Mathematische Zeitschrift (2019) 291:1419–1434
https://doi.org/10.1007/s00209-018-2213-z

Mathematische Zeitschrift

Sturm bounds for Siegel modular forms of degree 2 and odd weights

Toshiyuki Kikuta[1] · **Sho Takemori**[2]

Received: 19 August 2015 / Accepted: 4 December 2018 / Published online: 1 January 2019

Abstract
We correct the proof of the theorem in the previous paper presented by Kikuta, which concerns Sturm bounds for Siegel modular forms of degree 2 and of even weights modulo a prime number dividing $2 \cdot 3$. We give also Sturm bounds for them of odd weights for any prime numbers, and we prove their sharpness. The results cover the case where Fourier coefficients are algebraic numbers.

Keywords Siegel modular forms · Congruences for modular forms · Fourier coefficients · J. Sturm

Mathematics Subject Classification Primary 11F33 · Secondary 11F46

1 Introduction

Sturm [15] studied how many Fourier coefficients we need, when we want to prove that an elliptic modular form vanishes modulo a prime ideal. Its number is so called "Sturm bound". We shall explain it more precisely. For a modular form f, let Λ be the index set of the Fourier expansion of f. An explicitly given finite subset S of Λ is said to be a *Sturm bound* if vanishing modulo a prime ideal of Fourier coefficients of f at S implies vanishing modulo the prime ideal of all Fourier coefficients of f.

Poor–Yuen [11] studied initially Sturm bounds for Siegel modular forms of degree 2 for any prime number p. After their study, in [1], Choi, Choie and Kikuta gave other type bounds with simple descriptions for them in the case of $p \geq 5$. Moreover, Kikuta [7] attempted to supplement the case of $p \mid 2 \cdot 3$. However, there are some gaps in the proof (of Theorem 2.1 in Sect. 3.1, [7]). It seems that the method can only give worse bounds. Richter–Raum [13]

✉ Toshiyuki Kikuta
kikuta@fit.ac.jp

Sho Takemori
stakemorii@gmail.com

1 Department of Information and Systems Engineering, Faculty of Information Engineering, Fukuoka Institute of Technology, 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka 811-0295, Japan

2 Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

gave some bounds for any p in the case of general degree and any weight. However, their bounds seem not to be sharp in degree $n \geq 2$ excepted in the case $n = 2$, $p \geq 5$, and for even weights. An improvement of their bounds depends on the case of degree 2.

In this paper, we correct the proof of Theorem 2.1 in [7] by a new method. Namely we give sharp Sturm type bounds for Siegel modular forms of degree 2 and even weight in the case of $p = 2, 3$. Moreover we give also sharp bounds for odd weight and modulo any prime number p. It should be remarked that sharpness becomes important to confirm congruences between two modular forms by numerical experiments, as the weights grow larger. Finally, we remark also that our results cover the case where Fourier coefficients are algebraic numbers.

2 Statement of the results

In order to state our results, we fix notation. For a positive integer n, we define the Siegel modular group Γ_n of degree n by

$$\Gamma_n = \{\gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t\gamma J_n \gamma = J_n\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and 0_n (resp. 1_n) is the zero matrix (resp. the identify matrix) of size n. For a positive integer N, we define the principal congruence subgroup $\Gamma^{(n)}(N)$ of level N by

$$\Gamma^{(n)}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n \;\middle|\; \begin{matrix} a \equiv d \equiv 1_n \mod N \\ b \equiv c \equiv 0_n \mod N \end{matrix} \right\}.$$

Here a, b, c, d are $n \times n$ matrices. A subgroup $\Gamma \subset \Gamma_n$ is said to be a congruence subgroup if there exists a positive integer N such that $\Gamma^{(n)}(N) \subset \Gamma \subset \Gamma_n$. For a congruence subgroup Γ, we define the level of Γ to be

$$N = \min\{m \in \mathbb{Z}_{\geq 1} \mid \Gamma^{(n)}(m) \subset \Gamma\}.$$

We define the Siegel upper half space $\mathbb{H}_n$ of degree n by

$$\mathbb{H}_n = \left\{ x + iy \mid x \in \mathrm{Sym}_n(\mathbb{R}),\ y \in \mathrm{Sym}_n(\mathbb{R}),\ y \text{ is positive definite} \right\},$$

where $\mathrm{Sym}_n(\mathbb{R})$ is a space of $n \times n$ symmetric matrices with entries in $\mathbb{R}$. For a congruence subgroup Γ and $k \in \mathbb{Z}_{\geq 0}$, a $\mathbb{C}$-valued holomorphic function f on $\mathbb{H}_n$ is said to be a (holomorphic) Siegel modular form of degree n, of weight k and of level Γ if $f((aZ+b)(cZ+d)^{-1}) = \det(cZ+d)^k f(Z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $n = 1$, we add the cusp condition. We denote by $M_k(\Gamma)$ the space of Siegel modular forms of weight k and of level Γ.

Any f in $M_k(\Gamma)$ has a Fourier expansion of the form

$$f(Z) = \sum_{0 \leq T \in \frac{1}{N}\Lambda_n} a_f(T) q^T, \quad q^T := e^{2\pi i \mathrm{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where T runs over all positive semi-definite elements of $\frac{1}{N}\Lambda_n$, N is the level of Γ and

$$\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbb{Z}\}.$$

When $n = 2$, for simplicity we write $T = (m, r, n)$ for $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in \frac{1}{N}\Lambda_2$ and also $a_f(m, r, n)$ for $a_f(T)$.

Let R be a subring of $\mathbb{C}$ and $M_k(\Gamma)_R \subset M_k(\Gamma)$ the R-module of all modular forms whose Fourier coefficients lie in R.

Let $f_1,\ f_2$ be two formal power series of the forms $f_i = \sum_{0 \leq T \in \frac{1}{N}\Lambda_n} a_{f_i}(T) q^T$ with $a_i \in R$. For an ideal I of R, we write

$$f_1 \equiv f_2 \mod I,$$

if and only if $a_{f_1}(T) \equiv a_{f_2}(T) \mod I$ for all $T \in \frac{1}{N}\Lambda_n$ with $T \geq 0$. If $I = (r)$ is a principal ideal, we simply denote $f_1 \equiv f_2 \mod r$.

Let K be an algebraic number field and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ in $\mathcal{O}$, we denote by $\mathcal{O}_\mathfrak{p}$ the localization of $\mathcal{O}$ at $\mathfrak{p}$. Under these notation, we have

Theorem 2.1 *Let k be a non-negative integer, $\mathfrak{p}$ an any prime ideal and $f \in M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$. We put*

$$b_k = \begin{cases} \left[\frac{k}{10}\right] & \text{if } k \text{ is even,} \\ \left[\frac{k-5}{10}\right] & \text{if } k \text{ is odd.} \end{cases}$$

Here $[x]$ is the Gauss symbol of $x \in \mathbb{R}$, i.e., $[x] := \max\{n \in \mathbb{Z} \mid n \leq x\}$. For $\nu \in \mathbb{Z}_{\geq 1}$, assume that $a_f(m,r,n) \equiv 0 \bmod \mathfrak{p}^\nu$ for all $m,\ r,\ n \in \mathbb{Z}$ with

$$0 \leq m, n \leq b_k,$$

and $4mn - r^2 \geq 0$, then we have $f \equiv 0 \bmod \mathfrak{p}^\nu$.

Remark 2.2 1. If k is even and $\mathfrak{p} \nmid 2 \cdot 3$, then the statement of the theorem was essentially proved by Choi, Choie and Kikuta [1].
2. As mentioned in Sect. 1, in the case where $\mathfrak{p} \mid 2 \cdot 3$ and k is even, Kikuta stated the same property in [7]. However, the proof has some gaps and its method can give only larger bounds. We give a new proof in Sect. 5.1.
3. We note that $M_k(\Gamma_2) = \{0\}$ if k is odd and $k < 35$.
4. Other kind of bounds also were given in [8].

By the arguments of [1], we can prove the following.

