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An adelic arithmeticity theorem for lattices in products

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Abstract

In this paper we prove that, under mild assumptions, a lattice in a product of semi-simple Lie group and a totally disconnected locally compact group is, in a certain sense, arithmetic. We do not assume the lattice to be finitely generated or the ambient group to be compactly generated.

Keywords Lattice · Algebraic group · Arithmetic group · Adels · tdlc group

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1 Introduction and statement of the main result

1.1 Motivation

Our goal in this paper is to show that, under mild conditions, a lattice Γ in a product of a semi-simple Lie group H and a totally disconnected locally compact group D is—what we will call—*arithmetically constructed*. Under the assumption that $\Gamma < H \times D$ is irreducible and Γ is finitely generated or D is compactly generated such arithmeticity results are obtained by Caprace–Monod in [8]. In this work we drop these assumptions and obtain an adelic arithmeticity result for lattices in $H \times D$. An application will appear in [2].

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The exact formulation of the main result is given in Theorem 1.6 below. Before formally stating it we will provide some motivation and an idea of what *arithmetically constructed* means. A prototypical example is the lattice embedding

$$\mathbf{PGL}_2(\mathbb{Q}) < \mathbf{PGL}_2(\mathbb{A}) \cong \mathbf{PGL}_2(\mathbb{R}) \times \prod_p{}' \mathbf{PGL}_2(\mathbb{Q}_p).$$

Here the factor $\mathbf{PGL}_2(\mathbb{R})$ is a connected, center-free semi-simple, real Lie group. Further, $\mathbb{A}$ denotes the ring of adeles. By $\prod'$ we denote restricted products in the sense of adeles: for almost all places p the corresponding element lies in $\mathbf{PGL}_2(\mathbb{Z}_p)$. A similar example is

$$\mathbf{SL}_2(\mathbb{Q}) < \mathbf{SL}_2(\mathbb{A}) \cong \mathbf{SL}_2(\mathbb{R}) \times \prod_p{}' \mathbf{SL}_2(\mathbb{Q}_p).$$

Despite the similarity, note that the image of $\mathbf{SL}_2(\mathbb{Q})$ in $\mathbf{PGL}_2(\mathbb{A})$ under the obvious map

$$\mathbf{SL}_2(\mathbb{Q}) \to \mathbf{SL}_2(\mathbb{A}) \to \mathbf{PGL}_2(\mathbb{A})$$

is not a lattice. Indeed, it is of infinite index in the lattice $\mathbf{PGL}_2(\mathbb{Q})$, as can be seen by the exact sequence

$$1 \to \{+1, -1\} \to \mathbf{SL}_2(\mathbb{Q}) \to \mathbf{PGL}_2(\mathbb{Q}) \to \mathbb{Q}^*/(\mathbb{Q}^*)^2 \to 1.$$

Intermediate subgroups between $\mathbf{PGL}_2(\mathbb{Q})$ and the image of $\mathbf{SL}_2(\mathbb{Q})$ are arithmetically constructed lattices in the sense of Definition 1.4. A special case of our main result identifies, under mild conditions, an arbitrary lattice in $\mathbf{PGL}_2(\mathbb{R}) \times D$ with an arithmetically constructed lattice.

1.2 Notation and basic setup

Throughout, let K be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{V}$ be the set of inequivalent valuations (places) of K, let $\mathcal{V}^\infty$ denote the archimedean ones, and $\mathcal{V}^{\text{fin}} = \mathcal{V} - \mathcal{V}^\infty$ the non-archimedean ones (finite places). Given a subset $S \subset \mathcal{V}$ we set $S^\infty = S \cap \mathcal{V}^\infty$ and $S^{\text{fin}} = S \cap \mathcal{V}^{\text{fin}}$. We let

$$\mathcal{O}[S] := \{x \in K \mid |x|_\nu \le 1 \text{ for all } \nu \in \mathcal{V}^{\text{fin}} - S\}$$

be the ring of S-integers in K. For a semi-simple K-algebraic group $\mathbf{H}$ we denote by $\mathbf{H}_S$ the (truncated) S-adeles of $\mathbf{H}$, that is, the restricted product

$$\mathbf{H}_S := \Big\{(x_\nu) \in \prod_{\nu \in S} \mathbf{H}(K_\nu) \mid x_\nu \in \mathbf{H}(\mathcal{O}_\nu) \text{ for almost all } \nu \in S\Big\}.$$

We let $\mathbf{H}_S^\#$ be the closure in $\mathbf{H}_S$ of the image of $\mathbf{H}(K)$ under the diagonal embedding.

For a field extension k of K we denote by $\mathbf{H}(k)^+ < \mathbf{H}(k)$ the normal subgroup as defined in [5, Section 6]. Since k is perfect, $\mathbf{H}(k)^+$ is the subgroup generated by all unipotent elements and it has finite index in $\mathbf{H}(k)$. If $\nu \in \mathcal{V}^\infty$, then $\mathbf{H}(K_\nu)^+$ is just the connected component of the identity in the real Lie group $\mathbf{H}(K_\nu)$. We denote by $\mathbf{H}_S^+$ the restricted product

$$\mathbf{H}_S^+ := \Big\{(x_\nu) \in \prod_{\nu \in S} \mathbf{H}(K_\nu)^+ \mid x_\nu \in \mathbf{H}(\mathcal{O}_\nu) \text{ for almost all } \nu \in S\Big\}.$$

Note that if S is finite, then

$$\mathbf{H}_S = \prod_{\nu \in S} \mathbf{H}(K_\nu), \qquad \mathbf{H}_S^+ = \prod_{\nu \in S} \mathbf{H}(K_\nu)^+.$$

In this case $\mathbf{H}_S^+ < \mathbf{H}_S$ has finite index. In general, this is false for infinite S.

Definition 1.1 In the above context S is said to be *compatible* with $\mathbf{H}$ if the following hold:

(1) For every $v \in S$, $\mathbf{H}$ is K_v-isotropic.
(2) S contains all $v \in \mathcal{V}^\infty$ for which $\mathbf{H}$ is K_v-isotropic.
(3) S contains at least one finite and one infinite place.

The following group-theoretic notion plays an important role.

Definition 1.2 Two subgroups of an (abstract) group are *commensurable* if their intersection has finite index in both subgroups. A subgroup H of a group Λ is *commensurated* if H and any conjugate $\lambda H \lambda^{-1}$ are commensurable.

One easily sees that commensurability is an equivalence relation.

1.3 Arithmetically constructed lattices

The definition of arithmetically constructed lattices builds on the following proposition which we prove in Sect. 4.

Proposition 1.3 *For every $i \in \{1, \ldots, n\}$ let K_i be a number field, $\mathbf{H}_i$ be a connected, non-commutative, absolutely simple, adjoint K_i-group, and $S_i \subset \mathcal{V}_i$ be a (possibly infinite) subset of places of K_i which is compatible with $\mathbf{H}_i$. Then the following hold.*

(1) $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^+$ is an open subgroup of $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$ for every $i \in \{1, \ldots, n\}$.
(2) For every subgroup

$$A < \prod_{i=1}^n \left((\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#} \Big/ (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^+ \right)$$

let $D_A < \prod_{i=1}^n (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$ be the preimage of A under the quotient map, and let $\Gamma_A < \prod_{i=1}^n \mathbf{H}_i(\mathcal{O}_i[S_i])$ be the preimage of D_A under the product of diagonal embeddings $\mathbf{H}_i(\mathcal{O}_i[S_i]) \to (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$. The subgroup obtained via the diagonal embedding

$$\Gamma_A < \prod_{i=1}^n (\mathbf{H}_i)_{S_i^\infty} \times D_A$$

is a lattice.

Definition 1.4 A subgroup of $\prod_{i=1}^n \mathbf{H}_i(K_i)$ is an *arithmetically constructed lattice* if it contains the group Γ_A from Proposition 1.3 as a subgroup of finite index for some subgroup

$$A < \prod_{i=1}^n \left((\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#} \Big/ (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^+ \right).$$

Example 1.5 We continue the example from Sect. 1.1. Here $K = K_1 = \mathbb{Q}$, $n = 1$, $\mathbf{H} = \mathbf{H}_1 = \mathbf{PGL}_2$ and $S^{\mathrm{fin}} = S_1^{\mathrm{fin}}$ consists of all finite primes. Let $\mathbb{A}_{\mathrm{fin}} = \prod_{p \neq \infty}' \mathbb{Q}_p$ be the ring of finite rational adeles. The determinant morphism $\det \colon \mathbf{G}_m(\mathbb{A}_{\mathrm{fin}}) \to \mathbb{A}_{\mathrm{fin}}^* / (\mathbb{A}_{\mathrm{fin}}^*)^2$ modulo squares descends to a morphism from $\mathbf{PGL}_2(\mathbb{A}_{\mathrm{fin}})$. The group $\mathbb{A}_{\mathrm{fin}}^*$ of ideles is the subset of

$(x_p) \in \prod_{p \neq \infty} \mathbb{Q}_p^*$ such that $x_p \in \mathbb{Z}_p^*$ for almost all primes p. Identifying the p-adic units $\mathbb{Q}_p^*$ with $\mathbb{Z}_p^* \times \mathbb{Z}$ we obtain an isomorphism

$$\mathbf{G}_m(\mathbb{A}_{\mathrm{fin}}) \cong \prod_{p \neq \infty} \mathbb{Z}_p^* \times \bigoplus_{p \neq \infty} \mathbb{Z}$$

of topological groups and a commutative diagram of topological groups:

$$
\begin{array}{ccccc}
\mathbf{GL}_2(\mathbb{A}_{\mathrm{fin}}) & =\!=\!= & \prod' \mathbf{GL}_2(\mathbb{Q}_p) & \xrightarrow{\ \det\ } & \prod_{p \neq \infty} \mathbb{Z}_p^* \times \bigoplus_{p \neq \infty} \mathbb{Z} \\
\downarrow & & & & \downarrow \\
\mathbf{PGL}_2(\mathbb{A}_{\mathrm{fin}}) & =\!=\!= & \prod' \mathbf{PGL}_2(\mathbb{Q}_p) & \xrightarrow{\ \det\ } & \prod_{p \neq \infty} (\mathbb{Z}_p^*)/(\mathbb{Z}_p^*)^2 \times \bigoplus_{p \neq \infty} \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

In this example we have

$$\mathbf{H}_{S^{\mathrm{fin}}}^{\#} = \det^{-1}\Big(\{1\} \times \bigoplus_{p \neq \infty} \mathbb{Z}/2\mathbb{Z}\Big)$$

$$\mathbf{H}_{S^{\mathrm{fin}}}^{+} = \ker \det .$$

Thus the quotient $\mathbf{H}_{S^{\mathrm{fin}}}^{\#}/\mathbf{H}_{S^{\mathrm{fin}}}^{+}$ appearing in the previous definition is here isomorphic to $\bigoplus_{p \neq \infty} \mathbb{Z}/2\mathbb{Z}$.

