



# Conjecture and improved extension theorems for paraboloids in the finite field setting

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## Abstract

We study the extension estimates for paraboloids in  $d$ -dimensional vector spaces over finite fields  $\mathbb{F}_q$  with  $q$  elements. We use the connection between  $L^2$  based restriction estimates and  $L^p \rightarrow L^r$  extension estimates for paraboloids. As a consequence, we improve the  $L^2 \rightarrow L^r$  extension results obtained by Lewko and Lewko (Proc Am Math Soc 140:2013–2028, 2012) in even dimensions  $d \geq 6$  and odd dimensions  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ . Our results extend the consequences for 3-D paraboloids due to Lewko (Adv Math 270(1):457–479, 2015) to higher dimensions. We also clarifies conjectures on finite field extension problems for paraboloids.

**Keywords** Restriction theorem · Extension theorem · Paraboloid · Finite field

**Mathematics Subject Classification** 42B05

## 1 Introduction

Let  $V \subset \mathbb{R}^d$  be a hypersurface which is endowed with a surface measure  $d\sigma$ . In the Euclidean setting, the extension problem is to determine the exponents  $1 \leq p, r \leq \infty$  such that the following inequality holds:

$$\|(f d\sigma)^\vee\|_{L^r(\mathbb{R}^d)} \leq C \|f\|_{L^p(V, d\sigma)},$$

where the constant  $C > 0$  is independent of functions  $f \in L^p(V, d\sigma)$ . By duality, this extension estimate is the same as the restriction estimate

$$\|\widehat{g}\|_{L^{p'}(V, d\sigma)} \leq C \|g\|_{L^{r'}(\mathbb{R}^d)}.$$

Here,  $p'$  and  $r'$  denote the Hölder conjugates of  $p$  and  $r$ , respectively (i.e.  $1/p + 1/p' = 1$ ). Therefore, the extension problem is also called the restriction problem. In 1967, E.M. Stein

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[15] introduced the restriction problem. This problem had been completely solved for the parabola and the circle in two dimensions, and the cones in three and four dimensions (see [1,20,21]). However, it is still open in other cases although improved results have been obtained by harmonic analysts. We refer readers to [2,6,16–18] for further information and recent developments on the restriction problem in the Euclidean setting. In addition, see [3,4] for the restriction problem in the integer setting.

In 2002, Mockenhaupt and Tao [14] initially posed and studied the extension problem for various varieties in  $d$ -dimensional vector spaces over finite fields. In order to formulate a finite field analogue of the extension problem, the real set is replaced by finite fields. We begin by reviewing the definition of the finite field extension problem. We denote by  $\mathbb{F}_q$  a finite field with  $q$  elements. Throughout this paper, we shall assume that  $q$  is a power of odd prime. Let  $\mathbb{F}_q^d$  be a  $d$ -dimensional vector space over the finite field  $\mathbb{F}_q$ . We endow the vector space  $\mathbb{F}_q^d$  with the counting measure  $dm$ . We write  $(\mathbb{F}_q^d, dm)$  to stress that the vector space  $\mathbb{F}_q^d$  is endowed with the counting measure  $dm$ . Since the vector space  $\mathbb{F}_q^d$  is isomorphic to its dual space as an abstract group, we identify the space  $\mathbb{F}_q^d$  with its dual space. However, a normalized counting measure  $d\xi$  is endowed with its dual space which will be denoted by  $(\mathbb{F}_q^d, d\xi)$ . We always use the variable  $m$  for an element of the vector space  $(\mathbb{F}_q^d, dm)$ . On the other hand, the variable  $\xi$  will be an element of the dual space  $(\mathbb{F}_q^d, d\xi)$ . For example, we simply write  $m \in \mathbb{F}_q^d$  and  $\xi \in \mathbb{F}_q^d$  for  $m \in (\mathbb{F}_q^d, dx)$  and  $\xi \in (\mathbb{F}_q^d, d\xi)$ , respectively. For a complex valued function  $g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}$ , the Fourier transform  $\widehat{g}$  on  $(\mathbb{F}_q^d, d\xi)$  is defined by

$$\widehat{g}(\xi) = \int_{\mathbb{F}_q^d} g(m)\chi(-m \cdot \xi) dm = \sum_{m \in \mathbb{F}_q^d} g(m)\chi(-m \cdot \xi)$$

where  $\chi$  denotes a nontrivial additive character of  $\mathbb{F}_q$  and the dot product is defined by  $m \cdot \xi = m_1\xi_1 + \dots + m_d\xi_d$  for  $m = (m_1, \dots, m_d)$ ,  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{F}_q^d$ . For a complex valued function  $f : (\mathbb{F}_q^d, d\xi) \rightarrow \mathbb{C}$ , the inverse Fourier transform  $f^\vee$  on  $(\mathbb{F}_q^d, dm)$  is given by

$$f^\vee(m) = \int_{\mathbb{F}_q^d} f(\xi)\chi(\xi \cdot m) d\xi = \frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} f(\xi)\chi(\xi \cdot m).$$

Using the orthogonality relation of the nontrivial character  $\chi$  of  $\mathbb{F}_q$ , we obtain the Plancherel theorem:

$$\|\widehat{g}\|_{L^2(\mathbb{F}_q^d, d\xi)} = \|g\|_{L^2(\mathbb{F}_q^d, dm)} \quad \text{or} \quad \|f\|_{L^2(\mathbb{F}_q^d, d\xi)} = \|f^\vee\|_{L^2(\mathbb{F}_q^d, dm)}.$$

Namely, the Plancherel theorem yields the following equation

$$\frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} |\widehat{g}(\xi)|^2 = \sum_{m \in \mathbb{F}_q^d} |g(m)|^2 \quad \text{or} \quad \frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} |f(\xi)|^2 = \sum_{m \in \mathbb{F}_q^d} |f^\vee(m)|^2.$$

Notice by the Plancherel theorem that if  $G, F \subset \mathbb{F}_q^d$ , then we have

$$\frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} |\widehat{G}(\xi)|^2 = |G| \quad \text{and} \quad \sum_{m \in \mathbb{F}_q^d} |F^\vee(m)|^2 = \frac{|F|}{q^d},$$

where  $|E|$  denotes the cardinality of a set  $E \subset \mathbb{F}_q^d$ . Here, and throughout this paper, we shall identify the set  $E \subset \mathbb{F}_q^d$  with the indicator function  $1_E$  on the set  $E$ . Namely, we shall write

$\widehat{E}$  for  $\widehat{1_E}$ , which allows us to use a simple notation. Given functions  $g_1, g_2 : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}$ , the convolution function  $g_1 * g_2$  on  $(\mathbb{F}_q^d, dm)$  is defined by

$$g_1 * g_2(n) = \int_{\mathbb{F}_q^d} g_1(n - m)g_2(m) dm = \sum_{m \in \mathbb{F}_q^d} g_1(n - m)g_2(m).$$

On the other hand, if  $f_1, f_2 : (\mathbb{F}_q^d, d\xi) \rightarrow \mathbb{C}$ , then the convolution function  $f_1 * f_2$  on  $(\mathbb{F}_q^d, d\xi)$  is given by

$$f_1 * f_2(\eta) = \int_{\mathbb{F}_q^d} f_1(\eta - \xi)f_2(\xi) d\xi = \frac{1}{q^d} \sum_{\xi \in \mathbb{F}_q^d} f_1(\eta - \xi)f_2(\xi).$$

Then it is not hard to see that

$$\widehat{g_1 * g_2} = \widehat{g_1}\widehat{g_2} \quad \text{and} \quad (f_1 * f_2)^\vee = f_1^\vee f_2^\vee.$$

Given an algebraic variety  $V \subset (\mathbb{F}_q^d, d\xi)$ , we endow  $V$  with the normalized surface measure  $d\sigma$  which is defined by the relation

$$\int_V f(\xi) d\sigma(\xi) = \frac{1}{|V|} \sum_{\xi \in V} f(\xi).$$

Notice that  $d\sigma(\xi) = \frac{q^d}{|V|} 1_V(\xi) d\xi$  and we have

$$(fd\sigma)^\vee(m) = \int_V f(\xi)\chi(m \cdot \xi) d\sigma(\xi) = \frac{1}{|V|} \sum_{\xi \in V} f(\xi)\chi(m \cdot \xi).$$

For each  $1 \leq p, r \leq \infty$ , we define  $R_V^*(p \rightarrow r)$  as the smallest positive real number such that the following extension estimate holds:

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} \leq R_V^*(p \rightarrow r) \|f\|_{L^p(V, d\sigma)} \quad \text{for all functions } f : V \rightarrow \mathbb{C}.$$

