



# De Rham theorem for Whitney functions

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Received: 27 October 2015 / Accepted: 16 January 2019 / Published online: 29 August 2019  
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## Abstract

Let  $M$  be a real analytic manifold,  $F$  a bounded complex of constructible sheaves. We show that the Whitney–de Rham complex associated to  $F$  is quasi-isomorphic to  $F$ .

## 1 Introduction

Let  $M$  be a real analytic manifold. The Poincaré Lemma implies that the de Rham complex with  $C^\infty$ -coefficients over  $M$  is isomorphic to  $\mathbb{C}_M$ . In this paper we show that  $\mathbb{C}_M$  is isomorphic to the de Rham complex with Whitney coefficients over  $M$  on the associated subanalytic site. Then, using sheaf theoretical arguments we can easily prove that, given a bounded complex of constructible sheaves  $F$ , the Whitney–de Rham complex associated to  $F$  is quasi-isomorphic to  $F$ . As a corollary we obtain a theorem of [3]. (Another proof was given in [4] using deep results on  $\mathcal{D}$ -modules.) We also obtain a de Rham theorem for Schwartz functions on open subanalytic subsets. As a further application, we extend the results of [5] to the case of subanalytic sets. Indeed our result implies that the Whitney–de Rham cohomology of a closed subanalytic set is isomorphic as a commutative graded algebra to its singular cohomology with real coefficients.

Our proof uses these three known facts:

- There exists a site, the so called subanalytic site, where Whitney functions form a sheaf.
- Locally on this site, the sections of this sheaf are nothing but  $C^\infty$ -functions with bounded derivatives.
- The homotopy axiom holds for de Rham cohomology with bounded derivatives.

Let us detail our arguments.

It is well known that Whitney  $C^\infty$ -functions on the real analytic manifold  $M$  do not satisfy the gluing conditions on open coverings. Hence, they do not form a sheaf with the usual topology. However, following [8], one can overcome this problem by associating to  $M$  a Grothendieck topology by choosing as open subsets subanalytic open subsets and as

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coverings locally finite ones. This is the subanalytic site  $M_{sa}$ , and here Whitney  $C^\infty$ -functions form a (subanalytic) sheaf denoted  $C_M^{\infty,w}$ .

On  $M_{sa}$ , one can also define the (subanalytic) sheaf of  $C^\infty$ -functions with bounded derivatives on bounded open subanalytic subsets, and prove that the associated de Rham complex is quasi-isomorphic to the constant sheaf on  $M$ .

$C^\infty$ -functions on  $U$  with bounded derivatives coincide with Whitney  $C^\infty$ -functions on  $\bar{U}$  when  $U$  is a 1-regular open subset of  $\mathbb{R}^n$ , i.e. there exists a constant  $C > 0$  such that any two points  $x$  and  $x'$  can be joined by a path of length  $L \leq C|x - x'|$  (see [15]). Thanks to a decomposition result of [11] (which uses techniques developed in [10]) one can prove that locally on  $M_{sa}$  the sheaf of bounded  $C^\infty$ -functions coincide with  $C_M^{\infty,w}$ .

Moreover, applying some (subanalytic) sheaf theoretical techniques one can link the de Rham complex with coefficients in  $C_M^{\infty,w}$  with the one with Whitney coefficients on any closed subanalytic subset  $Z$  of  $M$ . This is because, basically,  $\text{RHom}(D'\mathbb{C}_Z, C_M^{\infty,w})$  is quasi-isomorphic to the space of Whitney  $C^\infty$ -functions on  $Z$ . Here  $\mathbb{C}_Z$  denotes the constant sheaf on  $Z$  and  $D'\mathbb{C}_Z$  its dual  $\text{RHom}(\mathbb{C}_Z, \mathbb{C}_M)$ . This is done thanks to the Whitney tensor product introduced by Kashiwara-Schapira in [7]. When  $Z = \bar{U}$ , with  $U$  open 1-regular relatively compact and such that  $D'\mathbb{C}_U \simeq \mathbb{C}_Z$ , we obtain that  $\text{RHom}(D'\mathbb{C}_Z, C_M^{\infty,w})$  is the space of  $C^\infty$ -functions on  $U$  with bounded derivatives.