1.4 Main result

The main goal of this paper is to establish the following partial converse of Proposition 1.3 which is an arithmeticity theorem for lattices in products. We use the abbreviation *tdlc* for totally disconnected locally compact groups.

Theorem 1.6 *Let H be a connected, center-free, semi-simple, real Lie group without compact factors, D be a tdlc group and $\Gamma < H \times D$ be a lattice. Assume that*

(1) the projection $\Gamma \to D$ has a dense image,
(2) the projection $\Gamma \to H$ has a dense image,
(3) the projection $\Gamma \to H$ is injective.

Then there exist number fields $K_1, \ldots, K_n$, connected, non-commutative, absolutely simple, adjoint K_i-groups $\mathbf{H}_i$, and (possibly infinite) compatible subsets $S_i \subset \mathcal{V}_i$ of places of K_i (see Definition 1.1) with the following properties: There is a topological isomorphism

$$H \cong \prod_{i=1}^{n} (\mathbf{H}_i)_{S_i^{\infty}}^{0}$$

and a short exact sequence $1 \to C \to D \to Q \to 1$ of topological groups, where C is compact, and Q is a closed intermediate subgroup

$$\prod_{i=1}^{n} (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+} < Q < \prod_{i=1}^{n} (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$$

so that the inclusion $\Gamma < H \times D$ is contained in the image of

$$\prod_{i=1}^{n} \mathbf{H}_i(K_i) \to \prod_{i=1}^{n} (\mathbf{H}_i)_{S_i},$$

and, identifying Γ with a subgroup of $\prod_{i=1}^{n} \mathbf{H}_i(K_i)$, Γ is an arithmetically constructed lattice as in Definition 1.4.

Furthermore, compact generation of D, finite generation of Γ, and finiteness of all $S_i \subset \mathcal{V}_i$, are equivalent conditions. If they hold, each $(\mathbf{H}_i)_{S_i}^{+}$ has finite index in $(\mathbf{H}_i)_{S_i}$, so up to commensurability Γ is the product of S_i-arithmetic lattices $\prod_{i=1}^{n} \mathbf{H}_i(\mathcal{O}_i[S_i])$ diagonally embedded in

$$H \times D/C \simeq \prod_{i=1}^{n} \mathbf{H}_{S_i} \simeq \prod_{i=1}^{n} \mathbf{H}_{S_i}^{+}$$

where $\simeq$ means being commensurable.

Remark 1.7 This result is an important ingredient in our work on lattice envelopes [2] (see also the announcement [1]). See especially Sect. 1.4 in [2] for an overview of the proof of the main structural result in *loc. cit.* and how the previous theorem enters.

In this application finite generation and irreducibility of Γ are not known a priori.

For finitely generated Γ or compactly generated D, such results are proved by Caprace and Monod [8, Section 5]. Our approach is different except for step (i) below that uses Margulis' commensurator criterion for arithmeticity.

An adelic analog of Margulis' arithmeticity where one assumes the ambient group of algebraic origin is proved by Oh [13].

1.5 Strategy of proof

The proof of Theorem 1.6 can be roughly divided into the following steps:

(1) Choosing a compact open subgroup $U < D$, we observe that the projection Δ of $\Gamma \cap (H \times U)$ to H is a lattice in H, commensurated by the projection Λ of all of Γ to H. Then Margulis' commensurator superrigidity theorem provides the arithmetic structure of Δ yielding the algebraic groups $\mathbf{H}_1, \ldots, \mathbf{H}_n$, the product of which defines the arithmetic structure on H. (The possible multitude of these groups corresponds to the possibility that Δ is reducible).

(2) The commensurator of Δ in H can be identified with the image of $\prod \mathbf{H}_i(K_i)$, and $\Lambda \cong \Gamma$ is viewed as a subgroup of the latter. Considering the projections of Γ to various finite places we identify (see Theorem 3.7) the sets S_i^{fin} of the non-archimedean places, so that on one hand

$$\Gamma < \prod_{i=1}^{n} \mathbf{H}_i(\mathcal{O}_i[S_i]),$$

and on the other the closure $\overline{\Gamma}$ of the projection of Γ to $\prod_{i=1}^{n}(\mathbf{H}_i)_{S_i}$ contains $\prod_{i=1}^{n}(\mathbf{H}_i)_{S_i}^{+}$. (If S_i is finite then $(\mathbf{H}_i)_{S_i}^{+}$ has finite index in $(\mathbf{H}_i)_{S_i}$).

(3) Next we construct a continuous surjective homomorphism with compact kernel from D to the closure of the diagonal image of Γ in $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$. This step is conceptualized in the notion of *representations of commensurable pairs* (cf. Sect. 5). This identifies Γ as an arithmetically constructed lattice.

(4) Finally, finite generation of Γ, finiteness of each S_i, and compact generation of D turn out to be equivalent.

2 Generalities on lattices

For later use we collect here some lemmas about lattices in locally compact groups.

Lemma 2.1 [15, Theorem 1.13 on p. 23] *Let G be a locally compact second countable group and $N \lhd G$ a normal closed subgroup. The projection of a lattice $\Gamma < G$ to G/N is discrete if and only if $\Gamma \cap N$ is a lattice in N. If so, then the projection of Γ to G/N is also a lattice.*

Lemma 2.2 [15, Lemma 1.6 on p. 20] *Let G be a locally compact group, and $H < L < G$ closed subgroups. Then $H < G$ has finite covolume if and only if both $L < G$ and $H < L$ have finite covolumes.*

The following easy lemma is well known (cf. [7, 2.C]).

Lemma 2.3 *Let $\Gamma < G$ be a lattice in a locally compact second countable group. Let $U < G$ be an open subgroup. Then $\Gamma \cap U$ is a lattice in U.*

3 Around Margulis' super-rigidity for commensurators

Throughout, let K be a number field and $\mathbf{H}$ a connected almost simple K-algebraic group. We also retain the notation from Sect. 1.2.

Let ν be a place of K. Whenever we consider the $\mathcal{O}$-points or $\mathcal{O}_\nu$-points of $\mathbf{H}$ for some place ν we pick an embedding $\mathbf{H} < \mathbf{GL}_n$ in advance and set $\mathbf{H}(\mathcal{O}) = \mathbf{GL}_n(\mathcal{O}) \cap \mathbf{H}(K)$ and similarly for $\mathcal{O}_\nu$. Let $\tilde{\mathbf{H}}$ be the simply connected cover of $\mathbf{H}$ (in the algebraic sense) and $i : \tilde{\mathbf{H}} \to \mathbf{H}$ be the canonical central isogeny. Since i is defined over K but not necessarily over $\mathcal{O}_\nu$ for a place ν of K we do not necessarily have $i(\tilde{\mathbf{H}}(\mathcal{O}_\nu)) \subset \mathbf{H}(\mathcal{O}_\nu)$. But according to the following result there are only finitely many exceptional places.

Lemma 3.1 [12, Lemma 6.6 on p. 295] *For all but finitely many finite places ν, the central isogeny i is defined over $\mathcal{O}_\nu$, thus $i(\tilde{\mathbf{H}}(\mathcal{O}_\nu)) \subset \mathbf{H}(\mathcal{O}_\nu)$.*

Let E_1 be the set of exceptional places from the previous lemma, and let E_2 be the of exceptional places described by the next lemma.

Lemma 3.2 [16, 3.9.1 on p. 55] *For all but finitely many finite places ν, $\mathbf{H}(\mathcal{O}_\nu) < \mathbf{H}(K_\nu)$ is a maximal compact subgroup. Similarly for $\tilde{\mathbf{H}}$.*

Let $\nu \notin E_1 \cup E_2$. Then $i^{-1}(\mathbf{H}(\mathcal{O}_\nu))$ is a compact subgroup that contains $\tilde{\mathbf{H}}(\mathcal{O}_\nu)$. Hence it coincides with it. We record this for later use:

Corollary 3.3 *For all but finitely many finite places ν, $i^{-1}(\mathbf{H}(\mathcal{O}_\nu)) = \tilde{\mathbf{H}}(\mathcal{O}_\nu)$.*

Lemma 3.4 *Let $I \subset \mathcal{V}^{\mathrm{fin}}$ be the set of finite places ν of K such that $\mathbf{H}$ is K_ν-isotropic. Let $J \subset I$. Assume $\mathbf{H}_{\mathcal{V}^\infty}$ is not compact. Then:*

(1) The image of $i : \tilde{\mathbf{H}}_J \to \mathbf{H}_J$ equals $\mathbf{H}_J^+$.
(2) $\mathbf{H}_J^+ < \mathbf{H}_J$ is closed, and the map $\tilde{\mathbf{H}}_J \to \mathbf{H}_J^+$ is open.
(3) $\mathbf{H}_J^+$ is an open subgroup of $\mathbf{H}_J^\#$.

Proof (1) For every $v \in J$ by [11, Corollary I.2.3.1(a)] and [11, Corollary I.1.5.5], $i(\tilde{\mathbf{H}}(K_v)) = i(\tilde{\mathbf{H}}(K_v)^+) = \mathbf{H}(K_v)^+$. By [11, Corollary I.2.3.1(b)] $\mathbf{H}(K_v)^+$ is closed in $\mathbf{H}(K_v)$. With Lemma 3.1 we obtain that $i(\tilde{\mathbf{H}}_J) \subset \mathbf{H}_J^+$. For the reverse inclusion, consider an element $(x_v)_{v \in J} \in \mathbf{H}_J^+$. For each $v \in J$ let $y_v \in \tilde{\mathbf{H}}(K_v)$ be a pre-image of x_v, so $i(y_v) = x_v$. Since $x_v \in \mathbf{H}(\mathcal{O}_v)$ for all but finitely many v, we also have $y_v \in \tilde{\mathbf{H}}(\mathcal{O}_v)$ for all but finitely many v. In other words, $(y_v)_{v \in J} \in \tilde{\mathbf{H}}_J$, and we proved that $i(\tilde{\mathbf{H}}_J) = \mathbf{H}_J^+$.