By duality,  $R_V^*(p \rightarrow r)$  is also the smallest positive constant such that the following restriction estimate holds:

$$\|\widehat{g}\|_{L^{p'}(V, d\sigma)} \leq R_V^*(p \rightarrow r) \|g\|_{L^r(\mathbb{F}_q^d, dm)} \quad \text{for all functions } g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}.$$

The number  $R_V^*(p \rightarrow r)$  may depend on  $q$ , the size of the underlying finite field  $\mathbb{F}_q$ . The main question on the extension problem for  $V \subset \mathbb{F}_q^d$  is to determine  $1 \leq p, r \leq \infty$  such that the number  $R_V^*(p \rightarrow r)$  is independent of  $q$ . Throughout this paper, we shall use  $X \lesssim Y$  for  $X, Y > 0$  if there is a constant  $C > 0$  independent of  $q = |\mathbb{F}_q|$  such that  $X \leq CY$ . We also write  $Y \gtrsim X$  for  $X \lesssim Y$ , and  $X \sim Y$  means that  $X \lesssim Y$  and  $Y \lesssim X$ . In addition, we shall use  $X \lesssim_\varepsilon Y$  if for every  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that  $X \lesssim C_\varepsilon q^\varepsilon Y$ . This notation is powers of  $\log q$ . Using the notathandy for suppressing ion  $\lesssim$ , the extension problem for  $V$  is to determine  $1 \leq p, r \leq \infty$  such that  $R_V^*(p \rightarrow r) \lesssim 1$ .

Since the finite filed extension problem was addressed in 2002 by Mockenhaupt and Tao [14], it has been studied for several algebraic varieties such as paraboloids, spheres, and cones (see, for example, [8–11, 13]). In particular, very interesting results have been recovered for paraboloids. From now on, we restrict ourselves to the study of the extension problem for the paraboloid  $P \subset (\mathbb{F}_q^d, d\xi)$  defined as

$$P = \{\xi \in \mathbb{F}_q^d : \xi_d = \xi_1^2 + \dots + \xi_{d-1}^2\}. \tag{1.1}$$

This paper is written to achieve two main goals. One is to address clarified conjectures on the extension problem for paraboloids. The other is to improve the previously known  $L^2 \rightarrow L^r$  extension estimates for paraboloids in higher dimensions.

In Sect. 2, we shall introduce neat necessary conditions which we may conjecture as sufficient conditions for  $R_p^*(p \rightarrow r) \lesssim 1$ . In particular, by Lemma 2.3 in Sect. 2 it is natural to conjecture the following statement on the  $L^2 \rightarrow L^r$  extension problem for paraboloids.

**Conjecture 1.1** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid defined as in (1.1). Then we have*

- (1) *If  $d \geq 2$  is even, then  $R_p^*(2 \rightarrow r) \lesssim 1 \iff \frac{2d+4}{d} \leq r \leq \infty$*
- (2) *If  $d = 4\ell - 1$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then we have*

$$R_p^*(2 \rightarrow r) \lesssim 1 \iff \frac{2d + 6}{d + 1} \leq r \leq \infty$$

- (3) *If  $d = 4\ell + 1$  for  $\ell \in \mathbb{N}$ , then  $R_p^*(2 \rightarrow r) \lesssim 1 \iff \frac{2d+2}{d-1} \leq r \leq \infty$*
- (4) *If  $d \geq 3$  is odd, and  $-1 \in \mathbb{F}_q$  is a square number, then we have*

$$R_p^*(2 \rightarrow r) \lesssim 1 \iff \frac{2d + 2}{d - 1} \leq r \leq \infty.$$

In the conclusions of Conjecture 1.1, the statements for “ $\implies$ ” direction follow immediately from Lemma 2.3 in the following section. Hence, Conjecture 1.1 can be reduced to the following critical endpoint estimate, because  $R_p^*(2 \rightarrow r_1) \geq R_p^*(2 \rightarrow r_2)$  for  $1 \leq r_1 \leq r_2 \leq \infty$ .

**Conjecture 1.2** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid defined as in (1.1). Then we have*

- (1) *If  $d \geq 2$  is even, then  $R_p^*(2 \rightarrow \frac{2d+4}{d}) \lesssim 1$*
- (2) *If  $d = 4\ell - 1$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then  $R_p^*(2 \rightarrow \frac{2d+6}{d+1}) \lesssim 1$*
- (3) *If  $d = 4\ell + 1$  for  $\ell \in \mathbb{N}$ , then  $R_p^*(2 \rightarrow \frac{2d+2}{d-1}) \lesssim 1$*
- (4) *If  $d \geq 3$  is odd, and  $-1 \in \mathbb{F}_q$  is a square number, then  $R_p^*(2 \rightarrow \frac{2d+2}{d-1}) \lesssim 1$ .*

### 1.1 Statement of main results

By the Stein-Tomas argument, Mockenhaupt and Tao [14] already showed that the statements (3), (4) in Conjecture 1.2 are true. In fact, they proved that  $R_p^*(2 \rightarrow (2d + 2)/(d - 1)) \lesssim 1$  for all dimensions  $d \geq 2$  without further assumptions.

The statements (1), (2) in Conjecture 1.2 are very interesting in that the conjectured results are better than the Stein-Tomas inequality which is sharp in the Euclidean case. This is due to number theoretic issue which we can enjoy when we study harmonic analysis in finite fields. In dimension two, the statement (1) in Conjecture 1.2 was already proved by Mockenhaupt and Tao [14], but it is open in higher even dimensions. For higher even dimensions  $d \geq 4$ , Iosevich and Koh [7] proved that  $R_p^*(2 \rightarrow 2d^2/(d^2 - 2d + 2)) \lesssim 1$  which improves the Stein-Tomas inequality due to Mockenhaupt and Tao. This result was obtained by using a connection between  $L^p \rightarrow L^4$  extension results and  $L^2 \rightarrow L^r$  extension estimates. In [13], Lewko and Lewko improved the result of Iosevich and Koh by recovering the endpoint. They adapted the bilinear approach to derive the improved result,  $R_p^*(2 \rightarrow 2d^2/(d^2 - 2d + 2)) \lesssim 1$ . In this paper, we shall obtain further improvement in higher even dimensions  $d \geq 6$ . Our first main result is as follows.

**Theorem 1.3** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid defined as in (1.1). If the dimension  $d \geq 6$  is even, then for each  $\varepsilon > 0$  we have*

$$R_p^* \left( 2 \rightarrow \frac{6d + 8}{3d - 2} + \varepsilon \right) \lesssim 1.$$

Notice that if  $d \geq 6$ , then  $(6d + 8)/(3d - 2) < 2d^2/(d^2 - 2d + 2)$ , which implies that Theorem 1.3 is better than the result  $R_p^*(2 \rightarrow 2d^2/(d^2 - 2d + 2)) \lesssim 1$  due to Lewko and Lewko.

The statement (2) in Conjecture 1.2 has not been solved in any case. In the case when  $d = 3$  and  $q$  is a prime with  $q \equiv 3 \pmod{4}$ , Mockenhaupt and Tao [14] deduced the following extension result: for every  $\varepsilon > 0$ ,

$$R_p^* \left( 2 \rightarrow \frac{18}{5} + \varepsilon \right) \lesssim 1. \tag{1.2}$$

This was improved to  $R_p^*(2 \rightarrow \frac{18}{5}) \lesssim 1$  by Lewko and Lewko [13] (Bennett, Carbery, Garrigos, and Wright independently proved it in unpublished work). Recently, Lewko [11] discovered a nice connection between the finite field extension problem and the finite field Szemerédi-Trotter incidence problem. Using the connection with ingenious arguments, he obtained the currently best known result on extension problems for the 3-d paraboloid. More precisely, he proved that if the dimension  $d$  is three and  $-1 \in \mathbb{F}_q$  is not a square, then there exists an  $\varepsilon > 0$  such that

$$R_p^* \left( 2 \rightarrow \frac{18}{5} - \varepsilon \right) \lesssim 1. \tag{1.3}$$

Furthermore, assuming that  $q$  is a prime and  $-1 \in \mathbb{F}_q$  is not a square, he gave the following explicit result for  $d = 3$ :

$$R_p^* \left( 2 \rightarrow \frac{18}{5} - \frac{1}{1035} + \varepsilon \right) \lesssim 1 \text{ for any } \varepsilon > 0. \tag{1.4}$$

Although this result is still far from the conjectured result,  $R_p^*(2 \rightarrow 3) \lesssim 1$ , M. Lewko provided novel ideas useful in developing the finite field extension problem and we will also adapt many of his methods to deduce our improved results. In specific higher odd dimensions, Iosevich and Koh [7] proved that  $R_p^*(2 \rightarrow \frac{2d^2}{d^2 - 2d + 2}) \lesssim 1$  with the assumptions of the statement (2) in Conjecture 1.2. This result is also better than the Stein-Tomas inequality. A. Lewko and M. Lewko [13] obtained the endpoint estimate so that the result by Iosevich and Koh was improved to

$$R_p^* \left( 2 \rightarrow \frac{2d^2}{d^2 - 2d + 2} \right) \lesssim 1. \tag{1.5}$$

As our second result, we shall improve this result in the case when  $d = 4\ell - 1 \geq 7$  for  $\ell \in \mathbb{N}$ . More precisely, we have the following result.

