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Microglobal regularity and the global wavefront set

Gustavo Hoepfner[1] · Andrew Raich[2]

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Abstract

In this paper, we begin the study of regularity of partial differential equations in the space of global L^q Gevrey functions, recently introduced in Adwan et al. (J Geom Anal 27(3):1874–1913, 2017) and Hoepfner and Raich (Indiana Univ Math J, forthcoming) and in a generalized and new function space called the space of global L^q Denjoy–Carleman functions. We develop a wedge approach similar to Bony's theorem (Bony in Séminaire Goulaouic–Schwartz (1976/1977), Équations aux dérivées partielles et analyse fonctionnelle, Exp No 3. Centre Math, École Polytech, Palaiseau, 1977) and prove three main theorems. The first establishes the existence of boundary values of continuous functions on a wedge. Next, we borrow the FBI transform approach from Hoepfner and Raich (forthcoming) to define global wavefront sets and prove a relationship between the inclusion of a direction in the global wavefront set and the existence of boundary values of sums of weighted L^p functions defined in wedges. The final result is an application in which we prove a global version of a classical result: namely, the relationship between the global characteristic set of a partial differential operator P and the microglobal wavefront sets of u and Pu.

Keywords FBI transform · Wavefront set · Global wavefront set · Gevrey functions · Global L^q-Gevrey functions · Denjoy–Carleman functions · Global L^q Denjoy–Carleman functions · Ultradifferentiable functions · Ultradistributions · Almost analytic extensions

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✉ Gustavo Hoepfner
hoepfner@dm.ufscar.br

Andrew Raich
araich@uark.edu

1 Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP 13565-905, Brazil

2 Department of Mathematical Sciences, SCEN 309, 1 University of Arkansas, Fayetteville, AR 72701, USA

1 Introduction

The purpose of this paper is to explore a new concept of regularity that is well suited for the global L^q Denjoy–Carleman function spaces. These function spaces are generalized version of the L^q Gevrey function spaces introduced and studied in [4,13,25]. In this paper, we show that appropriate global versions of three classical results on the boundary values of ultradistributions, wavefront sets, and characteristic sets of constant coefficient partial differential operators hold for these global function spaces. The function classes are natural generalizations of the global L^q-Gevrey functions that we studied in [4,25], and the majority of results that hold for global L^q-Gevrey functions hold in this more general setting.

The origin of this study resides in the $\Box_b$-heat kernel estimates of Boggess and Raich, in which they recover exponential decay of the order $e^{-a|t|^{1/\beta}}$ via a Fourier transform [13]. This led to their development (in our language) of the notion of global L^q Gevrey functions and proofs of some basic implications. Although the Fourier transform is a powerful tool to recover smoothness and/or size estimates in many circumstances, we showed that it is a deficient tool to recover global L^p smoothness estimates [25]. A very satisfying replacement for the Fourier transform is the FBI transform. More precisely, in [25], we showed that certain estimates of the FBI transform exactly characterize the behavior of global L^q-Gevrey functions. The FBI transform first appeared in the work of Bros and Iagolnitzer [10] to study local analyticity and later was shown to be the right tool to study microlocal (hypo) regularity among many function classes, including (real) analytic, Gevrey, Denjoy–Carleman, and C^∞ (see [7,9,11,16,20,21,30]).

The topic of this work is microglobal regularity and the global wavefront set. Intuitively, microlocal analysis and the wavefront set capture directions (in the cotangent space) that prevent regularity of a function nearby a given point. The question, then, is the appropriate global object to capture global obstructions to regularity. For this, we take our lesson from the Fourier transform—smoothness and decay are interchanged under the action of the transform. This means that if a function lacks smoothness at *any* point, then its FBI (or Fourier) transform will lack decay. Similarly, if a function lacks decay in *any* direction, then its transform will lack smoothness. Consequently, the global behavior is determined by exactly which directions are in the wavefront set, regardless of where they occur. Thus, our global objects need to record the directions which are well-behaved at every point or poorly behaved at any point, and they need to be defined in terms of the FBI transform because of the well-documented problems with the Fourier transform.

One of the themes of this paper, and indeed of all of our work on global L^q Denjoy–Carleman functions, is that there are appropriate global versions to many of the powerful local theorems. In this paper, we prove two structure theorems and provide one application. The first of our main results, Theorem 2.2, establishes that continuous functions on a wedge which exhibit controlled growth in L^p have boundary values in the space of ultradistributions. These ultradistributions are exactly ones that are dual to the space of global L^q Denjoy–Carleman functions. Our second main theorem, Theorem 2.5, is a further exploration of boundary values and the global L^q Denjoy–Carleman function spaces. We prove a relationship between directions in the global wavefront set and boundary values of a sum of continuous functions, each of which is defined on a specific wedge. Our final result, Theorem 2.8, is a global version of the classical result that the wavefront set of Pu is contained in the wavefront set of u, which in turn is contained in the union of the wavefront set of Pu and the characteristic set of P. Here, P is a constant coefficient partial differential operator.

The local versions of our results appear in a variety of settings in the literature. For example, in Hormander [21–24] and the references therein, there are concise and complete proofs of the local results and their applications, including propagation of singularities, pseudodifferential operators, and extensions of CR functions. Hörmander proved the most classical case of Theorem 2.8 [21, Chapter 8] and versions appear more specific to Gevrey and ultradifferentiable functions in [29] and (for example) [18]. Later development in the local theory, such as the study the local and microlocal regularity of CR functions, solutions of more general vector fields, and even first order (system of) nonlinear partial differential equations appear in [1–3,6,8,8,9,12,14,15,17,20,26] and references therein.

We hope that this machinery can be used as a tool to study global PDE's in noncompact settings where compactness is of fundamental nature as, for instance, in the recent research article [5].

2 Definitions and statements of the main results

2.1 Global L^q Denjoy–Carleman functions and boundary values of their duals

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$ and fix a sequence $M = (M_j)_{j\in\mathbb{N}_0}$ of nonnegative numbers. Suppose $\Omega \subset \mathbb{R}^d$. Define $W^{k,q}(\Omega)$ as the space of k-times differentiable functions in $L^q(\Omega)$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers, positive constants $A, \beta > 0$, and $1 \leq q \leq \infty$, define the seminorm $\varrho_{\alpha,A,\Omega,q,M} : W^{|\alpha|,q}(\Omega) \to [0, \infty)$

$$\varrho_{\alpha,A,\Omega,M}(g) = \varrho_\alpha(g) = \frac{\|D^\alpha g\|_{L^q(\Omega)}}{A^{|\alpha|} M_{|\alpha|}}.$$

We suppress as many indices for ϱ_α as possible.

Definition 2.1 Let $1 \leq q \leq \infty$ and $M = (M_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. A function $g \in W^{\infty,q}(\Omega)$ is said to satisfy *global L^q Denjoy–Carleman estimates of order M* if there exist constants $A, C > 0$ so that for every d-tuple of nonnegative integers α

$$\|D^\alpha g\|_{L^q(\Omega)} \leq C A^{|\alpha|} M_{|\alpha|}.$$

We say that a function that satisfies global L^q Denjoy–Carleman estimates of order M is a *global L^q Denjoy–Carleman function of order M*. For a fixed $A > 0$, we set

$$\mathcal{E}_A^{q,M}(\Omega) = \left\{ g \in W^{\infty,q}(\Omega) : \{\varrho_{\alpha,A,\Omega,q,M}(g)\}_{|\alpha|\geq 0} \in \ell^q(\mathbb{Z}^d_{\geq 0}) \right\}$$

and

$$\mathcal{E}^{q,M}(\Omega) = \bigcup_{A>0} \mathcal{E}_A^{q,M}(\Omega).$$

The choice $M_j = (j!)^\beta$ yields the global L^q-Gevrey functions of order β, see [4]. For a given sequence $M = (M_\ell)_{\ell\in\mathbb{N}_0}$, define the *associated function* $M(t)$ by

$$M(t) := \sup_\ell \log \frac{t^\ell}{M_\ell} \tag{2.1}$$

and its Young conjugate by

$$M^*(s) = -\log \inf_{\ell\in\mathbb{N}} \left\{ \frac{s^\ell M_\ell}{\ell!} \right\}. \tag{2.2}$$

It is well known (see Lemma 5.6 in [28]) that M^* is comparable to the function $w^* : [0, \infty) \to [0, \infty]$ given by $w^*(r) = \sup_{t \geq 0}\{M(t) - rt\}$, in the sense that for every $H > 1$ there exists a positive constant C such that

$$M^*(Hs) - C \leq w^*(s) \leq M^*(s), \quad \text{for all } s > 0. \tag{2.3}$$

From now on the sequences $M = (M_j)_{j\in\mathbb{N}_0}$ will be assumed to satisfy the standard conditions (A.1), (A.2), (A.3), and (A.9) and we will refer to such M as *convenient sequences*.

The final piece of notation we need to state our first main theorem is that of a truncated cone. A set $\Gamma \subset \mathbb{R}^d$ will be called a cone if $x \in \Gamma$ implies $tx \in \Gamma$ for all $t > 0$. Given a cone $\Gamma \subset \mathbb{R}^d_y$, we will write $\Gamma_\delta := \Gamma \cap B_\delta(0)$ for the truncated cone of height δ. Let $S^{d-1} := \{y \in \mathbb{R}^d : |y| = 1\}$ so that $\Gamma_\delta = \{\tau y' : 0 < \tau < \delta \text{ and } y' \in \Gamma \cap S^{d-1}\}$.

Now we are ready to state the first main theorem of this paper. This result provides sufficient conditions to guarantee the existence of boundary values, bf, of certain continuous functions f defined on wedges $\mathcal{W} := \Omega \times \Gamma_\delta \subset \mathbb{R}^d_x \times \mathbb{R}^d_y$.

Theorem 2.2 *Let $f \in C(\mathcal{W}) \cap L^p(\mathcal{W})$ be a function satisfying the following: there exists a positive constant $C > 0$ such that*

(1) *for every $1 \leq j \leq d$ and for $\frac{1}{p} + \frac{1}{q} = 1$*

$$\sup_{y'\in\Gamma\cap S^{d-1}} \int_0^\delta \left\| \partial_{\bar{z}_j} f(\cdot + i\tau y') \right\|_{L^p(\Omega)} d\tau \leq C < \infty; \tag{2.4}$$

(2) *for each $x \in \Omega$ and for all $\lambda > 0$,*

$$\sup_{y'\in\Gamma\cap S^{n-1}} \int_0^\delta \left\{ \left\| f(\cdot + i\tau y') \right\|_{L^p(\Omega)} e^{-M^*(\tau/\lambda)} \right\} d\tau \leq C < \infty. \tag{2.5}$$

Then $\lim_{\Gamma\ni y\to 0} f(\cdot + iy)$ exists in $\mathcal{E}^{q,M}(\Omega)'$, that is

$$\langle bf, \varphi\rangle := \lim_{\Gamma_\delta\ni y\to 0} \int_\Omega f(x + iy)\varphi(x)\, dx \tag{2.6}$$

exists and defines a ultradistribution in $\mathcal{E}^{q,M}(\Omega)'$.