Corollary 2.3 *Let $\Gamma \subset \Gamma_2$ be a congruence subgroup with level N, $k \in \mathbb{Z}_{\geq 0}$ and $f \in M_k(\Gamma)_{\mathcal{O}_\mathfrak{p}}$. We put $i = [\Gamma_2 : \Gamma]$. For $\nu \in \mathbb{Z}_{\geq 1}$, assume that $a_f(m,r,n) \equiv 0 \bmod \mathfrak{p}^\nu$ for all $m,\ r,\ n \in \frac{1}{N}\mathbb{Z}$ with*

$$0 \leq m, n \leq b_{ki}.$$

and $4mn - r^2 \geq 0$, then we have $f \equiv 0 \bmod \mathfrak{p}^\nu$.

In the case of level 1 (i.e., $N = 1$), our bounds are sharp. More precisely, the following theorem holds.

Theorem 2.4 *Let $k \in \mathbb{Z}_{\geq 0}$ and p be a prime number. We assume $M_k(\Gamma_2) \neq \{0\}$. Then there exists $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $f \not\equiv 0 \mod p$ such that*

$$a_f(m,r,n) = 0 \quad \text{for all } m,\ n \leq b_k - 1.$$

3 Notation

For a prime number p and a $\mathbb{Z}_{(p)}$-module M, we put

$$\widetilde{M} = M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p.$$

For an element $x \in M$, we denote by $\widetilde{x}$ the image of x in $\widetilde{M}$. For a $\mathbb{Z}_{(p)}$-linear map $\varphi : M \to N$, we denote by $\widetilde{\varphi}$ the induced map from $\widetilde{M}$ to $\widetilde{N}$ by φ. For $n \in \mathbb{Z}_{\geq 1}$, let Γ be a congruence subgroup of Γ_n. We define $\widetilde{M}_k(\Gamma)_{\mathbb{Z}_{(p)}}$ by $\widetilde{M_k(\Gamma)_{\mathbb{Z}_{(p)}}}$. For a commutative ring R and an R-module M, we denote by $\mathrm{Sym}^2(M) \subset M \otimes_R M$ the R-module generated by elements $m \otimes m$ for $m \in M$. Let R be a $\mathbb{Z}_{(2)}$-algebra and M an R-module. We define an R-module $\wedge^2(M)$ by $\wedge^2(M) = \{x \in M \mid x^{\iota} = -x\}$.
Here ι is defined by $\iota(m \otimes n) = n \otimes m$ for $m,\ n \in M$. Let q_1, q_{12}, q_2 be variables and $S = \{q_1^m q_{12}^r q_2^n \mid m, n \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}\}$ be a set of Laurent monomials. We define an order on S so that $q_1^m q_{12}^r q_2^n \leq q_1^{m'} q_{12}^{r'} q_2^{n'}$ if and only if one of the following conditions holds.

1. $m < m'$.
2. $m = m'$ and $n < n'$.
3. $m = m'$ and $n = n'$ and $r \leq r'$.

Let K be a field and $f = \sum_{m,r,n} a_f(m, r, n) q_1^m q_{12}^r q_2^n \in K[q_{12}, q_{12}^{-1}][\![q_1, q_2]\!]$ a formal power series. If $f \neq 0$, let $q_1^{m_0} q_{12}^{r_0} q_2^{n_0}$ be the minimum monomial which appears in f, that is the minimum monomial of the set $\{q_1^m q_{12}^r q_2^n \mid a_f(m, r, n) \neq 0\}$. We define the leading term $\mathrm{ldt}(f)$ of f by $a_f(m_0, r_0, n_0) q_1^{m_0} q_{12}^{r_0} q_2^{n_0}$. We also define the leading term of an element of $K[\![q_1, q_2]\!] \setminus \{0\}$ by the inclusion $K[\![q_1, q_2]\!] \subset K[q_{12}, q_{12}^{-1}][\![q_1, q_2]\!]$. We regard $M_k(\Gamma_2)$ as a subspace of $\mathbb{C}[q_{12}, q_{12}^{-1}][\![q_1, q_2]\!]$ by

$$\sum_{T=(m,r,n)\in\Lambda_2} a_f(m, r, n) q^T \mapsto \sum_{m,r,n} a_f(m, r, n) q_1^m q_{12}^r q_2^n.$$

For $f \in M_k(\Gamma_2)$, we denote by $\mathrm{ldt}(f)$ the leading term of the Fourier expansion of f. For a field K, we regard $K[\![q]\!] \otimes_K K[\![q]\!]$ as a subspace of $K[\![q_1, q_2]\!]$ by $q \otimes 1 \mapsto q_1$ and $1 \otimes q \mapsto q_2$. For a subring R of $\mathbb{C}$ and a subset S of $\mathbb{C}[\![q_1, q_2]\!]$, we put

$$S_R = \left\{ f = \sum_{m,n} a_f(m, n) q_1^m q_2^n \in S \;\middle|\; a_f(m, n) \in R \right\}.$$

4 Witt operators

For the proof of the main results, we use Witt operators. In this section, we define Witt operators and introduce basic properties of them.

4.1 Elliptic modular forms

Since images of Witt operators can be expressed in terms of elliptic modular forms, we introduce some notation for elliptic modular forms.

For $k \in 2\mathbb{Z}$ with $k \geq 4$, we denote by $e_k \in M_k(\Gamma_1)$ the Eisenstein series of degree 1 and weight k. We normalize e_k so that the constant term is equal to 1. We define Eisenstein series e_2 of degree 1 and weight 2 by

$$e_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where $\sigma_1(n)$ is the sum of all positive divisors of n. As is well known, e_2 satisfies the following identify:

$$\tau^{-2}e_2(-\tau^{-1}) = \frac{12}{2\pi i\tau} + e_2(\tau).$$

We put $\Delta = 2^{-6}\cdot 3^{-3}(e_4^3 - e_6^2)$. Then Δ is the Ramanujan's delta function.

For $k \geq 2$, we define $N_k(\Gamma_1)$ as the space of $\mathbb{C}$-valued holomorphic functions f on $\mathbb{H}_1$ that satisfies the following three conditions:

1. $f(\tau+1) = f(\tau)$.
2. There exists $g \in M_{k-2}(\Gamma_1)$ such that
$$\tau^{-k}f(-\tau^{-1}) = \frac{1}{2\pi i\tau}g(\tau) + f(\tau) \quad \text{for } \tau \in \mathbb{H}_1.$$
3. f is holomorphic at the cusp $i\infty$.

Since $f - e_2 g/12 \in M_k(\Gamma_1)$ for f as avobe, we have the following lemma.

Lemma 4.1 *We have*

$$N_k(\Gamma_1) = M_k(\Gamma_1) \oplus e_2 M_{k-2}(\Gamma_1).$$

For $M = M_k(\Gamma_1)$ or $N_k(\Gamma_1)$, we regard M as a subspace of $\mathbb{C}[\![q]\!]$ via the Fourier expansion.

For $k = 2, 4, 6, 12$, we define elements of $\mathrm{Sym}^2\,(N_k(\Gamma_1))_{\mathbb{Z}}$ as follows:

$$x_k = e_k \otimes e_k, \text{ for } k = 2, 4, 6, \qquad x_{12} = \Delta\otimes\Delta, \qquad y_{12} = e_4^3 \otimes \Delta + \Delta \otimes e_4^3. \tag{4.1}$$

We define $\alpha_{36} \in \wedge^2(M_{36}(\Gamma_1))_{\mathbb{Z}}$ by

$$\alpha_{36} = x_{12}^2(\Delta \otimes e_4^3 - e_4^3 \otimes \Delta).$$

4.2 Definition of Witt operators

For $k \in \mathbb{Z}_{\geq 0}$ and $f \in M_k(\Gamma_2)$, we consider the following Taylor expansion

$$f(Z) = W(f)(\tau_1,\tau_2) + 2W'(f)(\tau_1,\tau_2)\,(2\pi i\tau_{12}) + W''(f)(\tau_1,\tau_2)\,(2\pi i\tau_{12})^2 + O(\tau_{12}^3),$$

where $Z = \begin{pmatrix}\tau_1 & \tau_{12}\\ \tau_{12} & \tau_2\end{pmatrix} \in \mathbb{H}_2$. We put $q_1 = \mathbf{e}(\tau_1), q_2 = \mathbf{e}(\tau_2)$ and $q_{12} = \mathbf{e}(\tau_{12})$. By definition, the following properties hold (see [16, Sect. 9]).