**Theorem 1.4** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid defined as in (1.1). If  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then for every  $\varepsilon > 0$ , we have*

$$R_p^* \left( 2 \rightarrow \frac{6d + 10}{3d - 1} + \varepsilon \right) \lesssim 1.$$

**Table 1** Progress on the finite field extension problem for paraboloids in lower dimensions

Dimension $d$ , Field $\mathbb{F}_q$	$R_p^*(p \rightarrow r) \lesssim 1$	Authors
$d = 2$ , general $q$	$p = 2, r = 4$ (S-T)	Mockenhaupt and Tao [14] (solution)
$d = 3, -1$ a square	$p = 2, r = 4$ (S-T)	Mockenhaupt and Tao [14] (sharp)
	$p = 2.25, r = 3.6$	Lewko [12] (sharp)
	$p = \frac{18-5\varepsilon}{8-5\varepsilon}, r = 3.6 - \varepsilon$	Lewko [12] (sharp)
	for some $\varepsilon > 0$	
	$p = 3, r = 3$	(conjectured)
$d = 3, -1$ not a square (prime $q$ )	$p = 2, r > 3.6$	Mockenhaupt and Tao [14]
	$p > 1.6, r = 4$	Mockenhaupt and Tao [14]
	$p = 2, r = 3.6$	Lewko and Lewko [13]
	$p = 1.6, r = 4$	Lewko and Lewko [13] (sharp)
	$p = 2, r > 3.6 - \frac{1}{1035}$	Lewko [11]
	$p = 2, r = 3$	(conjectured)
$d = 3, -1$ not a square	$p = 2, r = 3.6 - \varepsilon$	Lewko [11]
	for some $\varepsilon > 0$	
	$p = 2, r = 3$	(conjectured)

Notice that Theorem 1.4 is superior to the result (1.5) due to A. Lewko and M. Lewko. If one could obtain the exponent in Theorem 1.4 for  $d = 3$ , we could have  $R_p^*(2 \rightarrow \frac{7}{2} + \varepsilon) \lesssim 1$ , which is much better than the best known result (1.4) due to M. Lewko. Unfortunately, our result does not cover the case of three dimensions and it only improves the previous known results in specific higher odd dimensions.

This paper will be organized as follows. In Sect. 2, we deduce the necessary conditions for  $R_p^*(p \rightarrow r)$  bound from which we make a conjecture on extension problems for paraboloids. In Sect. 3, we collect several lemmas which are essential in proving our main results, Theorem 1.3 and Theorem 1.4. In the final section, we give the complete proofs of our main theorems. In addition, we shall provide summary of progress on the finite field extension problems for paraboloids (Tables 1, 2).

## 2 Conjecture on extension problems for paraboloids

In [14], Mockenhaupt and Tao observed that if  $|V| \sim q^{d-1}$ , then the necessary conditions for  $R_V^*(p \rightarrow r) \lesssim 1$  are given by

$$r \geq \frac{2d}{d-1} \quad \text{and} \quad r \geq \frac{pd}{(p-1)(d-1)}. \tag{2.1}$$

In particular, when the variety  $V$  contains an affine subspace  $\Omega$  with  $|\Omega| = q^k$  for  $0 \leq k \leq d-1$ , the above necessary conditions can be improved to the conditions

$$r \geq \frac{2d}{d-1} \quad \text{and} \quad r \geq \frac{p(d-k)}{(p-1)(d-1-k)}. \tag{2.2}$$

Now, let us observe the necessary conditions for  $R_p^*(p \rightarrow r)$  bound where the paraboloid  $P \subset \mathbb{F}_q^d$  is defined as in (1.1). To find more exact necessary conditions for  $R_p^*(p \rightarrow r) \lesssim 1$ ,

**Table 2** Progress on the finite field extension problem for paraboloids in higher dimensions

Dimension $d$ , Field $\mathbb{F}_q$	$R_p^*(p \rightarrow r) \lesssim 1$	Authors
$d \geq 4$ even, general $q$	$p = 2, r = \frac{2d+2}{d-1}$ (S-T)	Mockenhaupt and Tao [14]
	$p = 2, r > \frac{2d^2}{d^2-2d+2}$	Iosevich and Koh [7]
	$p > \frac{4d}{3d-2}, r = 4$	Iosevich and Koh [7]
	$p = 2, r = \frac{2d^2}{d^2-2d+2}$	Lewko and Lewko [13]
	$p = \frac{4d}{3d-2}, r = 4$	Lewko and Lewko [13] (sharp)
	$p = 2, r > \frac{6d+8}{3d-2}$	Theorem 1.4
	$p = \frac{2d^2}{d^2-d+2}, r = \frac{2d}{d-1}$	(conjectured)
	$p = 2, r = \frac{2d+4}{d}$	(conjectured best $r$ for $p = 2$ )
$d \geq 5$ odd, $-1$ a square	$p = 2, r = \frac{2d+2}{d-1}$ (S-T)	Mockenhaupt and Tao [14] (sharp)
	$p = \frac{2d+2}{d-1}, r = \frac{2d+2}{d-1} - \varepsilon_d$ for some $\varepsilon_d > 0$	Lewko [12]
	$p = \frac{2d}{d-1}, r = \frac{2d}{d-1}$	(conjectured)
$d = 4\ell + 1$ for $\ell \in \mathbb{N}$ , $-1$ not a square	$p = 2, r = \frac{2d+2}{d-1}$ (S-T)	Mockenhaupt and Tao [14] (sharp)
	$p = \frac{2d}{d-1}, r = \frac{2d}{d-1}$	(conjectured)
$d = 4\ell + 3$ for $\ell \in \mathbb{N}$ , $-1$ not a square	$p = 2, r = \frac{2d+2}{d-1}$ (S-T)	Mockenhaupt and Tao [14]
	$p = 2, r > \frac{2d^2}{d^2-2d+2}$	Iosevich and Koh [7]
	$p > \frac{4d}{3d-2}, r = 4$	Iosevich and Koh [7]
	$p = 2, r = \frac{2d^2}{d^2-2d+2}$	Lewko and Lewko [13]
	$p = \frac{4d}{3d-2}, r = 4$	Lewko and Lewko [13]
	$p = 2, r > \frac{6d+10}{3d-1}$	Theorem 1.5
	$p = \frac{2d^2+2d}{d^2+3}, r = \frac{2d}{d-1}$	(conjectured)
	$p = 2, r = \frac{2d+6}{d+1}$	(conjectured best $r$ for $p = 2$ )

it is essential to know the size of subspaces lying on the paraboloid  $P \subset \mathbb{F}_q^d$ . To this end, we need the following lemma which is a direct consequence of Lemma 2.1 in [19].

**Lemma 2.1** *Let  $S_0 = \{(x_1, \dots, x_{d-1}) \in \mathbb{F}_q^{d-1} : x_1^2 + \dots + x_{d-1}^2 = 0\}$  be a variety in  $\mathbb{F}_q^{d-1}$  with  $d \geq 2$ . Denote by  $\eta$  the quadratic character of  $\mathbb{F}_q$ . If  $W$  is a subspace of maximal dimension contained in  $S_0$ , then we have the following facts:*

- (1) *If  $d - 1$  is odd, then  $|W| = q^{\frac{d-2}{2}}$*
- (2) *If  $d - 1$  is even and  $(\eta(-1))^{\frac{d-1}{2}} = 1$ , then  $|W| = q^{\frac{d-1}{2}}$*
- (3) *If  $d - 1$  is even and  $(\eta(-1))^{\frac{d-1}{2}} = -1$ , then  $|W| = q^{\frac{d-3}{2}}$ .*

Observe from Lemma 2.1 that  $\Omega := W \times \{0\} \subset \mathbb{F}_q^{d-1} \times \mathbb{F}_q$  is a subspace contained in the paraboloid  $P \subset \mathbb{F}_q^d$ . Since  $|\Omega| = |W|$ , we have the following result from Lemma 2.1.

