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Strongly semistable sheaves and the Mordell–Lang conjecture over function fields

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Abstract
We give a new proof of the Mordell–Lang conjecture in positive characteristic, in the situation where the variety under scrutiny is a smooth subvariety of an abelian variety. Our proof is based on the theory of semistable sheaves in positive characteristic, in particular on Langer's theorem that the Harder–Narasimhan filtration of sheaves becomes strongly semistable after a finite number of iterations of Frobenius pull-backs. The interest of this proof is that it provides simple effective bounds (depending on the degree of the canonical line bundle) for the degree of the isotrivial finite cover whose existence is predicted by the Mordell–Lang conjecture. We also present a conjecture on the Harder–Narasimhan filtration of the cotangent bundle of a smooth projective variety of general type in positive characteristic and a conjectural refinement of the Bombieri–Lang conjecture in positive characteristic.

1 Introduction

Let B be an abelian variety over an algebraically closed field F of characteristic $p > 0$. Let Y be an integral closed subscheme of B. Let $\Lambda \subseteq B(F)$ be a subgroup. Suppose that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module (here, as is customary, we write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime p). Suppose that $\mathrm{Stab}(Y) = 0$. Here $\mathrm{Stab}(Y) = \mathrm{Stab}_B(Y)$ is the translation stabilizer of Y. This is the closed subgroup scheme of B, which is characterized uniquely by the fact that for any scheme S and any morphism $b\colon S \to B$, translation by b on the product $B \times_F S$ maps the subscheme $Y \times S$ to itself if and only if b factors through $\mathrm{Stab}_B(Y)$. Its existence is proven in [3, exp. VIII, Ex. 6.5 (e)].

The Mordell–Lang conjecture for Y and B is the following statement.

Theorem 1.1 (Mordell–Lang conjecture for abelian varieties; Hrushovski [5]) *Suppose that $Y \cap \Lambda$ is Zariski dense in Y. Then there is a projective variety Y' over a finite subfield $\mathbb{F}_{p^r} \subseteq F$ and a finite and surjective morphism $h\colon Y'_F \to Y$.*

Theorem 1.1 was first proven by Hrushovski in [5] using model-theoretic methods and other proofs were given in [2,10,15].

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The main results of the present article are the following results, which we shall prove simultaneously.

Theorem 1.2 *Suppose that $Y \cap \Lambda$ is Zariski dense in Y. Suppose that Y is smooth over F. If*

$$p > \dim(Y)^2 \int_Y \mathrm{c}^1(\Omega_Y)^{\dim(Y)}$$

then there is a projective variety Y' defined over a finite subfield $\mathbb{F}_{p^r} \subseteq F$ and an isomorphism $h\colon Y'_F \simeq Y$.

Remark 1.3 A weaker (but more cumbersome) inequality than $p > \dim(Y)^2 \int_Y \mathrm{c}^1(\Omega_Y)^{\dim(Y)}$, which also implies the conclusion of Theorem 1.2 is given in (11) below.

Remark 1.4 If B is an ordinary abelian variety then h can be taken to be an isomorphism in Theorem 1.1 (without any further assumptions on p or Y). This can be deduced from Theorem 1.1 and from the properties of the Albanese variety of Y and of the $F|\bar{\mathbb{F}}_p$-trace of B. So Theorem 1.2 is only interesting if B is not ordinary.

Theorem 1.5 *Suppose that $Y \cap \Lambda$ is Zariski dense in Y. Suppose that Y is smooth over F. If Ω_Y is strongly semistable with respect to* $\det(\Omega_Y)$ *then there is a projective variety Y' defined over a finite subfield $\mathbb{F}_{p^r} \subseteq F$ and an isomorphism $h\colon Y'_F \simeq Y$.*

Note that $\det(\Omega_Y)$ is ample by [1, Lemma 6]. A vector bundle V on Y is strongly semistable if $F_Y^{*,or} V$ is semistable for all $r \geq 0$. Here F_Y is the absolute Frobenius endomorphism of Y. See Sect. 3 below for more details.

In other words, if Y is smooth and either the inequality $p > \dim(Y)^2 \int_Y \mathrm{c}^1(\Omega_Y)^{\dim(Y)}$ holds or the vector bundle Ω_Y is strongly semistable with respect to Ω_Y, then h may be taken to be an isomorphism in Theorem 1.1. In particular, the morphism h may be taken to be an isomorphism in Theorem 1.1 if $\dim(Y) = 1$. This is an old result of Samuel (see [11]). Note that the conclusion of Theorem 1.2 (resp. Theorem 1.5) is stronger than the conclusion of Theorem 1.1. Indeed, the geometry of the finite morphism h appearing in Theorem 1.1 is difficult to understand and Theorem 1.1 says little about the structure of h.

Remark 1.6 If S is a smooth closed subvariety of an abelian variety over a field of characteristic 0 such that $\mathrm{Stab}(S) = 0$ then Ω_S is semistable with respect to $\det(\Omega_S)$ by a classical result of Yau. Thus Theorem 1.5 may be viewed as an exact positive characteristic counterpart (when Y is smooth) of the Mordell–Lang conjecture in characteristic 0, because in the latter the analog of the morphism h can always be taken to be an isomorphism.

Remark 1.7 Theorem 1.2 has the following interesting consequence. Suppose that we are given a smooth projective subvariety with trivial stabiliser inside an abelian variety over the function field of a variety defined over a number field L. If this situation is reduced modulo a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, then for all but a finite number of such ideals, the Mordell–Lang conjecture 1.1 holds for the reduced situation and h can be taken to be an isomorphism (for any choice of Λ).

Our proof of Theorems 1.2 and 1.5 does not rely on existing proofs of Theorem 1.1. In particular, we provide in this text a complete proof of Theorem 1.1 in the situation where Y is smooth. Our method of proof does not rely on the differential techniques of [5] or on the Galois-theoretic techniques of [2,10,15]. It is purely geometric and uses the theory of semistable sheaves on varieties of dimension > 1, in particular on Langer's theorem that the Harder–Narasimhan filtration of sheaves becomes strongly semistable after a finite number of iterations of Frobenius pull-backs (see Theorem 3.1 below). The possibility of giving a geometric condition on the prime number p for the morphism h to be an isomorphism is intrinsic to our method and the previous methods of proof of Theorem 1.1 do not naturally lead to such bounds (be it only because the quantity $\int_Y \mathrm{c}^1(\Omega_Y)^{\dim(Y)}$ never appears in them).