1. $W'(f) = 0$ if k is even and $W(f) = W''(f) = 0$ if k is odd.
2. $W(f) \in \mathrm{Sym}^2(M_k(\Gamma_1))$ if k is even and $W'(f) \in \wedge^2(M_{k+1}(\Gamma_1))$ if k is odd. Here we identify q_1 with $q \otimes 1$ and q_2 with $1 \otimes q$.
3. For $f \in M_k(\Gamma_2)$ and $g \in M_l(\Gamma_2)$, we have
$$W(fg) = W(f)W(g), \quad W'(fg) = W'(f)W(g) + W(f)W'(g).$$
Assume k and l are both even. Then we have
$$W''(fg) = W''(f)W(g) + W(f)W''(g). \tag{4.2}$$
4. For $f = \sum_{m,r,n} a_f(m,r,n)q_1^m q_{12}^r q_2^n \in M_k(\Gamma_2)$, we have
$$W(f) = \sum_{m,r,n} a_f(m,r,n)q_1^m q_2^n, \quad W'(f) = \frac{1}{2}\sum_{m,r,n} r a_f(m,r,n)q_1^m q_2^n,$$
$$W''(f) = \frac{1}{2}\sum_{m,r,n} r^2 a_f(m,r,n)q_1^m q_2^n.$$

Let k be even and $f \in M_k(\Gamma_2)$. Then we have

$$\tau_1^{-k-2} W''(f)(\tau_1^{-1}, \tau_2) = -\frac{1}{2\pi i}\theta_2 W(f)(\tau_1, \tau_2)\tau_1^{-1} + W''(f)(\tau_1, \tau_2),$$
$$W''(f)(\tau_1, \tau_2) = W''(f)(\tau_2, \tau_1).$$

Here $\theta_2 = \frac{1}{2\pi i}\frac{d}{d\tau_{12}}$. Therefore by Lemma 4.1, we have the following lemma.

Lemma 4.2 *Let $k \in 2\mathbb{Z}_{\geq 0}$ and $f \in M_k(\Gamma_2)$. Then we have $W''(f) \in \mathrm{Sym}^2(N_{k+2}(\Gamma_1))$.*

Let R be a subring of $\mathbb{C}$. If k is even and $f \in M_k(\Gamma_2)_R$, then we have

$$W''(f) = \sum_{\substack{m,r,n \\ r>0}} r^2 a_f(m, r, n) q_1^m q_{12}^r q_2^n,$$

since $a_f(m, -r, n) = a_f(m, r, n)$. Thus we have $W''(f) \in \mathrm{Sym}^2(M_k(\Gamma_1))_R$. By a similar reason, we have $W'(f) \in M_{k+1}(\Gamma_2)_R$ for $f \in M_k(\Gamma_2)_R$ with odd k. For $k \in \mathbb{Z}_{\geq 0}$, we define the *Witt operators* as the R-linear maps induced by W, W' and W'' as follows.

$$W_{R,2k} : M_{2k}(\Gamma_2)_R \to \mathrm{Sym}^2(M_{2k}(\Gamma_1))_R,$$
$$W'_{R,2k-1} : M_{2k-1}(\Gamma_2)_R \to \wedge^2(M_{2k}(\Gamma_1))_R,$$
$$W''_{R,2k} : M_{2k}(\Gamma_2)_R \to \mathrm{Sym}^2(N_{2k+2}(\Gamma_1))_R.$$

4.3 Igusa's generators and their images

Let $X_4,\ X_6,\ X_{10},\ X_{12}$ and X_{35} be generators of $\bigoplus_{k\in\mathbb{Z}} M_k(\Gamma_2)$ given by Igusa [4,5]. Here X_4 and X_6 are Siegel–Eisenstein series of weight 4 and 6 respectively. And X_{10}, X_{12} and X_{35} are cusp forms of weight 10, 12 and 35 respectively. We normalize these modular forms so that

$$\mathrm{ldt}(X_4) = \mathrm{ldt}(X_6) = 1, \quad \mathrm{ldt}(X_{10}) = \mathrm{ldt}(X_{12}) = q_1 q_{12}^{-1} q_2, \quad \mathrm{ldt}(X_{35}) = q_1^2 q_{12}^{-1} q_2^3.$$

Here we note that $a_{X_{35}}(1, r, n) = 0$ for all $n, r \in \mathbb{Z}$, because any weak Jacobi form of index 1 and weight 35 vanishes. We also introduce $Y_{12} \in M_{12}(\Gamma_2)_{\mathbb{Z}}$ and $X_k \in M_k(\Gamma_2)_{\mathbb{Z}}$ for $k = 16$, 18, 24, 28, 30, 36, 40, 42 and 48. Then by Igusa [6],

$$\left\{X_k \;\middle|\; k = 4, 6, 10, 12, 16, 18, 24, 28, 30, 36, 40, 42, 48\right\} \cup \{Y_{12}\}$$

is a minimal set of generators of $\bigoplus_{k\in 2\mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}}$ as a $\mathbb{Z}$-algebra and we have

$$M_k(\Gamma_2)_{\mathbb{Z}} = X_{35} M_{k-35}(\Gamma_2)_{\mathbb{Z}}$$

for odd k.

Igusa [6] computed $W(X_4), \ldots, W(X_{48})$ and $W(Y_{12})$, we introduce some of them.

$$W(X_4) = x_4, \quad W(X_6) = x_6, \quad W(X_{10}) = 0,$$
$$W(X_{12}) = 2^2 \cdot 3x_{12}, \quad W(Y_{12}) = y_{12}, \quad W(X_{16}) = x_4 \cdot x_{12} \tag{4.3}$$

and

$$W(X_{12i}) = d_i x_{12}^i, \quad \text{for } i = 1, 2, 3, 4. \tag{4.4}$$

Here x_4, x_6, x_{12} and y_{12} are defined by (4.1), and d_i is defined by $12/\gcd(12, i)$.

Images of W' and W'' for some of the generators are given as follows.

Lemma 4.3 *We have*

$$W'(X_{35}) = \alpha_{36}$$

and

$$W''(X_{10}) = x_{12}, \quad W''(X_{12i}) = x_2 x_{12}^i, \quad \textit{for } i = 1, 2, 3, 4.$$

Proof By $\mathrm{ldt}(X_{35}) = q_1^2 q_{12}^{-1} q_2^3$ and $\wedge^2(M_k(\Gamma_1)) = (\Delta \otimes e_4^3 - e_4^3 \otimes \Delta)\mathrm{Sym}^2(M_{k-12}(\Gamma_1))$, we see that $W'(X_{35})$ is a constant multiple of α_{36}. Since $a_{X_{35}}(2, r, 3) = 0$ if $r \neq \pm 1$, we have $W'(X_{35}) = \alpha_{36}$. Igusa computed $W''(X_{10})$ and $W''(X_{12})$ (see [6, Lemma 12]). Note that our notation is different from his notation. We denote his W' by W''. By this result, we can compute $W''(X_{12i})$ for $i = 2, 3, 4$. □

4.4 Kernel of Witt operator modulo a prime

Let p be a prime number and k even. We consider the kernel of the Witt operator modulo p:

$$\widetilde{W}_{\mathbb{Z}_{(p)},k} : \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \to \mathrm{Sym}^2(M_k(\Gamma_2))_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p.$$

First we consider the case when $p \geq 5$. This case is easier.

