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A Lefschetz fibration on minimal symplectic fillings of a quotient surface singularity

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Abstract
In this article, we construct a genus-0 or genus-1 positive allowable Lefschetz fibration on any minimal symplectic filling of the link of non-cyclic quotient surface singularities. As a byproduct, we also show that any minimal symplectic filling of the link of quotient surface singularities can be obtained from a sequence of rational blowdowns from its minimal resolution.

Keywords Lefschetz fibration · Quotient surface singularity · Symplectic filling

Mathematics Subject Classification 57R17 · 53D05 · 14E15 · 14J17

1 Introduction

Ever since Donaldson [5] showed that any closed symplectic 4-manifold admits a Lefschetz pencil and that a Lefschetz fibration can be obtained from a Lefschetz pencil by blowing-up the base loci, the study of Lefschetz fibrations has become an important theme for topologically understanding symplectic 4-manifolds. In fact, Lefschetz pencils and Lefschetz fibrations have been studied extensively by algebraic geometers and topologists in the complex category, and these notions can be extended to the symplectic category. It is also known that an isomorphism class of Lefschetz fibrations is characterized by the monodromy factorization, an ordered sequence of right-handed Dehn twists, up to Hurwitz equivalence and global conjugation equivalence.

On the other hand, a main research topic in symplectic 4-manifold topology focuses on classifying symplectic fillings of certain 3-manifolds equipped with a contact structure. Among them, people have classified symplectic fillings of the link of a quotient surface sin-

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gularity. Note that the link of a quotient surface singularity admits a natural contact structure, called the *Milnor fillable contact structure*. For example, Lisca [17] classified symplectic fillings of cyclic quotient singularities whose corresponding link is a lens space, and Bhupal and Ono [2] listed all possible symplectic fillings of non-cyclic quotient surface singularities. Furthermore, the second author together with Park et al. [20] constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of non-cyclic quotient surface singularities. Note that the last result above implies that every minimal symplectic filling of a quotient surface singularity is in fact a Stein filling.

Although the existence of a (positive allowable) Lefschetz fibration, called briefly PALF, on a Stein filling is well known in general [1,18], it is a somewhat different problem to find an explicit monodromy description for the Lefschetz fibration on a given Stein filling. In this article, we investigate the problem for minimal symplectic fillings of the link of quotient surface singularities. Bhupal and Ozbagci [3] found an algorithm to present each minimal symplectic filling of a cyclic quotient surface singularity as an explicit genus-0 positive allowable Lefschetz fibration over the disk. Furthermore, they also showed that such a PALF can be obtained topologically from the minimal resolution by monodromy substitutions corresponding to rational blowdowns. The main goal of this article is to generalize their result for the non-cyclic quotient surface singularity cases. Thus, we obtain the following result.

Theorem 1.1 *Every minimal symplectic filling of the link of non-cyclic quotient surface singularities admits a genus-*0 *or genus-*1 *positive allowable Lefschetz fibration over the disk. Furthermore, each symplectic filling can be also obtained by rational blowdowns from the minimal resolution of its singularity.*

***Remark* 1.1** Note that a genus of the PALF in Theorem 1.1 above is determined only by the existence of a *bad vertex* (refer to Sect. 2.1 for a definition) in the minimal resolution graph of the corresponding singularity. Explicitly, a genus of the PALF is 0 if the minimal resolution graph of a quotient surface singularity has no bad vertex, and a genus is 1 otherwise.

In order to prove Theorem 1.1 above, we first construct a PALF on the minimal resolution graph of a non-cyclic quotient surface singularity: If there is no bad vertex in the minimal resolution graph, we follow the idea of Gay and Mark in [11], where they initially constructed a genus-0 PALF on the minimal resolution graph. If there is a bad vertex, then we construct a genus-1 PALF, which is a special case of open book decompositions on the boundary of plumbings obtained by Etnyre and Ozbagci [8]. Next, we show that the induced contact structure on the boundary is the Milnor fillable contact structure, which can be obtained by computing the first Chern class in terms of vanishing cycles and the rotation number of these vanishing cycles. Then, we construct a PALF on any minimal symplectic filling via the corresponding P-resolution. Since every Milnor fiber, hence every minimal symplectic filling, of a quotient surface singularity can be obtained topologically by rationally blowing down the corresponding P-resolution, it is sufficient to construct a PALF on the general fiber of P-resolutions. Finally we show that a Lefschetz fibration of any minimal symplectic filling can be obtained by monodromy substitutions from the minimal resolution of the corresponding singularity by adapting the same technique that Endo et al. [7].

This article is organized as follows: We briefly review some generalities on quotient surface singularities, including minimal resolutions and P-resolutions, and the relation between monodormy substitutions and rational blowdowns in Sect. 2. We introduce Lisca's classification result on symplectic fillings and Bhupal–Ozbagci's algorithm for finding a PALF on the cyclic cases in Sect. 3. We subsequently explain how to construct a genus-0 or genus-1

Lefschetz fibration on the minimal resolutions and we show that the induced contact structure on the boundary is indeed Milnor fillable in Sect. 4. Finally, we provide an explicit algorithm for a PALF on any minimal symplectic filling by investigating a PALF on each P-resolution in Sect. 5.

2 Generalities on quotient surface singularities

In this section we briefly recall some basics on quotient surface singularities (refer to [20] for details). Let $(X, 0) = (\mathbb{C}^2/G, 0)$ be a germ of a quotient surface singularity, where G is a finite subgroup of $GL(2, \mathbb{C})$ without reflections. Since $(\mathbb{C}^2/G_1, 0)$ is analytically isomorphic to $(\mathbb{C}^2/G_2, 0)$ if and only if G_1 is conjugate to G_2, it is enough to classify finite subgroups of $GL(2, \mathbb{C})$ without reflections up to conjugation when classifying quotient surface singularities $(\mathbb{C}^2/G, 0)$. We may assume that $G \subset U(2)$ because G is finite. The action of G on $\mathbb{C}^2$ then lifts to an action on the blow-up of $\mathbb{C}^2$ at the origin. Thus, G acts on the exceptional divisor $E \cong \mathbb{CP}^1$, where the action is induced by $G \subset U(2) \to PU(2) \cong SO(3)$. The image of G in $SO(3)$ is either a (finite) cyclic subgroup, a dihedral group, tetrahedral group, octahedral group, or icosahedral group. Therefore quotient surface singularities are divided into five classes: *cyclic* quotient surface singularities, dihedral singularities, tetrahedral singularities, octahedral singularities, and icosahedral singularities. We call the last four cases *non-cyclic* quotient surface singularities.

2.1 Symplectic fillings and Milnor fibers

Let $(X, 0) = (\mathbb{C}^2/G, 0)$ be a germ of a quotient surface singularity, where G is a finite subgroup of $U(2)$ without reflections. Assume that $(X, 0) \subset (\mathbb{C}^N, 0)$, which is always possible for a normal surface singularity. If $B \subset \mathbb{C}^N$ is a small ball centered at the origin, then a small neighborhood $X \cap B$ of the singularity is homeomorphic to the cone over its boundary $L := X \cap \partial B$. The smooth compact 3-manifold L is called the *link* of the singularity. It is well known that the topology of the germ $(X, 0)$ is completely determined by its link L and the link L admits a natural contact structure ξ_{st}, so-called *Milnor fillable contact structure* $\xi_{\text{st}} = TL \cap JTL$, where J is an induced complex structure along L. A *(strong) symplectic filling* of $(X, 0)$ is a symplectic 4-manifold (W, ω), where the boundary $\partial W = L$ satisfies the compatibility condition $\omega = d\alpha_{\text{st}}$ near L, and where α_{st} is a 1-form defining the contact structure $\xi_{\text{st}} = \ker \alpha_{\text{st}}$ on L. One may also define a so-called *weak* symplectic filling. However, it is known that two notions of symplectic fillings coincide in our case because the link L is a rational homology sphere. So we simply call them *symplectic fillings*.

Next, we call W a *Stein filling* of $(X, 0)$ if it is a Stein manifold W with L as its strictly pseudoconvex boundary and ξ_{st} is the set of complex tangencies to L. It is clear that Stein fillings are minimal symplectic fillings of the link L of $(X, 0)$.