Thanks to these considerations, one can reduce the isomorphism between the cohomology of  $Z$  and the Whitney de Rham complex to the quasi-isomorphism between  $\mathbb{C}_M$  and the de Rham complex with coefficients in  $C_M^{\infty,w}$ , which is locally the homotopy axiom for de Rham cohomology with bounded derivatives and can be obtained as in [2].

The statement still holds if we replace  $\mathbb{C}_Z$  with a bounded complex of constructible sheaves  $F$  and Whitney  $C^\infty$ -functions with the Whitney tensor product  $F \otimes^w C_M^{\infty,w}$ . This allows to consider different kinds of de Rham theorems, as the one for Schwartz functions when  $F = \mathbb{C}_U$ ,  $U$  open subanalytic.

As a further application, one can extend the results of [5] to the case of subanalytic sets. Indeed one can deduce that the Whitney–de Rham cohomology of a closed subanalytic set  $Z$  is isomorphic as a commutative graded algebra (CGA for short) to its singular cohomology with real coefficients and these isomorphisms are compatible with the CGA structures.

## 2 Subanalytic sheaves

The following results on subanalytic sheaves are extracted from [8] (see also [12]). We refer to [6] for classical sheaf theory.

Let  $M$  be a real analytic manifold. Denote by  $\text{Op}(M_{sa})$  the category of open subanalytic subsets of  $M$ . One endows  $\text{Op}(M_{sa})$  with the following topology:  $S \subset \text{Op}(M_{sa})$  is a covering of  $U \in \text{Op}(M_{sa})$  if  $\bigcup_{V \in S} V = U$  and for any compact  $K$  of  $M$  there exists a finite subset  $S_0 \subset S$  such that  $K \cap \bigcup_{V \in S_0} V = K \cap U$ . We will call  $M_{sa}$  the subanalytic site.

Let  $\text{Mod}(\mathbb{C}_{M_{sa}})$  (resp.  $D^b(\mathbb{C}_{M_{sa}})$ ) denote the category (resp. bounded derived category) of sheaves of  $\mathbb{C}$ -vector spaces on  $M_{sa}$  and let  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  (resp.  $D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ ) be the abelian category of  $\mathbb{R}$ -constructible sheaves (resp. bounded derived category of sheaves with  $\mathbb{R}$ -constructible cohomology) on  $M$ .

We denote by  $\rho : M \rightarrow M_{sa}$  the natural morphism of sites. We have functors

$$\begin{aligned} \rho_* &: \text{Mod}(\mathbb{C}_M) \rightarrow \text{Mod}(\mathbb{C}_{M_{sa}}) \\ \rho^{-1} &: \text{Mod}(\mathbb{C}_{M_{sa}}) \rightarrow \text{Mod}(\mathbb{C}_M) \end{aligned}$$

$$\rho_! : \text{Mod}(\mathbb{C}_M) \rightarrow \text{Mod}(\mathbb{C}_{M_{sa}})$$

The functors  $\rho^{-1}$  and  $\rho_*$  are the functors of inverse image and direct image associated to  $\rho$ . The functor  $\rho_!$  is left adjoint to  $\rho^{-1}$ . It is fully faithful and exact. If  $F \in \text{Mod}(\mathbb{C}_{M_{sa}})$ ,  $\rho_! F$  is the sheaf associated to the presheaf  $U \mapsto \Gamma(\bar{U}; F)$ . The functor  $\rho_*$  is fully faithful and exact on  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  and we identify  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  with its image in  $\text{Mod}(\mathbb{C}_{M_{sa}})$  by  $\rho_*$ .

### 3 Whitney tensor product

Let  $M$  be a real analytic manifold. We denote by  $\mathcal{C}_M^\infty$  the sheaf of complex valued  $C^\infty$ -functions on  $M$  and by  $\mathcal{D}_M$  the sheaf of differential operators on  $M$  with coefficients in complex valued real analytic functions. We denote by  $\text{Mod}(\mathcal{D}_M)$  (resp.  $D^b(\mathcal{D}_M)$ ) the category (resp. bounded derived category) of sheaves of  $\mathcal{D}_M$ -modules. References for Whitney functions are made to [9] and to [7] for a complete exposition on formal cohomology.