Lemma 4.4 *Let p be a prime number with $p \geq 5$. Then we have*

$$\bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} \mathrm{Sym}^2 (M_k(\Gamma_1))_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}[x_4,\ x_6,\ x_{12}].$$

Proof It is easy to see that $\mathrm{Sym}^2(M_k(\Gamma_1)_{\mathbb{Z}_{(p)}}) = \mathrm{Sym}^2 (M_k(\Gamma_1))_{\mathbb{Z}_{(p)}}$ (see the remark after Theorem 5.12 of [11]). Since $p \geq 5$, we have $\bigoplus_{k\in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_1)_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}[e_4,\ e_6]$ (see [14]). We note that $\bigoplus_{k\in 2\mathbb{Z}_{\geq 0}} \mathrm{Sym}^2(M_k(\Gamma_1)_{\mathbb{Z}_{(p)}})$ is generated by $x_4,\ x_6$ and $e_4^3 \otimes e_6^2 + e_6^2 \otimes e_4^3$ as an algebra over $\mathbb{Z}_{(p)}$. Then the assertion of the lemma follows from the equation

$$2^{12} \cdot 3^6 x_{12} = x_4^3 + x_6^2 - (e_4^3 \otimes e_6^2 + e_6^2 \otimes e_4^3).$$

□

The following is a key lemma for the proof of Theorem 2.1 for $\mathfrak{p} \nmid 2 \cdot 3$. This lemma was also used in [1].

Lemma 4.5 *Let $p \geq 5$ be a prime number and $k \in 2\mathbb{Z}_{\geq 0}$. Then we have*

$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},k}) = \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$

Proof This lemma seems well-known. But for the sake of completeness, we give a proof. The inclusion $\widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}} \subset \ker(\widetilde{W}_{\mathbb{Z}_{(p)},k})$ is obvious, because $W(X_{10}) = 0$. Take $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $W_{\mathbb{Z}_{(p)},k}(f) \equiv 0 \bmod p$. By (4.3) and Lemma 4.4, $W_{\mathbb{Z}_{(p)},k}$ is surjective. Take $g \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ so that $W_{\mathbb{Z}_{(p)},k}(f) = pW_{\mathbb{Z}_{(p)},k}(g)$. Then by [9, Corollary 4.2], there exists $h \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f - pg = X_{10}h$. This completes the proof. □

Remark 4.6 Since $W(X_{12}) = 12x_{12}$ and $M_2(\Gamma_2) = \{0\}$, the assertion of the lemma does not hold if $p = 2, 3$.

Next we consider the case where $p = 2, 3$. We recall the structure of the ring $\bigoplus_{k\in 2\mathbb{Z}} \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$.

Theorem 4.7 (Nagaoka [10], Theorem 2) *Let $p = 2, 3$. For $f \in \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, there exists a unique polynomial $Q \in \mathbb{F}_p[x, y, z]$ such that*

$$\widetilde{f} = Q(\widetilde{X}_{10}, \widetilde{Y}_{12}, \widetilde{X}_{16}).$$

The above Q for Igusa's generators are given as follows.

Lemma 4.8 (Nagaoka [10], proof of Lemma 1, Lemma 2)

1. *Suppose $p = 2$, then we have*

$$\begin{aligned} &X_4 \equiv X_6 \equiv 1 \mod p, && X_{12} \equiv X_{10} \mod p,\\ &X_{18} \equiv X_{16} \mod p, && X_{24} \equiv X_{10}X_{16} \mod p,\\ &X_{28} \equiv X_{30} \equiv X_{16}^2 \mod p, && X_{36} \equiv X_{10}X_{16}^2 \mod p,\\ &X_{40} \equiv X_{42} \equiv X_{16}^3 \mod p, && X_{48} \equiv X_{16}^4 + X_{10}X_{16}^3 + X_{10}^4 Y_{12} \mod p,\\ &X_{35}^2 \equiv X_{10}^2 Y_{12}^2 X_{16}^2 + X_{10}^6 \mod p. \end{aligned}$$

2. *Suppose $p = 3$, then we have*

$$\begin{aligned} &X_4 \equiv X_6 \equiv 1 \mod p, && X_{12} \equiv X_{10} \mod p,\\ &X_{18} \equiv X_{16} \mod p, && X_{24} \equiv X_{10}X_{16} \mod p,\\ &X_{28} \equiv X_{30} \equiv X_{16}^2 \mod p, && X_{36} \equiv X_{16}^3 + 2X_{10}^3 Y_{12} + X_{10}X_{16}^2 \mod p,\\ &X_{40} \equiv X_{16}^3 + 2X_{10}^3 Y_{12} \mod p, && X_{42} \equiv X_{16}^3 + X_{10}^3 Y_{12} \mod p,\\ &X_{48} \equiv X_{10}X_{16}^3 + 2X_{10}^4 Y_{12} \mod p, \end{aligned}$$

   and

$$\begin{aligned} X_{35}^2 \equiv\ & 2X_{10}X_{16}^4 + X_{10}Y_{12}^2 X_{16}^3\\ & + 2X_{10}^2 X_{16}^3 + X_{10}^2 Y_{12}^2 X_{16}^2 + 2X_{10}^3 Y_{12} X_{16}^2\\ & + 2X_{10}^4 Y_{12}^3 + X_{10}^4 X_{16}^2 + 2X_{10}^7 \mod p. \end{aligned}$$

For later use, we prove the following lemma.

Lemma 4.9 *Let $p = 2, 3$ and $k \in 2\mathbb{Z}_{\geq 0}$ with $12 \nmid k$. Then we have*

$$\widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \subset \widetilde{M}_{k+2}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$

Proof Take $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$. We show that there exists $g \in M_{k+2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv g \mod p$. We may assume f is an isobaric monomial of Igusa's generators of even weights, that is $X_4, \ldots, X_{48}$ and Y_{12}. If $f = X_k$ with $12 \nmid k$, then by Lemma 4.8, we have $\widetilde{f} \in \widetilde{M}_{k+2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. In fact, we have $X_{18} \equiv X_4 X_{16} \mod p$, $X_{42} \equiv X_{16}X_{28} \mod 2$, $X_{40} \equiv X_{42} + X_{10}^3 Y_{12} \mod 3$ and $X_{42} \equiv X_{16}X_{28} + X_{10}^2 X_{12} Y_{12} \mod 3$. If f is an isobaric monomial of weight k, then f contains some X_k with $12 \nmid k$. Therefore we have the assertion of the lemma. □

Let $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $p = 2, 3$. As we remarked before, $W(f) \equiv 0 \mod p$ does not imply the existence of $g \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10} g \mod p$. Instead of Lemma 4.5, we have the following proposition.

Proposition 4.10 *Let $p = 2, 3$ and $k \in 2\mathbb{Z}_{\geq 0}$.*

1. *Suppose $12 \nmid k$. Then we have*
$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},k}) = \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$
2. *Suppose $k = 12n$ with $n \in \mathbb{Z}$ and $p = 2$. For $0 \leq i \leq n$ with $4 \nmid i$, we put $i = 4s + t$ with $t \in \{1, 2, 3\}$ and $m_i = X_{12t}X_{48}^{s}Y_{12}^{n-i}$. Then we have*
$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},k}) = \bigoplus_{\substack{0\leq i\leq n\\ 4\nmid i}} \mathbb{F}_p\widetilde{m}_i \oplus \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$
3. *Suppose $k = 12n$ with $n \in \mathbb{Z}$ and $p = 3$. For $0 \leq i \leq n$ with $3 \nmid i$, we put $i = 3s + t$ with $t \in \{1, 2\}$ and $m_i = X_{12t}X_{36}^{s}Y_{12}^{n-i}$. Then we have*
$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},k}) = \bigoplus_{\substack{0\leq i\leq n\\ 3\nmid i}} \mathbb{F}_p\widetilde{m}_i \oplus \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$

Moreover, if $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $12 \mid k$ and
$$W(f) \equiv W''(f) \equiv 0 \mod p,$$
then there exists $g \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}^2 g \mod p$.