Third, we call a proper flat map $\pi : \mathcal{X} \to \Delta$ with $\Delta = \{t \in \mathbb{C} : |t| < \epsilon\}$ a *smoothing* of $(X, 0)$ if it satisfies $\pi^{-1}(0) = X$ and $\pi^{-1}(t)$ is smooth for all $t \neq 0$. The *Milnor fiber* M of a smoothing π of $(X, 0)$ is defined as a general fiber $\pi^{-1}(t)$ $(0 < t < \epsilon)$. It is known that the Milnor fiber M is a compact 4-manifold with link L as its boundary and the diffeomorphism type depends only on the smoothing π. Furthermore, M has a natural Stein (hence symplectic) structure, thus it provides an example of a Stein (and minimal symplectic) filling of (L, ξ_{st}). Recall that, as mentioned in the introduction, Park et al. [20] constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of quotient surface singularities. Hence, it is now a well-known fact that every minimal symplectic filling of a quotient surface singularity is a Stein filling and a Milnor fiber of the singularity.

2.2 Minimal resolutions

We first denote the *Hirzebruch-Jung continued fraction* by $[c_1, \ldots, c_t](c_i \geq 1)$, which is defined recursively as follows:

$$[c_t] = c_t, \text{ and } [c_i, c_{i+1}, \ldots, c_t] = c_i - \frac{1}{[c_{i+1}, \ldots, c_t]}.$$

Since a continued fraction $[c_1, c_2, \ldots, c_t]$ often describes a chain of smooth rational curves on a complex surface whose dual graph is given by

we use by analogy the term 'blowing up' for the following operations and the term 'blowing down' for their inverses:

$$[c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_t] \to [c_1, \ldots, c_{i-1} + 1, 1, c_{i+1} + 1, \ldots, c_t]$$
$$[c_1, \ldots, c_{t-1}] \to [c_1, \ldots, c_{t-1} + 1, 1].$$

Now we describe the (dual graph of) the minimal resolution of quotient surface singularities. In the resolution graph, note that a vertex v corresponds to the irreducible component E_v of the exceptional divisor E, and the edges correspond to the intersections of the irreducible components E_v. We call the number of edges connected to the vertex v the *valence* of v and the self-intersection of E_v the *degree* of v. If the absolute value of the degree of v is strictly less than the valence of v, we call the vertex v a *bad vertex*.

***Example* 2.1** The following figures show the cases of minimal resolution graphs with and without a bad vertex. A central vertex (vertex with valence 3) in the right-handed figure is a bad vertex.

(a) No bad vertex case (b) Bad vertex case

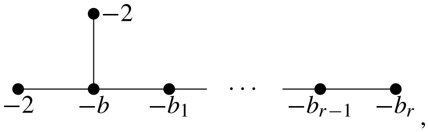
Cyclic singularities $A_{n,q}$

A *cyclic quotient surface singularity* $(X, 0)$ *of type* $\frac{1}{n}(1, q)$ with $1 \leq q < n$ and $(n, q) = 1$ is a quotient surface singularity, where a cyclic group $\mathbb{Z}_n$ acts by $\zeta \cdot (x, y) = (\zeta x, \zeta^q y)$. Then, the minimal resolution graph of $(X, 0)$ is given by

where $\dfrac{n}{q} = [b_1, b_2, \ldots, b_{r-1}, b_r]$ with $b_i \geq 2$ for all i.

Dihedral singularities $D_{n,q}$

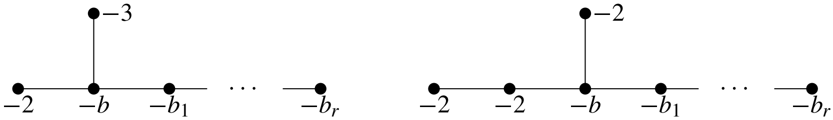
Let $(X, 0)$ be a dihedral singularity of type $D_{n,q}$, where $1 < q < n$ and $(n, q) = 1$. The minimal resolution graph of $(X, 0)$ is given by



where $\dfrac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$ with $b \geq 2$ and $b_i \geq 2$ for all i.

Other cases

For a tetrahedral, octahedral, or icosahedral singularity, the minimal resolution has a central curve C_0 with $C_0 \cdot C_0 = -b$ $(b \geq 2)$ and three arms, which can be divided into *type* (3, 1) and *type* (3, 2):



(a) type (3, 1) (b) type (3, 2)

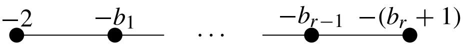
2.3 *P*-resolutions

Definition 2.1 A normal surface singularity is *of class* T if it is a rational double point singularity or a cyclic quotient surface singularity of type $\frac{1}{dn^2}(1, dna - 1)$ with $d \geq 1$, $n \geq 2$, $1 \leq a < n$, and $(n, a) = 1$. Equivalently, it is a quotient surface singularity which admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing [16].

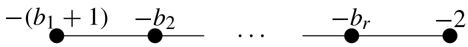
Note that one-parameter $\mathbb{Q}$-Gorenstein smoothing of a singularity of class T is interpreted topologically as a rational blowdown surgery defined by Fintushel and Stern [10], and later extended by Park [22]. Furthermore, thanks to Wahl [26], a cyclic quotient surface singularity of class T can be recognized from its minimal resolution as follows:

Proposition 2.2 *(1) The singularities*  *and*  *are of class* T.

(2) If  *is of class* T*, so are*



and

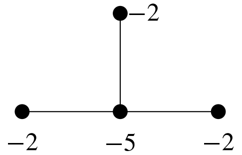


(3) Every singularity of class T that is not a rational double point can be obtained directly from one of the singularities described in (1) and by iterating through the steps described in (2) above.

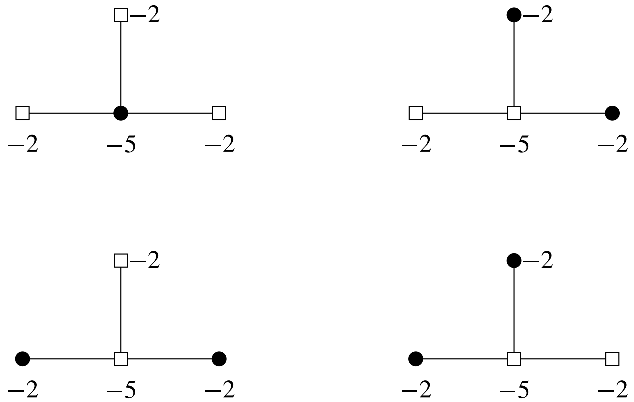
Definition 2.3 A *P-resolution* $f : (Y, E) \to (X, 0)$ of a quotient surface singularity $(X, 0)$ is a partial resolution such that Y has at most rational double points or singularities of class T and K_Y is ample relative to f.

We usually describe a P-resolution $Y \to X$ as the minimal resolution $\pi : Z \to Y$ of Y with π-exceptional divisors. Note that the ampleness condition in the definition of a P-resolution can be checked on Z: Every (-1) curve on Z must intersect two curves E_1 and E_2, which are exceptional for singularities of class T on Y. In addition, the sum of the k_i coefficients of E_i in the canonical divisor K_Z is less than -1. According to Kollar and Shepherd-Barron [16], there is a one-to-one correspondence between the set of all irreducible components of the versal deformation space of a quotient surface singularity $(X, 0)$ and the set of all P-resolutions of $(X, 0)$. Hence, since the Milnor fibers are invariants of the irreducible components of the versal deformation space of $(X, 0)$, there is a one-to-one correspondence between the Milnor fibers and the P-resolutions. Furthermore, Stevens [25] also showed how to find all P-resolutions of quotient surface singularities.

***Example* 2.2** Let $(X, 0)$ be a dihedral singularity of type $D_{9,2}$. Since $9/2 = [5, 2]$, the minimal resolution of $(X, 0)$ is given by

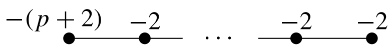


We have the following four P-resolutions of $(X, 0)$: Here, a linear chain of vertices decorated by a rectangle □ denotes curves on the minimal resolution of a P-resolution, which are contracted to a singularity of class T on the P-resolution. Note that there are certain symmetries in the list of P-resolutions.



2.4 Monodromy substitutions and rational blowdowns

In [10], Fintushel and Stern introduced the following rational blowdown surgery: Let C_p be a smooth 4-manifold obtained from plumbing disk bundles over 2-sphere according to the following linear diagram.