(2) Note that $\mathbf{H}_J$ as a topological space is the colimit of $\mathbf{H}_S^+ \times \prod_{J-S} \mathbf{H}(\mathcal{O}_v)$ running over finite subsets $S \subset J$. The subgroup $\mathbf{H}_J^+ < \mathbf{H}_J$ is closed since for every finite subset $S \subset J$ the intersection

$$\mathbf{H}_J^+ \cap \left(\mathbf{H}_S^+ \times \prod_{J-S} \mathbf{H}(\mathcal{O}_v) \right) = \mathbf{H}_S^+ \times \left(\prod_{J-S} \mathbf{H}(\mathcal{O}_v) \cap \mathbf{H}(K_v)^+ \right)$$

is closed being the product of a closed and compact groups. By the open mapping theorem [6, IX. §5.3] we conclude that the map $\tilde{\mathbf{H}}_J \to \mathbf{H}_J^+$ is open.

(3) By [12, Theorem 7.12] the simply connected group $\tilde{\mathbf{H}}$ satisfies the strong approximation property. Hence the image of $\tilde{\mathbf{H}}(K)$ in $\tilde{\mathbf{H}}_J$ is dense which implies that $\mathbf{H}_J^+$ is a subgroup of $\mathbf{H}_J^{\#}$. To show that it is an open subgroup we first consider the case $J = \mathcal{V}^{\text{fin}}$:

Let $U = \prod_{v \in \mathcal{V}^{\text{fin}}} \mathbf{H}(\mathcal{O}_v)$; it is a compact open subgroup of $\mathbf{H}_{\mathcal{V}^{\text{fin}}}$. By Corollary 3.3 there is a finite subset $E \subset \mathcal{V}^{\text{fin}}$ of exceptional places and compact open subgroups $C_v < \tilde{\mathbf{H}}(K_v)$ for $v \in E$ so that

$$i^{-1}(U) = \prod_{v \in \mathcal{V}^{\text{fin}} \setminus E} \tilde{\mathbf{H}}(\mathcal{O}_v) \times \prod_{v \in E} C_v$$

and $i(\tilde{\mathbf{H}}(\mathcal{O}_v)) = \mathbf{H}(\mathcal{O}_v)$ if $v \in \mathcal{V}^{\text{fin}} \setminus E$. In particular, $i^{-1}(U)$ and $\prod_{v \in \mathcal{V}^{\text{fin}}} \tilde{\mathbf{H}}(\mathcal{O}_v)$ are commensurable. Further, the groups $i(\tilde{\mathbf{H}}(\mathcal{O}))$ and $\mathbf{H}(\mathcal{O})$ are commensurable within $\mathbf{H}(K)$ by [11, Corollary I.3.2.9], and $\mathbf{H}(\mathcal{O})$ is dense in $\mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#} \cap \prod_{v \in \mathcal{V}^{\text{fin}}} \mathbf{H}(\mathcal{O}_v)$. We prove that $i(\tilde{\mathbf{H}}_{\mathcal{V}^{\text{fin}}}) = \mathbf{H}_{\mathcal{V}^{\text{fin}}}^+$ is open in $\mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#}$ by proving that the closed subgroup (see (2)) $\mathbf{H}_{\mathcal{V}^{\text{fin}}}^+ \cap U < \mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#} \cap U$ has finite index, thus is open. This follows from the following commutative diagram which collects the information obtained so far.

$$
\begin{array}{ccc}
\tilde{\mathbf{H}}(\mathcal{O}) & \hookrightarrow & \prod_{v \in \mathcal{V}^{\text{fin}}} \tilde{\mathbf{H}}(\mathcal{O}_v) \\
\downarrow & & \updownarrow \text{commen.} \\
& & i^{-1}(U) \hookrightarrow \tilde{\mathbf{H}}_{\mathcal{V}^{\text{fin}}} \\
& & \downarrow i \qquad\qquad \downarrow i \\
i(\tilde{\mathbf{H}}(\mathcal{O})) & & \mathbf{H}_{\mathcal{V}^{\text{fin}}}^+ \cap U \hookrightarrow \mathbf{H}_{\mathcal{V}^{\text{fin}}}^+ \\
\updownarrow \text{commen.} & & \downarrow \qquad\qquad \downarrow \\
\mathbf{H}(\mathcal{O}) & \xrightarrow{\text{dense}} & \mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#} \cap U \hookrightarrow \mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#}
\end{array}
$$

The diagram shows that there is a subgroup $L < i(\tilde{\mathbf{H}}(\mathcal{O})) \cap \mathbf{H}(\mathcal{O}) < \mathbf{H}(K)$ that has finite index in $\mathbf{H}(\mathcal{O})$ such that the induced homomorphism

$$\mathbf{H}(\mathcal{O})/L \to \mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#} \cap U / \mathbf{H}_{\mathcal{V}^{\text{fin}}}^+ \cap U$$

has dense image. Since the left hand side is finite, it is surjective and so the right hand side is finite too.

Next we show the openness statement for general J. Note that the projection $\mathbf{H}_{\mathcal{V}^{\text{fin}}} \to \mathbf{H}_J$, which is locally given as the proper homomorphism $\prod_{\nu \in \mathcal{V}^{\text{fin}}} \mathbf{H}(\mathcal{O}_\nu) \to \prod_{\nu \in J} \mathbf{H}(\mathcal{O}_\nu)$, sends closed groups to closed groups. It follows that the image of $\mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#}$ is $\mathbf{H}_J^{\#}$, and $p : \mathbf{H}_{\mathcal{V}^{\text{fin}}}^{\#} \to \mathbf{H}_J^{\#}$ is open by the open mapping theorem [6, IX. §5.3]. We conclude that of $\mathbf{H}_J^{+} = i(\tilde{\mathbf{H}}_J) = p(i(\tilde{\mathbf{H}}_{\mathcal{V}^{\text{fin}}})) < \mathbf{H}_J^{\#}$ is open.

Lemma 3.5 *Let K be a number field and $\mathbf{H}$ be a connected, almost simple K-algebraic group. Assume $\mathbf{H}_{\mathcal{V}^\infty}$ is not compact. Let $I \subset \mathcal{V}^{\text{fin}}$ be the set of non-archimedean places ν such that $\mathbf{H}$ is K_ν-isotropic. Let Σ be a subgroup of $\mathbf{H}(K)$ which contains a finite index subgroup of $\mathbf{H}(\mathcal{O})$. Then the closure of the image of Σ in $\mathbf{H}_I$ contains an open subgroup of $\mathbf{H}_I^{+}$.*

Proof We let $i : \tilde{\mathbf{H}} \to \mathbf{H}$ be the canonical central isogeny, where $\tilde{\mathbf{H}}$ denotes the simply connected cover of $\mathbf{H}$. By [12, Theorem 7.12] the simply connected group $\tilde{\mathbf{H}}$ satisfies the strong approximation property. In particular, the image of $\tilde{\mathbf{H}}(K)$ in $\tilde{\mathbf{H}}_I$ is dense. It follows that the image of $\tilde{\mathbf{H}}(\mathcal{O})$ is dense in $\prod_{\nu \in I} \tilde{\mathbf{H}}(\mathcal{O}_\nu)$. Therefore the image of any finite index subgroup of $\tilde{\mathbf{H}}(\mathcal{O})$ has an open closure in $\tilde{\mathbf{H}}_I$, and so is the closure of the image of $i^{-1}(\Sigma)$. By Lemma 3.4 the image of $\tilde{\mathbf{H}}_I \to \mathbf{H}_I$ is $\mathbf{H}_I^{+}$ and the map $\tilde{\mathbf{H}}_I \to \mathbf{H}_I^{+}$ is open, hence we are done by the commutativity of the obvious diagram

$$
\begin{array}{ccc}
\tilde{\mathbf{H}}(K) & \longrightarrow & \tilde{\mathbf{H}}_I \\
\downarrow & & \downarrow \\
\mathbf{H}(K) & \longrightarrow & \mathbf{H}_I
\end{array}
$$

$\square$

Theorem 3.6 *Let K be a number field and $\mathbf{H}$ be a connected, adjoint, almost simple K-algebraic group. Assume $\mathbf{H}_{\mathcal{V}^\infty}$ is not compact. Let Σ be a subgroup of $\mathbf{H}(K)$ which contains a finite index subgroup of $\mathbf{H}(\mathcal{O})$. Let S be the set of places ν of K such that Σ is unbounded in $\mathbf{H}(K_\nu)$. Then the following hold:*

(1) The closure of the projection of Σ to $\mathbf{H}_{S^{\text{fin}}}$ contains $\mathbf{H}_{S^{\text{fin}}}^{+}$ and it has no non-trivial compact normal subgroups.

(2) The closure of the projection of Σ to $\mathbf{H}_{\mathcal{V}-S}$ is compact.

(3) A finite index subgroup of Σ is contained in $\mathbf{H}(\mathcal{O}[S])$.

Proof In the proof we will use L_J to denote the closure of the diagonal embedding of Σ in $\mathbf{H}_J$ for $J \subset \mathcal{V}$. We will also use $L_\nu = L_{\{\nu\}}$.