Proof Suppose $12 \nmid k$. Then by [6, Lemma 13], $W_{\mathbb{Z},k}$ is surjective. Therefore, $W_{\mathbb{Z}_{(p)},k}$ is surjective. We can prove $\ker\big(\widetilde{W}_{\mathbb{Z}_{(p)},k}\big) = \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ by a similar argument to the proof of Lemma 4.5. Next assume $k = 12n$ with $n \in \mathbb{Z}$. For simplicity, we assume $p = 2$. We can prove the case when $p = 3$ in a similar way. Take $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $W(f) \equiv 0 \mod p$. Put $d_i = 12/\gcd(12, i)$. By [6, Lemma 13], there exist $a_{i,j}, b_i, c_i \in \mathbb{Z}_{(p)}$ such that
$$W(f) = \sum_{0\leq i\leq j<n} a_{i,j}x_4^{3(n-j)}x_{12}^{i}y_{12}^{j-i} + \sum_{\substack{0\leq i\leq n\\ 4\mid i}} b_i x_{12}^{i}y_{12}^{n-i} + \sum_{\substack{0\leq i\leq n\\ 4\nmid i}} c_i d_i x_{12}^{i}y_{12}^{n-i}.$$

By $x_4 \equiv 1 \mod p$ and $W(f) \equiv 0 \mod p$, we have $a_{i,j} \equiv b_i \equiv 0 \mod p$ for all i, j. Here we note that $\widetilde{x}_{12}$ and $\widetilde{y}_{12}$ are algebraically independent over $\mathbb{F}_p$. This is because $\mathrm{ldt}(x_{12}^{i}y_{12}^{j}) = q_1^{i}q_2^{i+j}$. By [6, Lemma 13], there exists $f' \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that
$$W(f') = \sum_{0\leq i\leq j<n} \frac{a_{i,j}}{p}x_4^{3(n-j)}x_{12}^{i}y_{12}^{j-i} + \sum_{\substack{0\leq i\leq n\\ 4\mid i}} \frac{b_i}{p}x_{12}^{i}y_{12}^{n-i}.$$

By (4.4), there exists $u_i \in \mathbb{Z}_{(p)}^{\times}$ such that $W(m_i) = u_i d_i x_{12}^{i}y_{12}^{n-i}$. Therefore, there exist $a_i \in \mathbb{Z}_{(p)}$ such that $W(f - pf' - \sum_{\substack{0\leq i\leq n\\ 4\nmid i}} a_i m_i) = 0$. By [9, Corollary 4.2], there exists $g \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $\widetilde{f} = \sum_i \widetilde{a}_i\widetilde{m}_i + \widetilde{X}_{10}\widetilde{g}$. Thus we have
$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},k}) = \sum_{\substack{0\leq i\leq n\\ 4\nmid i}} \mathbb{F}_p\widetilde{m}_i + \widetilde{X}_{10}\widetilde{M}_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}. \tag{4.5}$$

We show that the sum (4.5) is a direct sum. Let $a_i \in \mathbb{Z}_{(p)}$ for $0 \leq i \leq n$ with $4 \nmid i$ and $g \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. We put $f = \sum_i a_i m_i + X_{10}g$. By (4.2), we have
$$W''(m_i) \equiv W''(X_{12t})W(X_{48}^{s}Y_{12}^{n-i}) \equiv x_{12}^{i}y_{12}^{n-i} \mod p. \tag{4.6}$$

Here we use $W(X_{12t}) \equiv 0 \mod p$ for $t = 1, 2, 3$ and $x_2 \equiv 1 \mod p$. By Igusa's computation, images of 14 generators $X_4, \cdots, X_{48}$ by W can be written as $\mathbb{Z}$-coefficient polynomials of x_4, x_6, x_{12} and y_{12}. By Lemma 4.3, we have $W''(X_{10}) = x_{12}$. Thus there exist $\alpha_{a,b,c,d} \in \mathbb{Z}_{(p)}$ such that

$$W''(X_{10}g) = x_{12}W(g) = \sum_{a,b,c,d} \alpha_{a,b,c,d} x_4^a x_6^b x_{12}^c y_{12}^d,$$

where summation index runs over $\{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 \mid 4a + 6b + 12c + 12d = k + 2\}$. We assume $\widetilde{W}''_{\mathbb{Z}_{(p)},k}(\widetilde{f}) = \widetilde{W}''_{\mathbb{Z}_{(p)},k}(\sum_i \widetilde{a}_i \widetilde{m}_i + \widetilde{X}_{10}\widetilde{g}) = 0$. Then by (4.6) and $x_4 \equiv x_6 \equiv 1 \mod p$, we have

$$\sum_i \widetilde{a}_i \widetilde{x}_{12}^i \widetilde{y}_{12}^{n-i} + \sum_{a,b,c,d} \widetilde{\alpha}_{a,b,c,d} \widetilde{x}_{12}^c \widetilde{y}_{12}^d = 0.$$

Since $4a + 6b = 0$ or $4a + 6b \geq 4$, the isobaric degree of $\widetilde{x}_{12}^c \widetilde{y}_{12}^d$ is not equal to k. Therefore we have $\widetilde{a}_i = 0$ for all i. This shows that the sum (4.5) is a direct sum. This also shows that if $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $12 \mid k$ satisfies $W(f) \equiv W''(f) \equiv 0 \mod p$, then there exists $h \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}h \mod p$. By $W''(f) \equiv 0 \mod p$, we have $W(h) \equiv 0 \mod p$. Since $12 \nmid k - 10$, there exists $h' \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $h \equiv X_{10}h'$ mod p. Therefore we have $f \equiv X_{10}^2 h' \mod p$. This completes the proof. □

Corollary 4.11 *Let $p = 2, 3$ and $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $12 \mid k$. If $W(f) \equiv 0 \mod p$, then there exists $g \in M_{k-8}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}g \mod p$.*

Proof By Lemma 4.8, the statement for $f = m_i$ is true for all i. Then by Proposition 4.10, we have $f \equiv X_{10}(g + h) \mod p$ with $g \in M_{k-8}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ and $h \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. By Lemma 4.9, we have our assertion. □

5 Proof of the main results

In this section, we give proofs of Theorem 2.1, Corollary 2.3 and Theorem 2.4.

We have $\widetilde{M}_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}} = \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{F}_p} \mathcal{O}_\mathfrak{p}/\mathfrak{p}$. Therefore Theorem 2.1 is reduced to the case of $\mathcal{O}_\mathfrak{p} = \mathbb{Z}_{(p)}$, where p is a prime number. We also note that the statement of Theorem 2.1 for $\nu \geq 2$ is reduced to the case of $\nu = 1$ by repeatedly using the result. This method was used in [12].

As we remarked before, the statement of Theorem 2.1 was proven in [1] for k even and $p \geq 5$. Thus in this section, we assume $k \equiv 0 \mod 2$, $p = 2, 3$ or $k \equiv 1 \mod 2$.

First, we introduce the following notation, which is similar to mod p diagonal vanishing order defined by Richter and Raum [13]. Let $\widetilde{f}$ be a $\mathbb{F}_p$-coefficients formal power series as follows;

$$\widetilde{f} = \sum_{\substack{m,r,n \in \mathbb{Q} \\ m,n,4mn-r^2 \geq 0}} \widetilde{a}_f(m, r, n) q_1^m q_{12}^r q_2^n \in \bigcup_{N \in \mathbb{Z}_{\geq 1}} \mathbb{F}_p[q_{12}^{1/N}, q_{12}^{-1/N}][\![q_1^{1/N}, q_2^{1/N}]\!].$$

We define $v_p(\widetilde{f})$ by

$$v_p(\widetilde{f}) = \sup \left\{ A \in \mathbb{R} \;\middle|\; \begin{array}{l} \widetilde{a}_f(m, r, n) = 0, \\ \text{for all } m, r, n \in \mathbb{Q} \text{ with } 0 \leq m, n < A \end{array} \right\}.$$

By definition, we have

$$v_p(\widetilde{f}\widetilde{g}) \geq \max\{v_p(\widetilde{f}), v_p(\widetilde{g})\}, \tag{5.1}$$

for $\widetilde{f}, \widetilde{g} \in \bigcup_{N \in \mathbb{Z}_{\geq 1}} \mathbb{F}_p[q_{12}^{1/N}, q_{12}^{-1/N}][\![q_1^{1/N}, q_2^{1/N}]\!]$. We note that $v_p(\widetilde{f}) > A$ is equivalent to $\widetilde{a}_f(m, r, n) = 0$ for all $m, n \leq A$, where $A \in \mathbb{R}$.

For the proof of Theorem 2.1, we introduce the following three lemmas.