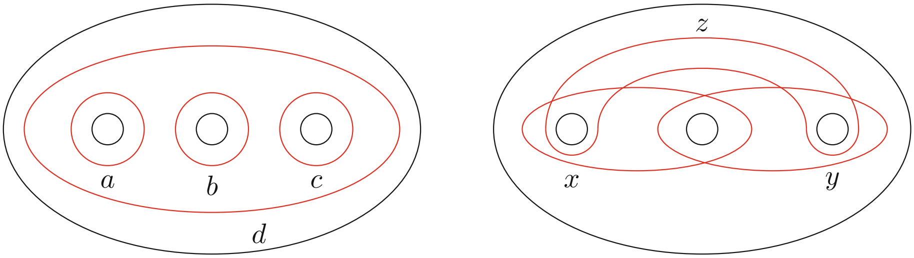


Fig. 1 Lantern relation

Then the boundary of C_p is a lens space $L(p^2, p-1)$, which bounds a rational ball B_p, i.e., $H_*(B_p; \mathbb{Q}) = H_*(D^4; \mathbb{Q})$. Therefore, if there exists an embedding C_p in a smooth 4-manifold X, one can construct a new smooth 4-manifold X_p by replacing C_p with B_p. This procedure is called a *rational blowdown* surgery and we say that X_p is obtained by rationally blowing down X. Furthermore, Symington [23] proved that a rational blowdown 4-manifold X_p admits a symplectic structure in some cases. For example, if X is a symplectic 4-manifold containing a configuration C_p such that all 2-spheres in C_p are symplectically embedded and intersect positively, then the rational blowdown 4-manifold X_p also admits a symplectic structure. Later, the Fintushel–Stern's rational blowdown surgery is generalized by Park [22] using a configuration $C_{p,q}$ obtained from plumbing disk bundles over a 2-sphere according to the dual resolution graph of $L(p^2, pq-1)$, which also bounds a rational ball $B_{p,q}$ as follows:

Definition 2.4 Suppose X is a smooth 4-manifold containing a configuration $C_{p,q}$. Then one can construct a new smooth 4-manifold $X_{p,q}$, called a *(generalized) rational blowdown* of X, by replacing $C_{p,q}$ with a rational ball $B_{p,q}$. We also call this a *(generalized) rational blowdown* surgery.

Next, we introduce a notion of monodromy substitution that is closely related to a rational blowdown surgery. That is, we briefly explain how to replace a rational blowdown surgery with a monodromy substitution in some cases.

Suppose that a symplectic 4-manifold X with a possibly non-empty boundary admits a Lefschetz fibration characterized by a monodromy factorization $\mathcal{W}_X$. Assume that W and W' are distinct products of right-handed Dehn twists which yield the same element as a global monodromy in the mapping class group of the fiber. If there is a partial monodromy factorization equal to W in the monodromy factorization $\mathcal{W}_X$ of X, then we can obtain a Lefschetz fibration on a new symplectic 4-manifold X' whose monodromy factorization $\mathcal{W}_{X'}$ is obtained by replacing W with W'. Note that the diffeomorphism types and the induced contact structures of ∂X and $\partial X'$ are the same. We call this procedure a *monodromy substitution*. For example, a famous lantern relation yields a rational blowdown surgery involving the lens space $L(4, 1)$ [6]: the PALF with monodromy $abcd$ yields a configuration C_2, while the PALF with monodromy xyz yields a rational ball B_2. As another example, the *daisy relation*, introduced in [7], yields a monodromy substitution for a configuration C_p and a rational ball B_p. One can also find a monodromy substitution for a (generalized) rational blowdown surgery in [7] (Fig. 1).

3 Review for the cyclic singularity cases

We briefly review Lisca's classification of minimal symplectic fillings and Bhupal–Ozbagci's algorithm of positive allowable Lefschetz fibrations for the minimal symplectic fillings of a cyclic quotient surface singularity in this section.

We first review Lisca's classification (refer to [17] for details): Let $(X, 0)$ be a cyclic quotient surface singularity of type $\frac{1}{n}(1, q)$ with $(n, q) = 1$ whose link is the lens space $L(n, q)$. P. Lisca [17] parametrized all minimal symplectic fillings of $(X, 0)$ by a set $\mathcal{Z}_e\left(\frac{n}{n-q}\right)$ of certain sequences of integers $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{N}^e$ (see Definition 3.1 below). That is, he constructed a compact oriented symplectic 4-manifold $W_{n,q}(\mathbf{n})$ with boundary $L(n, q)$ that is parametrized by $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$ using surgery diagrams. He also showed that $W_{n,q}(\mathbf{n})$ is in fact a Stein filling of $L(n, q)$. Finally, he proved that any symplectic filling of $L(n, q)$ is orientation-preserving diffeomorphic to a manifold obtained by blow-ups from one of $W_{n,q}(\mathbf{n})$'s. Hence, every minimal symplectic filling is diffeomorphic to one of the $W_{n,q}(\mathbf{n})$'s. On the other hand, Christophersen [4] and Stevens [24] parametrized all reduced irreducible components of the versal deformation space of $(X, 0)$ using the same set $\mathcal{Z}_e\left(\frac{n}{n-q}\right)$ but with different methods. Thus, it was a natural conjecture that every Milnor fiber of $(X, 0)$ is diffeomorphic to a $W_{n,q}(\mathbf{n})$, which are parametrized by the same element in $\mathcal{Z}_e\left(\frac{n}{n-q}\right)$. The conjecture was proven true by Nemethi and Popescu-Pampu [19].

Definition 3.1 An e-tuple of nonnegative integers $(n_1 \ldots, n_e)$ is called *admissible* if every denominator in the continued fraction $[n_1 \ldots, n_e]$ is positive. It is easy to see that an admissible e-tuple of nonnegative integers is either 0 or only consists of positive integers. Let $\mathcal{Z}_e$ be the set of all admissible e-tuples such that $[n_1 \ldots, n_e] = 0$, i.e., let $\mathcal{Z}_e$ be the set of all e-tuples of integers which can be obtained via a sequence of blow-ups from (0). For $\frac{n}{n-q} = [a_1, \ldots, a_e]$, we define

$$\mathcal{Z}_e\left(\frac{n}{n-q}\right) := \{(n_1, \ldots, n_e) \in \mathcal{Z}_e |\ 0 \le n_i \le a_i, \text{ for } i = 1, \ldots, e\}.$$

Lisca constructed a smooth 4-manifold $W_{n,q}(\mathbf{n})$ for each e-tuple $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$ whose boundary is diffeomorphic to the link of a cyclic singularity of type $A_{n,q}$, also known as the lens space $L(n, q)$ using a corbodism $C_{n,q}(\mathbf{n})$ between $S^1 \times S^2$ and $L(n, q)$: First consider a linear chain consisting of e number of unknots in S^3 with framings $n_1, \ldots, n_e$, respectively. Let $N(\mathbf{n})$ be a 3-manifold obtained by Dehn surgery on this framed link. Since $[n_1, \ldots, n_e] = 0$, it is clear that $N(\mathbf{n})$ is diffeomorphic to $S^1 \times S^2$. Then, using a framed link L in $N(\mathbf{n})$ as shown in Fig. 2, one can obtain a cobordism $C_{n,q}(\mathbf{n})$ by attaching 4-dimensional 2-handles to the $L \subset S^1 \times S^2 \times \{1\} \subset S^1 \times S^2 \times I$. Finally, choosing a diffeomorphism $\varphi : N(\mathbf{n}) \to S^1 \times S^2$ again, one can construct a desired smooth (in fact symplectic) 4-manifold

$$W_{n,q}(\mathbf{n}) := C_{n,q}(\mathbf{n}) \cup_\varphi S^1 \times D^3.$$

Note that, since any self-diffeomorphism φ of $S^1 \times S^2$ extends to $S^1 \times D^3$, the diffeomorphism type of $W_{n,q}(\mathbf{n})$ is independent of the choice of φ. According to P. Lisca [17], any symplectic

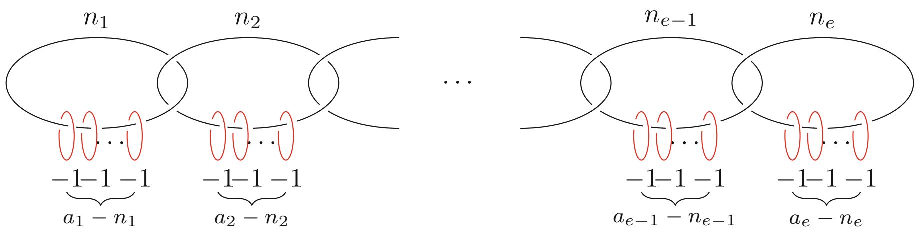


Fig. 2 The framed link $L \subset N(\mathbf{n})$

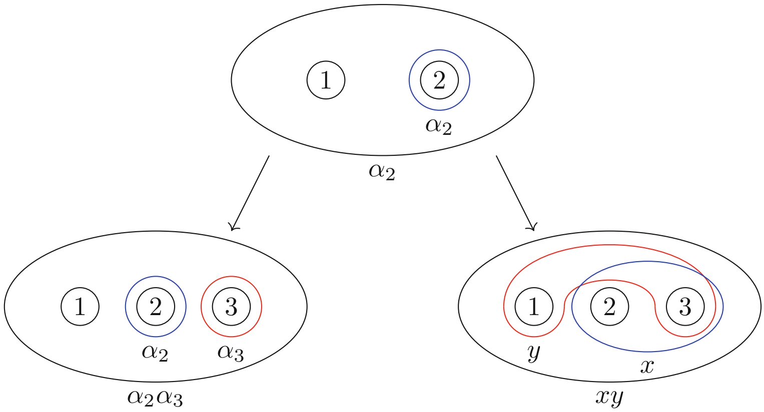


Fig. 3 PALF on $S^1 \times D^3$ corresponds to (1, 1), (1, 2, 1) and (2, 1, 2)

filling of $(L(n,q), \xi_{st})$ is orientation-preserving diffeomorphic to a blow-up of $W_{n,q}(\mathbf{n})$ for some $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$.

