: $\mathrm{Add}(K)$ is trivial or not. If $\mathrm{Add}(K)$ is trivial, then ${}_R\mathcal{SF}$ covering implies ${}_R\mathcal{SF} = {}_R\mathcal{F}$ by Proposition 5.16. But every cyclic (=finitely generated) left ideal of R is flat (= projective), so ${}_RJ$ must be flat, hence strongly flat (see for example [28, Theorem 39.12(2)]). Thus J must be finitely generated by Theorem 5.14, which is a contradiction. Now assume that $\mathrm{Add}(K)$ is not trivial. By Lemma 5.17, there exists a uniserial module V that is not quasismall and every element in $\mathrm{Add}(K)$ is of the form $K^{(Y)} \oplus V^{(X)}$ for suitable sets X and Y. Let $0 \to C \to S \to J \to 0$ be a strongly flat cover of J. By Lemma 5.15, S is also h-reduced, and so C is an h-reduced flat left module. Assume $C \neq 0$, and let $0 \to C' \to S' \to C \to 0$ be a strongly flat cover of C. Then S' is Matlis-cotorsion h-reduced strongly flat. By the left version of [13, Theorem 4.5], we have an exact sequence $0 \to S' \to \mathrm{Hom}(K, K \otimes S') \to \mathrm{Ext}^1(Q, S') \to 0$. So $S' \cong \mathrm{Hom}(K, K \otimes S')$. Since S' is strongly flat, $K \otimes S'$ is a direct summand of a direct sum of copies of K. Therefore there exist sets X and Y such that $K \otimes S' \cong K^{(Y)} \oplus V^{(X)}$. Thus $S' \cong \mathrm{Hom}(K, K \otimes S') \cong \mathrm{Hom}(K, K^{(Y)}) \oplus \mathrm{Hom}(K, V^{(X)})$. As we saw in the proof of Theorem 5.16, if Y is non-empty, we can consider the exact sequence $0 \to R^Y \to \mathrm{Hom}(K, K \otimes R^{(Y)}) \to \mathrm{Ext}^1_R(Q, R^{(Y)}) \to 0$, and conclude that $J\,\mathrm{Hom}(K, K^{(Y)}) \neq \mathrm{Hom}(K, K^{(Y)})$. Similarly, by Lemma 5.19, if X is non-empty, $J\,\mathrm{Hom}(K, V^{(X)}) \neq \mathrm{Hom}(K, V^{(X)})$. Consequently, $JS' \neq S'$. By Lemma 5.5, considering the pure exact sequence $0 \to C \to S \to J \to 0$, we see that $JC = C$. By Lemma 5.5 again, from the exact sequence $0 \to C' \to S' \to C \to 0$, we get that $JS' = S'$, which is a contradiction. This proves that $C = 0$, so that J is strongly flat, which contradicts Theorem 5.14.

Step 3: The ring R is invariant and ${}_R\mathcal{SF} = {}_R\mathcal{F}$.

By Steps 1 and 2, the ring R must be invariant. Therefore the endomorphism ring of every uniserial module is local (the proof is similar to the commutative case, because, like in the proof of [14, Corollary 3], every uniserial module is unshrinkable, and so the endomorphism ring of every uniserial module is local like in the proof of [12, Example 2.3(e)]). Thus $\mathrm{End}({}_RK)$ is local and every direct summand of a direct sum of copies of K is isomorphic to a direct sum of copies of ${}_RK$ because ${}_RK$ is uniserial [10, Proposition 2.2]. Thus ${}_R\mathcal{SF} = {}_R\mathcal{F}$ by Proposition 5.16. □

We conclude with an example concerning right noetherian right chain domains. In a right noetherian right chain domain R, all right ideals are principal and two-sided [7, Lemma 3.2]. In particular, $J = pR$ for some $p \in R$. The right noetherian right chain domain R is said to be *of type* ω [7, p. 26 and Lemma 3.4] if its chain of right ideals (=two-sided ideals) is the chain

$$R = p^0R \supset J = pR \supset p^2R \supset \cdots \supset 0 = \bigcap_{n \geq 0} p^nR.$$

Thus for every non-zero right ideal I of R, we have that $\mathrm{End}(R_R/I) \cong R/I$ is a right artinian ring, hence a perfect ring. In the next example, we show that this is also true for every non-zero principal left ideal I of a right noetherian right chain domain R of type ω which is not left Ore. Notice that for any right noetherian right chain domain R of type ω which is not left Ore, the ring is not left chain (otherwise R would be left Ore) and is not left noetherian [7, Proposition 3.7]. The main example of such a ring R can be constructed with the skew polynomial ring with coefficients in a field F, where F has an endomorphism that is not an automorphism.

Example 5.21 Let R be a right noetherian right chain domain of type ω which is not left Ore. For every non-zero principal left ideal I of R, the endomorphism ring $\mathrm{End}({}_RR/I)$ is a perfect ring.

Proof For every non-zero element $x \in R$, we have that $xR = p^n R$ for some $n \geq 0$. Therefore $x = p^n u$ for some invertible element $u \in R$. Right multiplication by u induces an isomorphism $R/Rp^n \to R/Rx$. Hence it suffices to show that $\mathrm{End}(R/Rp^n)$ is right and left perfect for $n \geq 1$. Notice that $Rp^n \subseteq p^n R$. Set $S := \mathrm{End}(R/Rp^n) \cong E/Rp^n$, where $E := \{r \in R \mid p^n r \in Rp^n\}$ denotes the idealizer of Rp^n in R, and set $K := \{r \in R \mid p^n r \in J(R)p^n\}$. By [1, Theorem 2.1], S has at most two maximal ideals, the ideals K/Rp^n and $(J(R) \cap E)/Rp^n$. Let us show that $K \subseteq J(R)$. Assume the contrary, so that K contains a unit u of R. Therefore $p^n u = rp^n$ for some $r \in J(R)$. Then $r = p^j v$ for some unit v of R and some $j \geq 1$. Thus $p^n = p^j vp^n u^{-1}$. If $j \geq n$, then $1 = p^{j-n} vp^n u^{-1}$, which implies $J(R) = R$, a contradiction. If $j < n$, then $p^{n-j} = vp^n u^{-1}$. Thus p^{n-j} belongs to the two-sided ideal $p^n R$ of R, which is a contradiction because $n - j < n$. Therefore $K \subseteq J(R)$, so S is local with maximal ideal $(J(R) \cap E)/Rp^n$. We claim that if $y \in R$ and $p^n y \in E$, then $p^n y \in Rp^n$. To prove the claim, assume that $p^n y \in E$. Then there exists $s \in R$ such that $p^n p^n y = sp^n$. Similarly, there exists $i \geq 0$ and a unit u in R such that $s = p^i u$. If $i \geq n$, then we are done, the claim is proved. Otherwise, if $i < n$, by supposing that $y = p^j v$ for some unit v, we get that $u^{-1} p^l v = p^n$, so $l > n$, which is a contradiction by [1, Lemma 2.3].

Therefore $(J(R) \cap E)^n \subseteq J(R)^n \cap E = p^n R \cap E \subseteq Rp^n$, so that the Jacobson radical $J(S) = (J(R) \cap E)/Rp^n$ of the local ring S is nilpotent. It follows that $S = \mathrm{End}(R/Rp^n)$ is a right and left perfect ring. □

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