Lemma 5.1 *Let p be a prime number and $\widetilde{f} \in \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $k \in \mathbb{Z}_{\geq 0}$. Then we have $v_p(\widetilde{X}_{10}\widetilde{f}) = v_p(\widetilde{f}) + 1$ and $v_p(\widetilde{X}_{35}\widetilde{f}) \geq v_p(\widetilde{f}) + 2$.*

Proof We regard $\widetilde{X}_{10}$ and $\widetilde{X}_{35}$ as images in the ring of formal power series $\mathbb{F}_p(q_{12})[\![q_1, q_2]\!]$. Recall that the descriptions of X_{10} and X_{35} as the Borcherds products are given by

$$X_{10} = q_1 q_{12} q_2 \prod_{\substack{m,r,n \in \mathbb{Z} \\ (m,r,n) > 0}} (1 - q_1^m q_{12}^r q_2^n)^{c(4mn - r^2)},$$

$$X_{35} = q_1^2 q_{12} q_2^2 (q_1 - q_2) \prod_{\substack{m,r,n \in \mathbb{Z} \\ (m,r,n) > 0}} (1 - q_1^m q_{12}^r q_2^n)^{d(4mn - r^2)},$$

where $c(M)$, $d(M)$ ($M \in \mathbb{Z}$) are certain integers determined by the Fourier coefficients of the weak Jacobi form of weight 0 with index 1. For more details, see [2,3]. By this formula for X_{10}, we have $\widetilde{X}_{10} = q_1 q_2 u$ where u is a unit in $\mathbb{F}_p(q_{12})[\![q_1, q_2]\!]$. Similarly, we have $\widetilde{X}_{35} = q_1^2 q_2^2 (q_1 - q_2) v$ for some unit v in $\mathbb{F}_p(q_{12})[\![q_1, q_2]\!]$. The assertion of the lemma follows from these facts. □

Remark 5.2 It is not easy to give an upper bound for $v_p(\widetilde{X}_{35}\widetilde{f}) - v_p(\widetilde{f})$ because of the factor $q_1 - q_2$ in the Borcherds product of X_{35}.

Lemma 5.3 *Let p be a prime number and*

$$f = \sum_{m,n \geq 0} a_f(m,n) q_1^m q_2^n \in (M_k(\Gamma_1) \otimes M_k(\Gamma_1))_{\mathbb{Z}_{(p)}}.$$

If $a_f(m, n) \equiv 0 \mod p$ for all $m, n \leq [k/12]$, then $f \equiv 0 \mod p$. In particular, for $g \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we have $W(g) \equiv 0 \mod p$ if $v_p(\widetilde{g}) > [k/12]$ and $W'(g) \equiv 0 \mod p$ if $v_p(\widetilde{g}) > [(k+1)/12]$.

Proof By the original Sturm's theorem [15], the map

$$\widetilde{M}_k(\Gamma_1)_{\mathbb{Z}_{(p)}} \hookrightarrow \mathbb{F}_p[\![q]\!]/(q^{[k/12]+1})$$

is injective. Therefore we have the following injective map

$$\begin{aligned} \mathrm{Sym}^2(M_k(\Gamma_1))_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p &= \mathrm{Sym}^2(\widetilde{M}_k(\Gamma_1)_{\mathbb{Z}_{(p)}}) \\ &\hookrightarrow F_p[\![q]\!]/(q^{[k/12]+1}) \otimes_{\mathbb{F}_p} F_p[\![q]\!]/(q^{[k/12]+1}). \end{aligned}$$

Here we note that $\mathrm{Sym}^2(M_k(\Gamma_1)_{\mathbb{Z}_{(p)}}) = \mathrm{Sym}^2(M_k(\Gamma_1))_{\mathbb{Z}_{(p)}}$, as we remarked in the proof of Lemma 4.4. Since the image of $\widetilde{f}$ by this map vanishes, we have $\widetilde{f} = 0$. □

Lemma 5.4 *We define $f_k \in M_k(\Gamma_2)_{\mathbb{Z}}$ for $k = 35, 39, 41, 43$ and 47 as follows.*

$$f_{35} = X_{35}, \quad f_{39} = X_4 X_{35}, \quad f_{41} = X_6 X_{35}, \quad f_{43} = X_4^2 X_{35}, \quad f_{47} = X_{12} X_{35}.$$

Then $\mathrm{ldt}(f_k) = q_1^2 q_{12}^{-1} q_2^3$ *for* $k = 35, 39, 41, 43$ *and* $\mathrm{ldt}(f_{47}) = q_1^3 q_{12}^{-2} q_2^4$.

Proof This follows from $\mathrm{ldt}(X_4) = \mathrm{ldt}(X_6) = 1$, $\mathrm{ldt}(X_{12}) = q_1 q_{12}^{-1} q_2$ and $\mathrm{ldt}(X_{35}) = q_1^2 q_{12}^{-1} q_2^3$. □

5.1 Proof of Theorem 2.1 for $p = 2, 3$ and even k

Let $p = 2, 3$, $k \in 2\mathbb{Z}_{\geq 0}$ and $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$. We assume

$$v_p(\widetilde{f}) > b_k, \tag{5.2}$$

where b_k is given in Theorem 2.1. We prove the statement of Theorem 2.1 by induction on k. First, we assume $k < 10$. Then the statement is true because $M_k(\Gamma_2)$ for $k = 4, 6, 8$ is one-dimensional and $\mathrm{ldt}(X_4) = \mathrm{ldt}(X_6) = \mathrm{ldt}(X_4^2) = 1$.

Next, we assume $k \geq 10$ and the statement is true if the weight is strictly less than k. By (5.2) and Lemma 5.3, we have $W(f) \equiv 0 \mod p$. If $12 \nmid k$, then by Proposition 4.10, there exists $g \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}g \mod p$. By (5.2) and Lemma 5.1, we have $v_p(\widetilde{g}) > b_{k-10}$. By the induction hypothesis, we have $g \equiv 0 \mod p$. Thus we have the assertion of Theorem 2.1 in this case. Next we assume $12 \mid k$. Then by Corollary 4.11, there exists $g \in M_{k-8}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}g \mod p$. Since $b_{k-10} \geq [(k-8)/12]$ for $k \geq 10$, we have $W(g) \equiv 0 \mod p$ by (5.2), Lemmas 5.1 and 5.3. Therefore $W''(f) \equiv x_{12}W(g) \equiv 0 \mod p$. By Proposition 4.10, there exists $h \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f \equiv X_{10}^2 h \mod p$. Since $v_p(\widetilde{h}) > b_{k-20}$, we have $h \equiv 0 \mod p$ by the induction hypothesis. Thus we have $f \equiv 0 \mod p$. This completes the proof. □

5.2 Proof of Theorem 2.1 for the case $p \nmid 2 \cdot 3$ and odd k

Let p be a prime number with $p \geq 5$ and $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with k odd. We assume

$$v_p(\widetilde{f}) > b_k. \tag{5.3}$$

We prove the theorem by induction on k. Note that $M_k(\Gamma_2) = \{0\}$ if k is odd and $k < 35$ or $k = 37$. First assume that $0 \leq k - 35 < 10$ with $k \neq 37$. Then $M_k(\Gamma_2)$ is one-dimensional and spanned by f_k given in Lemma 5.4. By Lemma 5.4, the assertion of the theorem holds if $k - 35 < 10$.

Next we assume $k - 35 \geq 10$ and the assertion of the theorem holds if the weight is strictly less than k. By Igusa [6], there exists $g \in M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f = X_{35}g$. By Lemma 4.3, we have

$$W'(f) = W'(X_{35})W(g) = \alpha_{36}W(g). \tag{5.4}$$

By $[(k+1)/12] \leq b_k$ and Lemma 5.3, we have $W'(f) \equiv 0 \bmod p$. Therefore, we have $W(g) \equiv 0 \bmod p$ by (5.4). Then by Lemma 4.5, there exists $g' \in M_{k-45}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $g \equiv X_{10}g' \bmod p$. We put $f' = X_{35}g'$. Then we have $f \equiv X_{10}f' \bmod p$. By (5.3) and Lemma 5.1, we have $v_p(\widetilde{f'}) > b_{k-10}$. By the induction hypothesis, we have $f' \equiv 0 \bmod p$. Thus $f \equiv 0 \bmod p$. This completes the proof. □

5.3 Proof of Theorem 2.1 for $p = 2, 3$ and odd k

In this subsection, we assume $p = 2, 3$ and k is odd. Since the case when $k = 48 + 35 = 83$ is special in our proof, we prove the following two lemmas first.