Next, for each $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$, M. Bhupal and B. Ozbagci constructed a genus-0 PALF on $S^1 \times D^3$ so that the attaching circles of (-1)-framed 2-handles in $W_{n,q}(\mathbf{n})$ lie on a generic fiber (refer to [3] for details).

One can construct a PALF on $S^1 \times D^3$ over the disk corresponding to each $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$. Note that this depends on a blow-up sequence from (0). For each $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$, a generic fiber $F_{\mathbf{n}}$ is the disk with e holes. We may assume the holes in the disk are ordered linearly from left to right, as shown in Fig. 3. If $\mathbf{n} \in \mathcal{Z}_e\left(\frac{n}{n-q}\right)$ is obtained from $\mathbf{n}' \in \mathcal{Z}_{e-1}$ by blowing up the j^{th} term $(1 \leq j \leq e-2)$, we construct a generic fiber $F_{\mathbf{n}}$ to be a surface obtained from $F_{\mathbf{n}'}$ by splitting the $(j+1)^{\text{th}}$ hole so that vanishing cycles $\{x_i \mid i = 1, 2, \ldots, e-2\}$ for $\mathbf{n}'$ are naturally extended to $\{\widetilde{x_i} \mid i = 1, 2, \ldots, e-2\}$ in $F_{\mathbf{n}}$. The monodromy factorization subsequently changes from $x_1 x_2 \ldots x_{e-2}$ to $\widetilde{x_1}\widetilde{x_2} \cdots \widetilde{x}_{e-2}\beta_j$, where β_j is a curve on $F_{\mathbf{n}}$ that encircles the $1, \ldots j, (j+2)$-labelled holes while skipping the $(j+1)$-labelled hole. For a blowing up of the $(e-1)$th term, we just add the eth hole to $F_{\mathbf{n}'}$ at the right of the $(e-1)$th hole and add a Dehn twist on a curve encircling the eth hole. In this way, we obtain a genus-0 PALF on $W_{n,q}(\mathbf{n})$ such that, if the attaching circle of

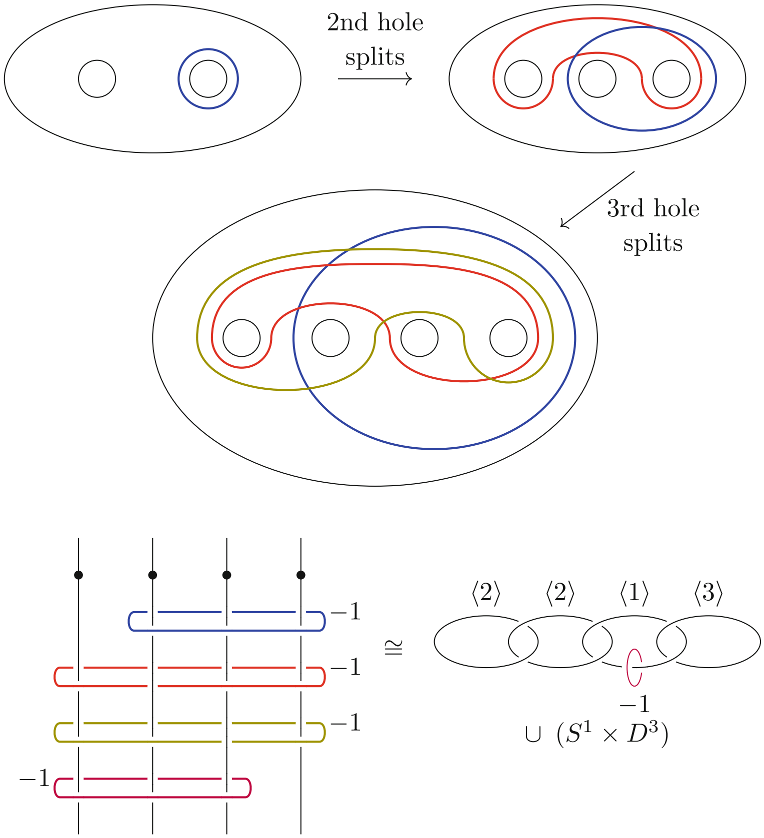


Fig. 4 PALF on $W_{9,2}((2, 2, 1, 3))$

a (-1)-framed 2-handle h in $W_{n,q}(\mathbf{n})$ is the meridian of a n_i-framed unknot, the 2-handle h corresponds to a Dehn twist on a curve γ_i encircling the first i holes. We refer to Fig. 4 below for an example.

Bhupal and Ozbagci [3] also showed that the monodromy factorization for each minimal symplectic filling of the lens space $L(n, q)$ can be obtained by a sequence of monodromy substitutions that can be interpreted as a sequence of rational blowdowns from the minimal resolution of the corresponding singularity.

4 Lefschetz fibrations on minimal resolutions

In this section, as a first step towards proving our main theorem (Theorem 1.1), we construct a genus-0 or genus-1 positive allowable Lefschetz fibarion (PALF) on each minimal resolution of non-cyclic quotient surface singularities. Note that a genus of the PALF is determined only by the existence of a bad vertex in the minimal resolution graph of the corresponding singularity. That is, a genus of the PALF is 0 if the minimal resolution graph has no bad vertex, and a genus is 1 otherwise. We subsequently check that a contact structure on the boundary induced from the PALF is the Milnor fillable contact structure so that every PALF obtained via monodromy substitutions is also a Stein filling of (L, ξ_{st}).

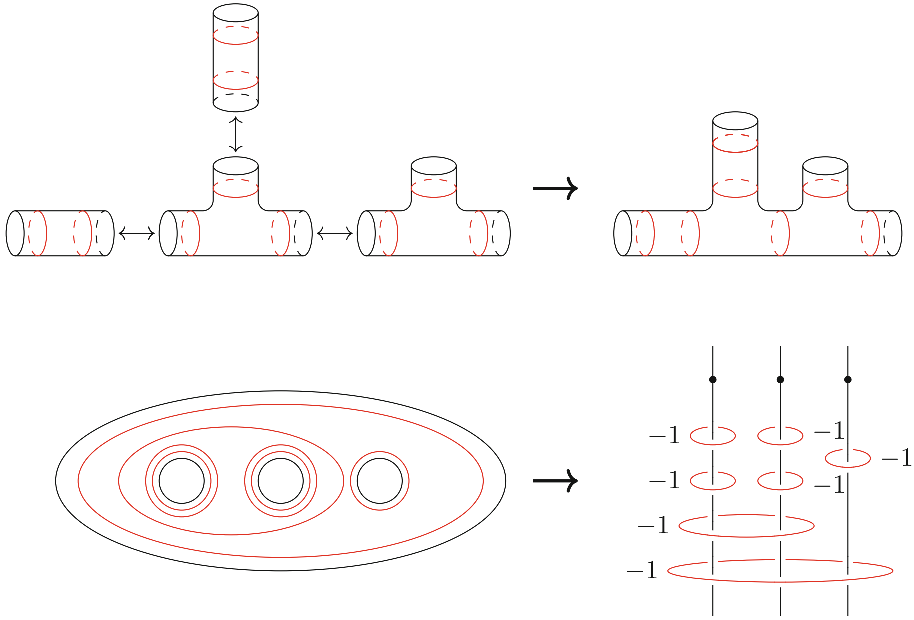


Fig. 5 A genus-0 PALF on minimal resolution of $D_{8,3}$

4.1 No bad vertex cases

If the minimal resolution graph Γ of a quotient surface singularity does not have a bad vertex, then there is a well-known genus-0 PALF on the minimal resolution Γ, as demonstrated by Gay and Mark [11]. We consider the 2-sphere Σ_i with b_i holes for each vertex v_i with degree $-b_i$. Then the fiber surface Σ is obtained by gluing Σ_i along their boundaries according to Γ and the vanishing cycles are the set of curves parallel to the boundary of each Σ_i. Note that we end up with only one right-handed Dehn twist on the connecting neck. We refer to Fig. 5 below for an example. Note that this PALF is compatible with the symplectic structure ω given by a convex plumbing X_Γ of symplectic surfaces, where each vertex represents a symplectic surface with self-intersection $-b_i$ that intersect each other ω-orthogonally according to Γ. Thus, the induced contact structure ξ on the boundary ∂X_Γ is compatible with the open book decomposition coming from the aforementioned PALF. Park and Stipsicz [21] showed that ξ is indeed the Milnor fillable contact structure. In fact, their argument holds for any negative-definite intersection matrix of Γ.