2.2 Global wavefront sets and the FBI transform

As we showed [25], the Fourier transform is not suitable to characterize the global behavior of L^p functions. Rather the FBI transform serves as a fitting substitute. We use of the FBI transform by Sjöstrand [30], as written by Christ [16].

For $y \in \mathbb{R}^d$, set

$$\langle y\rangle = \sqrt{1 + |y|^2}$$

and define the function $\alpha(x, \xi)$ and the form ω by

$$\begin{aligned}\omega &= dx_1 \wedge \cdots \wedge dx_d \wedge d(\xi_1 + ix_1\langle\xi\rangle) \wedge \cdots \wedge d(\xi_d + ix_d\langle\xi\rangle)\\ &= \alpha(x, \xi)\, dx_1 \wedge \cdots \wedge dx_d \wedge d\xi_1 \wedge \cdots \wedge d\xi_d\end{aligned}$$

where $\xi \in \Gamma$, and Γ is a conic neighborhood of $\mathbb{R}^d$ in which $\langle\cdot\rangle$ is a holomorphic function. Note that

$$\alpha(-x, -\xi) = (-1)^d \alpha(x, \xi). \tag{2.7}$$

Given two global L^q Denjoy–Carleman function class $\mathcal{E}^{q,M}(\Omega)$ and $\mathcal{E}^{q,M'}(\Omega)$, we say that $\mathcal{E}^{q,M}(\Omega)$ *completely contains* $\mathcal{E}^{q,M'}(\Omega)$ if $\mathcal{E}^{q,M}_A(\Omega) \supset \mathcal{E}^{q,M'}(\Omega)$ for all $A > 0$, and we will denote complete containment by $\mathcal{E}^{q,M}(\Omega) \succcurlyeq \mathcal{E}^{q,M'}(\Omega)$. Moreover, it is equivalent to the following (see, for instance, [27], for the class of local L^∞ Denjoy–Carleman functions)

$$\text{for all } \epsilon > 0, \text{ there exists } C_\epsilon > 0 \text{ such that } M'_j \le C_\epsilon \epsilon^j M_j, \quad \forall j \in \mathbb{N}_0. \tag{2.8}$$

Let $\mathcal{E}^{q,M}(\mathbb{R}^d)$ be a global L^q Denjoy–Carleman function class that completely contains $\mathcal{G}^{q,\frac{1}{2}}(\Omega)$, the global L^q Gevrey functions of order $\frac{1}{2}$. Then for a distribution $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$, define the *FBI transform* of u by

$$\mathcal{F}u(x,\xi) = \left\langle u, e^{i(x-\cdot)\cdot\xi - \langle\xi\rangle(x-\cdot)^2}\alpha(x-\cdot,\xi)\right\rangle.$$

The function $\mathcal{F}u$ is well defined for $u \in \mathcal{E}^{q,M}_A(\mathbb{R}^d)'$ since the exponential function $e^{-a|x|^2}$ is in $\mathcal{E}^{q,M}_A(\mathbb{R}^d)$ [4, Section 4.1]. We can extend [25, Theorem 2.2] to $\mathcal{E}^{q,M}(\Omega)$.

Theorem 2.3 *Let $\mathcal{E}^{q,M}(\mathbb{R}^d)$ completely contain $\mathcal{G}^{q,\frac{1}{2}}(\mathbb{R}^d)$. Suppose $A > 0$ and $u \in \mathcal{E}^{q,M}_A(\mathbb{R}^d)$. Then there exist positive constants A_0, a, and c that do not depend on u, and a positive constant C so that for any multiindex $J \in \mathbb{N}_0^d$, and any r satisfying $q \le r \le \infty$, $\mathcal{F}u(\cdot,\xi)$ is in $\mathcal{E}^{r,M}_{A_0}(\mathbb{R}^d)$ and satisfies the estimates*

$$\|D_x^J \mathcal{F}u(x,\xi)\|_{L^r(\mathbb{R}^d)} \le C A_0^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \tag{2.9}$$

for any $u \in \mathcal{E}^{q,M}_A(\mathbb{R}^d)$. Conversely, to each $A_0 > 0$, there exists $A = A(A_0)$ so that for any $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ such that $\mathcal{F}u(\cdot,\xi) \in \mathcal{E}^{q,\beta}_{A_0}(\mathbb{R}^d)$ and (2.9) holds, then u is a function and $u \in \mathcal{E}^{q,M}_A(\mathbb{R}^d)$.

Proof The proof is a straight forward adaptation of the proof of [25, Theorem 2.2]. □

Definition 2.4 Let $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ and $\xi^0 \in \mathbb{R}^d$. We say that u is $\mathcal{E}^{q,M}$*-microglobal regular at* $\mathbb{R}^d \times \{\xi^0\}$ (or simply ξ^0) if there exist a conic neighborhood Γ_0 of ξ^0 in $\mathbb{R}^d \setminus \{0\}$ and constants $c, C > 0$ such that for each $q \le r \le \infty$,

$$\|D_x^J \mathcal{F}u(x,\xi)\|_{L^r(\mathbb{R}^d)} \le C A_0^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)}, \quad \forall \xi \in \Gamma_0. \tag{2.10}$$

We define the $\mathcal{E}^{q,M}$*-wave front set of* u as the complement of the set of the directions ξ in which u is $\mathcal{E}^{q,M}$-microglobal regular, that is

$$WF_{\mathcal{E}^{q,M}}(u) := \{\xi \in \mathbb{R}^d : u \text{ is not } \mathcal{E}^{q,M}\text{-microglobal regular at } \xi\}.$$

Our second main theorem relates directions *not* in the global wavefront set and boundary of certain functions in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$.

Theorem 2.5 *Suppose that $\mathcal{E}^{q,M}(\mathbb{R}^d) \succcurlyeq \mathcal{G}^{q,\frac{1}{2}}(\mathbb{R}^d)$, $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$, and $\xi^0 \in \mathbb{R}^d$. The vector $\xi^0 \notin WF_{\mathcal{E}^{q,M}}(u)$ if and only if there exist open and acute cones $\Gamma_1, \dots, \Gamma_k \subset \mathbb{R}^d \setminus \{0\}$ and $\delta > 0$ so that*

(1) *For each $j \in \{1, \dots, k\}$, $\xi^0 \cdot \Gamma_j < 0$;*
(2) *For each $j \in \{1, \dots, k\}$, there exist functions f_j on $\mathbb{R}^d \times (\Gamma_j)_\delta$ satisfying* (1) *from Theorem* 2.2;

(3) *There exists* $a > 0$ *so that for all* $p \le r \le \infty$ *and all* $\lambda > 0$,

$$\sup_{y\in(\Gamma_j)_\delta} \left\{ \| f_j(x+iy)\|_{L^r(\mathbb{R}^d)} e^{-aM^*(|y|/\lambda)} \right\} \le A_{\lambda,r}, \tag{2.11}$$

for some $A_{\lambda,r} > 0$.

(4) *So that* bf_j *exists in* $\mathcal{E}^{q,M}(\mathbb{R}^d)'$ *and*

$$u - \sum_{j=1}^{k} bf_j \in \mathcal{E}^{q,M}(\mathbb{R}^d).$$

Remark 2.6 The functions f_j that we construct satisfy a much stronger estimate than (2.11). Namely, there exist $A, C > 0$ so that for all multi-indices J,

$$\sup_{y\in\Gamma_j} \left\{ \left\| D^J_{(x,y)} f_j(\cdot + iy)\right\|_{L^p(\mathbb{R}^d)} e^{-aM^*(|y|/\lambda)} \right\} \le AC^{|J|+1} M_{|J|}. \tag{2.12}$$

Therefore, we can think of Theorem 2.5 as a self-improving theorem. In one direction, we start with functions and cones that satisfy (2.11) and conclude that $\xi_0 \notin WF_{\mathcal{E}^{q,M}}(u)$. Once we have that $\xi_0 \notin WF_{\mathcal{E}^{q,M}}(u)$, we then apply Theorem 2.5 again and conclude that there exist new functions $\{f_j\}$ and cones $\{\Gamma_j\}$ on which f_j satisfies a much stronger estimate, namely (2.12).

2.3 Application: global characteristic sets of linear partial differential operators

Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j} = -i\frac{\partial}{\partial x_j}$.

Definition 2.7 Let $P = \sum_{\ell=0}^m P_\ell(x, D)$ be a differential operator with C^∞ coefficients where each $P_\ell(x,\xi)$ is a polynomial of degree ℓ in ξ and smooth in x. The *characteristic set* Char P is defined to be

$$\text{Char } P = \{(x,\xi) \in T^*(X) \setminus \{0\} : P_m(x,\xi) = 0\}.$$

The *global characteristic set* $\text{Char}_G\, P$ is defined to be

$$\text{Char}_G\, P = \{\xi : (x,\xi) \in \text{Char } P\}.$$

Theorem 2.8 *Let* P *be a constant coefficient differential operator of order* m *and* M *be a sequence so that* $\mathcal{G}^{q,1}(\mathbb{R}^d) \preccurlyeq \mathcal{E}^{q,M}(\mathbb{R}^d)$. *Then*

$$WF_{\mathcal{E}^{q,M}}(Pu) \subset WF_{\mathcal{E}^{q,M}}(u) \subset WF_{\mathcal{E}^{q,M}}(Pu) \cup \text{Char}_G\, P.$$

3 Properties of $\mathcal{E}^{q,M}(\Omega)$

The function spaces $\mathcal{E}^{q,M}(\Omega)$ share many properties with $\mathcal{G}^{q,\beta}(\Omega)$ that can be proven by the same techniques as in [4,25].

Proposition 3.1 $\mathcal{E}^{q,M}(\Omega)$ *is a non-quasianalytic DFS-space that is invariant under differentiation and composition.*

A very useful result is the characterization of the dual spaces.

Proposition 3.2 *Let* $1 \leq q < \infty$ *and* p *be the dual exponent of* q, *i.e.*, $\frac{1}{p} + \frac{1}{q} = 1$. *For a convenient sequence* $M = (M_j)_{j\in\mathbb{N}_0}$ *of nonnegative numbers, the dual of* $\mathcal{E}^{q,M}(\mathbb{R}^d)$, $\mathcal{E}^{q,M}(\mathbb{R}^d)'$, *can be identified with the following space*

$$\left\{ f = \sum_{\alpha\in\mathbb{N}_0^d} f_\alpha^{(\alpha)} : \ f_\alpha \in L^p(\mathbb{R}^d) \ \text{and} \ \forall A > 0, \sum_{\alpha\in\mathbb{N}_0^d} \frac{M_{|\alpha|}}{A^{|\alpha|}} \| f_\alpha \|_{L^p(\mathbb{R}^d)} < \infty \right\}. \quad (3.1)$$

Proof It is similar to the one given in the case where $M_j = j!^s$, $s > 1$, see [4,25]. □

3.1 Generalized Carleman's problem for $\mathcal{E}^{q,M}$ functions

Fix convenient sequences $M = (M_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$.