(1) For $\nu \in S$, L_ν is unbounded. By [11, Corollary I.2.3.1(c2)], $\mathbf{H}(K_\nu)/\mathbf{H}(K_\nu)^{+}$ is finite, thus $L_\nu \cap \mathbf{H}(K_\nu)^{+}$ is unbounded. By Lemma 3.5, $L_\nu \cap \mathbf{H}(K_\nu)^{+}$ is open in $\mathbf{H}(K_\nu)^{+}$. It follows from [14, Theorem T] that $\mathbf{H}(K_\nu)^{+} < L_\nu$. The group generated by the groups $\mathbf{H}(K_\nu)^{+}$ for $\nu \in S$ is dense in $\mathbf{H}_S^{+}$. Since L_S contains an open subgroup of $\mathbf{H}_S^{+}$ by Lemma 3.5, it actually contains $\mathbf{H}_S^{+}$, that is $\mathbf{H}_S^{+} < L_S < \mathbf{H}_S$. Assume $N < L_S$ is a non-trivial compact normal subgroup. Then there exists $\nu \in S$ such that the image of N, $\bar{N}$, is non trivial under the projection $\mathbf{H}_S \to \mathbf{H}(K_\nu)$. By [11, Theorem I.1.5.6(i)], either $\mathbf{H}(K_\nu)^{+} < \bar{N}$ or $\bar{N}$ is central. Both cases are impossible, the first by the compactness of N and the second by $\mathbf{H}$ being adjoint.

(2) First we define two finite exceptional sets $F_1, F_2 \subset \mathcal{V}^{\text{fin}}$ of finite places. We let $i : \tilde{\mathbf{H}} \to \mathbf{H}$ again denote the canonical central isogeny of the simply connected form $\tilde{\mathbf{H}}$ of $\mathbf{H}$.

According to Lemma 3.2 and Corollary 3.3 there is a finite set F_1 of finite places such that for $\nu \in \mathcal{V}^{\text{fin}} - F_1$ we have that $i^{-1}(\mathbf{H}(\mathcal{O}_\nu)) = \tilde{\mathbf{H}}(\mathcal{O}_\nu)$ and both $\tilde{\mathbf{H}}(\mathcal{O}_\nu)$ and $\mathbf{H}(\mathcal{O}_\nu)$ are

maximal compact subgroups in $\tilde{\mathbf{H}}(K_\nu)$ and $\mathbf{H}(K_\nu)$, respectively. For $\nu \in \mathcal{V}^{\mathrm{fin}} - F_1$, $\mathbf{H}(\mathcal{O}_\nu)$ is the unique compact open subgroup of $\mathbf{H}(K_\nu)$ that contains $i(\tilde{\mathbf{H}}(\mathcal{O}_\nu))$. To see this consider the Bruhat-Tits building of $\mathbf{H}(K_\nu)$ with its $\tilde{\mathbf{H}}(K_\nu)$-action via i. The subgroup $\tilde{\mathbf{H}}(\mathcal{O}_\nu)$ has a unique fixed point, hence it is contained in a unique maximal compact of $\mathbf{H}(K_\nu)$.

Next we define F_2. By [12, Theorem 7.12] the simply connected group $\tilde{\mathbf{H}}$ satisfies the strong approximation property. In particular, the image of $\tilde{\mathbf{H}}(K)$ in $\tilde{\mathbf{H}}_{\mathcal{V}^{\mathrm{fin}}}$ is dense. It follows that the image of $\tilde{\mathbf{H}}(\mathcal{O})$ is dense in $\prod_{\nu \in \mathcal{V}^{\mathrm{fin}}} \tilde{\mathbf{H}}(\mathcal{O}_\nu)$. Therefore the image of any finite index subgroup of $\tilde{\mathbf{H}}(\mathcal{O})$ has an open closure in $\tilde{\mathbf{H}}_{\mathcal{V}^{\mathrm{fin}}}$. Hence $\overline{i^{-1}(\Sigma)} < \tilde{\mathbf{H}}_{\mathcal{V}^{\mathrm{fin}}}$ is open. It follows that $\overline{i^{-1}(\Sigma)}$ contains a finite index subgroup of the compact group $\prod_{\mathcal{V}^{\mathrm{fin}}} \tilde{\mathbf{H}}(\mathcal{O}_\nu)$. Hence there exists a finite set $F_2 \subset \mathcal{V}^{\mathrm{fin}}$ such that for every $\nu \in \mathcal{V}^{\mathrm{fin}} - F_2$, $\tilde{\mathbf{H}}(\mathcal{O}_\nu) < \overline{i^{-1}(\Sigma)}$.

We now fix $\nu \in \mathcal{V}^{\mathrm{fin}} - S - F_1 - F_2$ and claim that $L_\nu < \mathbf{H}(\mathcal{O}_\nu)$. Observe first that $i^{-1}(L_\nu) = \tilde{\mathbf{H}}(\mathcal{O}_\nu)$. Indeed, $i^{-1}(L_\nu)$ is a compact group because of $\nu \notin S$ that contains $\tilde{\mathbf{H}}(\mathcal{O}_\nu)$ because of $\nu \notin F_2$. Hence it coincides with it because of $\nu \notin F_1$. It follows $i(\tilde{\mathbf{H}}(\mathcal{O}_\nu)) < L_\nu$, and the claim follows by the fact that for $\nu \in \mathcal{V}^{\mathrm{fin}} - F_1$, $\mathbf{H}(\mathcal{O}_\nu)$ is the unique maximal compact subgroup which contains $i(\tilde{\mathbf{H}}(\mathcal{O}_\nu))$.

We conclude that $L_{\mathcal{V}-S}$ is contained in the compact group

$$\left(\prod_{(\mathcal{V}-S) \cap (\mathcal{V}^\infty \cup F_1 \cup F_2)} L_\nu \right) \times \left(\prod_{\mathcal{V}-S-(\mathcal{V}^\infty \cup F_1 \cup F_2)} \mathbf{H}(\mathcal{O}_\nu) \right),$$

hence it is compact, proving our second point.

(3) By compactness a finite index subgroup of $L_{\mathcal{V}^{\mathrm{fin}}-S}$ is contained $\prod_{\mathcal{V}^{\mathrm{fin}}-S} \mathbf{H}(\mathcal{O}_\nu)$, hence its preimage in Σ is contained in $\mathbf{H}(\mathcal{O}[S])$. □

Note that in the setting of Theorem 3.6, $\mathbf{H}_{S^\infty}$ is a center-free, semisimple real Lie group without compact factors, the image of $\mathbf{H}(\mathcal{O})$ in $\mathbf{H}_{S^\infty}$ under the diagonal embedding is an irreducible lattice which is commensurated by the image of Σ. Moreover, $S^{\mathrm{fin}} \neq \emptyset$ if and only if $\mathbf{H}(\mathcal{O})$ is of infinite index in Σ. The following is a sort of a converse to this situation.

Theorem 3.7 *(Commensurator Super-Rigidity) Let H be a connected, center-free, semisimple real Lie group without compact factors. Let $\Delta < H$ be an irreducible lattice. Let Λ be a group such that $\Delta < \Lambda < \mathrm{Commen}_H(\Delta)$ and $[\Lambda : \Delta] = \infty$. Then there exist a number field K, a set of places S, a connected, adjoint, almost simple K-algebraic group $\mathbf{H}$ and a subgroup $\Sigma < \mathbf{H}(K)$ and an isomorphism of topological groups $\phi : \mathbf{H}_{S^\infty}^0 \xrightarrow{\cong} H$ so that*

(1) $\phi^{-1}(\Lambda)$ equals the diagonal image of Σ in $\mathbf{H}_{S^\infty}$,
(2) $\phi^{-1}(\Delta)$ in Σ is commensurated with $\mathbf{H}(\mathcal{O})$.
(3) a finite index subgroup of Σ is contained in $\mathbf{H}(\mathcal{O}[S])$.
(4) The closure of the projection of Σ to $\mathbf{H}_{S^{\mathrm{fin}}}$ contains $\mathbf{H}_{S^{\mathrm{fin}}}^+$ and it has no non-trivial compact normal subgroups.
(5) S is compatible with $\mathbf{H}$ (see Definition 1.1).

Proof Let $\mathbf{G}$ be the Zariski closure of the image of H in its adjoint representation. Then $\mathbf{G}$ is a real algebraic group and $\mathrm{Ad}(H) = \mathbf{G}(\mathbb{R})^0$ (see the beginning of Section 6 of [11, IX]). By the center-free assumption on H, $H \cong \mathrm{Ad}(H)$. Thus we may and will identify H with $\mathbf{G}(\mathbb{R})^0$.

We apply [11, Theorem IX.6.5] and get that Δ is arithmetic in $\mathbf{G}$. For the definition of arithmeticity see [11, Definitions IX.1.4]. According to this definition there exists a number field K, a connected non-commutative absolutely almost simple K-algebraic group $\tilde{\mathbf{H}}$, a finite set of places $T \subset \mathcal{V}$ containing all archimedean places ν for which $\tilde{\mathbf{H}}$ is K_ν-isotropic

and a continuous homomorphism $\tilde{\phi}\colon \tilde{\mathbf{H}}_T \to \mathbf{G}(\mathbb{R})$ such that $\tilde{\phi}(\tilde{\mathbf{H}}(\mathcal{O}[T]))$ is commensurable with Δ, where $\mathcal{O}$ is the ring of integers in K. By [11, Remark IX.1.6(i)] we may and will assume that $\tilde{\mathbf{H}}$ is simply connected (justifying our choice of notation).

Working through the intricate definition of the class Ψ appearing in [11, Definitions IX.1.4], which is given in [11, IX.1.3], and taking into account that $\mathbf{G}$ is an $\mathbb{R}$-group, we get that $T \subset \mathcal{V}^\infty$. In particular, $\mathcal{O}[T] = \mathcal{O}$ and $\tilde{\phi}(\tilde{\mathbf{H}}(\mathcal{O}))$ is commensurable with Δ. By [11, Remark IX.1.3(ii)], $\mathrm{Ker}(\tilde{\phi})$ is finite and $H = \mathbf{G}(\mathbb{R})^0 < \tilde{\phi}(\tilde{\mathbf{H}}_T)$. The facts above are also explained in [11, IX.1.5(*)].