Here is a more detailed outline of the proof. We first show that if the assumptions of the Mordell–Lang conjecture are verified, then one can construct an infinite tower of torsors under vector bundles over X, which is trivialised by an infinite tower of finite surjective base-changes. This first step already appears in [10]. We then use the theory of semistable sheaves in positive characteristic, as developed by Langer in [7], to show that this tower can be trivialised by a single finite purely inseparable morphism (see the proof of Theorem 2.2 below). Here we need some simple facts about the slopes of the sheaf of differentials of a smooth subvariety of an abelian variety whose stabiliser is trivial (see Lemma 3.8 below) and the key input from Langer's theory is Theorem 3.1 below. We also need a cohomological result of Szpiro and Lewin-Ménégaux (see Proposition 3.10 below), which appears in their partial proof of the Kodaira vanishing theorem in positive characteristic. With this trivialisation in hand, we construct the variety Y' whose existence is asserted in Theorem 1.2, by using Grothendieck's formal GAGA theorem. This method of descent is also used in [4]. The bound given in Theorem 1.2 is deduced from a result of Langer (see Theorem 3.2 below), which compares the slopes of a torsion free sheaf with the slopes of its Frobenius pull-backs.

Finally, we would like to state the following conjectures, which are suggested by our proof of Theorem 1.2.

Conjecture 1.8 *Let Z be a projective variety over F. Suppose that Z is smooth and that $\det(\Omega_Z)$ is an ample line bundle. Suppose also that $H^0(X, \Omega_Z^\vee) = 0$. Then $\bar{\mu}_{\min,\det(\Omega_Z)}(\Omega_Z) > 0$.*

Here $\bar{\mu}_{\min,\det(\Omega_Z)}(\cdot)$ refers to the Frobenius-stabilised minimal slope with respect to $\det(\Omega_Z)$. See Sect. 3 below for the definition.

Remark 1.9 Lemma 3.8 below shows that Conjecture 1.8 is verified if X can be embedded in an abelian variety. Also, note that it seems likely that there are "many" varieties X satisfying the assumptions of Conjecture 1.8, such that Ω_Z is strongly semistable with respect to $\det(\Omega_Z)$ (see the beginning of Sect. 3 for this notion), which is a condition stronger than $\bar{\mu}_{\min,\det(\Omega_Z)}(\Omega_X) > 0$). Indeed, recall that the cotangent bundle Ω_S of a smooth and projective variety S over $\mathbb{C}$ is semistable with respect to $\det(\Omega_S)$, if $\det(\Omega_S)$ is ample. This is a consequence of the main result of [14]. On the other hand, there is speculation (see for example [12] and the references therein) that in many situations the reduction modulo a prime number p of a semistable sheaf is strongly semistable for "most" prime numbers p.

Another conjecture concerns a possible generalisation of Theorem 1.2 to a more general geometrical context.

Conjecture 1.10 *Let Z be a smooth and projective variety over a finitely generated field F_0 of characteristic p. Suppose that* $\det(\Omega_Z)$ *is an ample line bundle. Suppose also that $Z(F_0)$ is Zariski dense in Z. If*

$$p > \dim(Z)^2 \int_Z \mathrm{c}^1(\Omega_Z)^{\dim(Z)}$$

then there is a projective variety Z' over a finite subfield $\mathbb{F}_{p^r} \subseteq \bar{F}_0$ and an isomorphism

$$h\colon Z'_{\bar{F}_0} \simeq Z_{\bar{F}_0}.$$

Theorem 1.2 and the Lang–Néron theorem show that Conjecture 1.10 holds if Z can be embedded over F_0 into an abelian variety over F_0. In [4], using some of the results of the present text, we show that Conjecture 1.10 holds if Ω_Z is ample and F_0 has transcendence degree one over its prime field. Conjecture 1.10 is an attempt to make the Bombieri–Lang over function fields in positive characteristic (see the introduction of [4] for a discussion) more precise when the variety under scrutiny is smooth.

The structure of the text is the following. In Sect. 2, we shall formulate three more technical results, from which Theorem 1.2 will be deduced. In particular, Corollary 2.3 gives a strengthening of Theorem 1.2 but is more complicated to formulate. The proofs of these results are given in Sect. 4. In Sect. 3, we shall review the results from Langer's theory that we shall need, derive some simple consequences from it and also formulate the cohomological result of Szpiro and Lewin-Ménégaux alluded to above. Finally in Sect. 5, the proof of Theorem 1.2 is given.

The basic definitions for this article will be fixed in the next section.

Basic notational conventions If Z is a scheme of characteristic p, we write $F_Z\colon Z \to Z$ for the absolute Frobenius endomorphism of Z. The short-hand w.r.o.g. refers to "without restriction of generality".

2 Main results

The definitions and notations given in this section will be used throughout the article. They differ from those used in the introduction, which will not be used again in the text. The results in this section will be proven in Sect. 4.

Let k_0 be an algebraically closed field of characteristic $p > 0$ and let U be a smooth variety over k_0. Let $\mathcal{A}$ be an abelian scheme over U and let $\mathcal{X} \hookrightarrow \mathcal{A}$ be a closed subscheme. We let K_0 be the function field of U and let $A := \mathcal{A}_{K_0}$ (resp. $X := \mathcal{X}_{K_0}$) be the generic fibre of $\mathcal{A}$ (resp. $\mathcal{X}$).

For all $n \geqslant 0$, we define

$$\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_*(J^n(\mathcal{A}/U)) \cap J^n(\mathcal{X}/U).$$

Here $J^n(\bullet/U)$ refers to the n-th jet scheme of $\bullet$ over U. See [10, par. 2] for this and some more explanations. The scheme $J^n(\mathcal{A}/U)$ is naturally a commutative group scheme over U and $[p^n]$ refers to the multiplication-by-p^n-morphism. The notation $[p^n]_*(J^n(\mathcal{A}/U))$ refers to the scheme-theoretic image of $J^n(\mathcal{A}/U)$ by $[p^n]$. There is a natural projective system of U-schemes

$$\cdots \to J^n(\mathcal{X}/U) \overset{\Lambda^{\mathcal{X}}_{n,n-1}}{\to} J^{n-1}(\mathcal{X}/U) \to \cdots \to J^0(\mathcal{X}/U) = \mathcal{X}.$$

If $\mathcal{X}$ is smooth over U then the $J^{n-1}(\mathcal{X}/U)$-scheme $J^n(\mathcal{X}/U)$ carries a natural structure of torsor under the vector bundle $\Omega^\vee_{\mathcal{X}/U} \otimes \mathrm{Sym}^n(\Omega_{U/k_0})$, where the vector bundles $\Omega^\vee_{\mathcal{X}/U}$ and $\mathrm{Sym}^n(\Omega_{U/k_0})$ have been pulled back to $J^{n-1}(\mathcal{X}/U)$ via the natural morphisms.