**Corollary 2.2** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid. Then the following statements hold:*

- (1) If  $d \geq 2$  is even, then the paraboloid  $P$  contains a subspace  $\Omega$  with  $|\Omega| = q^{\frac{d-2}{2}}$
- (2) If  $d = 4\ell - 1$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then the paraboloid  $P$  contains a subspace  $\Omega$  with  $|\Omega| = q^{\frac{d-3}{2}}$
- (3) If  $d = 4\ell + 1$  for  $\ell \in \mathbb{N}$ , then the paraboloid  $P$  contains a subspace  $\Omega$  with  $|\Omega| = q^{\frac{d-1}{2}}$
- (4) If  $d \geq 3$  is odd, and  $-1 \in \mathbb{F}_q$  is a square number, then the paraboloid  $P$  contains a subspace  $\Omega$  with  $|\Omega| = q^{\frac{d-1}{2}}$ .

Applying Corollary 2.2 to (2.2), the necessary conditions for  $R_p^*(p \rightarrow r) \lesssim 1$  are given as follows:

**Lemma 2.3** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid defined as in (1.1). Assume that  $R_p^*(p \rightarrow r) \lesssim 1$  for  $1 \leq p, r \leq \infty$ . Then the following statements are true:*

- (1) If  $d \geq 2$  is even, then  $(1/p, 1/r)$  must be contained in the convex hull of points

$$(1, 0), (0, 0), \left(0, \frac{d-1}{2d}\right), \text{ and } P_1 := \left(\frac{d^2-d+2}{2d^2}, \frac{d-1}{2d}\right).$$

- (2) If  $d = 4\ell - 1$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then  $(1/p, 1/r)$  lies on the convex hull of points

$$(1, 0), (0, 0), \left(0, \frac{d-1}{2d}\right), \text{ and } P_2 := \left(\frac{d^2+3}{2d^2+2d}, \frac{d-1}{2d}\right).$$

- (3) If  $d = 4\ell + 1$  for  $\ell \in \mathbb{N}$ , then  $(1/p, 1/r)$  must be contained in the convex hull of points  $(1, 0), (0, 0), \left(0, \frac{d-1}{2d}\right)$ , and  $P_3 := \left(\frac{d-1}{2d}, \frac{d-1}{2d}\right)$ .
- (4) If  $d \geq 3$  is odd, and  $-1 \in \mathbb{F}_q$  is a square number, then  $(1/p, 1/r)$  must be contained in the convex hull of points  $(1, 0), (0, 0), \left(0, \frac{d-1}{2d}\right)$ , and  $\left(\frac{d-1}{2d}, \frac{d-1}{2d}\right)$ .

We may conjecture that the necessary conditions for  $R_p^*(p \rightarrow r) \lesssim 1$  in Lemma 2.3 are in fact sufficient. For this reason, we could settle the extension problem for paraboloids if we could obtain the critical endpoints  $P_1, P_2, P_3$  in the statement of Lemma 2.3. In conclusion, to solve the extension problem for paraboloids, it suffices to establish the following conjecture on critical endpoints.

**Conjecture 2.4** *The following statements hold:*

- (1) If  $d \geq 2$  is even, then  $R_p^*\left(\frac{2d^2}{d^2-d+2}, \frac{2d}{d-1}\right) \lesssim 1$
- (2) If  $d = 4\ell - 1$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then  $R_p^*\left(\frac{2d^2+2d}{d^2+3}, \frac{2d}{d-1}\right) \lesssim 1$
- (3) If  $d = 4\ell + 1$  for  $\ell \in \mathbb{N}$ , then  $R_p^*\left(\frac{2d}{d-1}, \frac{2d}{d-1}\right) \lesssim 1$
- (4) If  $d \geq 3$  is odd, and  $-1 \in \mathbb{F}_q$  is a square number, then  $R_p^*\left(\frac{2d}{d-1}, \frac{2d}{d-1}\right) \lesssim 1$ .

### 3 Preliminary lemmas

In this section, we collect several lemmas which shall be used to prove our main results. As we shall see, both Theorems 1.3 and 1.4 will be proved in terms of the restriction estimates (dual extension estimate). Thus, we start with lemmas about the restriction operators associated with paraboloids. We shall write  $R_p(p \rightarrow r)$  for  $R_p^*(r' \rightarrow p')$  for  $1 \leq p, r \leq \infty$ . Namely,



$R_P(p \rightarrow r)$  is the smallest positive real number such that the following restriction estimate holds:

$$\|\widehat{g}\|_{L^r(P, d\sigma)} \leq R_P(p \rightarrow r) \|g\|_{L^p(\mathbb{F}_q^d, dm)} \quad \text{for all functions } g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}.$$

The following definition was given in [11].

**Definition 3.1** Let  $G \subset \mathbb{F}_q^d$ . For each  $a \in \mathbb{F}_q$ , define a level set

$$G_a = \{(m_1, \dots, m_{d-1}, m_d) \in G : m_d = a\}.$$

In addition, define

$$L_G = \{a \in \mathbb{F}_q : |G_a| \geq 1\}.$$

We say that the set  $G$  is a regular set if

$$\frac{|G_a|}{2} \leq |G_{a'}| \leq 2|G_a| \quad \text{for } a, a' \in L_G.$$

Finally, the function  $g : \mathbb{F}_q^d \rightarrow \mathbb{C}$  is called a regular function if the function  $g$  is supported on a regular set  $G$  and  $\frac{1}{2} \leq |g(m)| \leq 1$  for  $m \in G$ .

Notice that if  $G$  is a regular set, then  $|G| \sim |G_a||L_G|$  for all  $a \in L_G$ . By the the dyadic pigeonhole principle, the following lemma was given by M. Lewko (see Lemma 14 in [11]).

**Lemma 3.2** *If the restriction estimate*

$$\|\widehat{g}\|_{L^r(P, d\sigma)} \leq R_P(p \rightarrow r) \|g\|_{L^p(\mathbb{F}_q^d, dm)}$$

*holds for all regular functions  $g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}$ , then for each  $\varepsilon > 0$ ,*

$$R_P(p - \varepsilon \rightarrow r) \lesssim 1.$$

Working on regular test functions, we lose the endpoint result but our analysis becomes extremely simplified. When the size of the support  $G$  of a regular function  $g$  is somewhat big, we shall invoke the following restriction estimate.

**Lemma 3.3** *Let  $g$  is a regular function on  $(\mathbb{F}_q^d, dm)$  with  $\text{supp}(g) = G$ . Then we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \leq q^{\frac{1}{2}} |G|^{\frac{1}{2}}.$$

**Proof** By the Plancherel theorem, we see that

$$\|(fd\sigma)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = q^{\frac{1}{2}} \|f\|_{L^2(P, d\sigma)} \quad \text{for all functions } f : P \rightarrow \mathbb{C}.$$

By duality, it is clear that

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \leq q^{\frac{1}{2}} \|g\|_{L^2(\mathbb{F}_q^d, dm)} \leq q^{\frac{1}{2}} \|G\|_{L^2(\mathbb{F}_q^d, dm)} = q^{\frac{1}{2}} |G|^{\frac{1}{2}},$$

where the last inequality follows from the property of the regular function  $g$  (namely,  $\frac{1}{2} \leq |g| \leq 1$  on its support  $G$ ). □

The following result is well known in [14] (see also [7]).

**Lemma 3.4** *Let  $d\sigma$  be the normalized surface measure on the paraboloid  $P \subset (\mathbb{F}_q^d, d\xi)$ . For each  $m = (\underline{m}, m_d) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q$ , we have*

$$(d\sigma)^\vee(m) = \begin{cases} q^{-(d-1)} \chi\left(\frac{\|\underline{m}\|}{-4m_d}\right) \eta^{d-1}(m_d) G_1^{d-1} & \text{if } m_d \neq 0 \\ 0 & \text{if } m_d = 0, m \neq (0, \dots, 0) \\ 1 & \text{if } m = (0, \dots, 0). \end{cases}$$

where  $\|\underline{m}\| := m_1^2 + \dots + m_{d-1}^2$ ,  $\eta$  denotes the quadratic character of  $\mathbb{F}_q^*$ , and  $G_1$  denotes the standard Gauss sum with  $|G_1| = |\sum_{s \neq 0} \eta(s)\chi(s)| = q^{\frac{1}{2}}$ .

When a regular function  $g$  is supported on a small set  $G$ , the following result will be useful to deduce a good  $L^2$  restriction estimate.