4.2 Bad vertex cases

If the minimal resolution graph Γ of a non-cyclic quotient surface singularity has a bad vertex, we now construct a genus-1 PALF on the minimal resolution Γ as follows: First we construct a PALF on X_L, where X_L is the minimal resolution of a cyclic singularity determined by a maximal linear subgraph Γ_L of Γ. We consider a 4-dimensional Kirby diagram of X_L, which can be easily obtained from the PALF of X_L. We could subsequently obtain a Kirby diagram of X_Γ by adding a 2-handle h or two 2-handles $\{h_1, h_2\}$ to that of X_L, depending on which type of arm is not in Γ_L. After introducing a cancelling 1-handle/2-handle pair, the 2-handles not coming from the Kirby diagram of X_L can be thought of as vanishing cycles of a new fiber F, which is obtained by attaching a 1-handle to the surface F_L. Note that the new fiber F is a genus-1 surface with holes. We refer to Fig. 6 below for an example.

Next, we check that a contact structure on the boundary induced from the PALF constructed above is the Milnor fillable contact structure. First, recall that, the 2-plane field ξ induces a $Spin^c$ structure t_ξ on L for a contact 3-manifold (L, ξ). Furthermore, if (W, J) is a Stein filling of (L, ξ), then t_ξ is a restriction of $Spin^c$ structure $\mathcal{S}$ on W to $\partial W = L$ induced by its complex structure J on W. On the other hand, there is a theorem of Gay–Stipsicz [12] that characterizes the contact structure on the link of a quotient surface singularity.

Theorem 4.1 [12] *Suppose that a small Seifert* 3*-manifold* $M = M(s_0; r_1, r_2, r_3)$ *satisfies* $s_0 \leq -2$ *and* M *is an L-space. Then two tight contact structures* ξ_1, ξ_2 *on* M *are isotopic if and only if* $t_{\xi_1} = t_{\xi_2}$.

Theorem 4.2 *The contact structure on the link of non-cyclic quotient surface singularities induced by the PALF constructed above is Milnor fillable.*

Proof First note that, since a convex plumbing X_Γ of the minimal resolution graph Γ of a quotient singularity is simply connected, the $Spin^c$ structure $\mathcal{S}$ on X_Γ is determined by the first Chern class $c_1(\mathcal{S})$. On the other hand, $t_{\xi_{st}}$ is a restriction of $\mathcal{S}$ whose first Chern class $c_1(\mathcal{S})$ satisfies the adjunction equality on each vertex in Γ. Hence, according to Theorem 4.1 above, a PALF on X_Γ induces the Milnor fillable contact structure on the boundary if and only if $c_1(J)$ satisfies the adjunction equality for each vertex in Γ, where J is a complex structure coming from the Stein structure of the PALF. From the PALF on X_Γ constructed above, we can compute the first Chern class $c_1(J)$ in terms of vanishing cycles C_i: $c_1(J)$ is represented by a co-cycle whose value on the 2-handle corresponding to C_i is the rotation number $r(C_i)$ [13], which can be computed once we fix a trivialization of the tangent bundle of a page [9]. The vertices in Γ_L satisfy the adjunction equality because the PALF for the no bad vertex cases induces the Milnor fillable contact structure [21]. The homology classes of the vertices not in Γ_L can be represented by new vanishing cycles together with some vanishing cycles in Γ_L, thus we can check whether they satisfy the adjunction equality by computing the rotation number of the vanishing cycles. Note that all non-cyclic quotient singularities can be divided into the following three cases: Dihedral singularities, singularities of type (3, 1), and singularities of type (3, 2).

Dihedral singularities and singularities of type (3, 2): There is only one degree -2 vertex v which is not in Γ_L for dihedral cases. If we construct a genus-1 PALF on the minimal resolution as shown above, then the global monodromy of the PALF should contain $C_{\text{red}} C_{\text{blue}} C_{\text{orange}}^2$ as a subword and v is homologically equal to $C_{\text{blue}} - C_{\text{red}} + 2C_{\text{orange}}$. Once we fix a trivialization of the tangent bundle of fiber F as a natural extension of a trivialization of the tangent bundle of $\mathbb{R}^2$, the rotation numbers of blue, red, and orange vanishing cycles are -1, $+1$, and $+1$, respectively. This means that the -2 vertex v satisfies the adjunction equality. The only difference between the dihedral case and the type (3, 2) case is that there is another degree -2 vertex v' connected to v which is not in Γ_L for the (3, 2) case. Hence, for type (3, 2) singularities, the global monodromy of the PALF on the minimal resolution should contain $C_{\text{red}} C_{\text{blue}}^2 C_{\text{orange}}^2$ as a subword. The vertex v' is also homologically $C_{\text{blue}} - C_{\text{blue}}$, meaning every vertex in Γ satisfies the adjunction equality. See Fig. 7.

Singularities of type (3,1): Consider a maximal linear subgraph Γ_L starting with a degree -3 vertex v. Since Γ_L starts with a degree -3 vertex v, the global monodromy of the PALF should contain $C_{\text{red}} C_{\text{blue}} C_{\text{olive}} C_{\text{orange}}$ as a subword. A similar argument shows that the