Definition 3.3 Let $\Omega \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets, and $1 \leq q, \widetilde{q} \leq \infty$. We define $\mathcal{E}_A^{q,M}(\Omega; \mathcal{E}_{A'}^{\widetilde{q},N}(U))$ to be the space of functions $f(x,t) \in C^\infty(\Omega \times U)$ for which there exist constants $C, C', A, A' > 0$ so that

(1) $\left\| D_t^{\alpha'} f(x,\cdot) \right\|_{L^{\widetilde{q}}(U)} \leq C'(A')^{|\alpha'|} N_{|\alpha'|}$;
(2) $\left\| \| D_x^{\alpha} D_t^{\alpha'} f(x,\cdot) \|_{L^{\widetilde{q}}(U)} \right\|_{L^q(\Omega)} \leq C(A')^{|\alpha'|} A^{|\alpha|} M_{|\alpha|} N_{|\alpha'|}$.

Define

$$\mathcal{E}^{q,M}(\Omega; \mathcal{E}^{\widetilde{q},N}(U)) = \bigcup_{A,A'>0} \mathcal{E}_A^{q,M}(\Omega; \mathcal{E}_{A'}^{\widetilde{q},N}(U)).$$

Definition 3.4 Let $\Omega \subset \mathbb{R}^d$ be an open set and $M = (M_\ell)_{\ell\in\mathbb{N}_0}$ a convenient sequence. Given $f = f(x) \in \mathcal{E}^{q,M}(\Omega)$ we say that a smooth function $u = u(x,y)$ defined in a neighborhood $\Omega \times V$ of Ω in $\mathbb{R}^d \times \mathbb{R}^d$, is an $\mathcal{E}^{q,M}$*-almost analytic extension of* f if the following is true:

(1) $u \in \mathcal{E}^{q,M}(\Omega; \mathcal{E}^{\infty,M}(V))$
(2) $u(x,0) = f(x)$ for all $x \in \Omega$; and
(3) there exists a positive constant λ such that

$$\sup_y \left\{ \left\| \partial_{\bar{z}_j} u(\cdot, y) \right\|_{L^q(\Omega)} e^{M^*(|y|/\lambda)} \right\} \leq C < \infty, \quad \forall j = 1, \ldots, n. \quad (3.2)$$

Here, we write $z_j = x_j + iy_j$, $\partial_{\bar{z}_j} = \frac{1}{2}\left(\partial_{x_j} + i\partial_{y_j}\right)$ for $j = 1, ..., m$ as usual.

Theorem 3.5 (Almost analytic extensions, [4]) *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and* $M = (M_\ell)_{\ell\in\mathbb{N}_0}$ *a convenient sequence. Then every* $f \in \mathcal{E}^{q,M}(\Omega)$ *has a* $\mathcal{E}^{q,M}$*-almost analytic extension.*

Definition 3.6 Let $\Omega, V \subset \mathbb{R}^d$ be an open sets such that $0 \in \overline{V}$ and $M = (M_\ell)_{\ell\in\mathbb{N}_0}$ a convenient sequence. A function $F \in \mathcal{E}^{q,M}(\Omega; \mathcal{E}^{\infty,M}(V))$ satisfying (3.2) will be called an $\mathcal{E}^{q,M}$*-almost analytic function.*

4 Existence of traces: the proof of Theorem 2.2

Proof of Theorem 2.2 As in [19] the proof will be divided into 3 steps.

Step 1. We first claim that the limit in (2.6) exists along a fixed direction $y' \in \Gamma \cap S^{n-1}$ when $f \in C(\mathcal{W}) \cap W^{1,p}(\mathcal{W})$.

Fix $y' = (y'_1, \dots, y'_d) \in \Gamma \cap S^{d-1}$ and consider the complex vector field

$$\partial' := y'_1 \partial_{\bar{z}_1} + \dots y'_n \partial_{\bar{z}_n}$$

and the $(m+1)$ dimensional submanifold

$$\Pi' := \Omega \times \{\tau y'\} := \{(x, \tau y') : \ x \in \Omega, \ \tau \in (0, 2\delta)\} \subset \mathbb{R}^d_x \times \mathbb{R}^d_y.$$

Let $\Pi = \Omega \times [0, 2\delta)$ and $f'(x,\tau) : \Pi \to \mathbb{C}$ be the function defined by restricting $f(x, y)$ to Π', namely, $f'(x,\tau) := f(x, \tau y')$, $0 < \tau < 2\delta$. Writing

$$\partial' := i\partial_\tau + \sum_{k=1}^{n} y'_k \partial_{x_k}$$

and observe that differentiating $y = \tau y'$ with respect to τ gives

$$\partial_\tau = y'_1 \partial_{y_1} + \dots + y'_n \partial_{y_n}.$$

Consequently, we may regard ∂' as a single globally integrable vector field in $(m+1)$ variables $(x, \tau) \in \Pi$, with first integrals

$$Z'(x,\tau) := (Z'_1(x,\tau), \dots, Z'_n(x,\tau)), \quad \text{where } Z'_j(x,\tau) := x_j + i\tau y'_j, \ j = 1, \dots, d.$$

Therefore, by (2.4),

$$\partial' f'(x,\tau) = \sum_j y'_j (\partial_{\bar{z}_j} f)(x, \tau y') \in L^p(\Omega \times (0,\delta)). \tag{4.1}$$

For $0 < \varepsilon < \delta/2$, let $f'_\varepsilon(x,\tau) := f'(x, \varepsilon + \tau)$ then, it follows from (2.4) and (2.5) that

$$\begin{aligned}
\sup_{t' \in \Gamma \cap S^{n-1}} \int_0^{\delta/2} \left\| \partial' f'_\varepsilon(\cdot, \tau) \right\|_{L^p(\Omega)} d\tau &\le \sum_j \sup_{t' \in \Gamma \cap S^{n-1}} \int_0^{\delta/2} \left\| (\partial_{\bar{z}_j} f)(\cdot, (\tau + \varepsilon) y') \right\|_{L^p(\Omega)} d\tau \\
&\le \sum_j \sup_{t' \in \Gamma \cap S^{n-1}} \int_0^{\delta} \left\| (\partial_{\bar{z}_j} f)(\cdot, \tau y') \right\|_{L^p(\Omega)} d\tau \\
&\le C < \infty
\end{aligned} \tag{4.2}$$

and since $M^*(\tau/\lambda)$ is decreasing in $\tau > 0$,

$$\begin{aligned}
&\int_0^{\delta/2} \sup_{t' \in \Gamma \cap S^{n-1}} \left\| f'_\varepsilon(\cdot, \tau) e^{-M^*(\tau/\lambda)} \right\|_{L^p(\Omega)} d\tau \\
&\quad \le \int_0^{\delta/2} \sup_{t' \in \Gamma \cap S^{n-1}} \left\| f(\cdot, (\tau+\varepsilon) y') e^{-M^*((\tau+\varepsilon)/\lambda)} \right\|_{L^p(\Omega)} d\tau \\
&\quad \le \int_0^{\delta} \sup_{t' \in \Gamma \cap S^{n-1}} \left\| f(\cdot, \tau y') e^{-M^*(\tau/\lambda)} \right\|_{L^p(\Omega)} d\tau \le C < \infty
\end{aligned} \tag{4.3}$$

independently of $0 < \varepsilon < \delta/2$.

We will now prove the theorem, continuing our assumption that $f \in C(\mathcal{W}) \cap W^{1,p}(\mathcal{W})$. Fix $\varphi \in \mathcal{E}^{q,M}(\Omega)$ and let $\Psi(x, y) \in \mathcal{E}^{q,M}(\Omega; \mathcal{E}^{\infty,M}(V))$ be the almost analytic extension of φ given by Theorem 3.5 and satisfying (1)–(3) from Definition 3.4. Define $\Psi'(x,\tau) = \Psi(x,\tau t')$. It follows from Eq. (3.2) that

$$\begin{aligned}\left\|\partial'\Psi'(\cdot,\tau)e^{M^*(\tau/\lambda)}\right\|_{L^q(\Omega)} &\le \sum_j \left\|(\partial_{\bar z_j}\Psi)(\cdot,\tau y')e^{M^*(\tau/\lambda)}\right\|_{L^q(\Omega)}\\ &\le \sum_j \sup_{y\in\Gamma_\delta}\left\|(\partial_{\bar z_j}\Psi)(\cdot,y)e^{M^*(|y|/\lambda)}\right\|_{L^q(\Omega)} \le C<\infty. \quad (4.4)\end{aligned}$$

Also, for any $g(x,\tau)\in W^{1,p}(\Pi)$, we have

$$\mathrm{d}g(x,\tau) = \partial' g(x,\tau)\,\mathrm{d}\tau + \sum_{k,j=1}^{m} \partial_{x_j} g(x,\tau)\,\mathrm{d}Z'_k(x,\tau).$$

Let $\mathrm{d}Z'(x,\tau) = \mathrm{d}Z'_1\wedge\cdots\wedge\mathrm{d}Z'_d$ the volume element generated by the first integrals. Then if $g(x,\tau) = f'_\varepsilon(x,\tau)\Psi'(x,\tau)$, $0<\varepsilon<\delta/2$, and $\omega(x,\tau) = g(x,\tau)\,\mathrm{d}Z'(x,\tau)$, it follows that

$$\mathrm{d}\omega = f'_\varepsilon(x,\tau)\partial'\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z' + (\partial' f'_\varepsilon)(x,\tau)\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z'.$$

Using Stokes Theorem, for $\delta'<\delta/2$, we get

$$\int_\Omega\int_0^{\delta'}\mathrm{d}\omega(x,t) = \int_\Omega \omega(x,\delta') - \int_\Omega\omega(x,0).$$

Writing things out explicitly, we obtain

$$\begin{aligned}\int_\Omega f'(x,\varepsilon)\,\varphi(x)\,\mathrm{d}x = &\int_\Omega f'(x,\delta'+\varepsilon)\Psi'(x,\delta')\,\mathrm{d}Z'(x,\delta')\\ &-\int_0^{\delta'}\int_\Omega \partial' f'_\varepsilon(x,\tau)\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z'(x,\tau)\\ &-\int_0^{\delta'}\int_\Omega f'_\varepsilon(x,\tau)\partial'\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z'(x,\tau) \quad (4.5)\end{aligned}$$

We now show that the limit as $\epsilon\to 0$ exists for each of the integrals on the right-hand side of (4.5). Since we are assuming that f is continuous, the function $f'(x,\delta'+\epsilon)$ is well-defined and the L^p-assumption, a priori defined for almost all ϵ is actually continuous in ϵ. Consequently, the integral, as a function of ϵ is continuous and defined on a compact set (in ϵ) and hence attains its max. We can now use the Dominated Convergence Theorem to establish the limit as $\epsilon\to 0$. For the first double integral on the right hand-side of (4.5) we note that, in view of (4.2) and (1) in Definition 3.4, we have

$$\begin{aligned}&\int_0^{\delta'}\left|\int_\Omega \partial' f'_\varepsilon(x,\tau)\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z'(x,\tau)\right|\\ &\le C\int_0^{\delta}\left\|\partial' f'(\cdot,\tau)\right\|_{L^p(\Omega)} d\tau\cdot\sup_{y\in\Gamma}\left\|\Psi(\cdot,y)\right\|_{L^q(\Omega)} \le C'<\infty \quad (4.6)\end{aligned}$$

independently of ε and t' and we again use the Dominated Convergence Theorem to send $\varepsilon\to 0$. For the second double integral on the right hand-side of (4.5) we will use Eq. (3.2) in Definition 3.4 together (4.3) and (4.4) and observe

$$\left|\int_0^{\delta'}\int_\Omega f'_\varepsilon(x,\tau)\partial'\Psi'(x,\tau)\,\mathrm{d}\tau\wedge\mathrm{d}Z'(x,\tau)\right|$$

$$\leq C \int_0^{\delta'} \int_\Omega |f'_\varepsilon(x,\tau)| e^{M^*(\tau/\lambda)-M^*(\tau/\lambda)} |\partial' \Psi'(x,\tau)|\, dx\, d\tau$$
$$\leq C \int_0^{\delta'} \left\| f_\varepsilon(\cdot, \tau y') e^{-M^*(\tau/\lambda)} \right\|_{L^p(\Omega)} d\tau \cdot \sup_{y\in\Gamma} \left\| \partial' \Psi'(\cdot, y)\, e^{M^*(|y|/\lambda)} \right\|_{L^q(\Omega)} \leq C < \infty.$$