Denoting by $I \subset \mathcal{V}^\infty$ the set of archimedean places ν such that $\tilde{\mathbf{H}}$ is K_ν-isotropic, we get $I \subset T \subset \mathcal{V}^\infty$. We denote by $\mathbf{H}$ the adjoint form of $\tilde{\mathbf{H}}$. Let p be the composition of the natural projection $\tilde{\mathbf{H}}_T \to \tilde{\mathbf{H}}_I$ and the isogeny $\tilde{\mathbf{H}}_I \to \mathbf{H}_I$. By the choice of I, p has a compact kernel. By [11, Theorem I.1.2.3.1(a')], $\tilde{\mathbf{H}}_I = \tilde{\mathbf{H}}_I^+$ and by [11, Theorem I.1.5.5] its image in $\mathbf{H}_I$ is $\mathbf{H}_I^+$, which equals $\mathbf{H}_I^0$ by [11, Theorem I.1.2.3.1(a)2)]. Note that $H = \mathbf{G}(\mathbb{R})^0 = \mathbf{G}(\mathbb{R})^+$ has no non-trivial compact normal subgroups, by assumption. It follows from [11, Theorem I.1.5.6(i)] that also $\tilde{\phi}(\tilde{\mathbf{H}}_T)$ has no normal subgroup, as it contains $\mathbf{G}(\mathbb{R})^+$. From the compactness of $\mathrm{Ker}(p)$ we conclude that $\tilde{\phi}$ factors through the image of p, namely $\mathbf{H}_I^0$. We denote the map thus obtained by ϕ. Since the image of ϕ is a connected subgroup of $\mathbf{G}(\mathbb{R})$ containing H, it must be H, and since $\mathbf{H}_I^0$ has no compact normal subgroups, we may view ϕ as a continuous isomorphism, $\phi\colon \mathbf{H}_I^0 \cong H$. By a standard Baire category argument ϕ is also a homeomorphism. In particular, $I \neq \emptyset$.

By [11, Corollary I.3.2.9] the image of $\tilde{\mathbf{H}}(\mathcal{O})$ is commensurated with $\mathbf{H}(\mathcal{O})$ in $\mathbf{H}(K)$, which we now embed diagonally in $\mathbf{H}_I$. Since $\tilde{\phi}(\mathbf{H}(\mathcal{O}))$ is commensurated with Δ in H we conclude that $\phi^{-1}(\Delta)$ is commensurated with $\mathbf{H}(\mathcal{O})$ in $\mathbf{H}_I$.

Note that for every $\nu \in I$, $\mathbf{H}(K_\nu)$ is not compact. By Theorem A.6 the commensurator of $\mathbf{H}(\mathcal{O})$ in $\mathbf{H}_I$ is $\mathbf{H}(K)$. Thus, $\phi^{-1}(\Lambda)$ is contained in the diagonal image of $\mathbf{H}(K)$. We denote by $\Sigma < \mathbf{H}(K)$ its preimage. We can now consider the preimage of Δ in Σ and we conclude that it is commensurated with $\mathbf{H}(\mathcal{O})$. We are now in the setting of Theorem 3.6, and everything else follows. $\square$

4 Arithmetically constructed lattices

Throughout, we retain the notation of Sect. 1.2 and fix number fields $K_1, \ldots, K_n$, connected, non-commutative, absolutely simple K_i-groups $\mathbf{H}_i$, and (possibly infinite) compatible subsets $S_i \subset \mathcal{V}_i$ of places of K_i (see Definition 1.1). By convention, unspecified products are to be taken from 1 to n, e.g by $\prod K_i$ we mean $\prod_{i=1}^n K_i$.

Lemma 4.1 $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$ *contains* $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$ *as an open normal subgroup, and the discrete group* $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#} / \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$ *is a torsion abelian group.*

Proof By Theorem 3.6, $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$ contains $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$. Normality is proved in [11, I.2.3.1 (b)]. Openness follows from Lemma 3.4. The quotient is a torsion abelian group by [11, I.2.3.1(c)].

Proof of Proposition 1.3 By Lemma 4.1 we are left to show that

$$\Gamma_A < \prod(\mathbf{H}_i)_{S_i^\infty} \times D_A$$

is indeed a lattice. By reduction theory,

$$\prod \mathbf{H}_i(\mathcal{O}_i[S_i]) < \prod (\mathbf{H}_i)_{S_i} = \prod (\mathbf{H}_i)_{S_i^\infty} \times \prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$$

is a lattice. By Lemma 2.2, $\prod \mathbf{H}_i(\mathcal{O}_i[S_i])$ is a lattice in $\prod (\mathbf{H}_i)_{S_i^\infty} \times \prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#}$. The claim follows from Lemma 2.3 and the openness statement in Lemma 4.1. $\qquad\square$

Lemma 4.2 *Let Γ be a subgroup of $\prod \mathbf{H}_i(K_i)$ and D be the closure of the diagonal embedding of Γ to $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$. We assume:*

(1) Γ contains a finite index subgroup of $\prod \mathbf{H}_i(\mathcal{O}_i)$.
(2) A finite index subgroup of Γ is contained in $\prod \mathbf{H}_i(\mathcal{O}_i[S_i])$.
(3) $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$ is contained in the projection of D to $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ for each $i \in \{1, \dots, n\}$.

Then Γ is an arithmetically constructed lattice.

Proof We first claim that $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+} < D$. Denote by $\tilde{D}$ the preimage of D in $\prod (\tilde{\mathbf{H}}_i)_{S_i^{\mathrm{fin}}}$. Then $\tilde{D}$ is open since it contains a finite index subgroup of the image of $\prod \tilde{\mathbf{H}}_i(\mathcal{O}_i)$ which is dense in an open subgroup by strong approximation [12, Theorem 7.12]. For each i and $\nu \in S_i^{\mathrm{fin}}$, $\tilde{D} \cap \tilde{\mathbf{H}}_i(K_\nu)$ is open in $\tilde{\mathbf{H}}_i(K_\nu)$ and normal in $\tilde{D}$. Hence it is also normal in $\tilde{\mathbf{H}}_i(K_\nu)$ as $\tilde{D} \to (\tilde{\mathbf{H}}_i)_{S_i^{\mathrm{fin}}} \to \tilde{\mathbf{H}}_i(K_\nu)$ is surjective by the assumption that the projection $D \to (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ contains $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$. By [11, Theorem I.1.5.6(i)] we get that $\tilde{\mathbf{H}}_i(K_\nu) < \tilde{D}$. The group generated by these is dense. Hence $\tilde{D} = \prod (\tilde{\mathbf{H}}_i)_{S_i^{\mathrm{fin}}}$. By Lemma 3.4 we conclude that $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+} < D$.

Denote $\Gamma_0 = \Gamma \cap \prod \mathbf{H}_i(\mathcal{O}_i[S_i])$ and let D_0 be the closure of the projection of Γ_0 to $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$. Then $\Gamma_0 < \Gamma$ and $D_0 < D$ are of finite index.

We claim that the group $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$ has no proper closed finite index subgroup. Indeed, for every i and $\nu \in S_i^{\mathrm{fin}}$, by [11, Corollary I.1.5.7], the groups $\mathbf{H}_i(K_\nu)^{+}$ has no proper finite index subgroup, and the group generated by these is dense in $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$. We conclude that $\prod (\mathbf{H}_i)_{S_i}^{+} < D_0$.

By Lemma 4.1 $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#} / \prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$ is discrete. Since the image of Γ_0 in D_0 is dense, they have the same image in $\prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{\#} / \prod (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^{+}$, which we denote A. With the notation of Proposition 1.3, clearly $D_0 = D_A$ and $\Gamma_0 = \Gamma_A$. So Γ is arithmetically constructed. $\qquad\square$

5 Representations of commensurable pairs

An *invariant commensurability class* in a group is a commensurability equivalence class that is closed under conjugation.

Example 5.1 Let G be a topological group. Then the class $\mathcal{K}_G$ of compact open subgroups of G is an invariant commensurability class.

A pair (Λ, C) consisting of an abstract group Λ and an invariant commensurability class C in Λ is called a *commensurability pair*. If (Λ, C) is an invariant commensurability pair then every $\Delta \in C$ is a commensurated subgroup in Λ; conversely, the commensurability class C of any commensurated subgroup $\Delta < \Lambda$ defines an invariant commensurability pair (Λ, C). Note also that under a group homomorphism, the preimage of a commensurated subgroup is commensurated. This proves:

Lemma 5.2 *Let G be a locally compact group and let $f : \Lambda \to G$ be a homomorphism. Let $U < G$ be a compact open subgroup. Then $f^{-1}(U)$ is commensurated by Λ.*

Definition 5.3 A *tdlc representation* of a commensurability pair (Λ, C) is a tdlc group G together with a homomorphism $\phi : \Lambda \to G$ such that $\phi^{-1}(K) \in C$ for some (hence any) $K \in \mathcal{K}_G$. If, in addition, $\mathrm{im}(\phi)$ is dense in G we call it a *dense tdlc representation*.

A *morphism* of tdlc representations $\phi : \Lambda \to G$ and $\phi' : \Lambda \to G'$ is a continuous homomorphism $\psi : G \to G'$ such that $\psi \circ \phi = \phi'$.

Let $\phi : \Lambda \to G$ be a tdlc representation of a commensurability pair (Λ, C). For $\Delta \in C$, the closure of $\phi(\Delta)$ is compact in G, and it is also open, thus $\overline{\phi(\Delta)} \in \mathcal{K}_G$, provided ϕ is dense. Further, for any open group $H < G$, $H = \phi(\phi^{-1}(H))$. We summarise:

Lemma 5.4 *For a dense tdlc representation $\phi : \Lambda \to G$ of a commensurability pair (Λ, C), C is the commensurability class of any preimage $\phi^{-1}(K)$ with $K \in \mathcal{K}_G$ and $\{\overline{\phi(\Delta)} \mid \Delta \in C\} = \mathcal{K}_G$.*

Because of this lemma we may and will drop C from the notation when dealing with dense tdlc representations.

Lemma 5.5 *Let $\psi : G \to G'$ be a morphism of dense tdlc representations from $\phi : \Lambda \to G$ to $\phi' : \Lambda \to G'$. Then ψ is surjective with compact kernel.*

Proof Let $K' \in \mathcal{K}_{G'}$. Let $\Delta = \phi'^{-1}(K')$ and $K = \overline{\phi(\Delta)}$. By the previous lemma $K \in \mathcal{K}_G$. We have

$$\psi(K) = \overline{\psi(K)} = \overline{\psi(\phi(\Delta))} = \overline{\phi'(\Delta)} = K'$$

and

$$\psi^{-1}(K') = \overline{\phi(\phi^{-1}(\psi^{-1}(K')))} = \overline{\phi(\phi'^{-1}(K'))} = \overline{\phi(\Delta)} = K.$$

In particular, $\mathrm{Ker}(\psi) < K$ and

$$G' = K'\phi'(\Lambda) = \psi(K)\psi(\phi(\Lambda)) = \psi(K\phi(\Lambda)) = \psi(G).$$

$\square$

The following theorem is well known (however we believe our easy proof below is novel). It will not be used in these notes, it is only given for context.