In particular, we have projective system of U-schemes

$$\cdots \to \mathrm{Crit}^2(\mathcal{X}, \mathcal{A}) \to \mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \to \mathcal{X}.$$

and one can show that the connecting morphisms in this system are finite. See [10, par. 3.1] for this. We let $\mathrm{Exc}^n(\mathcal{A},\mathcal{X}) \hookrightarrow \mathcal{X}$ be the scheme-theoretic image of $\mathrm{Crit}^n(\mathcal{A},\mathcal{X})$ in $\mathcal{X}$. We let $\mathrm{Crit}^n(A,X)$ (resp. $\mathrm{Exc}^n(A,X) \hookrightarrow X$) be the generic fibre of $\mathrm{Crit}^n(\mathcal{A},\mathcal{X})$ (resp. $\mathrm{Exc}^n(\mathcal{A},\mathcal{X}) \hookrightarrow \mathcal{X}$).

Now fix once a for all an ample line bundle M on $X_{\bar{K}_0}$. If X is smooth over K_0 and $\mathrm{Stab}(X)=0$, a natural choice of an ample line bundle is $\det(\Omega_{X_{\bar{K}_0}})$. See [1, Lemma 6] for this.

Lemma-Definition 2.1 *Suppose that X is smooth and geometrically connected over K_0 and that* $\mathrm{Stab}(X)=0$. *Then* $\bar{\mu}_{\min}(\Omega_{X_{\bar{K}_0}})>0$ *and*

$$\mathfrak{DB}(X) := p^{\sup\{n\in\mathbb{N}\ \mid\ H^0(X,F_X^{*,\circ n}\Omega^\vee_{X/K_0}\otimes\Omega_{X/K_0})\neq 0\}} \leqslant \frac{\bar{\mu}_{\max}(\Omega_{X_{\bar{K}_0}})}{\bar{\mu}_{\min}(\Omega_{X_{\bar{K}_0}})}.$$

Here again $\bar{\mu}_{\min}(\cdot) = \bar{\mu}_{\min,M}(\cdot)$ (resp. $\bar{\mu}_{\max}(\cdot) = \bar{\mu}_{\max,M}(\cdot)$) refers to the Frobenius-stabilised minimal (resp. maximal) slope with respect to M. See Sect. 3 below for the definition.

Let now Γ be a subgroup of $A(\bar{K}_0)$. Suppose that

$$\Gamma = \mathrm{Div}^p(\Gamma_0) := \{\gamma \in A(\bar{K}_0) \mid \exists n \text{ a positive integer such that } (n,p)=1 \text{ and } n\cdot\gamma\in\Gamma_0\}$$

where Γ_0 is a finitely generated subgroup of $A(K_0)$. In particular, $\Gamma\otimes\mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module.

Theorem 2.2 *Suppose that $\mathcal{X}$ is smooth over U with geometrically connected fibres and suppose that* $\mathrm{Stab}(X)=0$. *Consider the statements*:

(a) *For any $n\geqslant 0$ there is a $Q=Q(n)\in\Gamma_0$ such that* $\mathrm{Exc}^n(A,X^{+Q}) \hookrightarrow X$ *is an isomorphism.*
(b) *For any closed point $u_0\in U$, there is an $n_0=n_0(u_0)$ such that $p^{n_0}\leqslant\mathfrak{DB}(X)$ and a finite and surjective morphism of $\widehat{\mathcal{O}}_{u_0}$-schemes*

$$\iota=\iota_{u_0}\colon \mathcal{X}_{u_0}^{p^{-n_0}}\times_{k_0}\widehat{\mathcal{O}}_{u_0}\to\mathcal{X}_{\widehat{\mathcal{O}}_{u_0}}$$

of degree equal to $p^{\dim(X)n_0}$.

Then (a) *implies* (b).

Here U_{u_0} is the spectrum of the local ring of U at u_0 and $\widehat{U}_{u_0}$ is its completion. The notation X^{+Q} refers to the pushforward by the addition-by-Q morphism of the subscheme X of A. The scheme $\mathcal{X}_{u_0}$ is the k_0-scheme, which is the fibre of $\mathcal{X}$ at u_0. The symbol $\mathcal{X}_{u_0}^{p^{-r}}$ refers to the scheme obtained from $\mathcal{X}_{u_0}$ by composing the structure map of $\mathcal{X}_{u_0}$ with the n-th power $\mathrm{Frob}_{k_0}^{-1,\circ n}$ of the inverse of the absolute Frobenius morphism Frob_{k_0} of $\mathrm{Spec}\ k_0$ (recall that Frob_{k_0} is an automorphism because k_0 is perfect).

Notice that the morphism ι must be flat by "miracle flatness" (see [9, Th. 23.1]), since both source and target of ι are regular schemes. By the degree of ι, we mean as usual

$$\deg(\iota) := \mathrm{rk}\left(\iota_*\left(\mathcal{O}_{\mathcal{X}_{u_0}^{p^{-n_0}}\times_{k_0}\widehat{\mathcal{O}}_{u_0}}\right)\right),$$

noting that $\iota_*(\mathcal{O}_{\mathcal{X}_{u_0}^{p^{-n_0}}\times_{k_0}\widehat{\mathcal{O}}_{u_0}})$ is a locally free sheaf, since ι is flat.

Corollary 2.3 *Suppose that $X_{\bar{K}_0} \cap \Gamma$ is dense in $X_{\bar{K}_0}$. Suppose also that X is smooth and geometrically connected over K_0.*

Then there exists a smooth projective variety X' over k_0 and a finite and surjective K_0^{sep}-morphism

$$h\colon X'_{K_0^{\mathrm{sep}}} \to (X/\mathrm{Stab}(X))_{K_0^{\mathrm{sep}}}$$

such that

$$\deg(h) \leqslant \mathfrak{DB}(X/\mathrm{Stab}(X))^{\dim(X/\mathrm{Stab}(X))}.$$

3 The geometry of vector bundles in positive characteristic

Let L be an ample line bundle on a smooth and projective variety Y over an algebraically closed field l_0. If V is a torsion free coherent sheaf on Y, we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V)/\mathrm{rk}(V)$$

for the slope of V (with respect to L). Here $\mathrm{rk}(V)$ is the rank of V, which is the dimension the stalk of V at the generic point of Y. Furthermore,

$$\deg_L(V) := \int_X \mathrm{c}_1(V)\cdot \mathrm{c}_1(L)^{\dim(Y)-1}.$$

Here $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral sign $\int_X$ is a short-hand for the push-forward morphism to Spec l_0 in that theory.

Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V, we have $\mu(W) \leqslant \mu(V)$. The torsion free sheaf V is called strongly semistable if $\mathrm{char}(l_0) > 0$ and $F_Y^{*,\circ n}V$ is semistable for all $n \geqslant 0$.

In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i/V_{i-1} are all semistable and such that the slopes $\mu(V_i/V_{i-1})$ are strictly decreasing for $i \geqslant 1$. This filtration is unique and is called the Harder–Narasimhan (HN) filtration of V. We shall write

$$\mu_{\min}(V) := \inf\{\mu(V_i/V_{i-1})\}_{i\geqslant 1}$$

and

$$\mu_{\max}(V) := \sup\{\mu(V_i/V_{i-1})\}_{i\geqslant 1}.$$

An important consequence of the definitions is the following fact: if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\mathrm{Hom}_Y(V, W) = 0$.

For more on the theory of semistable sheaves, see the monograph [6].

The following theorem will be a key input in our proof of Theorem 2.2. For the proof see [7, Th. 2.7].

Theorem 3.1 (Langer) *If V is torsion free coherent sheaf on Y and $\mathrm{char}(l_0) > 0$, then there exists $n_0 \geqslant 0$ such that $F_Y^{*,\circ n}V$ has a strongly semistable HN filtration for all $n \geqslant n_0$.*

If V is a torsion free sheaf on Y and $\mathrm{char}(l_0) > 0$, we now define

$$\bar{\mu}_{\min}(V) := \lim_{r\to\infty} \mu_{\min}(F_Y^{*,\circ r} V)/\mathrm{char}(l_0)^r$$

and

$$\bar{\mu}_{\max}(V) := \lim_{r\to\infty} \mu_{\max}(F_Y^{*,\circ r} V)/\mathrm{char}(l_0)^r.$$

Note that Theorem 3.1 implies that the sequences $\mu_{\min}(F_Y^{*,\circ r} V)/\mathrm{char}(l_0)^r$ (resp. $\mu_{\max}(F_Y^{*,\circ r} V)/\mathrm{char}(l_0)^r$) become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense.

One can show that the sequence $\mu_{\min}(F_Y^{*,\circ r} V)$ (resp. the sequence $\mu_{\max}(F_Y^{*,\circ r} V)$) is decreasing (resp. increasing).

We shall also need the following numerical estimate in the proof of Theorem 1.2. This is again a result of Langer, proved in [7, Cor. 6.2].

To formulate it, let

$$\alpha(V) := \max\{\mu_{\min}(V) - \bar{\mu}_{\min}(V), \bar{\mu}_{\max}(V) - \mu_{\max}(V)\}$$

when $\mathrm{char}(l_0) > 0$.

Theorem 3.2 (Langer) *If* $\mathrm{char}(l_0) > 0$, *we have*

$$\alpha(V) \leqslant \frac{\mathrm{rk}(V) - 1}{\mathrm{char}(l_0)} \max\{\bar{\mu}_{\max}(\Omega_Y), 0\}.$$

Lemma-Definition 3.3 *Suppose that* $\mathrm{char}(l_0) > 0$. *Suppose that* $\bar{\mu}_{\min}(V) > 0$. *Then the quantity*

$$\mathfrak{DB}(V) := p^{\sup\{n\in\mathbb{N} \mid H^0(X, F_Y^{*,\circ n} V^\vee \otimes \Omega_Y) \neq 0\}}$$

is finite and we have

$$\mathfrak{DB}(V) \leqslant \frac{\bar{\mu}_{\max}(\Omega_Y)}{\bar{\mu}_{\min}(V)}.$$

Proof Notice that

$$H^0(X, F_Y^{*,\circ n} V^\vee \otimes \Omega_Y) \simeq \mathrm{Hom}_X(F_Y^{*,\circ n} V, \Omega_Y)$$

and furthermore, for any $r \geqslant 0$, there is a natural inclusion

$$\mathrm{Hom}_X(F_Y^{*,\circ n} V, \Omega_Y) \subseteq \mathrm{Hom}_X(F_Y^{*,\circ(n+r)} V, F_Y^{*,\circ r} \Omega_Y)$$

given by pulling back morphisms of vector bundles by $F_Y^{*,\circ r}$. Now by Theorem 3.1, we may choose r sufficiently large so that $F_Y^{*,\circ r} V$ and $F_Y^{*,\circ r} \Omega_Y$ have Harder–Narasimhan filtrations with strongly semistable quotients. Then we have

$$\mu_{\min}(F_Y^{*,\circ(n+r)} V) = p^n \cdot \mu_{\min}(F_Y^{\circ r,*} V)$$

and

$$\mu_{\max}(F_Y^{*,\circ(n+r)} \Omega_Y) = p^n \cdot \mu_{\max}(F_Y^{\circ r,*} \Omega_Y).$$

Thus $\mathrm{Hom}_Y(F_Y^{*,\circ(n+r)} V, F_Y^{\circ r,*} \Omega_Y) = 0$ if

$$p^n \cdot \mu_{\min}(F_Y^{*,\circ r} V) > \mu_{\max}(F_Y^{*,\circ r} \Omega_Y).$$

Thus

$$\sup\{p^n \mid H^0(X, F_Y^{*,on} V^\vee \otimes V) \neq 0\}_{n\in\mathbb{N}} \leqslant \frac{\mu_{\max}(F_Y^{*,or}\Omega_Y)}{\mu_{\min}(F_Y^{*,or}V)} = \frac{\bar{\mu}_{\max}(\Omega_Y)}{\bar{\mu}_{\min}(V)}. \qquad \square$$

The next lemma is well-known but for lack of a bibliographical reference, we have included a proof.