Thus, it follows that the limit when $\varepsilon \to 0$ in the second double integral on the right hand-side of (4.5) also exists, independently the direction t' and hence $\lim_{\varepsilon\to 0^+} \int_\Omega f(x, \varepsilon y')\varphi(x)\, dx$ exists and

$$\lim_{\varepsilon\to 0^+} \int_\Omega f(x,\varepsilon y')\varphi(x)\, dx = \int_\Omega f'(x,\delta)\Psi'(x,\delta)\, \mathrm{d}Z'(x,\delta)$$
$$- \int_0^\delta \int_\Omega \left\{ (\partial' f')\Psi' + f'(\partial'\Psi') \right\}(x,\tau)\, \mathrm{d}\tau \wedge \mathrm{d}Z'(x,\tau) \tag{4.7}$$

Also, it follows from the proof that

$$\left| \int_\Omega f(x,\tau y')\varphi(x)\, dx \right| \leq C_\varphi. \tag{4.8}$$

Step 2. Assume that f is only of class $C(\mathcal{W}) \cap L^p(\mathcal{W})$. By regularizing f with a convolution of a $\phi \in \mathcal{E}^{1,M}(\Omega)$, with compact support and integral equal to one (see [4]), we may prove this step using the same ideas as in [19].

Step 3. The formula (4.7) is independent of the direction t'. In fact, fix $\varphi \in \mathcal{E}^{q,M}(\Omega)$ and consider the function

$$T(y) := \int_\Omega f(x,y)\varphi(x)\, dx, \quad y \in \Gamma_\delta.$$

One can use (4.8) to show that (see step 3 in [19]) $T(y)$ is a Lipschitz function and has a limit as $y \to 0$ in proper subcones of Γ_δ. In fact, in the sense of distributions, we have

$$\partial_{y_j} T(y) = -i \int_\Omega \partial_{\bar{z}_j} f(x,y)\varphi(x)\, dx - i \int_\Omega f(x,y)\partial_{x_j}\varphi(x)\, \mathrm{d}x \tag{4.9}$$

and it follows from (4.8) that $T(y)$ is a Lipschitz function and therefore using (4.7) we have that $bf(x)$ is an ultradistribution in $\mathcal{E}^{q,M}(\Omega)'$. □

5 Microglobal analysis: the global wavefront set

We first recall an inversion formula for the FBI transform proved in [25]. Let

$$u_\epsilon(x) = (2\pi)^{-d} \int_{|\xi|\leq\epsilon^{-1}} \mathcal{F}u(x,\xi)d\xi. \tag{5.1}$$

then [25, Theorem 2.1] (5.1) holds in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, for $\beta > \frac{1}{2}$. Specifically, if $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ and $\mathcal{E}^{q,M}(\mathbb{R}^d) \succcurlyeq \mathcal{G}^{q,\frac{1}{2}}(\mathbb{R}^d)$, then the limit defined in (5.1) converges in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$.

Proof of Theorem 2.5 Suppose that $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ and let $\xi^0 \in \mathbb{R}^d \setminus \{0\}$ be such that $\xi^0 \notin WF_{\mathcal{E}^{q,M}}(u)$. By Definition 2.4, there exist a conic neighborhood Γ_0 of ξ^0 in $\mathbb{R}^d \setminus \{0\}$ and constants $a_0, A_0 > 0$ such that, for each $q \leq r \leq \infty$, we have

$$\|D_x^J \mathcal{F}u(x,\xi)\|_{L^r(\mathbb{R}^d)} \leq A_0^{|J|+1} M_{|J|} e^{-\frac{1}{c}M(a_0|\xi|)}, \quad \forall \xi \in \Gamma_0. \tag{5.2}$$

Let $C_1, \ldots, C_k$ open acute cones in $\mathbb{R}^d$ satisfying,

$$\mathbb{R}^d \setminus \Gamma_0 = \bigcup_{j=1}^{k} \overline{C_j}, \quad C_j \cap C_i = \emptyset, \text{ for all } \; j \neq i \tag{5.3}$$

and, for some small positive constant c,

$$\Gamma_j = \{v \in \mathbb{R}^d : \xi \cdot v > c|\xi||v| \text{ for all } \xi \in C_j \text{ and } \xi_0 \cdot v < 0\} \tag{5.4}$$

are open acute and nonempty cones. Inspired by the FBI inversion formula (5.1), we define $u_j^\epsilon(x)$ for $\epsilon > 0$ and $j \in \{0, \ldots, k\}$ as

$$u_0^\epsilon(x) = (2\pi)^{-d} \int\limits_{\substack{\xi \in \Gamma_0 \\ |\xi| \le \epsilon^{-1}}} \mathcal{F}u(x, \xi) d\xi \tag{5.5}$$

and

$$u_j^\epsilon(x) = (2\pi)^{-d} \int\limits_{\substack{\xi \in C_j \\ |\xi| \le \epsilon^{-1}}} \mathcal{F}u(x, \xi) d\xi, \quad j \in \{1, \ldots, k\}. \tag{5.6}$$

By [25, Theorem 2.1], we have that

$$u = \lim_{\epsilon \to 0} \sum_{j=0}^{k} u_j^\epsilon(x), \quad \text{in } \mathcal{E}^{q,M}(\mathbb{R}^d)'.$$

Next we consider, for each $j \in \{1, \ldots, k\}$,

$$f_j(x+iy) := (2\pi)^{-d} \int\limits_{\xi \in C_j} \mathcal{F}u(x+iy, \xi) d\xi, \quad x+iy \in \mathbb{R}^d + i\Gamma_j. \tag{5.7}$$

Note that, in view of hypothesis (5.2), we have

$$\lim_{\epsilon \to 0} u_0^\epsilon(x) := (2\pi)^{-d} \int\limits_{\xi \in \Gamma_0} \mathcal{F}u(x, \xi) d\xi \quad \text{in } \mathcal{E}^{q,M}(\mathbb{R}^d). \tag{5.8}$$

Therefore, it follows that $u_0 \in \mathcal{E}^{q,M}(\mathbb{R}^d)$. Hence, the proof will be completed once we show that for each $j \in \{1, \ldots, k\}$ the following hold true:

(1) for every $p \le r \le \infty$ there exist positive constants a and δ such that the function $f_j(x+iy)$ is in the weighted space $\mathcal{E}^{r,M}(\mathbb{R}^d; \mathcal{E}^{\infty,M}_{e^{-aM^*(|y|)}}(\Gamma_j)_\delta)$, meaning that for every $\lambda > 0$ there exist positive constants C and $A = A(r, \lambda, d)$ such that (2.12) holds.
(2) the function f_j is of exponential M^* growth, that is, it satisfies hypothesis (1) and (2) from Theorem 2.2 for $\Omega = \mathbb{R}^d$ and $\mathcal{W} = \mathbb{R}^d + i\Gamma_j$; and
(3) $bf_j = \lim_{\epsilon \to 0} u_j^\epsilon$ in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$.

In fact, fix $j \in \{1, \ldots, k\}$.

Proof of (1) Since $\mathcal{F}u(z, \xi)$ is a holomorphic function in $z = x+iy$, (1) will be a consequence of Minkowski inequality for integrals and the following fact: there exists $a, C > 0$ and $F_j(\xi) \in L^1(C_j)$ such that for all $\lambda > 0$ and $p \le r \le \infty$ there is a positive constant $A = A(r, \lambda, d)$ such that the following inequality holds true:

$$\left\| D^J_{(x,y)} \mathcal{F}u(x+iy, \xi) \right\|_{L^r(\mathbb{R}^d)} \le AC^{|J|+1} M_{|J|} e^{aM^*(|y|/\lambda)} F_j(\xi), \quad \forall J \in \mathbb{N}_0^d. \tag{5.9}$$

First, the function $-M^*(\cdot)$ is an increasing function, so we only need to worry about proving (5.9) for values of $\lambda \in (0,1)$. Second, note that since

$$d(\xi_j + i x_j \langle\xi\rangle) = d\xi_j + i x_j \sum_{k=1}^{d} \frac{\xi_k}{\langle\xi\rangle}\, d\xi_k$$

we see that $\alpha(x,\xi)$ is a sum of terms of the form $(ix)^{\beta_1}\big(\frac{\xi}{\langle\xi\rangle}\big)^{\beta_2}$ where $|\beta_1| = |\beta_2| \le d$.

Next, we use the decomposition for u given by (3.1), to write $u = \sum_{\gamma\in\mathbb{N}_0^d} u_\gamma^{(\gamma)}$ in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$, with $u_\gamma \in L^p(\mathbb{R}^d)$ and $\sum_{\gamma\in\mathbb{N}_0^d} \frac{M_{|\gamma|}}{A^{|\gamma|}} \|u_\gamma\|_{L^p(\mathbb{R}^d)} < \infty$ for every $A > 0$. We therefore, can write $\mathcal{F}u$ as a sum in β_1, β_2, γ, (finite in β_1, β_2), of $U_{\beta_1,\beta_2,\gamma}$, where

$$U_{\beta_1,\beta_2,\gamma}(x+iy,\xi) := c_\gamma \int_{\mathbb{R}^d} u_\gamma(t)\, D_t^\gamma\big\{e^{i(x+iy-t)\cdot\xi - \langle\xi\rangle(x+iy-t)^2}(x+iy-t)^{\beta_1}\big(\tfrac{\xi}{\langle\xi\rangle}\big)^{\beta_2}\big\}\, dt. \tag{5.10}$$

Since derivatives in y in (5.10) can be replaced by derivatives in x multiplying by a constant with absolute value equals to one, to prove (2.12) it will be enough to differentiate only in x. The same remark holds for the derivatives D_t^γ. Thus differentiating in x, $J \in \mathbb{N}_0^d$ times, we have

$$\begin{aligned} &D_x^J U_{\beta_1,\beta_2,\gamma}(x+iy,\xi)\\ &= c_\gamma \big(\tfrac{\xi}{\langle\xi\rangle}\big)^\beta \textstyle\int_{\mathbb{R}^d} u_\gamma(t)\,(-1)^{|\gamma|} D_x^{J+\gamma}\big\{e^{i(x+iy-t)\cdot\xi-\langle\xi\rangle(x+iy-t)^2}(x+iy-t)^\alpha\big\}\, dt\\ &= c_\gamma(-1)^{|\gamma|}\big(\tfrac{\xi}{\langle\xi\rangle}\big)^\beta \textstyle\int_{\mathbb{R}^d} u_\gamma(t) \sum_{L\le J+\gamma} \binom{\gamma+J}{L} D_x^L\big\{e^{i(x+iy-t)\cdot\xi-\langle\xi\rangle(x+iy-t)^2}\big\}\\ &\qquad \times D_x^{J+\gamma-L}\big\{(x+iy-t)^\alpha\big\}\, dt. \end{aligned} \tag{5.11}$$