Theorem 5.6 *The category of tdlc representations of the pair (Λ, C) has an initial object, which is a dense representation.*

Proof The collection of isomorphism classes of dense tdlc representations of (Λ, C) forms a set Σ. Fix some $\Delta \in C$. For each dense representation $\phi : \Lambda \to G$ in Σ we pick the closure of the image of Δ in G as a distinguished subgroup of G. Then we take the restricted product inside $\prod_{(\phi : \Lambda \to G) \in \Sigma} G$ with respect to these distinguished subgroups. Finally, we take the closure $\hat{G}$ of the image of Λ under the diagonal homomorphism. We claim that the resulting representation $\hat{\phi} : \Lambda \to \hat{G}$ is an initial object in the category of tdlc representations.

Let $\phi : \Lambda \to G$ be a tdlc representation. By construction, $\hat{G}$ admits a projection onto an isomorphic copy of $\overline{\phi(\Lambda)}$. Composing this projection with the inverse of the alluded isomorphism and the inclusion of $\overline{\phi(\Lambda)}$ in G we get $\psi : \hat{G} \to G$, which is easily seen to be a morphism of representations. The uniqueness of this morphism is clear by the density of $\hat{\phi}$. $\square$

In the next theorem we consider the category of dense tdlc representations, which is a full subcategory of the category of tdlc representations.

Theorem 5.7 *Let Λ a countable group, $\check{\phi}\colon \Lambda \to \check{G}$ be a dense tdlc representation, and assume that $\check{G}$ has no non-trivial compact normal subgroups. Then $\check{\phi}$ is a terminal object in the category of dense tdlc representations.*

Proof Let $\phi\colon \Lambda \to G$ be a dense tdlc representation. We only need to show the existence of a morphism $G \to \check{G}$ since there is at most one morphism between dense representations. We denote $G' := \overline{(\phi \times \check{\phi})(\Lambda)} < G \times \check{G}$ and let ϕ' denote the diagonal representation, with the codomain restricted to G'. Let us denote $\psi = \mathrm{pr}_1 |_{G'}$. By Lemma 5.5 ψ is surjective with compact kernel. Since $\mathrm{Ker}(\psi) \lhd \{e\} \times \check{G} \cong \check{G}$ we conclude that ψ is injective. Since Λ is countable, G', G, $\check{G}$ are second countable tdlc groups. By the open mapping theorem [6, IX. §5.3] we conclude that ψ is a topological group isomorphism. It is now easy to see that $\mathrm{pr}_2 \circ \psi^{-1}\colon G \to \check{G}$ is a morphism of tdlc representations. $\square$

The following lemma provides an example of a terminal representation appearing in an arithmetic group setting. It will be useful later.

Lemma 5.8 *We retain the setting and notation described at the start of Section 4. In addition we assume that*

(1) each $\mathbf{H}_i$ is adjoint,
(2) Σ contains a finite index subgroup of $\prod \mathbf{H}_i(\mathcal{O}_i)$,
(3) a finite index subgroup of Σ is contained in $\prod \mathbf{H}_i(\mathcal{O}_i[S_i])$,
(4) the closure of the projection $\Sigma \to (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ has no non-trivial compact normal subgroup for each $i \in \{1, \ldots, n\}$.

Let $Q < \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ be the closure of the image of Σ under the diagonal map, and let C be the commensurability class of $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i)$ in Σ.

Then the embedding $\Sigma \to Q$ is a dense tdlc representation of the commensurability pair (Σ, C) which is is a terminal object in the category of dense tdlc representations of (Σ, C).

Proof By Theorem A.6, $\prod \mathbf{H}_i(\mathcal{O}_i)$ is commensurated in $\prod \mathbf{H}_i(K_i)$. Thus the group $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i)$ is commensurated in Σ and C is invariant. For each i let $T_i = \mathcal{V}_i^{\mathrm{fin}} - S_i$. Observe that

$$\prod \mathbf{H}_i(\mathcal{O}_i[T_i]) \cap \prod \mathbf{H}_i(\mathcal{O}_i[S_i]) = \prod \mathbf{H}_i(\mathcal{O}_i).$$

Thus

$$\left(\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[T_i])\right) \cap \left(\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[S_i])\right) = \Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i).$$

Since $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[S_i]) < \Sigma$ is of finite index, $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[T_i])$ is commensurable with $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i)$, i.e $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[T_i]) \in C$.

We claim that the embedding $\phi\colon \Sigma \to \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ is a tdlc representation: Consider the compact open subgroup

$$U = \prod_i \prod_{v \in S_i^{\mathrm{fin}}} \mathbf{H}_i((\mathcal{O}_i)_v) < \prod_i (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$$

The preimage of U in $\prod \mathbf{H}_i(K_i)$ is exactly $\prod \mathbf{H}_i(\mathcal{O}_i[T_i])$, thus the preimage in Σ is $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i[T_i])$, which is in C, proving our claim.

Hence $\Sigma \to Q$ is a dense tdlc representation of the commensurability pair (Σ, C). By Theorem 5.7 this is a terminal object if Q has no non-trivial compact normal subgroup. The latter is due to assumption (4). $\qquad\square$
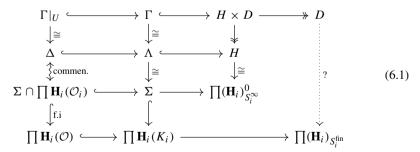
6 Proof of Theorem 1.6

Let $U < D$ be an open compact subgroup. We denote its preimage under $\Gamma \to D$ by $\Gamma|_U$. It is commensurated in Γ by Lemma 5.2. By Lemma 2.3 $\Gamma|_U$ is a lattice in $H \times U$ and by Lemma 2.1 its projection to H is a lattice in H. We denote $\Delta = \mathrm{pr}_H(\Gamma|_U)$ and $\Lambda = \mathrm{pr}_H(\Gamma)$. Thus $\Delta < \Lambda < H$, Δ is a lattice in H which is commensurated by Λ and Λ is dense in H.

The lattice $\Delta < H$ might be reducible. The decomposition into irreducible components gives a splitting $H = H_1 \times \cdots \times H_n$ into n factors and irreducible lattices $\Delta_i < H_i$, $i \in \{1, \ldots, n\}$ such that $\Delta_1 \times \cdots \times \Delta_n < \Delta$ is of finite index. Let $\Lambda_i = \mathrm{pr}_{H_i}(\Gamma)$, where pr_{H_i} is the projection $H \times D \to H \to H_i$. Clearly, Λ_i is dense in H_i, which implies $[\Lambda_i : \Delta_i] = \infty$, and Δ_i is commensurated by Λ_i.

We are in a position to apply Theorem 3.7 and obtain for each i: a number field K_i, a compatible set of places S_i, a connected, adjoint, almost simple K_i-algebraic group $\mathbf{H}_i$ and a subgroup $\Sigma_i < \mathbf{H}_i(K_i)$, satisfying the following properties:

- There exists an isomorphism of topological groups $\phi_i : (\mathbf{H}_i)^0_{S_i^\infty} \xrightarrow{\simeq} H_i$ such that $\phi_i^{-1}(\Lambda_i)$ equals the diagonal image of Σ_i in $(\mathbf{H}_i)_{S_i^\infty}$ and the preimage of Δ_i in Σ_i is commensurated with $\mathbf{H}_i(\mathcal{O}_i)$.
- Σ_i contains a finite index subgroup of $\mathbf{H}_i(\mathcal{O}_i)$ and a finite index subgroup of Σ_i is contained in $\mathbf{H}_i(\mathcal{O}_i[S_i])$.
- The closure of the projection of Σ_i to $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ contains $(\mathbf{H}_i)^+_{S_i^{\mathrm{fin}}}$ and it has no non-trivial compact normal subgroups.

The identification of H with a subgroup of $\prod(\mathbf{H}_i)_{S_i^\infty}$ identifies Λ with a subgroup Σ of the diagonal image of $\prod \mathbf{H}_i(K_i)$. We summarise some of the information we have so far in the following commutative diagram (arrows in this diagram are not named, but they are all explicit from our context):

$$
\begin{array}{ccccccc}
\Gamma|_U & \hookrightarrow & \Gamma & \longrightarrow & H \times D & \twoheadrightarrow & D \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \vdots \\
\Delta & \hookrightarrow & \Lambda & \longrightarrow & H & & \\
\updownarrow{\scriptstyle\mathrm{commen.}} & & \downarrow\cong & & \downarrow\cong & & ? \\
\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i) & \longrightarrow & \Sigma & \longrightarrow & \prod(\mathbf{H}_i)^0_{S_i^\infty} & & \\
\downarrow{\scriptstyle\mathrm{f.i}} & & \downarrow & & & & \vdots \\
\prod \mathbf{H}_i(\mathcal{O}) & \hookrightarrow & \prod \mathbf{H}_i(K_i) & \longrightarrow & & & \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}
\end{array}
\tag{6.1}
$$

where the twiddle vertical arrow between Δ and $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i)$ indicates that the images of these two groups in Σ are commensurated. The dotted arrow from D to $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ is not yet given – its existence is the heart of the theorem. This is what we are now after.