**Definition 3.1** Let  $Z$  be a closed subset of  $M$ . We denote by  $\mathcal{I}_{M,Z}^\infty$  the sheaf of complex valued  $C^\infty$ -functions on  $M$  vanishing up to infinite order on  $Z$ .

**Definition 3.2** A Whitney function on a closed subset  $Z$  of  $M$  is an indexed family  $F = (F^k)_{k \in \mathbb{N}^n}$  consisting of complex valued continuous functions on  $Z$  such that  $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^n, |k| \leq m, \forall x \in Z, \forall \varepsilon > 0$  there exists a neighborhood  $U$  of  $x$  such that  $\forall y, z \in U \cap Z$

$$\left| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon d(y, z)^{m-|k|},$$

where  $d$  is the Euclidean distance in a local chart. We denote by  $W_{M,Z}^\infty$  the space of Whitney  $C^\infty$ -functions on  $Z$ . We denote by  $\mathcal{W}_{M,Z}^\infty$  the sheaf  $U \mapsto W_{U \cap Z}^\infty$ .

Note that, once we fix the closed set  $Z, \mathcal{W}_{M,Z}^\infty$  is a sheaf on  $M$  with its usual topology, since continuous functions form a sheaf on  $Z$ . A result of [14] (see [9, Theorem 4.1]) asserts that, given a Whitney function  $F$  on  $Z$ , then there exist a neighborhood  $U$  of  $Z$  and a  $C^\infty$ -function  $f$  on  $U$  such that  $D^k f|_Z = F^k$ . Thanks to this result one can obtain an exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{M,Z}^\infty \rightarrow \mathcal{C}_M^\infty \rightarrow \mathcal{W}_{M,Z}^\infty \rightarrow 0.$$

In [7] the authors defined the functor

$$\cdot \overset{w}{\otimes} \mathcal{C}_M^\infty : \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \rightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way: let  $U$  be a subanalytic open subset of  $M$  and  $Z = M \setminus U$ . Then  $\mathbb{C}_U \overset{w}{\otimes} \mathcal{C}_M^\infty = \mathcal{I}_{M,Z}^\infty$ . As a consequence of [9, Theorem 4.1], one has  $\mathbb{C}_Z \overset{w}{\otimes} \mathcal{C}_M^\infty = \mathcal{W}_{M,Z}^\infty$ . The functor  $\cdot \overset{w}{\otimes} \mathcal{C}_M^\infty$  is exact and extends as a functor in the derived category, from  $D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$  to  $D^b(\mathcal{D}_M)$ . Moreover, the sheaf  $F \overset{w}{\otimes} \mathcal{C}_M^\infty$  is soft for any  $\mathbb{R}$ -constructible sheaf  $F$ .

### 4 The sheaf of Whitney functions

Here, we recall the definition and some properties of the subanalytic sheaf of Whitney functions. References are made to [8,12].

**Definition 4.1** One denotes by  $\mathcal{C}_M^{\infty,w}$  the sheaf on  $M_{sa}$  of Whitney  $C^\infty$ -functions defined as follows:

$$U \mapsto \Gamma(M; H^0 D'(\mathbb{C}_U) \overset{w}{\otimes} \mathcal{C}_M^\infty),$$

where  $D'(\cdot) = R\mathcal{H}om(\cdot, \mathbb{C}_M)$ .

Remark that  $\Gamma(U; \mathcal{C}_M^{\infty,w})$  is a  $\Gamma(\bar{U}; \mathcal{D}_M)$ -module for each  $U \in \text{Op}(M_{sa})$ , this implies that  $\mathcal{C}_M^{\infty,w}$  has a structure of  $\rho! \mathcal{D}_M$ -module. We have the following result. For each  $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$  one has the isomorphism

$$\rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty,w}) \simeq F \overset{w}{\otimes} \mathcal{C}_M^\infty.$$

Let us consider a locally cohomologically trivial (l.c.t.) subanalytic open subset, i.e.  $U \in \text{Op}(M_{sa})$  satisfying  $D'\mathbb{C}_U \simeq \mathbb{C}_{\bar{U}}$  so that  $D'\mathbb{C}_{\bar{U}} \simeq \mathbb{C}_U$ . Then

$$R\Gamma(U; \mathcal{C}_M^{\infty,w}) \simeq R\Gamma(M; \mathbb{C}_{\bar{U}} \overset{w}{\otimes} \mathcal{C}_M^\infty)$$

is concentrated in degree zero. It means that  $\mathcal{C}_M^{\infty,w}$  is the sheaf associated to the presheaf  $U \mapsto \{\text{Whitney functions on } \bar{U}\}$ . Indeed l.c.t. open subanalytic subsets form a basis for the topology of  $M_{sa}$  (i.e. every relatively compact subanalytic open subset has a finite cover consisting of l.c.t. subanalytic open subsets).