Lemma 3.4 *Let V be a torsion free sheaf on Y. Suppose that V is globally generated and of degree 0 with respect to L. Then there exists an isomorphism $V \simeq \mathcal{O}_Y^{\oplus \mathrm{rk}(V)}$.*

Proof Let $\phi\colon \mathcal{O}_Y^{\oplus l} \to V$ be a surjection, where l is chosen as small as possible. Suppose that ker $\phi \neq 0$ (otherwise the Lemma is proven). Let $V_0 = \ker\phi$. Then $\mu(V_0) = 0$ and furthermore, since $\mathcal{O}_Y^{\oplus l}$ is semistable, every semistable subsheaf of V_0 has slope $\leqslant 0$ and thus V_0 is also semistable. Now for any $i \in \{1,\ldots,l\}$, let $\pi_i : V_0 \to \mathcal{O}_Y$ be the projection on the i-th coordinate. Choose $i_0 \in \{1,\ldots,l\}$ so that π_{i_0} is non-vanishing. Then π_{i_0} is surjective in codimension 2, because otherwise, the degree of the image of π_{i_0} would be < 0, which would contradict the semistability of V_0. Now replace V_0 be a non-zero semistable subsheaf of $\ker\pi_{i_0}$ and repeat the above reasoning, unless π_{i_0} is an isomorphism outside a closed subset of codimension at least 2. Continuing in the same way, we end up with a semistable torsion free sheaf $M_0 \subseteq \ker\phi \subseteq \mathcal{O}^{\oplus l}$ of rank 1, endowed with an arrow $M_0 \to \mathcal{O}_Y$, which is an isomorphism outside a closed subset of codimension at least 2. We thus obtain a complex

$$\mathcal{O}_Y|_{Y\setminus Y_0} \to \mathcal{O}_Y^{\oplus l}|_{Y\setminus Y_0} \to V|_{Y\setminus Y_0},$$

where Y_0 is a closed subscheme of Y, which is of codimension at least 2. Since Y is normal, the arrow $\mathcal{O}_Y|_{Y\setminus Y_0} \to \mathcal{O}_Y^{\oplus l}|_{Y\setminus Y_0}$ extends uniquely to all of Y. We thus obtain a surjection $\mathcal{O}_Y^{\oplus l}/\mathcal{O}_Y \simeq \mathcal{O}_Y^{\oplus l-1} \to V$. This contradicts the minimality of l and proves the lemma. $\square$

Corollary 3.5 *Let V be a torsion free sheaf. Suppose that V is globally generated. Then $V \simeq V_0 \oplus \mathcal{O}_Y^l$ for some $l \geqslant 0$ and for some torsion sheaf V_0 such that $\mu_{\min}(V_0) > 0$.*

Proof Left as an exercise to the reader. $\square$

Corollary 3.6 *Let V be a vector bundle over Y. Suppose that*

- *for any surjective finite morphism $\phi\colon Y_0 \to Y$, we have $H^0(Y_0, \phi^*V) = 0$;*
- *$V^\vee$ is globally generated.*

Then for any surjective finite morphism $\phi\colon Y_0 \to Y$, such that Y_0 is smooth over l_0, we have $\mu_{\min}(\phi^*V^\vee) > 0$. In particular, if $\mathrm{char}(l_0) > 0$ *then $\bar{\mu}_{\min}(V^\vee) > 0$.*

Proof The bundle $V^\vee$ is globally generated so $\mu_{\min}(\phi^*V^\vee) \geqslant 0$. Now to obtain a contradiction, suppose that $\phi^*V^\vee$ has a non-zero semistable quotient Q of degree 0. Then we have $\phi^*V^\vee \simeq Q_0 \oplus \mathcal{O}_{Y_0}^{\oplus l}$ for some $l > 0$ by Corollary 3.5. This implies that ϕ^*V has a non-vanishing section, which contradicts the assumptions. $\square$

The following elementary lemma is crucial to this article. The assumption that Y is smooth over l_0 is not used in the next lemma.

Lemma 3.7 *Let*

$$0 \to V \to W \to N \to 0 \qquad (1)$$

be an exact sequence of vector bundles on Y.

Suppose that $W \simeq \mathcal{O}_Y^l$ *for some* $l > 0$.

Then $V^\vee$ *is globally generated and for any dominant proper morphism* $\phi\colon Y_0 \to Y$, *where* Y_0 *is integral, the morphism*

$$\phi^*\colon H^0(Y, V) \to H^0(Y_0, \phi^* V)$$

is an isomorphism.

Proof The fact that $V^\vee$ is globally generated follows from the fact that the natural dual map $W^\vee \to V^\vee$ is surjective (the surjectivity follows from the fact that the sequence (1) splits locally, because N is locally free). To prove the second statement, consider that we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(Y, V) & \longrightarrow & H^0(Y, W) & \longrightarrow & H^0(Y, N) \\ & \phi^*\downarrow & & \phi^*\downarrow & & \phi^*\downarrow \\ 0 \longrightarrow & H^0(Y_0, \phi^* V) & \longrightarrow & H^0(Y_0, \phi^* W) & \longrightarrow & H^0(Y_0, \phi^* N) \end{array}$$

In this diagram, all three vertical arrows are injective by construction. Furthermore, the middle vertical arrow is an isomorphism, also by construction. The five lemma now implies that the left vertical arrow is surjective. □

Lemma 3.8 *Suppose that there is a closed* l_0*-immersion* $i\colon Y \hookrightarrow B$, *where* B *is an abelian variety over* l_0. *Suppose that* $\mathrm{Stab}_B(Y) = 0$. *Then* $\Omega_Y^\vee$ *is globally generated and for any dominant proper morphism* $\phi\colon Y_0 \to Y$, *where* Y_0 *is integral, we have* $H^0(Y_0, \phi^*\Omega_Y^\vee) = 0$. *Furthermore, we have* $\mu_{\min}(\Omega_Y) > 0$ *and if* $\mathrm{char}(l_0) > 0$, *we have* $\bar{\mu}_{\min}(\Omega_Y) > 0$.

Proof We have an exact sequence

$$0 \to \Omega_Y^\vee \to i^*\Omega_B^\vee \to N_{Y/B} \to 0$$

where $N_{Y/B}$ is the normal bundle of Y in B. Furthermore, since $\mathrm{Stab}_B(Y) = 0$, we have $H^0(Y, \Omega_Y^\vee) = 0$. Remembering that Ω_B is a trivial bundle, the lemma now follows from Lemma 3.7 and Corollary 3.6. □

In the following lemma, the smoothness assumption on Y is not used either. The proof of the following lemma is extracted from [8, p. 49, before Prop. 3], where the argument is attributed to Moret-Bailly.