**Lemma 3.5** *If  $g$  is a regular function on  $(\mathbb{F}_q^d, dm)$  with  $\text{supp}(g) = G$ , then we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + q^{-\frac{d+1}{4}} |G|.$$

**Proof** It follows that

$$\begin{aligned} \|\widehat{g}\|_{L^2(P, d\sigma)}^2 &= \frac{1}{|P|} \sum_{\xi \in P} |\widehat{g}(\xi)|^2 = \frac{1}{q^{d-1}} \sum_{\xi \in P} \sum_{m, m' \in G} \chi(\xi \cdot (m - m')) g(m) \overline{g(m')} \\ &= q \sum_{m, m' \in G} P^\vee(m - m') g(m) \overline{g(m')} \leq q \sum_{m, m' \in G} |P^\vee(m - m')| \\ &= q \sum_{m \in G} |P^\vee(0, \dots, 0)| + q \sum_{m, m' \in G: m \neq m'} |P^\vee(m - m')| = \text{I} + \text{II}. \end{aligned}$$

Since  $P^\vee(0, \dots, 0) = \frac{|P|}{q^d} = \frac{1}{q}$ , we see that  $\text{I} = |G|$ . To estimate  $\text{II}$ , we observe from Lemma 3.4 that if  $w \neq (0, \dots, 0)$ ,

$$|P^\vee(w)| = \left| \frac{1}{q} (d\sigma)^\vee(w) \right| \leq q^{-\frac{d-1}{2}}.$$

Then it is clear that  $\text{II} \leq q^{-\frac{d+1}{2}} |G|^2$ . Putting all estimates together, we obtain the lemma.  $\square$

The improved  $L^P \rightarrow L^2$  restriction estimates for paraboloids have been obtained by extending the idea of Carbery [5] to the finite field setting. For instance, Mockenhaupt and Tao [14] observed that the restriction operator acting on a single vertical slice of  $\mathfrak{g}$ , say  $g_a$  for  $a \in \mathbb{F}_q$ , is closely related to the extension operator applied to a function  $h$  on  $P$ , which can be identified with the slice function  $g_a$ . In fact, they found the connection between the  $L^P \rightarrow L^2$  restriction estimate and the  $L^P \rightarrow L^4$  extension estimate obtained from the additive energy estimation. Recall that the additive energy  $\Lambda(E)$  for  $E \subset P$  is given by

$$\Lambda(E) := \sum_{x, y, z, w \in E: x+y=z+w} 1. \tag{3.1}$$

As a consequence, they obtained the extension result (1.2) for the 3-D paraboloid. Working with the restriction operator applied to regular test functions, M. Lewko [11] was able to achieve the further improved extension results for the 3-D paraboloid (see (1.3) and (1.4)). He also employed the relation between the  $L^P \rightarrow L^2$  restriction estimate and the  $L^P \rightarrow L^4$  extension result for the 3-D paraboloid. In this paper, we develop his work to higher dimensional cases. To estimate  $\|\widehat{g}\|_{L^2(P, d\sigma)}$ , we will invoke not only  $L^P \rightarrow L^4$  extension results

but also  $L^2 \rightarrow L^r$  extension results for paraboloids in higher dimensions. The following lemma can be obtained by a modification of the Mockenhaupt and Tao Machinery which explains the relation between the  $L^p \rightarrow L^2$  restriction estimate and the  $L^p \rightarrow L^4$  extension result for paraboloids.

**Lemma 3.6** *Let  $P \subset \mathbb{F}_q^d$  be the paraboloid. Then the following statements hold:*

- (1) *Let  $g$  be a regular function with the support  $G \subset (\mathbb{F}_q^d, dm)$ . For each  $a \in L_G$ , let  $h_a$  be a function on the paraboloid  $P \subset (\mathbb{F}_q^d, d\xi)$  such that  $\frac{1}{2} \leq |h_a(\xi)| \leq 1$  on  $\text{supp}(h_a)$  and  $|\text{supp}(h_a)| = |G_a|$ . In addition, assume that there exists a positive number  $U(|E|)$  depending on the size of a set  $E \subset P$  such that  $|E| \sim |\text{supp}(h_a)|$  for all  $a \in L_G$  and*

$$\max_{a \in L_G} \|(h_a d\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim U(|E|). \tag{3.2}$$

Then we have

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} (U(|E|))^{\frac{1}{2}}.$$

- (2) *If  $d \geq 4$  is even, or if  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$  and  $-1 \in \mathbb{F}_q$  is not a square number, then*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{d^2+d-1}{2d^2}} |L_G|^{\frac{1}{4}}$$

for all regular functions  $g$  on  $(\mathbb{F}_q^d, dm)$  with  $\text{supp}(g) = G$ .

**Proof** By duality, it follows that

$$\|\widehat{g}\|_{L^2(P, d\sigma)}^2 = \langle g, (\widehat{g} d\sigma)^\vee \rangle = \langle g, g * (d\sigma)^\vee \rangle.$$

Using the Bochner–Riesz kernel  $K$  which is defined by  $K(m) = (d\sigma)^\vee(m) - \delta_0(m)$  for  $m \in (\mathbb{F}_q^d, dm)$ , where  $\delta_0(m) = 1$  if  $m = (0, \dots, 0)$  and 0 otherwise, we can write from Hölder’s inequality that for  $1 \leq r \leq \infty$ ,

$$\begin{aligned} \|\widehat{g}\|_{L^2(P, d\sigma)}^2 &= \langle g, g * \delta_0 \rangle + \langle g, g * K \rangle \\ &\leq \|g\|_{L^2(\mathbb{F}_q^d, dm)}^2 + \|g\|_{L^{r'}(\mathbb{F}_q^d, dm)} \|g * K\|_{L^r(\mathbb{F}_q^d, dm)} \\ &\leq |G| + |G|^{\frac{1}{r'}} \|g * K\|_{L^r(\mathbb{F}_q^d, dm)}, \end{aligned} \tag{3.3}$$

where the last inequality follows from the property of a regular function  $g$  with  $\frac{1}{2} \leq g \leq 1$  on its support  $G$ . To estimate  $\|g * K\|_{L^r(\mathbb{F}_q^d, dm)}$ , define  $g_a$  for  $a \in L_G$  as the restriction of  $g$  to the hyperplane  $\{m = (m_1, \dots, m_d) \in \mathbb{F}_q^d : m_d = a\}$ . Notice that  $\text{supp}(g_a) = G_a$  for  $a \in L_G$ . It follows that

$$\|g * K\|_{L^r(\mathbb{F}_q^d, dm)} \leq \sum_{a \in L_G} \|g_a * K\|_{L^r(\mathbb{F}_q^d, dm)}. \tag{3.4}$$

By the definition of  $K$  and Lemma 3.4, we see that for each  $a \in L_G$ ,

$$\|g_a * K\|_{L^r(\mathbb{F}_q^d, dm)} = \left( \sum_{m \in \mathbb{F}_q^d} \left| \sum_{n \in \mathbb{F}_q^d} g_a(n) K(m - n) \right|^r \right)^{\frac{1}{r}}$$

$$= q^{-\frac{d+1}{2}} \left( \sum_{\underline{m} \in \mathbb{F}_q^{d-1}} \sum_{m_d \neq a} \left| \sum_{\underline{n} \in \mathbb{F}_q^{d-1}} g(\underline{n}, a) \chi \left( \frac{\|\underline{m} - \underline{n}\|}{-4(m_d - a)} \right) \right|^r \right)^{\frac{1}{r}},$$

where we define  $\|\underline{m} - \underline{n}\| = (m_d - n_d) \cdot (m_d - n_d)$ . After changing variables by letting  $s = -m_d + a$ , we use the change of variables one more by putting  $t = \frac{1}{4s}$  and  $\underline{u} = \frac{-\underline{m}}{2s}$ . Then it follows that

$$\begin{aligned} \|g_a * K\|_{L^r(\mathbb{F}_q^d, dm)} &= q^{-\frac{d+1}{2}} \left( \sum_{\underline{u} \in \mathbb{F}_q^{d-1}} \sum_{t \neq 0} \left| \chi \left( \frac{\underline{u} \cdot \underline{u}}{4t} \right) \sum_{\underline{n} \in \mathbb{F}_q^{d-1}} g(\underline{n}, a) \chi((\underline{u} \cdot \underline{n}) + t \underline{n} \cdot \underline{n}) \right|^r \right)^{\frac{1}{r}} \\ &= q^{-\frac{d+1}{2}} \left( \sum_{\underline{u} \in \mathbb{F}_q^{d-1}} \sum_{t \neq 0} \left| \sum_{\underline{n} \in \mathbb{F}_q^{d-1}} g(\underline{n}, a) \chi((\underline{u}, t) \cdot (\underline{n}, \underline{n} \cdot \underline{n})) \right|^r \right)^{\frac{1}{r}}. \end{aligned}$$

Now, for each  $a \in L_G$ , define  $h_a$  as a function on the paraboloid  $P$  given by

$$h_a(\underline{n}, \underline{n} \cdot \underline{n}) = g_a(n) = g(\underline{n}, a) \quad \text{for } n = (\underline{n}, n_d) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q. \tag{3.5}$$

Then we see that for each  $a \in L_G$ ,

$$\|g_a * K\|_{L^r(\mathbb{F}_q^d, dm)} \leq q^{\frac{d-1}{2}} \|(h_a d\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)}.$$

Hence, combining this with (3.4), the inequality (3.3) implies that

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{1}{2r}} q^{\frac{d-1}{4}} \left( \sum_{a \in L_G} \|(h_a d\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} \right)^{\frac{1}{2}}. \tag{3.6}$$

### 3.1 Proof of the statement (1) in Lemma 3.6

Since  $g$  is a regular function supported on the regular set  $G$ , it is clear from the definition of  $h_a$  that  $\frac{1}{2} \leq |h_a(\xi)| \leq 1$  on  $\text{supp}(h_a)$  and  $|\text{supp}(h_a)| = |\text{supp}(g_a)| = |G_a|$  for  $a \in L_G$ . Thus, using the assumption (3.2) with  $r = 4$ , the inequality (3.6) gives the desirable conclusion.