By the above Σ contains a finite index subgroup of $\prod \mathbf{H}_i(\mathcal{O}_i)$ and a finite index subgroup of Σ is contained in $\prod \mathbf{H}_i(\mathcal{O}_i[S_i])$. Also for each i, the closure of the projection $\Sigma \to (\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ has no non-trivial compact normal subgroup. Denote by $Q < \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ the closure of the

image of Σ. Denote by C the commensurability class of $\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i)$ in Σ. By Lemma 5.8 the embedding $\Sigma \to Q$ is a dense tdlc representation of the commensurability pair (Σ, C) and it is a terminal object in the category of dense tdlc representations of this pair (see §5 for definitions). Hence the path from Γ to Q in the commutative diagram below is a terminal dense tdlc representation of the pair $(\Gamma, [\Gamma|_U])$.

$$
\begin{array}{ccccc}
\Gamma|_U & \longrightarrow & \Gamma & \longrightarrow & D \\
\updownarrow{\scriptstyle\text{commen.}} & & \downarrow{\simeq} & & \vdots{\scriptstyle ?} \\
\Sigma \cap \prod \mathbf{H}_i(\mathcal{O}_i) & \longrightarrow & \Sigma & \longrightarrow & Q \hookrightarrow \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}
\end{array}
$$

By the property of terminal objects we can uniquely realise the dotted arrow in the above diagram. By Lemma 5.5 the dotted arrow is a continuous epimorphism with compact kernel C. The composition $D \to Q \to \prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ is thus a continuous homomorphism with closed image and compact kernel C which realises the dotted arrow in Diagram (6.1).

Note that Σ is a subgroup of $\prod \mathbf{H}_i(K_i)$ and Q is the closure of the projection of Σ to $\prod(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$ such that Σ contains a finite index subgroup of $\prod \mathbf{H}_i(\mathcal{O}_i)$, a finite index subgroup of Σ is contained in $\prod \mathbf{H}_i(\mathcal{O}_i[S_i])$, and for each i, $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}^+$ is contained in the projection of Q to $(\mathbf{H}_i)_{S_i^{\mathrm{fin}}}$. By Lemma 4.2 we conclude that Σ in is an arithmetically constructed lattice. The theorem is now proved by considering the commutative diagram:

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & H \times D/C \\
\downarrow{\cong} & & \downarrow{\cong} \\
\Sigma & \longrightarrow & \prod(\mathbf{H}_i)_{S_i^{\infty}}^0 \times Q \hookrightarrow \prod(\mathbf{H}_i)_{S_i}
\end{array}
$$

This completes the proof of Theorem 1.6.

Appendix A. A theorem of Borel revisited

In this appendix we generalise Borel's Theorem 3(b) from [3]. Our proof is almost identical to Borel's original proof and we have no doubt the sophisticated reader is well aware of this generalisation. Nevertheless, since the result had not been published elsewhere in the form that we need, we decided to add the short proof for the reader's convenience.

Theorem A.1 *Let K be a field of characteristic 0. Let $\mathbf{G}$ be a connected affine K-algebraic group and $\mathbf{Z}$ its center. Let $H < \mathbf{G}(K)$ be a Zariski dense subgroup. Let A be a K-algebra and let C be the commensurator of H in $\mathbf{G}(A)$. Then the image of C in $\mathbf{G}/\mathbf{Z}(A)$ is contained in $\mathbf{G}/\mathbf{Z}(K)$.*

Before proving the theorem, let us recall some basic facts about the functor of points of a K-algebraic group. For a K-algebraic group $\mathbf{G}$ and a K-algebra A we set $\mathbf{G}(A) = \mathrm{Hom}_{K\text{-alg}}(K[\mathbf{G}], A)$, where $K[\mathbf{G}]$ denotes the Hopf algebra of K-regular functions on $\mathbf{G}$. Using the coproduct $\Delta : K[\mathbf{G}] \to K[\mathbf{G}] \otimes_K K[\mathbf{G}]$ we get a product structure on $\mathbf{G}(A)$, by sending $x, y \in \mathbf{G}(A)$ to the map given by the following composition

$$
K[\mathbf{G}] \xrightarrow{\Delta} K[\mathbf{G}] \otimes_K K[\mathbf{G}] \xrightarrow{x \otimes y} A \otimes_K A \xrightarrow{m} A.
$$

Further, $\mathbf{G}(A)$ has an identity, given by the augmentation $f \mapsto f(e) \in K < A$, and the inverse of $x \in \mathbf{G}(A)$ is given by $x \circ S$, where $S : K[\mathbf{G}] \to K[\mathbf{G}]$ is the antipode,

$Sf(g) = f(g^{-1})$. Thus $\mathbf{G}(A)$ is a group. The association $A \rightsquigarrow \mathbf{G}(A)$ is called *the functor of points* associated with $\mathbf{G}$, see for example Waterhouse's book [17] for an account on algebraic group theory based on this point of view.

We set $A[\mathbf{G}] := A \otimes_K K[\mathbf{G}]$. Consider the conjugation action of $\mathbf{G}$ on itself. This is a K-morphic action which gives rise to a K-morphic action of $\mathbf{G}/\mathbf{Z}$ on $\mathbf{G}$. Consider the action map $a : \mathbf{G}/\mathbf{Z} \times \mathbf{G} \to \mathbf{G}$. For every $g \in \mathbf{G}/\mathbf{Z}(A)$ the map

$$K[\mathbf{G}] \xrightarrow{\ a^* \ } K[\mathbf{G}/\mathbf{Z}] \otimes_K K[\mathbf{G}] \xrightarrow{\ g \otimes \mathrm{id} \ } A \otimes_K K[\mathbf{G}] = A[\mathbf{G}]$$

extends uniquely to a A-algebra automorphism $A[\mathbf{G}] \to A[\mathbf{G}]$. By this we define an action of $\mathbf{G}/\mathbf{Z}(A)$ on $A[\mathbf{G}]$ by A-algebra automorphisms.

Lemma A.2 *Let K be a field of characteristic 0. Let $\mathbf{G}$ be an affine K-algebraic group and $\mathbf{Z}$ its center. Let A be a K-algebra. Consider the conjugation representation of $\mathbf{G}/\mathbf{Z}(A)$ on $A[\mathbf{G}]$. Considering $K[\mathbf{G}]$ as a subspace of $A[\mathbf{G}]$ and $\mathbf{G}/\mathbf{Z}(K)$ as a subgroup of $\mathbf{G}/\mathbf{Z}(A)$, we have*

$$\mathbf{G}/\mathbf{Z}(K) = \{g \in \mathbf{G}/\mathbf{Z}(A) \mid g(K[\mathbf{G}]) \subset K[\mathbf{G}]\}.$$

Proof Consider $K[\mathbf{G}/\mathbf{Z}]$ as a Hopf algebra and consider the comodule structure on $K[\mathbf{G}]$ associated with the conjugation representation. By [17, 3.3], $K[\mathbf{G}]$ is a direct union of finite dimensional subcomodules. We let $V < K[\mathbf{G}]$ be a finite dimensional subcomodule which generates $K[\mathbf{G}]$ as a K-algebra ($K[\mathbf{G}]$ is a finitely generated K-algebra). By the correspondence between subcomodules and subrepresentations, [17, 3.2], V is a representation of $\mathbf{G}/\mathbf{Z}$, in the sense that for every K-algebra B, $B \otimes_K V$ is a $\mathbf{G}/\mathbf{Z}(B)$-invariant subspace of $B[\mathbf{G}]$, and we obtain a natural family of group homomorphisms $\mathbf{G}/\mathbf{Z}(B) \to \mathrm{Aut}_B(B \otimes_K V)$. Considering the action of $\mathbf{G}/\mathbf{Z}(B)$ on $B[\mathbf{G}]$, it follows that the left hand side is included in the right hand side in the following equation:

$$\{g \in \mathbf{G}/\mathbf{Z}(B) \mid g(K[\mathbf{G}]) \subset K[\mathbf{G}]\} = \{g \in \mathbf{G}/\mathbf{Z}(B) \mid gV \subset V\}, \tag{A.1}$$

the other inclusion holds as the K-algebra generated in $B[\mathbf{G}]$ by V is $K[\mathbf{G}]$.

We now take $B = \bar{K}$, an algebraic closure of K. By definition of $\mathbf{Z}$ [4, 1.7], $\mathbf{G}/\mathbf{Z}(\bar{K})$ acts faithfully on $\mathbf{G}(\bar{K})$. It follows that its action on $\bar{K}[\mathbf{G}]$ is faithful as well, and by the fact that $\bar{K} \otimes_K V$ generates $\bar{K}[\mathbf{G}]$ as a $\bar{K}$-algebra, we get that $\mathbf{G}/\mathbf{Z}(\bar{K})$ acts faithfully on $\bar{K} \otimes_K V$. Thus the morphism $\mathbf{G}/\mathbf{Z}(\bar{K}) \to \mathbf{GL}(\bar{K} \otimes_K V)$ is injective. By [4, Cor 1.4(a)] its image is a Zariski closed K-subgroup of $\mathbf{GL}(\bar{K} \otimes_K V)$, and by [4, Proposition 6.4] this morphism is a K-isomorphism onto its image since it is separable by the characteristic 0 assumption on K.

As K-vector spaces, $V \cong K^n$. Thus we may identify $\mathbf{G}/\mathbf{Z}$ with a K-algebraic subgroup $\mathbf{H} < \mathbf{GL}_n$. As subgroups of $\mathbf{GL}_n(A)$ we have $\mathbf{H}(K) = \mathbf{H}(A) \cap \mathbf{GL}_n(K)$. We conclude the first equality in

$$\mathbf{H}(K) = \{g \in \mathbf{H}(A) \mid g(K^n) = K^n\} = \{g \in \mathbf{H}(A) \mid g(K^n) \subset K^n\}.$$

The second equality is a triviality since every g is regular. We thus obtain

$$\mathbf{G}/\mathbf{Z}(K) = \{g \in \mathbf{G}/\mathbf{Z}(A) \mid gV \subset V\}.$$

The proof is now complete by Equation (A.1). □

Proof of Theorem A.1 Let $g \in C$ be an element. We show that its image in $\mathbf{G}/\mathbf{Z}(A)$ – denoted also by g – lies in $\mathbf{G}/\mathbf{Z}(K)$.

We consider the conjugation action of $\mathbf{G}(A)$ on $A[\mathbf{G}]$ and claim that $gK[\mathbf{G}] \subset K[\mathbf{G}]$. By Lemma A.2 this suffices. Let $x \in K[\mathbf{G}]$. We write

$$gx = x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n, \quad \alpha_i \in A, \ x_i \in K[\mathbf{G}],$$

where $1, \alpha_1, \ldots, \alpha_n$ are independent over K and $x_i \neq 0$. If $n = 0$ then indeed $gx = x_0 \in K[\mathbf{G}]$.