Let

$$Q(t,x+iy,\xi) := i\xi\cdot(x+iy-t) - \langle\xi\rangle(x+iy-t)^2,$$

then

$$\begin{aligned} \Big|D_x^L\Big\{e^{Q(t,x+iy,\xi)}\Big\}\Big| &\le \sum_{L_1+L_2=L}\binom{L}{L_1}\big|D_x^{L_1}\{e^{i\xi\cdot(x+iy-t)}\}\big|\big|D_x^{L_2}\{e^{-\langle\xi\rangle(x+iy-t)^2}\}\big|\\ &\le \sum_{L_1+L_2=L}\binom{L}{L_1}|\xi|^{L_1}e^{-\xi\cdot y}\cdot\big|D_x^{L_2}\{e^{-\langle\xi\rangle(x+iy-t)^2}\}\big|. \end{aligned} \tag{5.12}$$

Also, for any $K \in \mathbb{N}_0^d$ with $K \le \beta_1$, we obtain

$$D_x^K\big\{(x+iy-t)^{\beta_1}\big\} = \frac{\beta_1!}{(\beta_1-K)!}\big\{(x+iy-t)^{\beta_1-K}\big\}. \tag{5.13}$$

Using (5.12) and (5.13) (with $K = J+\gamma-L \le \beta_1$) one can estimate $D_x^J U_{\beta_1,\beta_2,\gamma}(x+iy,\xi)$ given in (5.11) by

$$\begin{aligned} |D_x^J U_{\beta_1,\beta_2,\gamma}(x+iy,\xi)| &\le CC^{|J|+|\gamma|}e^{-\xi\cdot y}\sum_{\substack{L\subset\gamma+J\\ L_1+L_2=L}} |\xi|^{L_1}\\ &\qquad\times\Big\{|u_\gamma(\cdot)| * \big|(\cdot+iy)^{\beta_1+L-J-\gamma}\big(D_x^{L_2}\{e^{-\langle\xi\rangle(\cdot+iy)^2}\}\big)\big|\Big\}(x). \end{aligned} \tag{5.14}$$

Now, given $r \in [p, \infty]$, let $\tilde{r} \geq 1$ satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{\tilde{r}}$, we can integrate (5.14) with respect to $x \in \mathbb{R}^d$ and apply Young's inequality to obtain that for $(t, \xi) \in \bigcup_j (\Gamma_j \times C_j)$

$$\|D_x^J U_{\beta_1,\beta_2,\gamma}(\cdot + iy, \xi)\|_{L^r(\mathbb{R}^d)} \leq C C^{|J|+|\gamma|} e^{-\xi \cdot y} \|u_\gamma\|_{L^p} \times \sum_{\substack{L \leq \gamma + J \\ L_1 + L_2 = L}} |\xi|^{L_1} \left\| (\cdot + iy)^{\beta_1 + L - J - \gamma} \left(D_x^{L_2} e^{-\langle \xi \rangle (\cdot + iy)^2} \right) \right\|_{L^{\tilde{r}}}. \tag{5.15}$$

Assume, without loss of generality that $|\xi| \geq 1$ [the case where $|\xi| \leq 1$ is similar, see (5.19)]. Using Corollary B.4, Eq. (B.5), one can further bound the last expression by

$$\|D_x^J U_{\beta_1,\beta_2,\gamma}(\cdot + iy, \xi)\|_{L^r(\mathbb{R}^d)} \leq C C_d\, A^{|J|+|\gamma|}\, e^{-\xi \cdot y + \langle \xi \rangle |y|^2} \|u_\gamma\|_{L^p} \sum_{\substack{L \leq \gamma + J \\ L_1 + L_2 = L}} |\xi|^{L_1} \times \sum_{\substack{K_1 + K_2 = L_2 \\ 0 \leq \ell \leq d}} \binom{L_2}{K_1} |\xi|^{\frac{|K_1|}{2} + |K_2| - \frac{1}{2\tilde{r}} - \frac{\ell}{2}} K_1^{\frac{K_1}{2}}. \tag{5.16}$$

Thus, for $|y| < \delta$ sufficiently small one can use (5.4), and introducing a parameter $0 < \theta < 1$, we obtain

$$\begin{aligned}
\|D_x^J U_{\beta_1,\beta_2,\gamma}(\cdot + iy, \xi)\|_{L^r(\mathbb{R}^d)} &\leq C C_d \left(\frac{A}{\theta^{1/2}} \right)^{|J|+|\gamma|} e^{-\frac{c}{2}|\xi||y|} \|u_\gamma\|_{L^p} \sum_{\substack{L \leq \gamma + J \\ L_1 + L_2 = L}} |\xi|^{L_1} \\
&\quad \times \sum_{K_1 + K_2 = L_2} \binom{L_2}{K_1} \left(\frac{(\theta|\xi|)^{|K_1|}}{M_{|K_1|}} \right)^{\frac{1}{2}} M_{|K_1|} |\xi|^{|K_2|} \\
&\leq C C_d \left(\frac{A}{\theta} \right)^{|J|+|\gamma|} e^{-\frac{c}{2}|\xi||y|} e^{\frac{1}{2} M(\theta|\xi|)} \|u_\gamma\|_{L^p} \sum_{K_1 + K_2 = L \leq \gamma + J} |\xi|^{K_2} M_{|K_1|}
\end{aligned} \tag{5.17}$$

Now, for a given $0 < \lambda < 1$, fix $\theta = \frac{2c\lambda}{3}$, then we can further estimate $D_x^J U_{\beta_1,\beta_2,\gamma}$ by

$$\begin{aligned}
&\|D_x^J U_{\beta_1,\beta_2,\gamma}(\cdot + iy, \xi)\|_{L^r(\mathbb{R}^d)} \\
&\leq C_d \left(\frac{3A}{2c\lambda} \right)^{|J|+|\gamma|} e^{\frac{3}{4}\left(M(\frac{2c\lambda}{3}|\xi|) - \frac{2c\lambda|\xi|}{3} \frac{|y|}{\lambda} \right)} e^{-\frac{1}{4} M(\frac{2c\lambda}{3}|\xi|)} \|u_\gamma\|_{L^p} \sum_{K_1 + K_2 = L \leq \gamma + J} |\xi|^{K_2} M_{|K_1|} \\
&\leq C_d \left(\frac{3A}{2c\lambda} \right)^{|J|+|\gamma|} e^{\frac{3}{4} M^*\left(\frac{|y|}{\lambda} \right)} e^{-\frac{1}{4} M(\frac{2c\lambda}{3}|\xi|)} \|u_\gamma\|_{L^p} \sum_{K_1 + K_2 = L \leq \gamma + J} |\xi|^{K_2} M_{|K_1|} \\
&\leq C_d \left(\frac{3A}{2c\lambda} \right)^{|J|+|\gamma|} e^{\frac{3}{4} M^*\left(\frac{|y|}{\lambda} \right)} e^{-\frac{1}{8} M(\frac{2c\lambda}{3}|\xi|)} \|u_\gamma\|_{L^p} M_{|\gamma|} M_{|J|} \in L^1(\mathbb{R}^m_\xi)
\end{aligned} \tag{5.18}$$

where the last inequality is a consequence of Lemma B.1 with $\ell = 4$, $k = |L| \leq |\gamma + J|$ and $r = |K_2|$. Finally, in view of (3.1), this quantity is summable in γ. This proves (5.9) for $|\xi| \geq 1$. Note that for $|\xi| \leq 1$ one can rewrite estimate (5.16) as (taking into account that, when we apply Corollary B.4, $a = \langle \xi \rangle > 1$)

$$\begin{aligned}
&\|D_x^J U_{\beta_1,\beta_2,\gamma}(\cdot + iy, \xi)\|_{L^r(\mathbb{R}^d)} \\
&\quad \leq C C_d\, A^{|J|+|\gamma|}\, e^{-c|\xi||y|} \|u_\gamma\|_{L^p} \sum_{\substack{L \leq \gamma + J \\ L_1 + L_2 = L}} |\xi|^{|L_1|}
\end{aligned}$$

$$\times \sum_{\substack{K_1+K_2=L_2\\ 0\le\ell\le d}} \binom{L_2}{K_1}\langle\xi\rangle^{|K_1|/2+|K_2|}|K_1|^{\frac{|K_1|}{2}}$$

$$\le CC_d\left(\frac{A}{\theta}\right)^{|J|+|\gamma|} e^{-c|\xi||y|}e^{\frac{1}{2}M(\theta\langle\xi\rangle)}\,\|u_\gamma\|_{L^p}\sum_{K_1+K_2=L\le\gamma+J}\langle\xi\rangle^{|K_2|}M_{|K_1|} \tag{5.19}$$

and we can proceed as before. This shows claim (1).

Proof of (2) Note that hypothesis (2) of Theorem 2.2 follows from (5.9), while hypothesis (1) of Theorem 2.2 follows from the fact that f_j is a holomorphic function in $\mathbb{R}^d + i\Gamma_j$.

Proof of (3) Note that Theorem 2.2 implies that bf_j exists in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$ for each $j \in \{1,\dots,k\}$ and we can use Proposition 3.2, (3.1), for any $\varphi \in \mathcal{E}^{q,M}(\mathbb{R}^d)$, to write

$$\begin{aligned}
\langle bf_j,\varphi\rangle &= (2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\int_{\mathbb{R}^d}\int_{\xi\in C_j}\mathcal{F}u(x+iy,\xi)d\xi\,\varphi(x)\,dx \qquad (5.20)\\
&= (2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\int_{\mathbb{R}^d}\int_{\xi\in C_j}\langle u(t),e^{i(x+iy-t)\cdot\xi-\langle\xi\rangle(x+iy-t)^2}\alpha(x+iy-t,\xi)\rangle d\xi\,\varphi(x)\,dx\\
&= (2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\sum_\gamma(-1)^{|\gamma|}\\
&\quad\times\int_{\mathbb{R}^d}\int_{\xi\in C_j}\int_{\mathbb{R}^d}u_\gamma(t)D_t^\gamma\left\{e^{i(x+iy-t)\cdot\xi-\langle\xi\rangle(x+iy-t)^2}\alpha(x+iy-t,\xi)\right\}dt\,d\xi\,\varphi(x)\,dx\\
&= (2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\sum_\gamma(-1)^{|\gamma|}\\
&\quad\times\int_{\xi\in C_j}\int_{\mathbb{R}^d}u_\gamma(t)D_t^\gamma\left\{\int_{\mathbb{R}^d}\varphi(x)\,e^{i(x+iy-t)\cdot\xi-\langle\xi\rangle(x+iy-t)^2}\alpha(x+iy-t,\xi)\,dx\right\}dt\,d\xi.
\end{aligned}$$