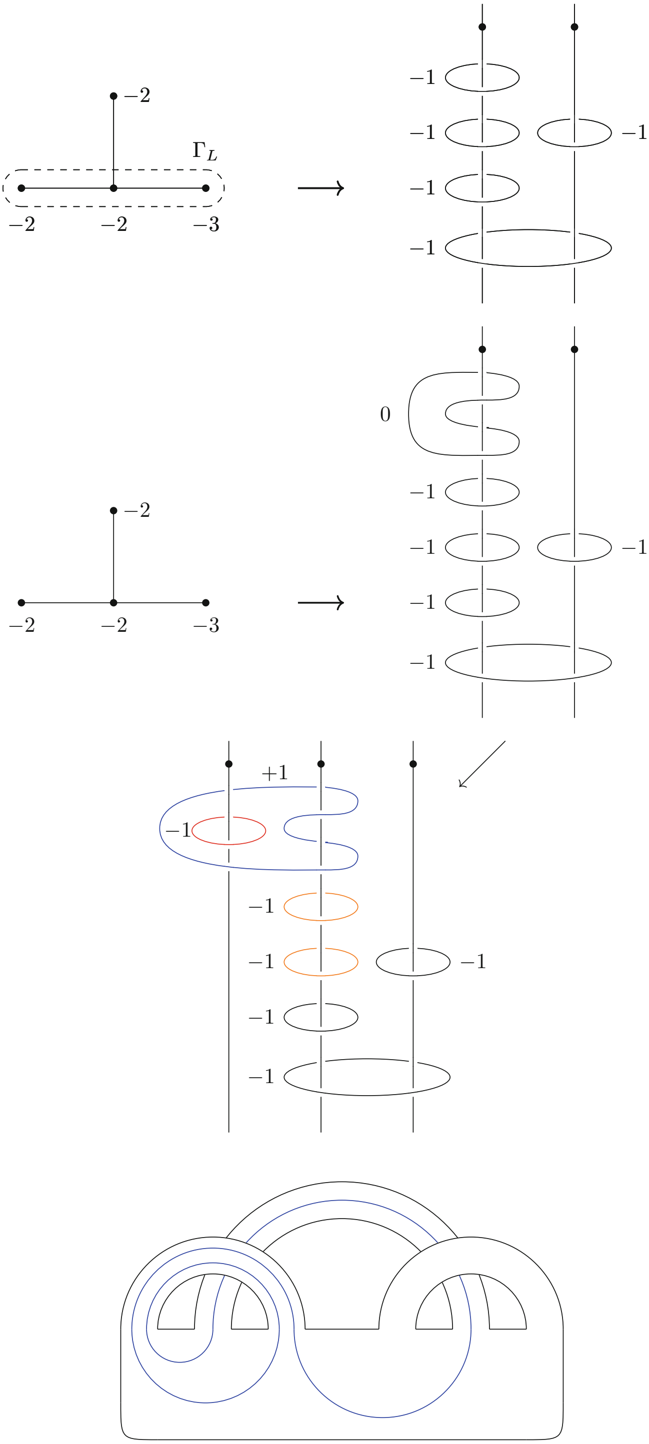


Fig. 6 A genus-1 PALF on the minimal resolution of $D_{5,3}$

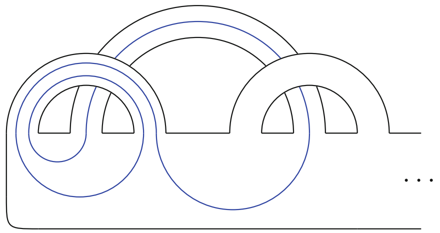


Fig. 7 A genus-1 PALF on the minimal resolution of dihedral singularities

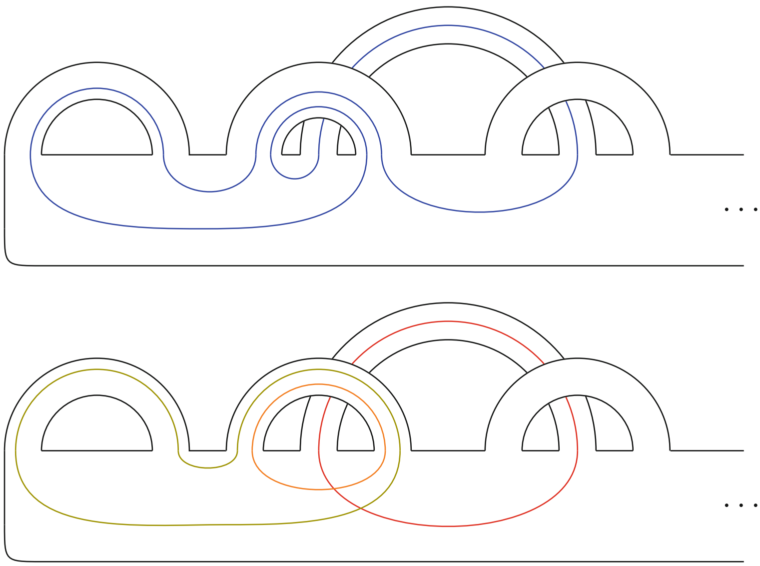


Fig. 8 A genus-1 PALF on the minimal resolution of type (3, 1) singularities

genus-1 PALF we constructed induces a Milnor fillable contact structure on the boundary. See Fig. 8. □

5 Lefschetz fibrations on minimal symplectic fillings

As mentioned in the Introduction, Bhupal and Ono [2] listed all possible minimal symplectic fillings for non-cyclic quotient surface singularities. In fact, they showed that each minimal symplectic filling of a non-cyclic singularity $(X, 0)$ is orientation-preserving diffeomorphic to $Z - \nu(E_\infty)$, where E_∞ is the *compactifying divisor* of X embedded in a rational symplectic 4-manifold Z. They also found all possible pairs of (Z, E_∞) for non-cyclic quotient singularities. On the other hand, Park et al. [20] observed that the number of P-resolutions in J. Stevens [25] and that of minimal symplectic fillings in Bhupal–Ono's list [2] are nearly equal. Since there is a one-to-one correspondence between Milnor fibers and P-resolutions for quotient singularities [16], it is natural to ask whether every minimal symplectic filling of non-cyclic quotient singularities is a Milnor fiber. This was proven in [20] using the corresponding complex model. In fact, they even constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of non-cyclic quotient surface singularities.

What follows is a strategy for proving our main theorem (Theorem 1.1). For a given P-resolution Y of a non-cyclic quotient singularity X, we construct a PALF on the minimal resolution of X, after appropriate monodromy substitutions, the resulting 4-manifold is diffeomorphic to a 4-manifold obtained by rationally blowing down all singularities of class T in the minimal resolution Z of Y. Since any Milnor fiber of quotient surface singularities can be obtained by rationally blowing down all singularities of class T in the minimal resolution of the corresponding P-resolution topologically, we would be done.

Now we construct a PALF on each minimal symplectic filling of a non-cyclic quotient singularity X via the corresponding P-resolution Y. In other words, we first construct a PALF on the minimal resolution of X and we find a suitable monodromy substitution to obtain a PALF on the 4-manifold obtained by rationally blowing down all singularities of class T in Z. Let Γ_Z be the dual graph of the minimal resolution Z of Y. For the sake of convenience, we divide all P-resolutions into the following two cases: those with and without a maximal linear subgraph Γ_L of Γ_Z containing all singularities of class T.

5.1 Case 1

Let Y be a P-resolution of a non-cyclic quotient singularity X whose minimal resolution graph Γ_Z has a maximal linear subgraph Γ_L containing all singularities of class T in Y. Note that the subgraph Γ_L becomes the minimal resolution graph of a P-resolution Y' for some cyclic quotient singularity X'. Then, by combining a PALF on the minimal resolution of X' and a technique developed in Section 4, we can construct a PALF on the minimal resolution of X based on the PALF of the minimal resolution of X'. Explicitly, starting from a PALF $(F_{X'}, y_1 y_2 \dots y_m)$ on the minimal resolution of X', we obtain a PALF $(F_X, x_1 \dots x_n \widetilde{y}_1 \widetilde{y}_2 \dots \widetilde{y}_m)$ on the minimal resolution of X, where the vanishing cycles $\widetilde{y}_i$'s in F_X are natural extensions of the corresponding vanishing cycles y_i's in $F_{X'}$, and the vanishing cycles $\{x_1, \dots, x_n\}$ come from the corresponding vertices in the arm which is not contained in Γ_L. Note that a genus of the generic fiber F_X depends on the existence of a bad vertex in the minimal resolution of X. We subsequently find a monodromy substitution that yields a PALF on Y. Since Y' contains all singularities of class T lying in Y, if a monodromy substitution of the form $y_1 y_2 \dots y_m = z_1 z_2 \dots z_l$ yields a PALF on Y', then $(F_X, x_1 \dots x_n \widetilde{z}_1 \widetilde{z}_2 \dots \widetilde{z}_l)$ yields a desired PALF on Y, where the vanishing cycles $\widetilde{z}_i$'s in F_X are natural extensions of the corresponding vanishing cycles z_i's in $F_{X'}$.

If the genus of generic fiber F_X is 1, then a monodromy substitution of the form $\widetilde{y}_1 \widetilde{y}_2 \dots \widetilde{y}_m = \widetilde{z}_1 \widetilde{z}_2 \dots \widetilde{z}_l$ still can be interpreted as a sequence of rational blowdowns. If the genus of the generic fiber F_X is 0, then there could be a vertex whose degree is strictly less than -2 in the arm not contained in Γ_L. In this case, a monodromy substitution of the form $\widetilde{y}_1 \widetilde{y}_2 \dots \widetilde{y}_m = \widetilde{z}_1 \widetilde{z}_2 \dots \widetilde{z}_l$ in F_X does not correspond to a sequence of rational blowdown surgeries because, in general, there could be more holes in F_X enclosed by $\widetilde{y}_i$ and $\widetilde{z}_j$ than holes in $F_{X'}$ enclosed by y_i and z_j for some $\widetilde{y}_i$ and $\widetilde{z}_j$. The construction of the PALF on the minimal resolution shows that there is a vanishing cycle x_i enclosing only the new hole for each new hole in F_X. Thus, after adding such $\{x_i\}$ to both sides, $x_{i_1} x_{i_2} \dots x_{i_k} \widetilde{y}_1 \widetilde{y}_2 \dots \widetilde{y}_m = x_{i_1} x_{i_2} \dots x_{i_k} \widetilde{z}_1 \widetilde{z}_2 \dots \widetilde{z}_l$ is a positive stabilization of $y_1 y_2 \dots y_m = z_1 z_2 \dots z_l$ which also can be interpreted as a sequence of rational blowdowns. See Example 5.2 below.