Lemma 3.9 *Suppose given a vector bundle* V *on* Y *with the following property: if* $\phi\colon Y_0 \to Y$ *is a surjective and finite morphism and* Y_0 *is integral, then we have* $H^0(Y_0, \phi^* V) = 0$.

Let $f\colon T \to Y$ *be a torsor under* V *and let* $Z \hookrightarrow T$ *be a closed immersion. Suppose that* $f|_Z\colon Z \to Y$ *is finite and surjective and that* Z *is integral.*

Then $f|_Z$ *is generically purely inseparable.*

Proof Let $f\colon T \times_Y T \to Y$. We consider the scheme $T \times_Y (T \times_Y T)$. Via the projection on the second factor $T \times_Y T$, this scheme is naturally a torsor under the vector bundle f^*V. This torsor has two sections:

- the section σ_1 defined by the formula $t_1 \times t_2 \mapsto t_1 \times (t_1 \times t_2)$;
- the section σ_2 defined by the formula $t_1 \times t_2 \mapsto t_2 \times (t_1 \times t_2)$.

Since $T \times_Y (T \times_Y T)$ is a torsor under f^*V, there is a section $s \in H^0(T \times_Y T, f^*V)$ such that $\sigma_1 + s = \sigma_2$ and by construction $s(t_1 \times t_2) = 0$ iff $t_1 = t_2$. In other words, s vanishes precisely on the diagonal of $T \times_Y T$.

Consider now the closed immersion $Z \times_Y Z \hookrightarrow T \times_Y T$. Suppose to obtain a contradiction that $f|_Z$ is not generically purely inseparable. Then there is an irreducible component C of $Z \times_Y Z$, which is not contained in the diagonal and such that $f|_C : C \to Y$ is dominant and hence surjective.

Indeed, if $f|_Z$ is not generically purely inseparable, then there is by constructibility an open subset $U \subseteq Y$, such that for any closed point $u \in U$, there is a point $P(u) \in Z_u \times_u Z_u$ such that $P(u)$ is not contained in the diagonal of $Z_u \times_u Z_u$. Hence there is an irreducible component of $Z \times_Y Z$, which does not coincide with the diagonal and furthermore there is one, which dominates U for otherwise not every $P(u)$ would be contained in an irreducible component of $Z \times_Y Z$.

Now consider $f|_C^* V$. By construction the section $s|_C \in H^0(C, f|_C^* V)$ does not vanish. This contradicts the assumption on V. □

We now quote a result proved in [13, exp. 2, Prop. 1].

Proposition 3.10 (Lewin-Ménégaux, Szpiro) *Suppose that* $\mathrm{char}(l_0) > 0$. *If* $H^0(Y, F_Y^*(V) \otimes \Omega_Y) = 0$ *then the natural map of abelian groups*

$$H^1(Y, V) \to H^1(Y, F_Y^* V)$$

is injective.

Corollary 3.11 *Suppose that* $\mathrm{char}(l_0) > 0$. *Let* V *be a vector bundle over* Y. *Suppose that*

- *for any surjective finite morphism* $\phi : Y_0 \to Y$, *where* Y_0 *is integral, we have* $H^0(Y_0, \phi^* V) = 0$;
- $V^\vee$ *is globally generated.*

Then there is an $n_0 \in \mathbb{N}$ *such that* $H^0(S, F_Y^{n,*}(V) \otimes \Omega_Y) = 0$ *for all* $n > n_0$ *and we might choose* $n_0 \leqslant \mathfrak{DB}(V)$.

Furthermore, let $T \to Y$ *be a torsor under* $F_Y^{n_0,*}(V)$. *Let* $\phi : Y' \to Y$ *be a finite surjective morphism and suppose that* Y' *is integral. Then the map*

$$H^1(Y, F_Y^{n_0,*}(V)) \to H^1(Y', \phi^*(F_Y^{n_0,*}(V)))$$

is injective.

Proof of Corollary 3.11 The existence of n_0 is a consequence of Corollary 3.6 and Lemma 3.3. The upper bound for n_0 is also a consequence of Lemma 3.3.

For the second assertion, notice that by Lemma 3.9, we may assume w.r.o.g. that ϕ is generically purely inseparable. Let H be the function field of Y and let $H'|H$ be the (purely inseparable) function field extension given by ϕ. Let $k_0 > 0$ be sufficiently large so that the extension $H'|H$ factors through the extension $H^{p^{-k_0}}|H$. We may suppose w.r.o.g. that Y' is a normal scheme, since we may replace Y' by its normalization without restriction of generality. On the other hand the morphism $F_Y^{k_0} : Y \to Y$ gives a presentation of Y as its own normalization in $H^{p^{-k_0}}$. Thus there is a natural factorization $Y \to Y' \xrightarrow{\phi} Y$, where the composition of the two arrows is given by $F_Y^{k_0}$. Now by Proposition 3.10 there is a natural injection $H^1(Y, F_Y^{k_0,*}(V)) \hookrightarrow H^1(Y, F_Y^{k_0,*}(F_Y^{n_0,*}(V)))$. Thus there is an injection $H^1(Y, F_Y^{n_0,*}(V)) \to H^1(Y', \phi^*(F_Y^{n_0,*}(V)))$. □

4 Proof of Lemma 2.1, Theorem 2.2 and Corollary 2.3

Proof of Lemma 2.1 Follows from Lemmas 3.3 and 3.8.

Proof of Theorem 2.2 Let $Q \in \mathcal{A}(U)$. Consider the infinite commutative diagram of $\mathcal{X}$-schemes

$$\begin{array}{ccccccc} \cdots \to & \mathrm{Crit}^2(\mathcal{X}^{+Q}, \mathcal{A}) & \to & \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A}) & \to & \mathcal{X} \\ & \downarrow & & \downarrow & & \downarrow = \\ \cdots \longrightarrow & J^2(\mathcal{X}/U) & \longrightarrow & J^1(\mathcal{X}/U) & \longrightarrow & \mathcal{X} \end{array}$$