Let $\Psi(x+iy)$ be an $\mathcal{E}^{q,M}$-almost analytic extension of φ granted by Proposition 3.5. Then

$$\begin{aligned}
\langle bf_j,\varphi\rangle &= (2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\sum_\gamma(-1)^{|\gamma|}\\
&\quad\times\int_{\xi\in C_j}\int_{\mathbb{R}^d}u_\gamma(t)D_t^\gamma\left\{\int_{\mathbb{R}^d}\Psi(x-iy)\,e^{i(x-t)\cdot\xi-\langle\xi\rangle(x-t)^2}\alpha(x-t,\xi)\,dx\right\}dt\,d\xi\\
&= (-2\pi)^{-d}\lim_{\Gamma_j\ni y\to 0}\sum_\gamma(-1)^{|\gamma|}\int_{\xi\in C_j}\int_{\mathbb{R}^d}u_\gamma(t)D_t^\gamma\left\{\mathcal{F}(\Psi(t-iy))(t,-\xi)\right\}dt\,d\xi\\
&= (-2\pi)^{-d}\sum_\gamma(-1)^{|\gamma|}\int_{\xi\in C_j}\int_{\mathbb{R}^d}u_\gamma(t)D_t^\gamma\left\{\mathcal{F}\varphi(t,-\xi)\right\}dt\,d\xi \qquad (5.21)
\end{aligned}$$

where the last equality is a consequence of the fact that $\Psi \in \mathcal{E}^{q,M}(\mathbb{R}^d;\mathcal{E}^{\infty,M}(\mathbb{R}^d))$ and that $\Psi(\cdot - iy) \to \varphi(\cdot)$ in the topology of $\mathcal{E}^{q,M}(\mathbb{R}^d)$ as $y \to 0$.

We will now treat the term $u_j^\epsilon(x)$. In view of Proposition 3.2, (3.1), we can rewrite $u_j^\epsilon(x)$, $j \in \{1,\dots,k\}$, given in (5.6) as

$$u_j^\epsilon(x) = (2\pi)^{-d} \sum_{\gamma\in\mathbb{N}_0^d} (-1)^{|\gamma|} \int_{\substack{\xi\in C_j \\ |\xi|\le\epsilon^{-1}}} \int_{\mathbb{R}^d} u_\gamma(t) D_t^\gamma \left\{ e^{i(x-t)\cdot\xi-\langle\xi\rangle(x-t)^2} \alpha(x-t,\xi)\right\} dt\, d\xi. \tag{5.22}$$

Now, one can use Eq. (2.7) to write, for any $\varphi \in \mathcal{E}^{q,M}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle u_j^\epsilon, \varphi\rangle &= (2\pi)^{-d} \sum_\gamma (-1)^{|\gamma|} \\
&\quad\times \int_{\mathbb{R}^d} \int_{\substack{\xi\in C_j \\ |\xi|\le\epsilon^{-1}}} \int_{\mathbb{R}^d} u_\gamma(t) D_t^\gamma \left\{ e^{i(x-t)\cdot\xi-\langle\xi\rangle(x-t)^2} \alpha(x-t,\xi)\right\} dt\, d\xi\, \varphi(x)\, dx \\
&= (-2\pi)^{-d} \sum_\gamma (-1)^{|\gamma|} \\
&\quad\times \int_{\substack{\xi\in C_j \\ |\xi|\le\epsilon^{-1}}} \int_{\mathbb{R}^d} u_\gamma(t) \int_{\mathbb{R}^d} \varphi(x)\, D_t^\gamma \left\{ e^{i(t-x)\cdot(-\xi)-\langle-\xi\rangle(t-x)^2} \alpha(t-x,-\xi)\right\} dx\, dt\, d\xi \\
&= (-2\pi)^{-d} \sum_\gamma (-1)^{|\gamma|} \\
&\quad\times \int_{\substack{\xi\in C_j \\ |\xi|\le\epsilon^{-1}}} \int_{\mathbb{R}^d} u_\gamma(t) D_t^\gamma \left\{ \int_{\mathbb{R}^d} \varphi(x)\, e^{i(t-x)\cdot(-\xi)-\langle-\xi\rangle(t-x)^2} \alpha(t-x,-\xi)\, dx \right\} dt\, d\xi \\
&= (-2\pi)^{-d} \sum_\gamma (-1)^{|\gamma|} \int_{\substack{\xi\in C_j \\ |\xi|\le\epsilon^{-1}}} \int_{\mathbb{R}^d} u_\gamma(t) D_t^\gamma \left\{ \mathcal{F}\varphi(t,-\xi)\right\} dt\, d\xi.
\end{aligned} \tag{5.23}$$

Hence, we can use Theorem 2.3 together Proposition 3.2 to conclude that

$$\lim_{\varepsilon\to 0} \langle u_j^\epsilon, \varphi\rangle = (-2\pi)^{-d} \sum_\gamma (-1)^{|\gamma|} \int_{\xi\in C_j} \int_{\mathbb{R}^d} u_\gamma(t) D_t^\gamma \left\{ \mathcal{F}\varphi(t,-\xi)\right\} dt\, d\xi. \tag{5.24}$$

Therefore, claim (3) follows from Eqs. (5.21) and (5.24).

For the converse, using the linearity of the FBI transform, it will be enough to prove the theorem under the following simpler assumption:

> If there exist an open acute cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ such that $\xi_0 \cdot \Gamma < 0$, and a function f defined on $\mathbb{R}^d \times \Gamma_\delta$ satisfying (1) from Theorem 2.2 and (2.11), and $u = bf$ in $\mathcal{E}^{q,M}(\mathbb{R}^d)'$, then there exists an open acute conic neighborhood of ξ_0, Γ_0, such that (2.10) is satisfied.

We first assume that $|\xi| \ge 1$. In this case, $\langle\xi\rangle \le \sqrt{2}|\xi|$. Next, we note that $\alpha(x,\xi)$ is a sum of terms of the form $i^\ell x^{\beta_1}\left(\frac{\xi}{\langle\xi\rangle}\right)^{\beta_2}$ where $|\beta_1|, |\beta_2| \le d$ as before. Therefore, to prove estimate (2.9), it suffices to obtain the same bound for

$$u^{\beta_1\beta_2}(x,\xi) := \left\langle u, e^{i(x-\cdot)\cdot\xi-\langle\xi\rangle(x-\cdot)^2} (x-\cdot)^{\beta_1} \left(\tfrac{\xi}{\langle\xi\rangle}\right)^{\beta_2} \right\rangle \tag{5.25}$$

in some conic neighborhood of ξ^0. To this end, we first will use the hypothesis that $u = bf$ to write

$$u^{\beta_1\beta_2}(x,\xi) = \lim_{\Gamma\ni y\to 0} \int_{\mathbb{R}^d} f(t+iy)e^{i(x-t)\cdot\xi-\langle\xi\rangle(x-t)^2}(x-t)^{\beta_1}\left(\tfrac{\xi}{\langle\xi\rangle}\right)^{\beta_2} dt. \tag{5.26}$$

Second, we may assume, without loss of generality (see the proof of Theorem 2.2), that $y = he_1$, for some fixed $0 < h < \delta/2$. We now apply Stokes Theorem in the complex variable $z_1 = t_1 + is$ to the function

$$f(z_1 + ih, t_2, \ldots, t_d)e^{i(x-z_1)\cdot\xi-\langle\xi\rangle(x-z_1)^2}(x-z_1)^{\beta_1}, \quad t_1 \in \mathbb{R},\ s \in [0,h]$$

and recall that for any C^1 function g, $dg\wedge dz_1 = \frac{\partial g}{\partial \bar z_1}\, d\bar z_1\wedge dz_1 = 2i\frac{\partial g}{\partial \bar z_1}\, dt_1\wedge ds$. Additionally, $dz_1 = dt_1$ when $z_1 \in \mathbb{R}$ or $\mathbb{R}+iy$. We may therefore rewrite $u^{\beta_1\beta_2}(x,\xi)$ given in (5.26) as

$$\begin{aligned} &u^{\beta_1\beta_2}(x,\xi) \\ &= \lim_{\Gamma\ni y\to 0}\Bigg\{c_1\int_{\mathbb{R}^d} f(t+i2y)e^{i(x-t-iy)\cdot\xi-\langle\xi\rangle(x-t-iy)^2}(x-t-iy)^{\beta_1}\left(\tfrac{\xi}{\langle\xi\rangle}\right)^{\beta_2} dt \\ &\quad + c_2\int_{\mathbb{R}^d}\int_0^h \frac{\partial f}{\partial \bar z_1}(t+i(s+h)e_1)e^{i(x-t-is)\cdot\xi-\langle\xi\rangle(x-t-is)^2}(x-t-is)^{\beta_1}\left(\tfrac{\xi}{\langle\xi\rangle}\right)^{\beta_2} ds\,dt\Bigg\} \end{aligned} \tag{5.27}$$

In the first integral in the right hand-side of (5.27), we use estimates (5.12), (5.13) and (5.18), and the fact that $\xi_0\cdot\Gamma < 0$. Note that (5.4) was fundamental to obtain (5.18) (see (5.17)), and in this situation, its substitute is $\xi\cdot\Gamma < 0$ for all ξ in a conic neighborhood Γ_0 of ξ_0 given by the hypothesis to obtain a conic neighborhood Γ_0 of ξ^0 such that one can interchange the derivatives in x with the integral and obtain a limits and obtain an estimate like (2.10) independent of y. For the second integral in the right hand-side of (5.27), we use that f satisfies (1) from Theorem 2.2 so that we can reason as in the proof of Theorem 2.2, see (4.6), together (5.18), to obtain the desired bounds for all derivatives in x uniformly of y. This allows us to apply D_x^J to $u^{\beta_1\beta_2}$, and proceed as in (5.15) (with the difference that now $\gamma = 0$) to obtain the desired estimate. □

6 Application: wavefront sets and constant coefficient PDE. Proof of Theorem 2.8

Proof of Theorem 2.8 We follow the general argument of [21, Theorem 8.3.1]. The inclusion $WF_{\mathcal{E}^{q,M}}(Pu) \subset WF_{\mathcal{E}^{q,M}}(u)$ is a consequence of the fact that $\mathcal{F}u(x,\xi)$ is defined in terms of a convolution in $x-y$ and (A.4). In particular, we will show that if u is $\mathcal{E}^{q,M}$-microglobal regular at ξ, then so is $D^\kappa u$ for any fixed κ. Supposing this, the inclusion $WF_{\mathcal{E}^{q,M}}(Pu) \subset WF_{\mathcal{E}^{q,M}}(u)$ is immediate. Therefore, assume that $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ is $\mathcal{E}^{q,M}$-microglobal regular at ξ^0. Then there exists a conic neighborhood Γ of ξ^0 in $\mathbb{R}^d\setminus\{0\}$ in which (2.10) holds. Let $\xi\in\Gamma$ and observe

$$\begin{aligned} D_x^J\,\mathcal{F}\{D^\kappa u\}(x,\xi) &= D_x^J\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}D_y^\kappa u(y)\alpha(x-y,\xi)\,dy \\ &= (-1)^{|\kappa|}D_x^J\int_{\mathbb{R}^d} u(y)D_y^\kappa\Big\{e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\Big\}\,dy \\ &= D_x^{J+\kappa}\int_{\mathbb{R}^d} u(y)e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\,dy = D^{J+\kappa}\,\mathcal{F}u(x,\xi) \end{aligned}$$

where the integration should be understood as a pairing a function with an element in its dual space. The estimate to show that $D^\kappa u$ is $\mathcal{E}^{q,M}$-microglobal regular at ξ^0 now follows from the estimate that u is $\mathcal{E}^{q,M}$-microglobal regular at ξ^0 and (A.4) (which allows us to bound $M_{|J|+|\kappa|}$ in terms of $M_{|J|}$ by paying a price of increasing the geometric constant).