Lemma 5.5 *Let $\widetilde{f} \in \widetilde{M}_{48}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $\widetilde{f} \neq 0$ and $\mathrm{ldt}(\widetilde{f}) = \alpha q_1^a q_{12}^b q_2^c$ be the leading term of $\widetilde{f}$. Here $\alpha \in \mathbb{F}_p^{\times}$. Assume $\widetilde{W}_{\mathbb{Z}_{(p)},48}(\widetilde{f}) = 0$. Then we have $a \leq 4$ and $c \leq 4$.*

Proof By Proposition 4.10, we have

$$\ker(\widetilde{W}_{\mathbb{Z}_{(p)},48}) = \bigoplus_i \mathbb{F}_p \widetilde{m}_i \oplus \widetilde{X}_{10}\widetilde{M}_{38}(\Gamma_2)_{\mathbb{Z}_{(p)}}.$$

Here $i = 1, 2, 3$ if $p = 2$ and $i = 1, 2, 4$ if $p = 3$. For $\widetilde{g} \in \widetilde{M}_{48}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, let $Q_g = \sum_{a,b,c} \gamma_{a,b,c} x^a y^b z^c$ be a $\mathbb{F}_p$-coefficients polynomial such that $\widetilde{g} = Q_g(\widetilde{X}_{10}, \widetilde{Y}_{12}, \widetilde{X}_{16})$ as in Theorem 4.7. Since

$$\mathrm{ldt}(\widetilde{X}_{10}^a \widetilde{Y}_{12}^b \widetilde{X}_{16}^c) = q_1^{a+c} q_{12}^{-a} q_2^{a+b+c}, \tag{5.5}$$

there exists a unique monomial $\widetilde{X}_{10}^{a_0} \widetilde{Y}_{12}^{b_0} \widetilde{X}_{16}^{c_0}$ with $\gamma_{a_0,b_0,c_0} \neq 0$ such that $\mathrm{ldt}(\widetilde{g}) = \mathrm{ldt}(\gamma_{a_0,b_0,c_0} \widetilde{X}_{10}^{a_0} \widetilde{Y}_{12}^{b_0} \widetilde{X}_{16}^{c_0})$. We put $\phi(\widetilde{g}) = \widetilde{X}_{10}^{a_0} \widetilde{Y}_{12}^{b_0} \widetilde{X}_{16}^{c_0}$. We define a set S' by

$$\begin{gathered}\{1, \widetilde{X}_{16}, \widetilde{Y}_{12}, \widetilde{X}_{10}, \widetilde{X}_{16}^2, \widetilde{Y}_{12}\widetilde{X}_{16}, \widetilde{Y}_{12}^2, \widetilde{X}_{10}\widetilde{X}_{16}, \widetilde{X}_{10}\widetilde{Y}_{12}, \widetilde{X}_{10}^2,\\ \widetilde{X}_{10}\widetilde{X}_{16}^2, \widetilde{X}_{10}\widetilde{Y}_{12}\widetilde{X}_{16}, \widetilde{X}_{10}\widetilde{Y}_{12}^2, \widetilde{X}_{10}^2\widetilde{X}_{16}, \widetilde{X}_{10}^2\widetilde{Y}_{12}, \widetilde{X}_{10}^3\}.\end{gathered}$$

Then S' forms a basis of $\widetilde{M}_{38}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. This follows from

$$\dim_{\mathbb{F}_p}(\widetilde{M}_{38}(\Gamma_2)_{\mathbb{Z}_{(p)}}) = \dim_{\mathbb{C}} M_{38}(\Gamma_2) = 16$$

and Lemma 4.8. We put $S = \{\widetilde{X}_{10}a \mid a \in S'\}$. We define the set T by

$$T = \begin{cases} \{\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3\} & \text{if } p = 2,\\ \{\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_4\} & \text{if } p = 3.\end{cases}$$

Then $S \cup T$ forms a basis of $\ker(\widetilde{W}_{\mathbb{Z}_{(p)},48})$. We have $\phi(s) = s$ except when $p = 3$ and $s = m_4$ for $s \in S \cup T$. If $p = 3$, we have $\phi(\widetilde{m}_4) = \widetilde{X}_{10}\widetilde{Y}_{12}\widetilde{X}_{16}^2$. Thus we see that ϕ is injective on $S \cup T$. Therefore if $\widetilde{f} \in \ker(\widetilde{W}_{\mathbb{Z}_{(p)},48})$ with $\widetilde{f} \neq 0$, then there exists a unique $s \in S \cup T$ such that $\mathrm{ldt}(\widetilde{f}) = \alpha\,\mathrm{ldt}(s)$ with $\alpha \neq 0$. Note that degrees of monomials $\{\phi(s) \mid s \in S \cup T\}$ are less than or equal to 4. Then by (5.5), we have the assertion of the lemma. □

Lemma 5.6 *Let $k = 83$, $\widetilde{f} = \widetilde{X}_{35}\widetilde{g} \in \widetilde{M}_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $g \in \widetilde{M}_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ and $\widetilde{W}_{\mathbb{Z}_{(p)},k-35}(\widetilde{g}) = 0$. Assume $v_p(\widetilde{f}) > b_k = 7$. Then we have $\widetilde{f} = 0$.*

Proof Assume $\widetilde{f} \neq 0$. We put $\mathrm{ldt}(\widetilde{g}) = \alpha q_1^a q_{12}^b q_2^c$, where $\alpha \in \mathbb{F}_p^\times$. Then by Lemma 5.5, we have $a, c \leq 4$. Since $\mathrm{ldt}(\widetilde{X}_{35}) = q_1^2 q_{12}^{-1} q_2^3$, we have $\mathrm{ldt}(\widetilde{f}) = \alpha q_1^{a+2} q_{12}^{b-1} q_2^{c+3}$. By $a+2 \leq 6$ and $c+3 \leq 7$, we have $v_p(\widetilde{f}) \leq 7$. □

Let k be odd and $f \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$. Assume that

$$v_p(\widetilde{f}) > b_k. \tag{5.6}$$

If $k < 45$, then the assertion follows from Lemma 5.4. Hence we suppose that $k \geq 45$. To apply an induction on k, suppose that the assertion is true for any weight strictly smaller than k.

We take $g \in M_{k-35}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $f = gX_{35}$. By (5.6), (5.4) and Lemma 5.3, we have $W(g) \equiv 0 \mod p$. Now we separate into four cases:

(1) If $k \not\equiv 11 \mod 12$, then there exists $g' \in M_{k-45}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $g \equiv X_{10}g' \mod p$, by Proposition 4.10. Then $f = X_{35}g = X_{35}X_{10}g'$. If we put $f' := X_{10}g' \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, then

$$b_k < v_p(\widetilde{f}) = v_p(\widetilde{X}_{10}\widetilde{f}') = 1 + v_p(\widetilde{f}').$$

This implies $v_p(\widetilde{f'}) > b_{k-10}$. By the induction hypothesis, we get $f' \equiv 0 \mod p$. Therefore we have $f \equiv 0 \mod p$.

(2) If $k \equiv 11 \mod 12$ and $k \equiv 1, 5, 7, 9 \mod 10$, then we have $b_k = b_{k-8} + 1$. By Corollary 4.11, there exists $g' \in M_{k-43}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $g \equiv X_{10}g' \mod p$. Put $f' = X_{35}g' \in M_{k-8}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. Then we have $v_p(\widetilde{f'}) = v_p(\widetilde{f}) - 1 > b_{k-8}$. By the induction hypothesis, we have $f' \equiv 0 \mod p$. Therefore we have $f \equiv 0 \mod p$.