For any $n \geqslant 0$, we shall write

$$\begin{array}{ccccccc} \cdots \to & \mathrm{Crit}^2(\mathcal{X}^{+Q}, \mathcal{A})^{(p^n)} & \to & \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})^{(p^n)} & \to & \mathcal{X} \\ & \downarrow & & \downarrow & & \downarrow = \\ \cdots \longrightarrow & J^2(\mathcal{X}/U)^{(p^n)} & \longrightarrow & J^1(\mathcal{X}/U)^{(p^n)} & \longrightarrow & \mathcal{X} \end{array}$$

for the diagram obtained by pulling back the original diagram by $F_{\mathcal{X}}^{*,\circ n}$. Let

$$n_0 := \sup\{n \in \mathbb{N}^* \mid H^0(X, F_X^{*,\circ n}\Omega^\vee_{X/K_0} \otimes \Omega_{X/K_0}) \neq 0\}.$$

Suppose that (a) in Theorem 2.2 is satisfied. We shall study diagram (4) in the case where $n = n_0$. Now fix any $m > 1$ and choose some $Q = Q(m) \in \mathcal{A}(U)$ such that $\mathrm{Exc}^m(A, X^{+Q}) \hookrightarrow X$ is an isomorphism. This is possible by assumption. By construction, the morphism

$$\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})} \to \mathcal{X}$$

is then surjective. Choose an irreducible component $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \hookrightarrow \mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})}$, which dominates $\mathcal{X}$. Endow $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$ with its induced reduced scheme structure and for any $l < m$, let $\mathrm{Crit}^l(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \hookrightarrow \mathrm{Crit}^l(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})}$ be the irreducible component obtained by direct image from $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$.

Now notice that by Corollary 3.11 and Lemma 3.8 the base-change of the $F_X^{*,\circ n_0}(\Omega^\vee_{X/K_0} \otimes \Omega_{K_0/k_0})$-torsor $J^1(X/K_0)^{(p^{n_0})} \to X$ to $\bar{K}_0$ is trivial and it is thus a trivial torsor. Let $\sigma : X \to J^1(X/K_0)^{(p^{n_0})}$ be a section. The datum of the composed morphism

$$\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})} \to X \xrightarrow{\sigma} J^1(X/K_0)^{(p^{n_0})}$$

is equivalent to the datum of a section of the pull-back of $\Omega^\vee_{X/k_0}$ to $\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})}$, which must vanish by Lemma 3.8 (note that Ω_{K_0/k_0} is a trivial bundle). Hence the morphism $\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})} \to X$ is an isomorphism and is the image of σ. In particular, if $J^1(X/K_0)^{(p^{n_0})} \to X$ has a section over X, this section is unique. Furthermore, by Zariski's main theorem, the morphism $\mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathcal{X}$ is an isomorphism. We now repeat this reasoning for the restriction to $\mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$ of the $F_X^{*,\circ n_0}(\Omega^\vee_{X/K_0} \otimes \mathrm{Sym}^2(\Omega_{K_0/k_0}))$-torsor $J^2(X/K_0)^{(p^{n_0})} \to J^1(X/K_0)^{(p^{n_0})}$ and we conclude that

$$\mathrm{Crit}^2(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$$

is an isomorphism. Continuing this way, we see that in the whole tower

$$\mathrm{Crit}^m(\mathcal{X}^{+Q},\mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^{m-1}(\mathcal{X}^{+Q},\mathcal{A})_0^{(p^{n_0})} \to \cdots \to \mathrm{Crit}^1(\mathcal{X}^{+Q},\mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q},\mathcal{A})_0^{(p^{n_0})} \to \mathcal{X}$$

the connecting morphisms are all isomorphisms. Letting $m \to \infty$, we obtain an infinite commutative diagram (4):

$$\begin{array}{ccccccccc} \cdots \to & \mathcal{X}^{m+1} & \to & \mathcal{X}^m & \to \cdots \to & \mathcal{X}^1 & \to & \mathcal{X} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow = \\ \cdots \to & J^{m+1}(\mathcal{X}/U) & \to & J^m(\mathcal{X}/U) & \to \cdots \to & J^1(\mathcal{X}/U) & \to & \mathcal{X} \end{array}$$

where all the morphisms $\mathcal{X}^m \to \mathcal{X}$ are isomorphisms.

Now choose a closed point $u_0 \in U$. View u_0 as a closed subscheme of U. For any $i \geqslant 0$, let u_i be the i-th infinitesimal neighborhood of $u_0 \simeq \operatorname{Spec} k_0$ in U (so that there is no ambiguity of notation for u_0). Notice that u_i has a natural structure of k_0-scheme. Recall that by the definition of the jet scheme (see [10, sec. 2]), the scheme $J^m(\mathcal{X}/U)_{u_0}$ represents the functor on k_0-schemes

$$T \mapsto \mathrm{Mor}_{u_m}(T \times_{k_0} u_m, \mathcal{X}_{u_m}).$$

Thus the infinite chain (4) gives rise to morphisms

$$\mathcal{X}_{u_0}^{(p^{-n_0})} \times_{k_0} u_m \to \mathcal{X}_{u_m} \tag{2}$$

compatible with each other under base-change. In particular, base-change to u_0 gives $F^{n_0}_{\mathcal{X}_{u_0}}$.

View the $\widehat{U}_{u_0}$-schemes $\mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0}$ and $\mathcal{X}_{\widehat{U}_{u_0}}$ as formal schemes over $\widehat{U}_{u_0}$ in the next sentence. The family of morphisms (2) provides us with a morphism of formal schemes

$$\mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0} \to \mathcal{X}_{\widehat{U}_{u_0}}$$

and since both schemes are projective over $\widehat{U}_{u_0}$, Grothendieck's GAGA theorem shows that this morphism of formal schemes comes from a unique morphism of schemes

$$\iota\colon \mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0} \to \mathcal{X}_{\widehat{U}_{u_0}}.$$

By construction the morphism ι specializes to $F^{n_0}_{\mathcal{X}_{u_0}}$ at the closed point u_0 of $\widehat{U}_{u_0}$. Since $F^{n_0}_{\mathcal{X}_{u_0}}$ has finite fibres and ι is proper (since both source and target are projective over $\widehat{U}_{u_0}$), the morphism ι is quasi-finite by semicontinuity of fibre dimension and it is thus finite by Zariski's main theorem. The morphism ι is also flat by "miracle flatness", since both source and target are regular (see [9, Th. 23.1]).

Thus

$$\deg(\iota) = p^{\dim(X)n_0}.$$

Finally, $p^{n_0} \leqslant \mathfrak{DB}(X)$ by Lemma 2.1.