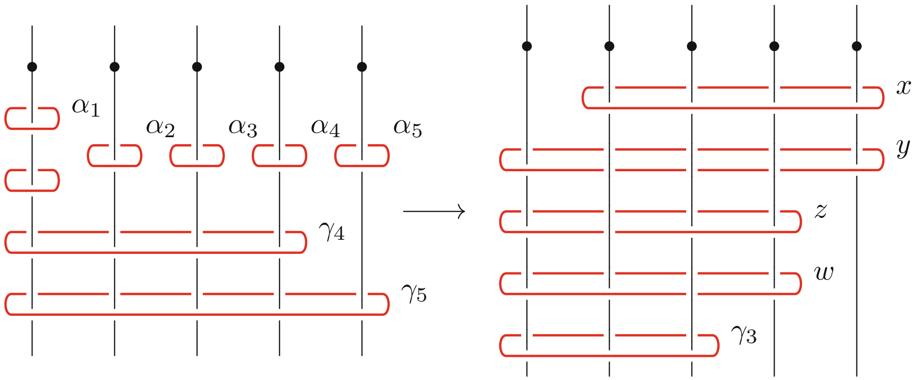
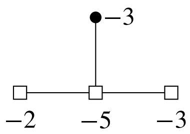
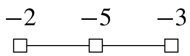


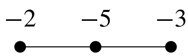
Fig. 9 A PALF on a P-resolution Y'

***Remark* 5.1** The minimal resolution graph of X shows that there is a symplectic cobordism between a lens space determined by X' and the link of X. Therefore, the minimal symplectic filling of X correponding to Y is obtained from the minimal symplectic filling of X' corresponding to Y' by symplectically gluing the cobordism. Since every minimal sympelctic filling of a lens space is obtained from a sequence of rational blowdowns from the minimal resolution, we can conclude that every minimal symplectic filling corresponding to a P-resolution Y in Case 1 is obtained via a sequence of rational blowdowns from the minimal resoluiton.

***Example* 5.2** Let $(X, 0)$ be a tetrahedral singularity of type $T_{6(5-2)+5}$ which has the following P-resolution Y:



Thus, a PALF on a P-resolution $Y' =$ 

of 

can be obtained via a monodromy substitution of the form

$$\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5\gamma_4\gamma_5 = xyzw\gamma_3$$

which is shown in Fig. 9 below. Hence, we obtain a PALF $\beta_1\beta_2\widetilde{x}\widetilde{y}\widetilde{z}\widetilde{w}\widetilde{\gamma}_3$ on the P-resolution Y from a PALF $\beta_1\beta_2\widetilde{\alpha}_1^2\widetilde{\alpha}_2\widetilde{\alpha}_3\widetilde{\alpha}_4\widetilde{\alpha}_5\widetilde{\gamma}_4\widetilde{\gamma}_5$ on the minimal resolution of X via a monodromy substitution of the form

$$\beta_2\widetilde{\alpha}_1^2\widetilde{\alpha}_2\widetilde{\alpha}_3\widetilde{\alpha}_4\widetilde{\alpha}_5\widetilde{\gamma}_4\widetilde{\gamma}_5 = \beta_2\widetilde{x}\widetilde{y}\widetilde{z}\widetilde{w}\widetilde{\gamma}_3,$$

which can be topologically interpreted as a rational blowdown surgery. See Fig. 10.

5.2 Case 2

In this subsection, we address a P-resolution Y of X such that any maximal linear subgraph Γ_L of the minimal resolution of Y cannot contain all singularities of class T lying in Y. If

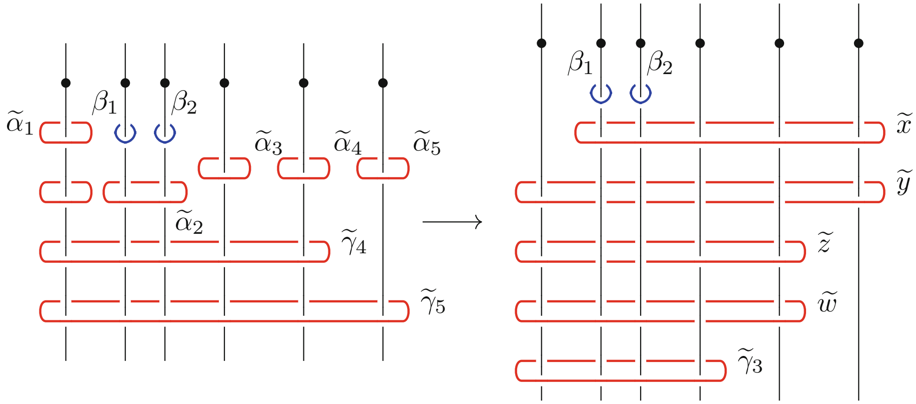


Fig. 10 A PALF on P-resolution Y

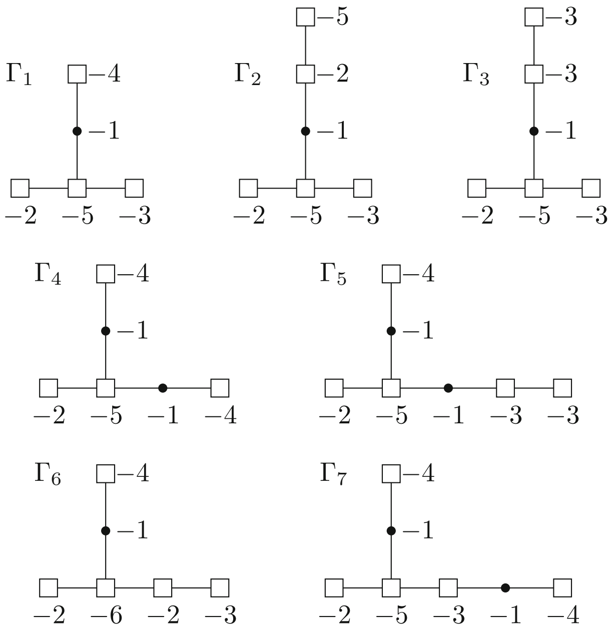


Fig. 11 Types of subgraph Γ_i

the genus of generic fiber is 0, then such a P-resolution Y contains one of the subgraphs Γ_i shown in Fig. 11, which will be discussed in 5.2.1. If the genus is 1, then there are two such P-resolutions, which will be discussed in 5.2.2. See [15] for the list of all P-resolutions for non-cyclic singularities.

5.2.1 Genus-0 cases

Note that each subgraph Γ_i in Fig. 11 represents a P-resolution Y_i of another quotient surface singularity, say X_i. Since the subgraphs in Fig. 11 contain all singularities of class T in Y,

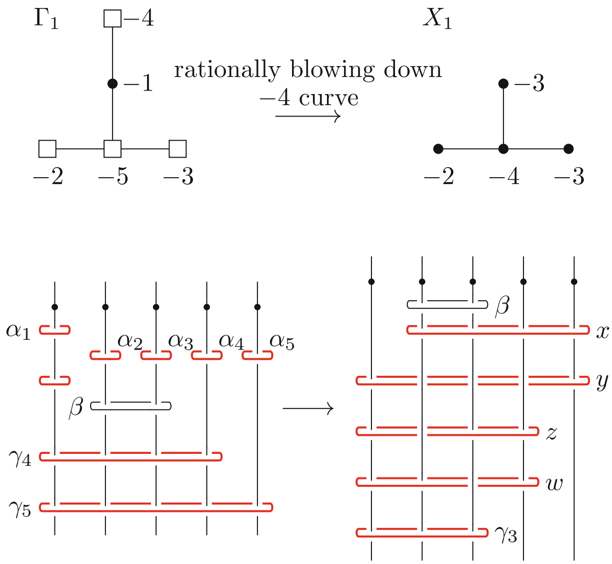


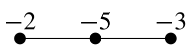
Fig. 12 A PALF on Γ_1

it suffices to find an explicit PALF on Y_i. Recall that the minimal symplectic filling of X_i corresponding to the P-resolution Y_i can be obtained as follows. First, we rationally blow down all singularities of class T lying in Y_i, except the one containing a central vertex. This yields a 4-manifold diffeomorphic to the minimal resolution of X_i that can be also obtained from X_{Γ_i} by blowing down all (-1)-spheres until there is no (-1)-sphere in the resulting plumbing graph. Since the blow-ups and blow-downs can be performed in symplectic category, the above argument implies that there is a convex plumbing of symplectic submanifolds of codimension 0 in the minimal resolution of X_i according to the dual graph of singularities of class T containing the central vertex of Γ_i. The desired minimal symplectic filling of X_i is obtained by rationally blowing down the convex plumbing. Hence, in order to obtain a PALF on Y_i, we only need to find a subword representing the convex plumbing from the monodormy factorization of the minimal resolution of X_i. This is always possible because we know explicitly how vanishing cycles (i.e., 2-handles) in the PALF on the minimal resolution of X_i correspond to vertices (i.e., embedded 2-spheres) in the minimal resolution graph of X_i.

***Example* 5.3** Figure 12 shows a PALF on the minimal resolution of X_1 whose monodromy factorization is given by

$$\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5\beta\gamma_4\gamma_5.$$

Note that the monodromy factorization without β above represents a 4-manifold diffeomorphic to a convex plumbing of a subgraph



lying in Γ_1. Hence, the right-hand side of Fig. 12 above yields a desired PALF $\beta xyzw\gamma_3$ on Y_1.