### 5 Whitney–de Rham complexes

Let  $\mathcal{A}_M$  be the sheaf of complex valued real analytic functions on  $M$ . We denote by  $\Omega_M^p$  the sheaf of  $p$ -differential forms with coefficients in  $\mathcal{A}_M$ . Let  $F \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ . We introduce the following two complexes: the Whitney–de Rham complex  $\mathcal{C}_M^{\infty,w,\bullet}$  with values in  $\text{Mod}(\mathbb{C}_{M_{sa}})$

$$\mathcal{C}_M^{\infty,w,\bullet} := 0 \rightarrow \mathcal{C}_M^{\infty,w} \rightarrow \mathcal{C}_M^{\infty,w,1} \rightarrow \dots \rightarrow \mathcal{C}_M^{\infty,w,\dim M} \rightarrow 0$$

and the Whitney–de Rham complex  $F \overset{w}{\otimes} \mathcal{C}_M^{\infty,\bullet}$  associated to the constructible sheaf  $F$  with values in  $\text{Mod}(\mathbb{C}_M)$

$$F \overset{w}{\otimes} \mathcal{C}_M^{\infty,\bullet} := 0 \rightarrow F \overset{w}{\otimes} \mathcal{C}_M^\infty \rightarrow F \overset{w}{\otimes} \mathcal{C}_M^{\infty,1} \rightarrow \dots \rightarrow F \overset{w}{\otimes} \mathcal{C}_M^{\infty,\dim M} \rightarrow 0.$$

Here  $\mathcal{C}_M^{\infty,w,p} = \mathcal{C}_M^{\infty,w} \overset{\rho!}{\otimes}_{\rho! \mathcal{A}_M} \Omega_M^p$ ,  $F \overset{w}{\otimes} \mathcal{C}_M^{\infty,p} = F \overset{w}{\otimes}_{\mathcal{A}_M} \mathcal{C}_M^\infty \otimes \Omega_M^p$  and the arrows are induced by the differentials  $d : \Omega_M^p \rightarrow \Omega_M^{p+1}$ .

Consider the following diagram

$$\begin{CD} C^b(\mathbb{C}_{M_{sa}}) @>Q>> D^b(\mathbb{C}_{M_{sa}}) \\ @VV\rho^{-1}V @VV\rho^{-1}V \\ C^b(\mathbb{C}_M) @>Q>> D^b(\mathbb{C}_M), \end{CD}$$

where  $C^b(\mathbb{C}_M)$  (resp.  $C^b(\mathbb{C}_{M_{sa}})$ ) denotes the category of bounded complexes of sheaves on  $M$  (resp.  $M_{sa}$ ) and  $Q$  is the localization functor. One has

$$Q(\mathcal{C}_M^{\infty,w,\bullet}) \simeq R\mathcal{H}om_{\rho! \mathcal{D}_M}(\rho! \mathcal{A}_M, \mathcal{C}_M^{\infty,w}),$$

$$Q(F \otimes^w C_M^{\infty, \bullet}) \simeq R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{A}_M, F \otimes^w C_M^{\infty}).$$

### 6 Statement and proof of the main theorem

In this section we study the cohomology of the Whitney–de Rham complex. In order to do this we need an homotopy axiom, which can be obtained in the same way as the classical one (see [2, Corollary 4.1.2]).

Let  $M$  be a real analytic manifold and let us consider a bounded interval  $(a, b) \subset \mathbb{R}$  containing  $[0, 1]$ . Let  $c \in (a, b)$  and let  $s_c : M \rightarrow M \times (a, b)$  be the section  $s_c(x) = (x, c)$ . It defines a map

$$s_c^* : \Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, \bullet}) \rightarrow \Gamma_b(M; C_M^{\infty, \bullet}),$$

where  $\Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, p})$  and  $\Gamma_b(M; C_M^{\infty, p})$  are spaces of  $p$ -forms whose coefficients are bounded with bounded derivatives.