We assume $n \geq 1$ and argue by contradiction. Note that $H \cap gHg^{-1} < H$ is of finite index, thus $H \cap gHg^{-1}$ is Zariski dense in $\mathbf{G}(K)$ since $\mathbf{G}$ is connected. Thus we can find $h \in H \cap gHg^{-1} < \mathbf{G}(K) = \mathrm{Hom}_{K\text{-alg}}(K[\mathbf{G}], K)$ such that $h(x_1) \neq 0$. From

$$h(x_0) + \alpha_1 h(x_1) + \cdots + \alpha_n h(x_n) = h(gx) = (g^{-1}hg)(x) \in K$$

we get a non-trivial linear combination of $1, \alpha_1, \ldots, \alpha_n$ over K. This gives the desired contradiction. $\qquad\qquad\square$

Remark A.3 Neither Theorem A.1 nor Lemma A.2 are correct as stated if K has positive characteristic. Take for example K to be the algebraic closure of $\mathbb{F}_2$, $\mathbf{G} = \mathbf{SL}_2$, and $A = K[t]/(t^2)$. Then $\mathbf{G}(K) = \mathbf{SL}_2(K)$ has trivial center $\mathbf{Z}$, thus $\mathbf{G}/\mathbf{Z} = \mathbf{G}$. However, $\mathbf{G}/\mathbf{Z}(A) = \mathbf{G}(A) \cong \mathbf{SL}_2(K) \ltimes \mathfrak{sl}_2(K)$ has a non-trivial center L corresponding to the scalar matrices in $\mathfrak{sl}_2(K)$. Then L is not contained in $\mathbf{G}/\mathbf{Z}(K)$, but

$$L < \{g \in \mathbf{G}/\mathbf{Z}(A) \mid g(K[\mathbf{G}]) < K[\mathbf{G}]\},$$

and for any H in $\mathbf{G}(K)$, L commensurates H in $\mathbf{G}/\mathbf{Z}(A)$.

Correct statements are obtained by replacing the smooth subgroup $\mathbf{Z}$ with a scheme theoretically defined center $\mathcal{Z}$. For example, for $\mathbf{G} = \mathbf{SL}_2$, regardless of the field of definition, $\mathbf{G}/\mathcal{Z} \cong \mathbf{PGL}_2$. In characteristic 0 we always have $\mathcal{Z} = \mathbf{Z}$. The proof of the corrected statement of the theorem follows formally from the corrected lemma. We will not give the proof of the corrected lemma but merely remark that it is verbatim the proof above, upon replacing the notion of "faithfulness of an action" by a scheme theoretical analogue. Moreover, there is no need to pass to an algebraic closure anymore.

Working with the scheme theoretical center has advantages even if one is concerned only with the classical setting. For example, if K is reduced then $\mathbf{G}/\mathcal{Z}(K) = \mathbf{G}/\mathbf{Z}(K)$. To see this one observes that the algebra representing $\mathbf{G}/\mathcal{Z}$, modulo its nil-radical, equals the algebra representing $\mathbf{G}/\mathbf{Z}$.

The following application of Theorem A.1 is needed in Sect. 3.

Theorem A.4 *We retain the setup in Sect. 1.2. Let $\mathbf{H}$ be a connected adjoint semi-simple K-algebraic group. Assume in addition that $\mathbf{H}(\mathcal{O}[S])$ is Zariski dense in $\mathbf{H}$. Then the commensurator of $\mathbf{H}(\mathcal{O}[S])$ in $\mathbf{H}_S$ (via the diagonal embedding) is exactly $\mathbf{H}(K)$.*

Remark A.5 The subgroup $\mathbf{H}(\mathcal{O}[S]) < \mathbf{H}(K)$ depends on a choice of embedding $\mathbf{H} < \mathbf{GL}_n$ but not its commensurability class. Since $\mathbf{H}$ is connected its Zariski-density is unambiguous.

Proof Consider the locally compact totally disconnected group given by the restricted product

$$\prod_{v \in \mathcal{V}-\mathcal{V}^\infty - S}' \mathbf{H}(K_v),$$

which is taken with respect to the compact open subgroups $\mathbf{H}(\mathcal{O}_v) < \mathbf{H}(K_v)$. Then

$$\prod_{v \in \mathcal{V}-\mathcal{V}^\infty - S} \mathbf{H}(\mathcal{O}_v)$$

is a compact open subgroup, hence it is commensurated. Its preimage under the diagonal embedding

$$\mathbf{H}(K) \to \prod_{\nu \in \mathcal{V} - \mathcal{V}^\infty - S}^{\prime} \mathbf{H}(K_\nu)$$

is $\mathbf{H}(\mathcal{O}[S])$. By Lemma 5.2, $\mathbf{H}(\mathcal{O}[S])$ is commensurated by $\mathbf{H}(K)$.

We need to show that the commensurator group of $\mathbf{H}(\mathcal{O}[S])$ inside $\mathbf{H}_S$ is contained in the diagonal image of $\mathbf{H}(K)$. Let A be the K-algebra of S-adeles, that is, the restricted product $\prod_{\nu \in S}^{\prime} K_\nu$ taken with respect to the subgroups $\mathcal{O}_\nu < K_\nu$. In [12, pp. 249–250] the identification $\mathbf{H}(A) \cong \mathbf{H}_S$ is explained. Under this identification, $\mathbf{H}(K)$ is identified with its diagonal image. Now apply Theorem A.1 and use the assumption that $\mathbf{H}$ is adjoint, hence has a trivial center.   □

For the special case where S is a set of archimedean places we get the following.

Theorem A.6 *We retain the setup in Sect. 1.2. Let $S \subset \mathcal{V}^\infty$ be a set of archimedean places. Let $\mathbf{H}$ be a connected adjoint semi-simple K-algebraic group. Assume that there exists $\nu \in \mathcal{V}^\infty$ such that the Lie group $\mathbf{H}(K_\nu)$ is not compact. Then the commensurator of $\mathbf{H}(\mathcal{O})$ in $\prod_{\nu \in S} \mathbf{H}(K_\nu)$ is exactly $\mathbf{H}(K)$.*

Proof This follows from Theorem A.4, once we show that $\mathbf{H}(\mathcal{O})$ is Zariski dense in $\mathbf{H}$. By [3, Theorem 3(a)] $\mathbf{H}(\mathcal{O})$ is Zariski dense if and only if $R_{K/\mathbb{Q}}\mathbf{H}(\mathbb{R})$ is non-compact where $R_{K/\mathbb{Q}}$ denotes the restriction of scalars from K to $\mathbb{Q}$. By applying [11, Theorem I.3.1.4(i)] to the real place,

$$R_{K/\mathbb{Q}}\mathbf{H}(\mathbb{R}) \cong \prod_{\nu \in V^\infty} \mathbf{G}(K_\nu).$$

The right hand side is non-compact by assumption.   □

In view of Remark A.3, Theorem A.4 has the following analogue in positive characteristic. The proof is the same as of Theorem A.4, taking into account that for adjoint groups the scheme theoretical center is trivial.

Theorem A.7 *Let K be field of rational functions over a finite field, $\mathcal{O}$ its ring of polynomials and $\mathcal{V}$ the set of places of k. We fix a subset $S \subset \mathcal{V}$. Let $\mathbf{H}$ be a connected adjoint semi-simple K-algebraic group. We assume that $\mathbf{H}(\mathcal{O}[S])$ is Zariski dense in $\mathbf{H}$. Then the commensurator of $\mathbf{H}(\mathcal{O}[S])$ in $\prod_{\nu \in S}^{\prime} \mathbf{H}(K_\nu)$ is exactly $\mathbf{H}(K)$.*

References

1. Bader, U., Furman, A., Sauer, R.: On the structure and arithmeticity of lattice envelopes. C. R. Math. Acad. Sci. Paris **353**(5), 409–413 (2015)
2. Bader, U., Furman, A., Sauer, R.: Lattice envelopes, available at arXiv:1711.08410,
3. Borel, A.: Density and maximality of arithmetic subgroups. J. Reine Angew. Math. **224**, 78–89 (1966)
4. Borel, A.: Linear Algebraic Groups, Graduate Texts in Mathematics, vol. 126, p. 2. Springer-Verlag, New York (1991). (xii+288)
5. Borel, A., Tits, J.: Homomorphismes "abstraits" de groupes algébriques simples. Ann. Math. **97**, 499–571 (1973). (2) (French)
6. Bourbaki, N.: General topology. Chapters 5–10, Elements of Mathematics (Berlin), Translated from the French; Reprint of the: English translation, p. 1998. Springer-Verlag, Berlin (1989)
7. Caprace, P.-E., Monod, N.: A lattice in more than two Kac-Moody groups is arithmetic. Israel J. Math. **190**, 413–444 (2012)

8. Caprace, P.-E., Monod, N.: Isometry groups of non-positively curved spaces: discrete subgroups. J. Topol. **2**(4), 701–746 (2009)
9. Margulis, G.A.: Discrete groups of motions of manifolds of nonpositive curvature, Russian, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2, , Canad. Math. Congress, Montreal, Que., 21–34, (1975)
10. Margulis, G.A.: Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1. Invent. Math. **76**(1), 93–120 (1984)
11. Margulis, G.A.: Discrete subgroups of semisimple Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], p. 17. Springer-Verlag, Berlin (1991)
12. Platonov, V., Rapinchuk, A.: Algebraic groups and number theory, Pure and Applied Mathematics, vol. 139. Academic Press Inc, Boston (1994). (xii+614)
13. Oh, Hee: Adelic version of Margulis arithmeticity theorem. Math. Ann. **321**(4), 789–815 (2001)
14. Prasad, G.: Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits. Bull. Soc. Math. France **110**(2), 197–202 (1982). (English, with French summary)
15. Raghunathan, M.S.: Discrete subgroups of Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg (1972). (ix+227)
16. Tits, J.: Reductive groups over local fields, Automorphic forms, representations and *L*-functions, Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977. In: Proceedings of the Symposium Pure Mathematics, XXXIII, American Mathematical Society, Providence, R.I., 29–69, (1979)
17. Waterhouse, W.C.: Introduction to affine group schemes, Graduate Texts in Mathematics, vol. 66. Springer-Verlag, New York-Berlin (1979). (xi+164)