We now establish the second inclusion. Since P has constant coefficients, its symbol does not depend on x, hence we may write $P_m(x,\xi) = P_m(\xi)$. We may assume that ξ^0 is such that $P_m(\xi^0) \neq 0$. Since P_m is a homogeneous polynomial of degree m and $P_m(\xi^0) \neq 0$, there exists an open cone $\Gamma_0 \subset \mathbb{R}^d \setminus \{0\}$ that contains ξ^0 and a constant $C > 0$ so that

$$|\xi|^m \leq C|P_m(\xi)| \quad \text{if } \xi \in \Gamma_0.$$

Given a suitable v, $Pu = f$ means that

$$(u, P^t v) = (Pu, v) = (f, v)$$

where the transpose operator, P^t, is given by

$$P^t = \sum_{|\kappa|\leq m} (-1)^{|\kappa|} a_\kappa D^\kappa.$$

For fixed $\xi \in \Gamma_0$ large and $x \in \mathbb{R}^d$, we want to find v so that

$$(u, P^t v) = \mathcal{F}u(x,\xi) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - \langle\xi\rangle(x-y)^2} \alpha(x-y,\xi) u(y)\, dy.$$

This means we need v to satisfy

$$P^t_y v(y) = e^{i(x-y)\cdot\xi - \langle\xi\rangle(x-y)^2} \alpha(x-y,\xi).$$

Suppose that

$$v(y) = w(x-y) e^{i(x-y)\cdot\xi} / P_m(\xi). \tag{6.1}$$

Then applying P^t to the expression for v defined in (6.1), shows that

$$\begin{aligned}
&\frac{1}{P_m(\xi)} P^t_y\big(w(x-y)e^{i(x-y)\cdot\xi}\big) \\
&= \frac{1}{P_m(\xi)} \sum_{|\kappa|\leq m} (-1)^{|\kappa|} a_\kappa D^\kappa_y \big(w(x-y)e^{i(x-y)\cdot\xi}\big) \\
&= e^{i(x-y)\cdot\xi} w(x-y) - \tilde{R}_y\big\{e^{i(x-y)\cdot\xi} w(x-y)\big\}
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
\tilde{R}_y\big\{e^{i(x-y)\cdot\xi} w(x-y)\big\} &= w(x-y)e^{i(x-y)\cdot\xi} - \frac{1}{P_m(\xi)} \sum_{|\kappa|\leq m} (-1)^{|\kappa|} a_\kappa D^\kappa_y\big(w(x-y)e^{i(x-y)\cdot\xi}\big) \\
&= e^{i(x-y)\cdot\xi} \frac{1}{P_m(\xi)} \sum_{|\kappa|\leq m} \sum_{\substack{\gamma\subseteq\kappa \\ |\gamma|\leq m-1}} \binom{\kappa}{\gamma} (-1)^{|\kappa|-|\gamma|} a_\kappa D^{\kappa-\gamma}_y w(x-y)\xi^\gamma \\
&:= e^{i(x-y)\cdot\xi} R_y w(x-y).
\end{aligned} \tag{6.3}$$

Additionally, we can write $R = R_1 + \cdots + R_m$ and R_j is a differential operator of order at most j and the coefficients of $R_j|\xi|^j$ are homogeneous functions of ξ of degree 0. Moreover, we have the relationship $e^{i(x-y)\cdot\xi} R_y = \tilde{R}_y e^{i(x-y)\cdot\xi}$ which means, of course, that

$$R^j_y = e^{-i(x-y)\cdot\xi} \tilde{R}^j_y e^{i(x-y)\cdot\xi}.$$

Solving

$$e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi) = w(x-y) - R_y w(x-y) = (I-R)w(x-y),$$

is unlikely to be straight forward since the sum $\sum_{k=0}^{\infty} R_y^k$ is unlikely to converge. Instead, set

$$w_N(x-y) = \sum_{k=0}^{N-1} R_y^k\{e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\}.$$

Then

$$w_N - Rw_N = e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi) - R^N\{e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\}.$$

We combine this equality with Eqs. (6.2) and (6.3) (with w_N replacing w) to observe that

$$\begin{aligned}&\frac{1}{P_m(\xi)}P^t\big(e^{i(x-y)\cdot\xi}w_N(x-y)\big)\\ &\quad= e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi) - e^{i(x-y)\cdot\xi}R^N\big\{e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\big\}\end{aligned}$$

or, by rearranging,

$$\begin{aligned}&e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\\ &\quad= \frac{1}{P_m(\xi)}P^t\big(e^{i(x-y)\cdot\xi}w_N(x-y)\big) + \tilde{R}^N\big\{e^{i(x-y)\cdot\xi}e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\big\}.\end{aligned}$$

Suppose now that u is an ultradistribution. It now follows that

$$\mathcal{F}u(x,\xi) = \big\langle u, \tilde{R}^N\big\{e^{i(x-y)\cdot\xi}e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\big\}\big\rangle + \Big\langle f, \frac{1}{P_m(\xi)}e^{i(x-y)\cdot\xi}w_N(x-y)\Big\rangle \tag{6.4}$$

where $f = Pu$ is $\mathcal{E}^{q,M}$-microglobal regular at ξ_0 and, without loss of generality, satisfies (2.10) for $\xi \in \Gamma_0$.

We start with the bound for the f term in (6.4), we compute

$$\begin{aligned}\frac{1}{P_m(\xi)}\langle f, e^{i(x-y)\cdot\xi}w_N(x-y)\rangle &= \frac{1}{P_m(\xi)}\sum_{k=0}^{N-1}\big\langle f, \tilde{R}_y^k\big\{e^{i(x-y)\cdot\xi-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)\big\}\big\rangle\\ &= \frac{1}{P_m(\xi)}\mathcal{F}f(x,\xi) + \sum_{k=1}^{N-1}\tilde{S}_x^k\,\mathcal{F}f(x,\xi)\end{aligned}$$

where the operator $\tilde{S} = \tilde{S}_1 + \cdots + \tilde{S}_m$ and $\tilde{S}_j$, $j \in \{1,\dots m\}$, is given by

$$\tilde{S}_j g(x,\xi) := \frac{1}{|\xi|^j}\sum_{|\kappa|\le j} a_{j,\kappa}(\xi)D_x^\kappa$$

and where $a_{j,\kappa}(\xi)$ is a function that is homogeneous of degree 0 in ξ. Thus, if

$$A' = \max_{1\le j\le m}\max_{|\xi|=1}\sum_{|\kappa|\le j}|a_{j,\kappa}(\xi)|$$

then

$$D_x^J \frac{1}{P_m(\xi)} \langle f, e^{i(x-y)\cdot\xi} w_N(x-y)\rangle = \frac{1}{P_m(\xi)} \sum_{k=0}^{N-1} \sum_{|\gamma|=k} \binom{k}{\gamma} D_x^J \tilde{S}_1^{\gamma_1} \cdots \tilde{S}_m^{\gamma_m} \mathcal{F} f(x,\xi).$$

Consequently, for some A that may increase between every step, we use (A.9) and observe

$$\begin{aligned}
&\frac{1}{P_m(\xi)} \left\| D_x^J \langle f, e^{i(x-y)\cdot\xi} w_N(x-y)\rangle \right\|_{L^r(\mathbb{R}^d)} \\
&\leq \frac{1}{P_m(\xi)} \sum_{k=0}^{N-1} \sum_{|\gamma|=k} \binom{k}{\gamma} \frac{(A')^k}{|\xi|^{\gamma\cdot\vec{m}}} M_{|J|+\gamma\cdot\vec{m}} e^{-\frac{1}{c}M(a|\xi|)} \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \sum_{k=0}^{N-1} \sum_{|\gamma|=k} \binom{k}{\gamma} \frac{(A')^k A^{\gamma\cdot\vec{m}}}{|\xi|^{\gamma\cdot\vec{m}}} M_{\gamma\cdot\vec{m}} \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \sum_{\ell=0}^{(N-1)m} \frac{A^\ell}{|\xi|^\ell} M_\ell
\end{aligned}$$

where $\vec{m}$ is the vector $\vec{m} = (1, 2, \dots, m)$. Since M_ℓ increases faster than $\ell!$ by (A.10), it follows that the function $\ell \mapsto \frac{A^\ell}{|\xi|^\ell} M_\ell$ has exactly one critical point which is a minimum (and the function has the minimum value $\exp(-M(\frac{|\xi|}{A})) \leq 1$). Consequently, using (A.9) in the final inequality and allowing A to grow as necessary (e.g., in the fourth line), we obtain (for some $H > 0$ which we allow to grow later, e.g., in (6.11), if necessary)

$$\begin{aligned}
&\frac{1}{P_m(\xi)} \left\| D_x^J \langle f, e^{i(x-y)\cdot\xi} w_N(x-y)\rangle \right\|_{L^r(\mathbb{R}^d)} \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \left(1 + \frac{A^{(N-1)m}}{|\xi|^{(N-1)m}} M_{(N-1)m} \sum_{\ell=0}^{(N-1)m} \frac{|\xi|^{(N-1)m-\ell}}{A_{(N-1)m-\ell} M_{(N-1)m-\ell}} \right) \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} Nm \left(1 + \frac{A^{(N-1)m}}{|\xi|^{(N-1)m}} M_{(N-1)m} \frac{1}{\inf_{\rho\in\mathbb{N}} \frac{A^\rho}{|\xi|^\rho} M_\rho} \right) \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} Nm \left(1 + \frac{A^{(N-1)m}}{|\xi|^{(N-1)m}} M_{(N-1)m} e^{M(\frac{1}{A}|\xi|)} \right) \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)+M(\frac{1}{A}|\xi|)} Nm \left(1 + \frac{A^{(N-1)m}}{|\xi|^{m(N-1)}} M_{m(N-1)} \right) \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)+M(\frac{1}{A}|\xi|)} \left(Nm + M_{mN} \left(\frac{|\xi|}{A}\right)^{-Nm} |\xi|^m \right) \\
&\leq C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|))} \left\{ Nm + \left[M_N \left(\frac{H}{|\xi|}\right)^N \right]^m |\xi|^m \right\}.
\end{aligned} \tag{6.5}$$

where the last inequality is obtained as a consequence of (B.3) to bound $e^{-\frac{1}{c}M(a|\xi|)} e^{M(\frac{|\xi|}{A})}$ by (possibly decreasing $\frac{1}{c}$ and increasing A) a constant times $e^{-\frac{1}{c}M(a|\xi|)}$. Choose N to minimize $M_N(\frac{H}{|\xi|})^N$. The fact that M_ℓ grows faster than $\ell!$ means that N is smaller than if M_ℓ were only

$\ell!$ (which would be approximately $\frac{|\xi|}{H}$). Plugging this information into (6.5) and recognizing that we can absorb the m term into the constant C, Then

$$\begin{aligned}
&\frac{1}{P_m(\xi)}\left\| D_x^J \langle f, e^{i(x-y)\cdot\xi} w_N(x-y)\rangle \right\|_{L^r(\mathbb{R}^d)} \\
&\quad \le C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \left(\frac{|\xi|}{H} + e^{-\frac{1}{c'}M(\frac{|\xi|}{A})} |\xi|^m \right) \\
&\quad \le C A^{|J|} M_{|J|} e^{-\frac{1}{c}M(a|\xi|)} \qquad (6.6)
\end{aligned}$$

with a slight decrease in $\frac{1}{c}$, see (B.1).