**Lemma 6.1** *Let  $c, c' \in (a, b)$ . The maps  $s_c^*$  and  $s_{c'}^*$  are homotopic ( $s_c^* \sim s_{c'}^*$  for short) and induce the same map in cohomology.*

**Proof** Let  $\pi : M \times (a, b) \rightarrow M$  be the projection  $\pi(x, t) = x$ . The maps  $s_c^*$  and  $\pi^*$  are inverse to each other in cohomology. We have trivially  $s_c^* \circ \pi^* = \text{id}$  being  $\pi \circ s_c = \text{id}$ . In order to prove that  $\pi^* \circ s_c^* \sim \text{id}$  one constructs a homotopy operator  $K$  on  $\Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, \bullet})$ .

We recall the construction of  $K$ . A form in  $M \times (a, b)$  is a combination of forms of this kind: (I)  $(\pi^* \phi) f(x, t)$  and (II)  $(\pi^* \phi) f(x, t) dt$  where  $\phi$  is a form on  $M$ . The map  $K : \Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, q}) \rightarrow \Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, q-1})$  is defined as follows:

$$(I) (\pi^* \phi) f(x, t) \mapsto 0 \quad \text{and} \quad (II) (\pi^* \phi) f(x, t) dt \mapsto (\pi^* \phi) \int_c^t f(x, t) dt.$$

One checks that  $1 - \pi^* \circ s_c^* = (-1)^{q-1} (dK - Kd)$  on  $\Gamma_b(M \times (a, b); C_{M \times (a,b)}^{\infty, q})$ . This implies that  $\pi^* \circ s_c^* \sim \text{id}$ . We refer to [2] for the proof.

Composing with  $s_{c'}^*$  we have  $s_c^* = s_{c'}^* \circ \pi^* \circ s_c^* \sim s_{c'}^*$ . □

**Lemma 6.2** *There is the following quasi-isomorphism*

$$C_M \xrightarrow{\sim} C_M^{\infty, w, \bullet}.$$

*In particular  $C_M \xrightarrow{\sim} R\mathcal{H}om_{\rho! \mathcal{D}_M}(\rho! \mathcal{A}_M, C_M^{\infty, w})$  in  $D^b(C_{M_{sa}})$ .*

**Proof** The natural arrow  $C_M \rightarrow C_M^{\infty, w}$  induces a morphism of complexes

$$C_M \rightarrow C_M^{\infty, w, \bullet}.$$

We have to show that the above morphism induces an isomorphism in cohomology. It follows from [11, Theorem 0.3] that every relatively compact subanalytic open subset has a finite cover consisting of contractible 1-regular open subanalytic subsets. Hence, they form a basis for the Grothendieck topology of  $M_{sa}$ .

A result of [15] asserts that on 1-regular open subsets Whitney functions are bounded  $C^\infty$ -functions with bounded derivatives. Then, it is enough to prove that, for any  $U$  contractible 1-regular open subanalytic subset of  $M$ ,

$$H^k(U; C_M^{\infty, w, \bullet}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is nothing but the homotopy axiom for de Rham cohomology with bounded derivatives. Let  $\text{id}_U, p_U : U \rightarrow U$  be respectively the identity of  $U$  and the projection to a point of  $U$ . Let  $h$  be a  $C^\infty$ -homotopy between  $\text{id}_U$  and  $p_U$ . We have

$$\text{id}_U^* = (h \circ s_0)^* = s_0^* \circ h^* \sim s_1^* \circ h^* = (h \circ s_1)^* = p_U^*,$$

where the homotopy  $s_0^* \sim s_1^*$  follows from Lemma 6.1. □

**Theorem 6.3** *Let  $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$ . There is the following isomorphism in  $D^b(\mathbb{C}_M)$*

$$F \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{A}_M, F \overset{w}{\otimes} C_M^\infty).$$

**Proof** Let us apply the functor  $\rho^{-1}R\mathcal{H}om(D'F, \cdot)$  to

$$\mathbb{C}_M \xrightarrow{\sim} R\mathcal{H}om_{\rho^*\mathcal{D}_M}(\rho^*\mathcal{A}_M, C_M^{\infty,w}).$$