### 3.2 Proof of the statement (2) in Lemma 3.6

We shall appeal the following  $L^2 \rightarrow L^r$  extension result obtained by A. Lewko and M. Lewko (see Theorem 2 in [13]).

**Lemma 3.7** *Let  $P$  be the paraboloid in  $(\mathbb{F}_q^d, d\xi)$ . If  $d \geq 4$  is even, or if  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$  and  $-1 \in \mathbb{F}_q$  is not a square number, then we have*

$$R_P^* \left( 2 \rightarrow \frac{2d^2}{d^2 - 2d + 2} \right) \lesssim 1.$$

Applying this lemma to the inequality (3.6) with  $r = \frac{2d^2}{d^2 - 2d + 2}$ , it follows

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{d^2 + 2d - 2}{4d^2}} q^{\frac{d-1}{4}} \left( \sum_{a \in L_G} \|h_a\|_{L^2(P, d\sigma)} \right)^{\frac{1}{2}}.$$

By the Cauchy-Schwarz inequality and the definition of  $h_a$  given in (3.5), we conclude that

$$\begin{aligned} \|\widehat{g}\|_{L^2(P,d\sigma)} &\lesssim |G|^{\frac{1}{2}} + |G|^{\frac{d^2+2d-2}{4d^2}} q^{\frac{d-1}{4}} |L_G|^{\frac{1}{4}} \left( \sum_{a \in L_G} \|h_a\|_{L^2(P,d\sigma)}^2 \right)^{\frac{1}{4}} \\ &= |G|^{\frac{1}{2}} + |G|^{\frac{d^2+2d-2}{4d^2}} q^{\frac{d-1}{4}} |L_G|^{\frac{1}{4}} \left( \sum_{a \in L_G} \frac{1}{q^{d-1}} \sum_{n \in P} |h_a(n)|^2 \right)^{\frac{1}{4}} \\ &= |G|^{\frac{1}{2}} + |G|^{\frac{d^2+2d-2}{4d^2}} |L_G|^{\frac{1}{4}} \left( \sum_{a \in L_G} \sum_{n \in \mathbb{F}_q^d} |g_a(n)|^2 \right)^{\frac{1}{4}} \\ &= |G|^{\frac{1}{2}} + |G|^{\frac{d^2+2d-2}{4d^2}} |L_G|^{\frac{1}{4}} \left( \sum_{n \in \mathbb{F}_q^d} |g(n)|^2 \right)^{\frac{1}{4}} \\ &\leq |G|^{\frac{1}{2}} + |G|^{\frac{d^2+2d-2}{4d^2}} |L_G|^{\frac{1}{4}} |G|^{\frac{1}{4}} \lesssim |G|^{\frac{d^2+d-1}{2d^2}} |L_G|^{\frac{1}{4}}, \end{aligned}$$

where the last line follows because  $\frac{1}{2} \leq |g(n)| \leq 1$  on its support  $G$ . □

### 4 Proof of main theorems

First, let us see basic ideas to deduce our main results. We want to improve Lemma 3.7 which is the previously best known result on extension problems for paraboloids in higher dimensions. By duality, Lemma 3.7 implies the following restriction estimate:

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim \|g\|_{L^{\frac{2d^2}{d^2+2d-2}}(\mathbb{F}_q^d, dm)}. \tag{4.1}$$

Now let us only consider the regular function  $g$  on its support  $G$ . Since  $\|g\|_{L^p(\mathbb{F}_q^d, dm)} \sim |G|^{\frac{1}{p}}$ , when  $|G|$  is much bigger than  $q^{\frac{d^2}{2d-2}}$ , Lemma 3.3 already gives us a better result than (4.1). On the other hand, when  $|G|$  is very small, Lemma 3.5 yields very strong results. Therefore, our main task is to obtain much better estimate than (4.1) for every set  $G$  with  $q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon}$  for some  $\delta, \varepsilon > 0$ . This will be successfully done by applying Lemma 3.6. In practice, we need to find a  $U(|E|)$  in the conclusion of the first part of Lemma 3.6. To do this, we shall invoke the following additive energy estimates due to Iosevich and Koh (see Lemma 7, Lemma 8, and Remark 4 in [7]).

**Lemma 4.1** *Let  $P$  be the paraboloid in  $(\mathbb{F}_q^d, d\xi)$ . Then the following statements hold:*

(1) *If the dimension  $d \geq 4$  is even and  $E \subset P$ , then we have*

$$\Lambda(E) \lesssim \min\{|E|^3, q^{-1}|E|^3 + q^{\frac{d-2}{4}}|E|^{\frac{5}{2}} + q^{\frac{d-2}{2}}|E|^2\}$$

(2) *If  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then we have*

$$\Lambda_4(E) \lesssim \min\{|E|^3, q^{-1}|E|^3 + q^{\frac{d-3}{4}}|E|^{\frac{5}{2}} + q^{\frac{d-2}{2}}|E|^2\},$$

where  $\Lambda(E)$  denotes the additive energy defined as in (3.1).

As we shall see, we only need the upper bound of  $\Lambda(E)$  for a restricted range of  $E \subset P$ . Considering the dominating value in terms of  $|E|$ , the following result is a simple corollary of the lemma above.

**Corollary 4.2** *For the paraboloid  $P \subset (\mathbb{F}_q^d, d\xi)$ , we have the following facts:*

(1) *If the dimension  $d \geq 4$  is even and  $E$  is any subset of  $P$  with  $q^{\frac{d-2}{2}} \leq |E| \leq q^{\frac{d+2}{2}}$ , then*

$$\Lambda(E) \lesssim q^{\frac{d-2}{4}} |E|^{\frac{5}{2}}$$

(2) *Suppose that  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number. Then, for any subset  $E$  of  $P$  with  $q^{\frac{d-2}{2}} \leq |E| \leq q^{\frac{d+1}{2}}$ , we have*

$$\Lambda(E) \lesssim q^{\frac{d-3}{4}} |E|^{\frac{5}{2}} + q^{\frac{d-2}{2}} |E|^2.$$

We can deduce the following result by applying Corollary 4.2 to the first part of Lemma 3.6.