(3) If $k \equiv 11 \mod 12$, $k \equiv 3 \mod 10$ and $k < 115$, then we have $k = 83$ because $k \geq 45$. Then by Lemma 5.6, we have $f \equiv 0 \mod p$.

(4) Finally, we assume $k \equiv 11 \mod 12$ and $k \geq 115$. To prove this case, we start with proving the following lemma.

Lemma 5.7 *Let $f = X_{35}g \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $W(g) \equiv 0 \mod p$. Assume $k \equiv 11 \mod 12$, $k \geq 115$ and (5.6). Then we have $W''(g) \equiv 0 \mod p$.*

Proof We show the statement only for $p = 2$. The case $p = 3$ also can be proved by a similar argument. By Corollary 4.11, there exists $g' \in M_{k-43}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $g \equiv X_{10}g' \mod p$. Then, it follows from Lemma 4.8 that

$$fX_{35} \equiv X_{10}X_{35}^2 g' \equiv g'X_{10}^3(Y_{12}^2 X_{16}^2 + X_{10}^4) \mod p.$$

By Lemma 5.1 and the assumption (5.6), we have

$$b_k + 2 < v_p(\widetilde{f}) + 2 \leq v_p(\widetilde{f}\widetilde{X}_{35}) = v_p(\widetilde{g'}\widetilde{X}_{10}\widetilde{X}_{35}^2) = 3 + v_p(\widetilde{g'}(\widetilde{Y}_{12}^2\widetilde{X}_{16}^2 + \widetilde{X}_{10}^4)).$$

This implies that

$$v_p(\widetilde{g'}(\widetilde{Y}_{12}^2\widetilde{X}_{16}^2 + \widetilde{X}_{10}^4)) > [(k-15)/10].$$

On the other hand, we have

$$W(g'(Y_{12}^2X_{16}^2 + X_{10}^4)) = W(g'Y_{12}^2X_{16}^2) \equiv W(g') \cdot y_{12}^2 \cdot x_{12}^2 \mod p,$$

where we used (4.3) and the fact $x_4 \equiv 1 \mod p$. By this congruence, $W(\widetilde{g'}(\widetilde{Y}_{12}^2\widetilde{X}_{16}^2 + \widetilde{X}_{10}^4))$ can be regarded as of weight $k - 43 + 48 = k + 5$. By $k \geq 115$, we have

$$v_p(\widetilde{g'}(\widetilde{Y}_{12}^2\widetilde{X}_{16}^2 + \widetilde{X}_{10}^4)) > [(k-15)/10] \geq [(k+5)/12].$$

Applying Lemma 5.3, we have

$$W(g'(Y_{12}^2X_{16}^2 + X_{10}^4)) \equiv W(g') \cdot y_{12}^2 \cdot x_{12}^2 \equiv 0 \mod p.$$

This implies that

$$W''(g) \equiv W(g') \cdot x_{12} \equiv 0 \mod p.$$

This completes the proof of the lemma. □

We shall return to proof of the case (4). Since $W(g) \equiv 0 \mod p$ and $W''(g) \equiv 0 \mod p$, there exists $h \in M_{k-55}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $g \equiv X_{10}^2 h \mod p$ by Proposition 4.10. Note that $f \equiv X_{10}^2 X_{35} h \mod p$. If we put $f' := X_{35}h \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, then

$$v_p(\widetilde{f}) = v_p(\widetilde{X}_{10}^2\widetilde{f'}) = 2 + v_p(\widetilde{f'}) > b_k.$$

This means that

$$v_p(\widetilde{f'}) > b_{k-20}.$$

By the induction hypothesis, we get $f' \equiv 0 \mod p$. Therefore we have $f \equiv 0 \mod p$. This completes the proof. □

5.4 Proof of Corollary 2.3

As explained in the beginning of this section, we may assume $\mathcal{O}_{\mathfrak{p}} = \mathbb{Z}_{(p)}$, where p is a prime number. Let $\Gamma \subset \Gamma_2$ be a congruence subgroup of level N and $f \in M_k(\Gamma)_{\mathbb{Z}_{(p)}}$. By the proof of [1, Theorem 1.3], there exists $g \in M_{k(i-1)}(\Gamma)_{\mathbb{Z}_{(p)}}$ such that

$$fg \in M_{ki}(\Gamma_2)_{\mathbb{Z}_{(p)}}, \text{ and } g \not\equiv 0 \mod p.$$

Here $i = [\Gamma_2 : \Gamma]$. We assume $v_p(\widetilde{f}) > b_{ki}$. Then by (5.1), we have

$$v_p(\widetilde{f}\widetilde{g}) \geq v_p(\widetilde{f}) > b_{ki}.$$

By Theorem 2.1, we have $\widetilde{f}\widetilde{g} = 0$. Since $\widetilde{g} \neq 0$, we have $\widetilde{f} = 0$, i.e., $f \equiv 0 \mod p$. This completes the proof. □

5.5 Proof of the sharpness

We prove Theorem 2.4. If k is even, then we can show the assertion of the theorem by a similar argument to [1] (Sect. 3.1, pp.135–136). For $k = 35, 39, 41, 43$ and 47, let f_k be modular forms given in Lemma 5.4. Then by Lemma 5.4, we have $\mathrm{ldt}(f_k X_{10}^i) = q_1^{2+i} q_2^{-1-i} q_3^{3+i}$ for $k = 35, 39, 41, 43$ and $\mathrm{ldt}(f_{47} X_{10}^i) = q_1^{3+i} q_2^{-2-i} q_3^{4+i}$. Thus we have the assertion of the theorem. □

Acknowledgements The authors would like to thank the referee for a detailed reading of the manuscript, for helpful advice that improved the presentation of this paper. Toshiyuki Kikuta is supported by JSPS Kakenhi JP18K03229. Sho Takemori is partially supported by JSPS Kakenhi 23224001.

References

1. Choi, D., Choie, Y., Kikuta, T.: Sturm type theorem for Siegel modular forms of genus 2 modulo p. Acta Arith. **158**(2), 129–139 (2013)
2. Gritsenko, V .A., Nikulin, V .V.: Igusa modular forms and 'the simplest' Lorentzian Kac–Moody algebras. Sb. Math. **187**(11), 1601 (1996)
3. Gritsenko, V.A., Nikulin, V.V.: Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras. Am. J. Math. **119**, 181–224 (1997)
4. Igusa, J.: On Siegel modular forms of genus two. Am. J. Math. **84**, 175–200 (1962)
5. Igusa, J.: On Siegel modular forms of genus two (II). Am. J. Math. **86**, 392–412 (1964)
6. Igusa, J.: On the ring of modular forms of degree two over $\mathbb{Z}$. Am. J. Math. **101**, 149–183 (1979)
7. Kikuta, T.: Remark on Sturm bounds for Siegel modular forms of degree 2. Proc. Jpn. Acad. Ser. A Math. Sci. **91**(6), 82–84 (2015)
8. Kikuta, T., Kodama, H., Nagaoka, S.: Note on Igusa's cusp form of weight 35. Rocky Mt. J. Math. **45**(3) (2015) (to appear)
9. Nagaoka, S.: Note on mod p Siegel modular forms. Math. Z. **235**(2), 405–420 (2000)
10. Nagaoka, S.: Note on mod p Siegel modular forms II. Math. Z. **251**(4), 821–826 (2005)
11. Poor, C., Yuen, D.: Paramodular cusp forms. Math. Comput. **84**(293), 1401–1438 (2015)
12. Rasmussen, J.: Higher congruences between modular forms. Ph.D. thesis, University of Copenhagen (2009)
13. Richter, O.K., Westerholt-Raum, M.: Sturm bounds for Siegel modular forms. Res. Number Theory **1**(1), 5 (2015)
14. Serre, J.P.: Formes modulaires et fonctions zêta p-adiques. In: Modular Functions of One Variable III. Springer, New York, pp 191–268 (1973)
15. Sturm, J.: On the congruence of modular forms. In: Number Theory. Springer, New York, pp 275–280 (1987)

16. Van Der Geer, G.: Siegel modular forms and their applications. In: The 1–2–3 of Modular Forms. Springer, New York, pp 181–245 (2008)