We have the chain of isomorphisms in  $D^b(\mathbb{C}_M)$

$$\begin{aligned} F &\simeq R\mathcal{H}om(D'F, \mathbb{C}_M) \\ &\xrightarrow{\sim} \rho^{-1}R\mathcal{H}om(D'F, R\mathcal{H}om_{\rho^*\mathcal{D}_M}(\rho^*\mathcal{A}_M, C_M^{\infty,w})) \\ &\simeq \rho^{-1}R\mathcal{H}om_{\rho^*\mathcal{D}_M}(\rho^*\mathcal{A}_M, R\mathcal{H}om(D'F, C_M^{\infty,w})) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{A}_M, \rho^{-1}R\mathcal{H}om(D'F, C_M^{\infty,w})) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{A}_M, F \overset{w}{\otimes} C_M^\infty), \end{aligned}$$

where the first isomorphism follows from the fact that  $D'D'F \simeq F$  if  $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$  and the second one from Lemma 6.2. □

**Corollary 6.4** (i) *When  $F \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ , there is the following isomorphism in  $D^b(\mathbb{C}_M)$*

$$F \xrightarrow{\sim} F \overset{w}{\otimes} C_M^{\infty,\bullet}.$$

(ii) *In particular, let  $Z$  be a closed subanalytic subset of  $M$ . There is the following isomorphism in  $D^b(\mathbb{C}_M)$*

$$\mathbb{C}_Z \xrightarrow{\sim} \mathcal{W}_{M,Z}^{\infty,\bullet}$$

where  $\mathcal{W}_{M,Z}^{\infty,\bullet}$  denotes the de Rham complex with coefficients in  $\mathcal{W}_{M,Z}^\infty$ , i.e. the Whitney–de Rham complex is quasi-isomorphic to  $\mathbb{C}_Z$ .

**Proof** (i) When  $F \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  one has  $R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{A}_M, F \overset{w}{\otimes} C_M^\infty) \simeq F \overset{w}{\otimes} C_M^{\infty,\bullet}$ . Then the result follows from Theorem 6.3.

(ii) Since  $\mathcal{W}_{M,Z}^\infty \simeq \mathbb{C}_Z \overset{w}{\otimes} C_M^\infty$ , we have  $\mathcal{W}_{M,Z}^{\infty,\bullet} \simeq \mathbb{C}_Z \overset{w}{\otimes} C_M^{\infty,\bullet}$ . Then the result follows from (i) with  $F = \mathbb{C}_Z$ . □

**Corollary 6.5** *Let  $Z$  be a closed subanalytic subset of  $M$ . For each  $k \in \mathbb{Z}$  there is the following isomorphism*

$$H^k(Z; \mathbb{C}_Z) \xrightarrow{\sim} H^k\Gamma(M; \mathcal{W}_{M,Z}^{\infty,\bullet}).$$

Hence, the cohomology of  $Z$  is isomorphic to the cohomology of the de Rham complex with Whitney coefficients on  $Z$ .

**Proof** Applying  $R\Gamma(M; \cdot)$  to the morphism of Corollary 6.4 (ii), we have the following chain of isomorphisms in  $D^b(\mathbb{C})$

$$R\Gamma(Z; \mathbb{C}_Z) \simeq R\Gamma(M; \mathbb{C}_Z) \simeq R\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet}),$$

where the first isomorphism follows from the definition of direct image and the second one follows applying  $R\Gamma(M; \cdot)$  to Corollary 6.4 (ii). The complex  $\mathcal{W}_{M,Z}^{\infty, \bullet}$  is a complex of soft sheaves, which are  $\Gamma(M; \cdot)$ -acyclic. Hence,

$$R\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet}) \simeq \Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet})$$

and the result follows. □

**Corollary 6.6** *Let  $U$  be an open subanalytic subset of  $M$ . For each  $k \in \mathbb{Z}$  there is the following isomorphism*

$$H_c^k(U; \mathbb{C}_U) \simeq H^k \Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet}),$$

where  $\mathcal{I}_{M, M \setminus U}^{\infty, \bullet}$  denotes the de Rham complex with coefficients in  $\mathcal{I}_{M, M \setminus U}^{\infty}$ . Hence, the cohomology with compact support of  $U$  is isomorphic to the cohomology with compact support of the de Rham complex with  $C^\infty$ -coefficients on  $M$  vanishing with all their derivatives outside  $U$ .