**Lemma 4.3** *Let  $g$  be a regular function with its support  $G \subset (\mathbb{F}_q^d, dm)$ . Then the following statements are valid:*

(1) *If the dimension  $d \geq 4$  is even and  $q^{\frac{d-2}{2}} \lesssim |G_a| \lesssim q^{\frac{d+2}{2}}$  for  $a \in L_G$ , then we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} |L_G|^{\frac{3}{16}} q^{\frac{-3d+6}{32}}$$

(2) *Assume that  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number. Then if  $q^{\frac{d-2}{2}} \lesssim |G_a| \lesssim q^{\frac{d+1}{2}}$  for  $a \in L_G$ , we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} |L_G|^{\frac{3}{16}} q^{\frac{-3d+5}{32}} + |G|^{\frac{5}{8}} |L_G|^{\frac{1}{4}} q^{\frac{-d+2}{16}}.$$

**Proof** For each  $a \in L_G$ , let  $h_a$  be the function on  $P$  given in the statement (1) of Lemma 3.6. For each  $a \in L_G$ , let  $H_a = \text{supp}(h_a)$ . Since  $\frac{1}{2} \leq |h_a| \leq 1$  on its support  $H_a$ , expanding  $L^4$  norm of  $(h_a d\sigma)^\vee$  gives

$$\|(h_a d\sigma)^\vee\|_{L^4(F_q^d, dm)} \leq \|(H_a d\sigma)^\vee\|_{L^4(F_q^d, dm)} = q^{\frac{-3d+4}{4}} (\Lambda(H_a))^{\frac{1}{4}}.$$

First, let us prove the first part of Lemma 4.3. Since  $|G_a| = |H_a|$  for  $a \in L_G$ , the first part of Corollary 4.2 and the above inequality yield

$$\|(h_a d\sigma)^\vee\|_{L^4(F_q^d, dm)} \lesssim q^{\frac{-3d+4}{4}} \left( q^{\frac{d-2}{4}} |H_a|^{\frac{5}{2}} \right)^{\frac{1}{4}} = q^{\frac{-11d+14}{16}} |H_a|^{\frac{5}{8}}.$$

By the definition of a regular set  $G$ , it is obvious that  $|G_a| \sim |G_{a'}|$  for  $a, a' \in L_G$ . Hence,  $|H_a| \sim |H_{a'}|$  for  $a, a' \in L_G$ . Thus, we can choose  $E \subset P$  such that  $|E| \sim |H_a|$  for all  $a \in L_G$ . It follows that

$$\max_{a \in L_G} \|(h_a d\sigma)^\vee\|_{L^4(F_q^d, dm)} \lesssim q^{\frac{-11d+14}{16}} |E|^{\frac{5}{8}} := U(|E|).$$

By applying the first part of Lemma 3.6 and observing that  $|G| \sim |G_a| |L_G| \sim |E| |L_G|$  for all  $a \in L_G$ , we conclude that

$$\begin{aligned} \|\widehat{g}\|_{L^2(P, d\sigma)} &\lesssim |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} \left( q^{\frac{-11d+14}{16}} |E|^{\frac{5}{8}} \right)^{\frac{1}{2}} \\ &\sim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} |L_G|^{\frac{3}{16}} q^{\frac{-3d+6}{32}}, \end{aligned}$$

which proves the first part of Lemma 4.3.

To prove the second part of Lemma 4.3, we use the same arguments as in the proof of the first part of Lemma 4.3. In this case, we just utilize the second part of Corollary 4.2 to see that

$$\begin{aligned} \max_{a \in L_G} \|(h_a d\sigma)^\vee\|_{L^4(F_q^d, dm)} &\lesssim q^{-\frac{3d+4}{4}} \left( q^{\frac{d-3}{4}} |E|^{\frac{5}{2}} + q^{\frac{d-2}{2}} |E|^2 \right)^{\frac{1}{4}} \\ &\sim q^{-\frac{3d+4}{4}} \left( q^{\frac{d-3}{16}} |E|^{\frac{5}{8}} + q^{\frac{d-2}{8}} |E|^{\frac{1}{2}} \right) \\ &= q^{-\frac{11d+13}{16}} |E|^{\frac{5}{8}} + q^{-\frac{5d+6}{8}} |E|^{\frac{1}{2}} := U(|E|). \end{aligned}$$

As before, we appeal the first part of Lemma 3.6 and use that  $|G| \sim |G_a| |L_G| \sim |E| |L_G|$  for all  $a \in L_G$ . Then the proof of the second part of Lemma 4.3 is complete as follows:

$$\begin{aligned} \|\widehat{g}\|_{L^2(P, d\sigma)} &\lesssim |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} \left( q^{-\frac{11d+13}{16}} |E|^{\frac{5}{8}} + q^{-\frac{5d+6}{8}} |E|^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\sim |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} \left( q^{-\frac{11d+13}{32}} |E|^{\frac{5}{16}} + q^{-\frac{5d+6}{16}} |E|^{\frac{1}{4}} \right) \\ &= |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} q^{-\frac{11d+13}{32}} |E|^{\frac{5}{16}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} q^{\frac{d-1}{4}} q^{-\frac{5d+6}{16}} |E|^{\frac{1}{4}} \\ &= |G|^{\frac{1}{2}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} |E|^{\frac{5}{16}} q^{-\frac{3d+5}{32}} + |G|^{\frac{3}{8}} |L_G|^{\frac{1}{2}} |E|^{\frac{1}{4}} q^{-\frac{d+2}{16}} \\ &\sim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} |L_G|^{\frac{3}{16}} q^{-\frac{3d+5}{32}} + |G|^{\frac{5}{8}} |L_G|^{\frac{1}{4}} q^{-\frac{d+2}{16}}. \end{aligned}$$

□

We are ready to complete the proof of our main theorems, Theorem 1.3 and Theorem 1.4, which will be proved in the following subsections.

### 4.1 Proof of Theorem 1.3

By duality and Lemma 3.2, it is enough to prove the following statement:

**Theorem 4.4** *If the dimension  $d \geq 6$  is even, then we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim \|g\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{F}_q^d, dm)}$$

for every regular function  $g$  supported on  $G \subset (\mathbb{F}_q^d, dm)$ .

**Proof** As mentioned in the beginning of this section, it is helpful to work on three kinds of regular functions  $g$  classified according to the following size of  $G = \text{supp}(g)$  : for some  $\varepsilon, \delta > 0$ ,

$$(1) \ 1 \leq |G| \leq q^{\frac{d^2}{2d-2}-\delta} \quad (2) \ q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon} \quad (3) \ q^{\frac{d^2}{2d-2}+\varepsilon} \leq |G| \leq q^d.$$

Notice that Lemma 3.2 yields much strong restriction inequality whenever  $|G|$  becomes larger. Thus, Lemma 3.2 is useful for the case (3). Also observe that Lemma 3.5 gives the better restriction inequality for smaller size of  $G$  and so it is helpful for the case (1). Thus, choosing big  $\varepsilon$  and  $\delta$  will yield good results for both the case (1) and the case (3). However, whenever  $\varepsilon$  and  $\delta$  become larger, the restriction estimate will be worse for the case (2). Hence, to deduce desirable results for all cases, our main task is to select optimal values of  $\varepsilon$  and  $\delta$ . Now, let

us see how to find the optimal  $\varepsilon$  and  $\delta$ . Let  $\varepsilon, \delta > 0$  which will be chosen later. Let  $g$  be a regular function with its support  $G$  such that

$$q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon}. \tag{4.2}$$

Let  $|L_G| = q^\alpha$  for  $0 \leq \alpha \leq 1$ . Since  $|G| \sim |G_a||L_G| = |G_a|q^\alpha$  for  $a \in L_G$ , it must follow that for every  $a \in L_G$ ,

$$q^{\frac{d^2}{2d-2}-\delta-\alpha} \lesssim |G_a| \lesssim q^{\frac{d^2}{2d-2}+\varepsilon-\alpha}.$$

In order to use the first part of Lemma 4.3, we need to choose  $\varepsilon, \delta > 0$  such that

$$q^{\frac{d-2}{2}} \leq q^{\frac{d^2}{2d-2}-\delta-\alpha} \lesssim |G_a| \lesssim q^{\frac{d^2}{2d-2}+\varepsilon-\alpha} \leq q^{\frac{d+2}{2}}.$$

Thus, if we select  $\varepsilon, \delta > 0$  satisfying that

$$\delta + \alpha \leq \frac{3d-2}{2d-2} \quad \text{and} \quad \varepsilon - \alpha \leq \frac{d-2}{2d-2}, \tag{4.3}$$

then the first part of Lemma 4.3 yields

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+12}{32}} \quad \text{for} \quad q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon}, \tag{4.4}$$

where we use the fact that  $|L_G| \leq q$ . Notice that this inequality gives worse restriction results whenever  $|G|$  becomes larger. Thus, comparing this inequality with Lemma 3.3 which gives better restriction inequality for big size of  $G$ , it is desirable to choose a possibly large  $\varepsilon > 0$  such that

$$|G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+12}{32}} \lesssim |G|^{\frac{1}{2}} q^{\frac{1}{2}} \quad \left(\text{namely, } |G| \lesssim q^{\frac{3d+4}{6}}\right) \quad \text{and} \quad |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon}.$$

For this reason, we take  $\varepsilon = \frac{d-4}{6d-6}$  which is positive for even  $d \geq 6$ . Then we can take  $\delta = \frac{d}{2d-2}$  so that the inequality (4.3) holds for all  $0 \leq \alpha \leq 1$ . Now we start proving Theorem 4.4.