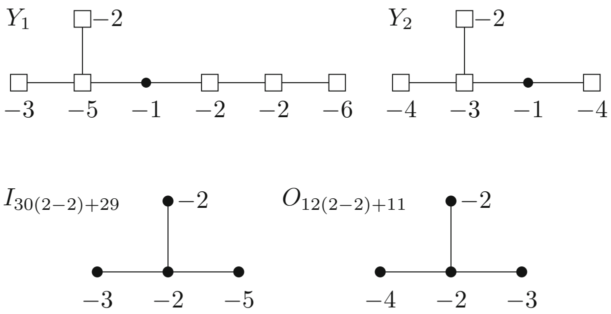


Fig. 13 Two genus-1 cases

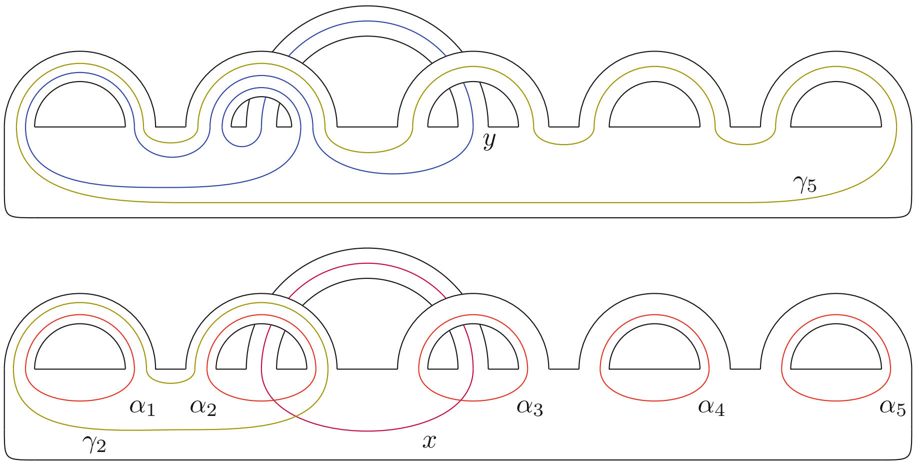


Fig. 14 A genus 1-PALF on the minimal resolution of $I_{30(2-2)+29}$

5.2.2 Genus-1 cases

There are two P-resolutions corresponding to genus 1 in Case 2, which come from an icosahedral singularity of type $I_{30(2-2)+29}$ and an octahedral singularity of type $O_{12(2-2)+11}$. See Fig. 13.

As shown in 5.2.1, we first rationally blow down all singularities of class T, except the one containing a central vertex such that the resulting 4-manifold is diffeomorphic to the minimal resolution. Since there is a bad vertex in the minimal resolution graph, we must consider a genus-1 PALF on this minimal resolution graph.

$I_{30(2-2)+29}$ *case*: Using the same technique yields a monodromy factorization of the following form for the minimal resolution of $I_{30(2-2)+29}$

$$xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5,$$

where α_i and γ_i are curves encircling the i^{th} hole and th first i holes, respectively (refer to Fig. 14). Note that, using Hurwitz moves and $y = t_{\alpha_2}(t_{\gamma_2}(x))$, we can change the monodromy factorization as follows:

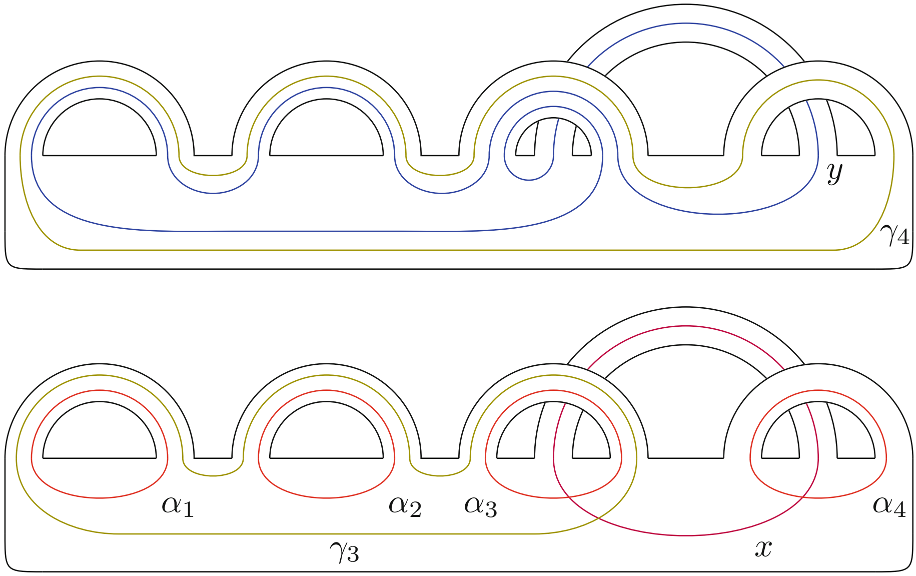


Fig. 15 A genus 1-PALF on the minimal resolution of $O_{12(2-2)+11}$

$$
\begin{aligned}
& xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5 \\
& \sim x\alpha_2\gamma_2 x\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
& \sim t_x(\alpha_2)t_x(\gamma_2)x^2\alpha_3\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
& \sim t_x(\alpha_2)t_x(\gamma_2)t_x^2(\alpha_3)x^2\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5 \\
& \sim t_x^2(\alpha_3)(t_x^2 \cdot t_{\alpha_3}^{-1} \cdot t_x^{-1})(\alpha_2)(t_x^2 \cdot t_{\alpha_3}^{-1} \cdot t_x^{-1})(\gamma_2)x^2\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5.
\end{aligned}
$$

Now, taking a global conjugation of each monodromy with $f = t_x \cdot t_{\alpha_3} \cdot t_x^{-2}$ and using a braid relation $t_x \cdot t_{\alpha_3} \cdot t_x = t_{\alpha_3} \cdot t_x \cdot t_{\alpha_3}$, we can show that the global monodromy factorization becomes

$$t_x(\alpha_3)\alpha_2\gamma_2\alpha_3^2\alpha_1\alpha_4\alpha_5 f(\gamma_2)\gamma_5.$$

Since the subword $\alpha_2\gamma_2\alpha_3^2\alpha_1\alpha_4\alpha_5\gamma_5$ in the monodromy factorization above corresponds to $\overset{-3}{\bullet}\!\!-\!\!\overset{-5}{\bullet}\!\!-\!\!\overset{-2}{\bullet}$ lying in the minimal resolution graph, we obtain a PALF on Y_1 by rationally blowing down it.

$O_{12(2-2)+11}$ *case*: Starting from a PALF on $\overset{-4}{\bullet}\!\!-\!\!\overset{-2}{\bullet}\!\!-\!\!\overset{-3}{\bullet}$, we obtain a monodromy factorization for the minimal resolution of $O_{12(2-2)+11}$ as follows (see Fig. 15 for vanishing cycles):

$$xy\gamma_3^2\alpha_1\alpha_2\alpha_3\alpha_4\gamma_4$$

A similar computation shows that above monodromy factorization is equivalent to

$$t_x(\alpha_4)\alpha_3\gamma_3\alpha_4^2\alpha_1\alpha_2 f(\gamma_3)\gamma_4,$$

where $f = t_x \cdot t_{\alpha_4} \cdot t_x^{-2}$. Now we can construct a PALF on Y_2 because the subword $\alpha_3\gamma_3\alpha_4^2\alpha_1\alpha_2\gamma_4$ corresponds to $\overset{-4}{\bullet}\!\!-\!\!\overset{-3}{\bullet}\!\!-\!\!\overset{-2}{\bullet}$.

Hence, summarizing all the arguments in this section, we conclude:

Theorem 5.1 *There is an explicit algorithm for a genus-*0 *or genus-*1 *PALF on any minimal symplectic filling of the link of non-cyclic quotient surface singularities.*

Recall that we divided all P-resolutions of X into two families in the construction of PALF on each P-resolution Y: those with and without a maximal subgraph Γ_L containing all singularities of class T in Y. The algorithm of PALF for the first family is essentially the same algorithm for cyclic cases, which means that the Milnor fiber corresponding to a P-resolution Y is obtained topologically via rational blowdowns from the minimal resolution of X [3]. On the other hand, we found a subword diffeomorphic to a convex neighborhood of a linear chain of 2-spheres in a smooth 4-manifold whose boundary is $L(p^2, pq-1)$ for the second family, which also can be rationally blowdown [7]. Hence we have:

Corollary 5.2 *Any Milnor fiber of the link of quotient surface singularities can be obtained, up to diffeomorphism, via a sequence of rational blowdowns from the minimal resolution of the singularity.*

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