**Proof** Applying  $R\Gamma_c(M; \cdot)$  to the morphism of Corollary 6.4 (i) (with  $F = \mathbb{C}_U$ ), we have the following chain of isomorphisms in  $D^b(\mathbb{C})$

$$R\Gamma_c(U; \mathbb{C}_U) \simeq R\Gamma_c(M; \mathbb{C}_U) \simeq R\Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet}),$$

where the first isomorphism follows from the definition of proper direct image and the second one follows applying  $R\Gamma_c(M; \cdot)$  to Corollary 6.4 (i). The complex  $\mathcal{I}_{M, M \setminus U}^{\infty, \bullet}$  is a complex of soft sheaves, which are  $\Gamma_c(M; \cdot)$ -acyclic, hence

$$R\Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet}) \simeq \Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet})$$

and the result follows. □

**Remark 6.7** As a consequence of Corollary 6.6, we obtain a result of [1]: the cohomology of the de Rham complex with Schwartz coefficients  $H^\bullet DR_S(M)$  of a Nash manifold  $M$  is isomorphic to the compactly supported cohomology of  $M$ . We first prove it for  $\mathbb{R}^n$ . We can see  $\mathbb{R}^n$  as an open subset of  $\mathbb{S}^n$  (the unit  $n$ -sphere). Let  $U$  be an open semialgebraic subset of  $\mathbb{R}^n$ . Then, for each  $k \in \mathbb{Z}$

$$H_c^k(U; \mathbb{C}_U) \simeq H^k \Gamma(\mathbb{S}^n; \mathcal{I}_{\mathbb{S}^n, \mathbb{S}^n \setminus U}^{\infty, \bullet}) \simeq H^k DR_S(U).$$

Using the fact that a Nash manifold has a finite cover consisting of open submanifolds Nash diffeomorphic to  $\mathbb{R}^n$  one obtains the result.

### 7 A CDGA quasi-isomorphism

The aim of this section is to extend to subanalytic subsets a result proved in [5] for semi-analytic subsets using different techniques.

**Theorem 7.1** *Let  $Z$  be a locally closed subanalytic subset of  $M$ . There is a canonical isomorphism between the Whitney–de Rham cohomology of  $Z$  and the singular cohomology of  $Z$  with complex coefficients, as graded algebras.*

**Proof** Let  $Z$  be a closed subanalytic subset of  $M$  and consider the Whitney–de Rham complex of sheaves

$$\mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}.$$

Let us consider a resolution of  $Z$

$$0 \rightarrow \oplus \mathbb{C}_{U_1} \rightarrow \dots \rightarrow \oplus \mathbb{C}_{U_k} \rightarrow \mathbb{C}_Z \rightarrow 0$$

made by subanalytic open subsets (we may also assume they are contractible and locally cohomologically trivial, see [13]), then  $\mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}$  is defined by a sequence of chain complexes whose entries are of this kind

$$\mathbb{C}_U \otimes^w \mathcal{C}_M^{\infty, \bullet},$$

with  $U$  locally cohomologically trivial. Now, by definition of  $\mathcal{C}_M^{\infty, w}$ , we have  $\mathbb{C}_U \otimes^w \mathcal{C}_M^{\infty} \simeq \rho^{-1} \mathcal{H}om(\mathbb{C}_{\bar{U}}, \mathcal{C}_M^{\infty, w})$  when  $U$  is locally cohomologically trivial. It follows from Lemma 6.2 that the chain complex of (subanalytic) sheaves

$$\mathbb{C}_M \rightarrow \mathcal{C}_M^{\infty, w, \bullet}$$

is exact. Moreover,  $\mathbb{C}_M$  and  $\mathcal{C}_M^{\infty, w}$  are acyclic with respect to  $\mathcal{H}om(\mathbb{C}_{\bar{U}}, \cdot)$  ( $U$  is locally cohomologically trivial) and  $\rho^{-1}$  is exact. Hence, the chain complex