**(Case I)** Assume that  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+4}{6}}$ , which is the case in (4.2) for  $\varepsilon = \frac{d-4}{6d-6}$  and  $\delta = \frac{d}{2d-2}$ . Then, by (4.4), we see that

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+12}{32}} \quad \text{for} \quad q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+4}{6}}.$$

By the direct comparison, it follows that for all  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+4}{6}}$ ,

$$|G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+12}{32}} \lesssim |G|^{\frac{3d+10}{6d+8}} = \|G\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{R}^d_q, dm)} \sim \|g\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{R}^d_q, dm)}.$$

Thus, the statement of Theorem 4.4 is valid for all regular functions  $g$  on  $(\mathbb{R}^d_q, dm)$  such that  $q^{\frac{d}{2}} \leq |\text{supp}(g)| = |G| \leq q^{\frac{3d+4}{6}}$ .

**(Case II)** Assume that  $1 \leq |G| \leq q^{\frac{d}{2}}$ . Applying Lemma 3.5, we obtain that

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim |G|^{\frac{1}{2}} + q^{\frac{-d+1}{4}} |G| \quad \text{for all } 1 \leq |G| \leq q^{\frac{d}{2}}.$$

In fact, this inequality gives much stronger restriction estimate than Theorem 4.4 for  $1 \leq |G| \leq q^{\frac{d}{2}}$ . By the direct comparison, if  $1 \leq |G| \leq q^{\frac{d}{2}}$ , then we have



$$|G|^{\frac{1}{2}} + q^{\frac{-d+1}{4}} |G| \lesssim |G|^{\frac{d+1}{2d}} = \|G\|_{L^{\frac{2d}{d+1}}(\mathbb{F}_q^d, dm)} \leq \|G\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{F}_q^d, dm)}.$$

Hence, Theorem 4.4 is proved in this case.

**(Case III)** Finally, assume that  $q^{\frac{3d+4}{6}} \leq |G| \leq q^d$ . In this case, by Lemma 3.3 and the direct comparison, the statement of Theorem 4.4 holds: for all  $q^{\frac{3d+4}{6}} \leq |G| \leq q^d$ ,

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim |G|^{\frac{1}{2}} q^{\frac{1}{2}} \lesssim \|G\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+8}{3d+10}}(\mathbb{F}_q^d, dm)}.$$

We has completed the proof. □

### 4.2 Proof of Theorem 1.4

Theorem 1.4 can be proved by following the same arguments as in the proof of Theorem 1.3 but we will need additional work to deal with a regular set  $G$  with middle size. The second part of Lemma 3.6 will make a crucial role in overcoming the problem. Now we start proving Theorem 1.4. By duality and Lemma 3.2, it suffices to prove the following statement:

**Theorem 4.5** *If  $d = 4\ell + 3$  for  $\ell \in \mathbb{N}$ , and  $-1 \in \mathbb{F}_q$  is not a square number, then we have*

$$\|\widehat{g}\|_{L^2(P, d\sigma)} \lesssim \|g\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)}$$

for every regular function  $g$  supported on  $G \subset (\mathbb{F}_q^d, dm)$ .

**Proof** As in the proof of Theorem 4.4, let  $g$  be a regular function supported on the set  $G \subset (\mathbb{F}_q^d, dm)$  satisfying that

$$q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon} \tag{4.5}$$

for some  $\varepsilon, \delta > 0$  which shall be selected as constants. Let  $|L_G| = q^\beta$  for  $0 \leq \beta \leq 1$ . Since  $|G| \sim |G_a||L_G| = |G_a|q^\beta$  for  $a \in L_G$ , it follows that for every  $a \in L_G$ ,

$$q^{\frac{d^2}{2d-2}-\delta-\beta} \lesssim |G_a| \lesssim q^{\frac{d^2}{2d-2}+\varepsilon-\beta}.$$

For such  $\varepsilon, \delta > 0$ , assume that for every  $a \in L_G$ ,

$$q^{\frac{d-2}{2}} \leq q^{\frac{d^2}{2d-2}-\delta-\beta} \lesssim |G_a| \lesssim q^{\frac{d^2}{2d-2}+\varepsilon-\beta} \leq q^{\frac{d+1}{2}}.$$

Namely, we assume that

$$\delta + \beta \leq \frac{3d-2}{2d-2} \quad \text{and} \quad \frac{1}{2d-2} \leq \beta - \varepsilon. \tag{4.6}$$

Then using the second part of Lemma 4.3, we have

$$\begin{aligned} \|\widehat{g}\|_{L^2(P, d\sigma)} &\lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} |L_G|^{\frac{3}{16}} q^{\frac{-3d+5}{32}} + |G|^{\frac{5}{8}} |L_G|^{\frac{1}{4}} q^{\frac{-d+2}{16}} \\ &\leq |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+11}{32}} + |G|^{\frac{5}{8}} q^{\frac{-d+6}{16}}, \end{aligned} \tag{4.7}$$

where we utilized the fact that  $|L_G| \leq q$ . As before, by comparing this estimate with Lemma 3.3, we select the  $\varepsilon > 0$  such that  $|G| \leq q^{\frac{3d+5}{6}} = q^{\frac{d^2}{2d-2}+\varepsilon}$ . Namely, we take  $\varepsilon = \frac{2d-5}{6d-6}$ . With this  $\varepsilon$ , if we choose  $\frac{1}{3} \leq \beta \leq 1$  and  $\delta = \frac{d}{2d-2}$ , then all conditions in (4.6) hold, because  $1 \leq |L_G| = q^\beta \leq q$ .

**Remark 4.6** In conclusion, we have seen that if  $g$  is a regular function with its support  $G \subset (\mathbb{F}_q^d, dm)$  such that  $q^{\frac{d^2}{2d-2}-\delta} \leq |G| \leq q^{\frac{d^2}{2d-2}+\varepsilon}$  and  $q^{\frac{1}{3}} \leq |L_G| \leq q$  for  $\varepsilon = \frac{2d-5}{6d-6}$  and  $\delta = \frac{d}{2d-2}$ , then the inequality (4.7) holds.

Now, we are ready to give the complete proof of Theorem 4.5.

**(Case 1)** Assume that  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+5}{6}}$  which is the case in (4.5) for  $\varepsilon = \frac{2d-5}{6d-6}$  and  $\delta = \frac{d}{2d-2}$ . In addition, assume that  $q^{\frac{1}{3}} \leq |L_G| \leq q$ . Then, by Remark 4.6 and the direct comparison, we see that if  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+5}{6}}$  and  $q^{\frac{1}{3}} \leq |L_G| \leq q$ , then for  $d \geq 7$ ,

$$\begin{aligned} \|\widehat{g}\|_{L^2(P,d\sigma)} &\lesssim |G|^{\frac{1}{2}} + |G|^{\frac{11}{16}} q^{\frac{-3d+11}{32}} + |G|^{\frac{5}{8}} q^{\frac{-d+6}{16}} \\ &\lesssim |G|^{\frac{3d+11}{6d+10}} = \|G\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)}. \end{aligned}$$

On the other hand, if  $1 \leq |L_G| \leq q^{\frac{1}{3}}$  and  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+5}{6}}$ , then we see from the second part of Lemma 3.6 and the direct comparison that

$$\begin{aligned} \|\widehat{g}\|_{L^2(P,d\sigma)} &\lesssim |G|^{\frac{d^2+d-1}{2d^2}} |L_G|^{\frac{1}{4}} \leq |G|^{\frac{d^2+d-1}{2d^2}} q^{\frac{1}{12}} \lesssim |G|^{\frac{3d^2+4d-3}{6d^2}} \\ &= \|G\|_{L^{\frac{6d^2}{3d^2+4d-3}}(\mathbb{F}_q^d, dm)} \leq \|G\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)}. \end{aligned}$$

Thus, Theorem 4.5 holds for all  $q^{\frac{d}{2}} \leq |G| \leq q^{\frac{3d+5}{6}}$ .

**(Case 2)** Assume that  $1 \leq |G| \leq q^{\frac{d}{2}}$ . In this case, Theorem 4.5 can be proved by using Lemma 3.5 and the direct comparison as follows:

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \lesssim |G|^{\frac{1}{2}} + q^{\frac{-d+1}{4}} |G| \lesssim |G|^{\frac{3d+11}{6d+10}} = \|G\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)}.$$

**(Case 3)** Assume that  $q^{\frac{3d+5}{6}} \leq |G| \leq q^d$ . In this case, the statement of Theorem 4.5 holds by Lemma 3.3 and the direct comparison as follows:

$$\|\widehat{g}\|_{L^2(P,d\sigma)} \leq q^{\frac{1}{2}} |G|^{\frac{1}{2}} \lesssim |G|^{\frac{3d+11}{6d+10}} = \|G\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)} \sim \|g\|_{L^{\frac{6d+10}{3d+11}}(\mathbb{F}_q^d, dm)}.$$

By Cases 1, 2, and 3, the proof of Theorem 4.5 is complete. □

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