Proof of Corollary 2.3 We may replace X by $X/\mathrm{Stab}(X)$ without restriction of generality in the statement of Corollary 2.3. Thus we may (and do) assume that $\mathrm{Stab}(X) = 0$. Notice that by construction, for any $n \geqslant 1$, the natural homomorphism of groups

$$\Gamma_0/p^n\Gamma_0 \to \Gamma/p^n\Gamma$$

is a surjection. Furthermore, $\Gamma_0/p^n\Gamma_0$ is finite since Γ_0 is finitely generated. Hence, using the assumptions of Corollary 2.3, we see that for any $n \geqslant 1$, there exists $Q = Q(n) \in \Gamma_0$, such that $X^{+Q(n)} \cap p^n\Gamma$ is dense in X^{+Q}. This implies that $\mathrm{Exc}^n(A, X^{+Q(n)}) \hookrightarrow X$ is an isomorphism (see [10, par. 3.2] for more details or this). Now applying Theorem 2.2(b), we obtain a surjective and finite morphism of $\widehat{\mathcal{O}}_{u_0}$-schemes

$$\mathcal{X}_{u_0}^{p^{-n_0}} \times_{k_0} \widehat{\mathcal{O}}_{u_0} \to \mathcal{X}_{\widehat{\mathcal{O}}_{u_0}}$$

for some closed point u_0 in U (in fact any will do) and some $n_0 \geqslant 0$ such that $p^{n_0} \leqslant \mathfrak{DB}(X)$. Let $\widehat{K}_0$ be the fraction field of $\widehat{\mathcal{O}}_{u_0}$.

Since k_0 is an excellent field, we know that the field extension $\widehat{K}_0|K_0$ is separable. On the other hand the just constructed finite and surjective morphism $\mathcal{X}_{u_0} \times_{k_0} \widehat{K}_0 \to X_{\widehat{K}_0}$ is defined over a finitely generated (as a field over K) subfield K_0' of $\widehat{K}$. The field extension $K_0'|K$ is then still separable (because the extension $\widehat{K}_0|K_0$ is separable) and thus by the theorem on separating transcendence bases, there exists a variety U'/K_0, which is smooth over K_0 and whose function field is K_0'. Furthermore, possibly replacing U' by one of its open subschemas, we may assume that the morphism $\mathcal{X}_{u_0} \times_k K_0' \to X_{K_0'}$ extends to a finite and surjective morphism

$$\alpha\colon \mathcal{X}_{u_0} \times_{k_0} U' \to X_{U'}.$$

Let $P \in U'(K_0^{\mathrm{sep}})$ be a K_0^{sep}-point over K (the set $U'(K_0^{\mathrm{sep}})$ is not empty because U' is smooth over K_0). The morphism α_P is the morphism h advertised in Theorem 2.2(b). The inequality $\deg(h) \leqslant \mathfrak{DB}(X)^{\dim(X)}$ is verified by construction.

5 Proof of Theorem 1.2

We shall use the shorthand $\Omega := \Omega_Y$. We shall derive Theorem 1.2 from the results of Sect. 2. We use the notation of that section. Let $k_0 := \bar{\mathbb{F}}_p$ and let K_0 be a finitely generated extension of k_0, such that Y admits a model over K_0. We define X to be such a model and we choose a smooth variety U over k_0, such that $\kappa(U) = K_0$ and such that X extends to a smooth an projective scheme $\mathcal{X}$ over U. We let $M := \det(\Omega)$ (recall that $\det(\Omega)$ is ample). Corollary 2.3 now implies Theorem 1.2 (resp. Theorem 1.5), provided we can show that $\mathfrak{DB} = 1$ under the assumptions of Theorem 1.2 (resp. Theorem 1.5). If Ω is strongly semistable then Lemma 3.3 immediately implies that $\mathfrak{DB} = 1$ so Theorem 1.5 is proven. In particular, to prove Theorem 1.2 we may assume w.r.o.g. that Ω is not strongly semistable.

We shall now use Langer's Theorem 3.2 to derive Theorem 1.2.

Note first that by Lemma 3.8, we have $\bar{\mu}_{\min}(\Omega) > 0$ and $\mu_{\min}(\Omega) > 0$. In particular, we have $\deg(\Omega) \geqslant 1$. In view of Lemma 3.3 again, it is now sufficient to show that the inequality

$$p > d^2 \deg(\Omega)$$

ensures that the inequality

$$\bar{\mu}_{\max}(\Omega) < p \cdot \bar{\mu}_{\min}(\Omega). \tag{3}$$

is verified. In view of Theorem 3.2, inequality (3) is implied by the inequality

$$\mu_{\max}(\Omega) + \alpha(\Omega) < p \cdot (\mu_{\min}(\Omega) - \alpha(\Omega)). \tag{4}$$

From the definitions, we have

$$\bar{\mu}_{\max}(\Omega) \leqslant \deg(\Omega)/2 \tag{5}$$

and

$$\mu_{\max}(\Omega) \leqslant \deg(\Omega) \tag{6}$$

(recall that $\bar{\mu}_{\min}(\Omega) \neq \bar{\mu}_{\max}(\Omega)$ since Ω is assumed not to be strongly semistable) and

$$\mu_{\min}(\Omega) \geqslant 1/d. \tag{7}$$

Thus by Theorem 3.2, inequality (4) is weaker than the inequality

$$\deg(\Omega) + \frac{(1+p)(d-1)}{2p} \deg(\Omega) < p/d \tag{8}$$

which can be rewritten as

$$2p^2 - (d^2+d)\deg(\Omega)p - (d^2-d)\deg(\Omega) > 0. \tag{9}$$

The roots of the equation in x

$$2x^2 - (d^2+d)\deg(\Omega)x - (d^2-d)\deg(\Omega) = 0 \tag{10}$$

are

$$\frac{1}{4}\left[(d^2+d)\deg(\Omega) \pm \sqrt{\deg(\Omega)^2(d^2+d)^2 + 8\deg(\Omega)(d^2-d)}\right].$$

One of these roots is $\leqslant 0$ and the other one is $\geqslant 0$. Thus the inequality (9) is equivalent to the inequality

$$p > \frac{1}{4}\left[(d^2+d)\deg(\Omega) + \sqrt{\deg(\Omega)^2(d^2+d)^2 + 8\deg(\Omega)(d^2-d)}\right]. \tag{11}$$

In particular, if

$$p > d^2 \deg(\Omega)$$

then inequality (11) is verified (to check this quickly, just notice that the function

$$2x^2 - (d^2+d)\deg(\Omega)x - (d^2-d)\deg(\Omega)$$

evaluated at $x = d^2 \deg(\Omega)$ is $\geqslant 0$).

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