$$\mathbb{C}_U \simeq \mathcal{H}om(\mathbb{C}_{\bar{U}}, \mathbb{C}_M) \rightarrow \rho^{-1} \mathcal{H}om(\mathbb{C}_{\bar{U}}, \mathcal{C}_M^{\infty, w, \bullet}) \simeq \mathbb{C}_U \otimes^w \mathcal{C}_M^{\infty, \bullet}$$

is exact. This implies that the chain complex

$$\mathbb{C}_Z \rightarrow \mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}$$

is exact as well. Let us consider the exact sequence of chain complexes

$$0 \rightarrow \mathbb{C}_{M \setminus Z} \otimes^w \mathcal{C}_M^{\infty, \bullet} \rightarrow \mathcal{C}_M^{\infty, \bullet} \rightarrow \mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet} \rightarrow 0$$

and apply the functor  $\Gamma(V; \cdot)$  ( $V \subseteq M$  open). The Whitney tensor product with a constructible sheaf being soft, we get an exact sequence of chain complexes

$$0 \rightarrow \Gamma(V; \mathbb{C}_{M \setminus Z} \otimes^w \mathcal{C}_M^{\infty, \bullet}) \rightarrow \Gamma(V; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \Gamma(V; \mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}) \rightarrow 0.$$

Remark that  $J$  is a CDGA (commutative differential graded algebra) morphism. By Corollary 6.4 we have

$$\begin{aligned} \Gamma(V; \mathbb{C}_{M \setminus Z} \otimes^w \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(V; \mathbb{C}_{M \setminus Z}), \\ \Gamma(V; \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(V; \mathbb{C}_M), \\ \Gamma(V; \mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(V; \mathbb{C}_Z). \end{aligned}$$

Suppose that  $Z$  is a deformation retract of  $V$ . Then

$$\begin{aligned} R\Gamma(V; \mathbb{C}_{M \setminus Z}) &\simeq 0, \\ R\Gamma(V; \mathbb{C}_M) &\simeq R\Gamma(Z; \mathbb{C}_Z), \\ R\Gamma(V; \mathbb{C}_Z) &\simeq R\Gamma(Z; \mathbb{C}_Z). \end{aligned}$$

Hence, for such a  $V$

$$\Gamma(V; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \Gamma(V; \mathbb{C}_Z \otimes^w \mathcal{C}_M^{\infty, \bullet}).$$



is a quasi-isomorphism of CDGA. Remark that, as a consequence of the triangulation theorem, a locally closed subanalytic subset  $Z$  is a deformation retract of an open subanalytic subset  $V$ . Furthermore,  $V$  is a smooth manifold, then its de Rham cohomology and singular cohomology are isomorphic.  $V$  is homotopy equivalent to  $Z$ , this implies that the singular cohomology of  $V$  and the one of  $Z$  are isomorphic.  $\square$

**Remark 7.2** Since every neighborhood of a subanalytic locally closed subset  $Z$  of  $M$  contains a neighborhood  $V$  which has a deformation retract to  $Z$ , we get a quasi-isomorphism of CDGA

$$\Gamma(M; \mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet}) \simeq \varinjlim_V \Gamma(V; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \varinjlim_V \Gamma(V; \mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet}) \simeq \Gamma(M; \mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet})$$

where  $V$  ranges through the family of neighborhoods having a deformation retract to  $Z$ . Thanks to this fact one can easily prove the quasi-isomorphism of sheaves of CDGA

$$\mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet} \xrightarrow{\sim} \mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet}.$$

**Remark 7.3** As a further application, one can extend the results of [5] (as pointed out by the authors in the introduction) to the case of subanalytic sets. One first remarks that all the previous results remain valid if we consider real valued functions and sheaves of real vector spaces. Hence, the singular cohomology of  $Z$  with real coefficients  $H^\bullet(Z, \mathbb{R})$  is isomorphic as a CGA to its Whitney–de Rham cohomology, i.e. for each  $k \in \mathbb{Z}$

$$H^k(Z, \mathbb{R}) \simeq H^k \Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet}),$$

and these isomorphisms are compatible with the CGA structures. Then, when the set  $Z$  is simply connected, one can prove that the de Rham complex with Whitney coefficients  $\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet})$  determines the real homotopy type of  $Z$ . This also implies that the Hochschild homology of the differential graded algebra  $\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet})$  is isomorphic to the cohomology of the free loop space  $\mathcal{L}Z$ .

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