We now turn to the u term in (6.4). The operator $\tilde{R}_y$ is a constant coefficient operator so $\tilde{R}_y^N\{e^{i(x-y)\cdot\xi}e^{-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi) \in \mathcal{G}^{p,1/2}(\mathbb{R}^d)$ for all $1 \le p \le \infty$. We use Proposition 3.2 to express the ultradistribution $u \in \mathcal{E}^{q,M}(\mathbb{R}^d)'$ as

$$u = \sum_{\kappa\in\mathbb{N}_0^d} \partial^\kappa u_\kappa$$

where $u_\kappa \in L^{q'}(\mathbb{R}^d)$, $\frac{1}{q} + \frac{1}{q'} = 1$. Consequently, for any multiindex J, if $1 \le p \le q$ and r satisfying $\frac{1}{p} + \frac{1}{q'} = 1 + \frac{1}{r}$, we use Young's inequality and estimate

$$\begin{aligned}
&\left\| D_x^J \langle u, \tilde{R}_y^N \{ e^{i(x-y)\cdot\xi} e^{-\langle\xi\rangle(x-y)^2} \alpha(x-y,\xi)\}\rangle \right\|_{L^r(\mathbb{R}^d)} \\
&\quad \le \sum_{\kappa\in\mathbb{N}_0^d} \|u_\kappa\|_{L^{q'}(\mathbb{R}^d)} \left\| D^J \partial^\kappa \tilde{R}^N \{ e^{i(x-y)\cdot\xi} e^{-\langle\xi\rangle(x-y)^2} \alpha(x-y,\xi)\} \right\|_{L^p(\mathbb{R}^d)}. \qquad (6.7)
\end{aligned}$$

Recall that $R = R_1 + \cdots + R_m$. We define operators $S, S_1, \ldots, S_m$ so that $(R_j)_y g(x-y) = (S_j)_x g(x-y)$. Then for any multiindex I, set $S^I = S_1^{I_1} \cdots S_m^{I_m}$. The operators $\tilde{S}_j$ defined above were defined above in an analogous manner so that $(\tilde{R}_j)_y g(x-y) = (\tilde{S}_j)_x g(x-y)$ We now compute

$$\begin{aligned}
&\left| D^J \partial^\kappa \tilde{S}^N \{ e^{ix\cdot\xi} e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right| = \left| D^J \partial^\kappa e^{ix\cdot\xi} S^N \{ e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right| \\
&\quad \le \sum_{|I|=N} \sum_{\gamma\subset J+\kappa} \binom{\kappa+J}{\gamma}\binom{N}{I} \left| \xi^{J+\kappa-\gamma} D^\gamma S^I \{ e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right|.
\end{aligned}$$

Observe that I has m components so that $\sum_{|I|=N}\binom{N}{I} \le m^N$. Similarly $\sum_{\gamma\subset J+\kappa}\binom{\kappa+J}{\gamma} \le d^{|\kappa|+|J|}$. Next, let $\vec{m}$ be the vector $\vec{m} = (1, 2, \ldots, m)$ as above, and recall that $\alpha(x,\xi)$ is a sum of terms of the form $x^{\mu'}\frac{\xi^\mu}{\langle\xi\rangle^{|\mu|}}$ for multiindices μ, μ' so that $|\mu'| = |\mu| \le d$. Thus, given the form of the operator S_j, there exists a constant $A > 0$ so that

$$\left| \xi^{J+\kappa-\gamma} D^\gamma S^I \{ e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right| \le |\xi|^{|J|+|\kappa|-|\gamma|-I\cdot\vec{m}} \sum_{\substack{0\le|\mu|\le d\\ 0\le\ell\le|\mu|}} \left|\nabla^\ell x^\mu\right| \left|\nabla^{|\gamma|+I\cdot\vec{m}-\ell} e^{-\langle\xi\rangle x^2}\right|.$$

By Corollary B.3,

$$\begin{aligned}
&\left\| \left|\nabla^\ell x^\mu\right| \left|\nabla^{|\gamma|+I\cdot\vec{m}-\ell} e^{-\langle\xi\rangle x^2}\right| \right\|_{L^p(\mathbb{R}^d)} \\
&\quad \le C A^{|\gamma|+I\cdot\vec{m}-\ell} \langle\xi\rangle^{\frac{1}{2}(|\gamma|+I\cdot\vec{m}-|\mu|-\frac{1}{p})} (|\gamma| + I\cdot m - \ell)^{\frac{1}{2}(|\gamma|+I\cdot\vec{m}-\ell)}.
\end{aligned}$$

Consequently, assuming that the constants A and C can grow from line to line, we obtain

$$\begin{aligned}
&\left|D^J \partial^\kappa \tilde{S}^N \{e^{ix\cdot\xi} e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right| \\
&\quad\le \sum_{|I|=N} \sum_{\gamma\subset J+\kappa} d^{|\kappa|+|J|} m^N |\xi|^{|J|+|\kappa|-|\gamma|-I\cdot\vec{m}} \sum_{\substack{0\le|\mu|\le d\\ 0\le\ell\le|\mu|}} \left\| \left|\nabla^\ell x^\mu\right| \left|\nabla^{|\gamma|+I\cdot\vec{m}-\ell} e^{-\langle\xi\rangle x^2}\right| \right\|_{L^p(\mathbb{R}^d)} \\
&\quad\le \sum_{|I|=N} \sum_{\gamma\subset J+\kappa} d^{|\kappa|+|J|} m^N |\xi|^{|J|+|\kappa|-|\gamma|-I\cdot\vec{m}} \\
&\qquad\times \sum_{\substack{0\le|\mu|\le d\\ 0\le\ell\le|\mu|}} A^{|\gamma|+I\cdot\vec{m}-\ell} \langle\xi\rangle^{\frac{1}{2}(|\gamma|+I\cdot\vec{m}-|\mu|-\frac{1}{p})} (|\gamma|+I\cdot\vec{m}-\ell)^{\frac{1}{2}(|\gamma|+I\cdot\vec{m}-\ell)} \\
&\quad\le \sum_{|I|=N} \sum_{\gamma\subset J+\kappa} d^{|\kappa|+|J|} m^N |\xi|^{|J|+|\kappa|-\frac{1}{2}|\gamma|-\frac{1}{2}I\cdot\vec{m}-\frac{1}{2p}} A^{|\gamma|+I\cdot\vec{m}} (|\gamma|+I\cdot\vec{m})^{\frac{1}{2}(|\gamma|+I\cdot\vec{m})} \\
&\quad\le \sum_{|I|=N} A^N (I\cdot\vec{m})^{\frac{1}{2}I\cdot\vec{m}} |\xi|^{-\frac{1}{2}I\cdot\vec{m}-\frac{1}{2p}} \sum_{\gamma\subset J+\kappa} A^{|\kappa|+|J|} |\xi|^{|J|+|\kappa|-\frac{1}{2}|\gamma|} |\gamma|^{\frac{|\gamma|}{2}}.
\end{aligned} \tag{6.8}$$

We need to sum in κ so we decompose the sum over $J+\kappa$ into two sums – one over J and one over κ. Then (6.8) becomes

$$\begin{aligned}
&\left|D^J \partial^\kappa \tilde{S}^N \{e^{ix\cdot\xi} e^{-\langle\xi\rangle x^2} \alpha(x,\xi)\}\right| \\
&\quad\le \sum_{|I|=N} A^N (I\cdot\vec{m})^{\frac{1}{2}I\cdot\vec{m}} |\xi|^{-\frac{1}{2}I\cdot\vec{m}-\frac{1}{2p}} \sum_{\gamma_1\subset J} A^{|J|} |\xi|^{|J|-\frac{1}{2}|\gamma_1|} |\gamma_1|^{\frac{|\gamma_1|}{2}} \sum_{\gamma_2\subset\kappa} A^{|\kappa|} |\xi|^{|\kappa|-\frac{1}{2}|\gamma_2|} |\gamma_2|^{\frac{|\gamma_2|}{2}}.
\end{aligned} \tag{6.9}$$

We first sum in κ. Our choice of M_ℓ forces there to exist B' and C' so that

$$M_\ell \ge C' B'^{\ell} \ell^\ell.$$

Consequently, we now recall the sum in κ from (6.7). In the estimate below, we assume $|\xi| \ge 1$ as the $|\xi| \le 1$ case is simpler. We also assume $|\gamma|$ is even for simplicity, the $|\gamma|$ odd calculation requires a simple modification. For $B > 1$ to be chosen later and C and A which may grow with each line (though they are required to be independent of $|\xi|$ and $|\kappa|$) and estimate

$$\begin{aligned}
&\sum_{\kappa\subset\mathbb{N}_0^d} \left[\|u_\kappa\|_{L^{q'}(\mathbb{R}^d)} A^{|\kappa|} B^{|\kappa|} M_{|\kappa|} |\xi|^{|\kappa|} \sum_{\gamma_2\subset\kappa} \frac{|\gamma_2|^{\frac{|\gamma_2|}{2}}}{B^{|\kappa|} |\xi|^{\frac{1}{2}|\gamma_2|} M_{|\kappa|}} \right] \\
&\quad< \sum_{\kappa\subset\mathbb{N}_0^d} \left[\|u_\kappa\|_{L^{q'}(\mathbb{R}^d)} A^{|\kappa|} B^{|\kappa|} M_{|\kappa|} \sum_{\gamma_2\subset\kappa} \frac{1}{B^{\frac{1}{2}|\gamma_2|}} \frac{|\xi|^{|\kappa|-\frac{1}{2}|\gamma_2|}}{B^{|\kappa|-\frac{1}{2}|\gamma_2|} M_{|\kappa|-\frac{1}{2}|\gamma|}} \right] \\
&\quad< \sum_{\kappa\subset\mathbb{N}_0^d} \left[\|u_\kappa\|_{L^{q'}(\mathbb{R}^d)} A^{|\kappa|} B^{|\kappa|} M_{|\kappa|} \sum_{\gamma\in\mathbb{N}_0^d} \frac{1}{B^{\frac{|\gamma|}{2}}} \left(\sup_{\ell\ge 0} \frac{|\xi|^\ell}{B^\ell M_\ell} \right) \right] \\
&\quad\le C_{AB} e^{M(\frac{|\xi|}{B})}
\end{aligned} \tag{6.10}$$

where the sum is finite because $B > 1$ and it is geometric. Next, we investigate the behavior in N and the sum in I and observe that since we are assuming $|\xi| \geq 1$ that

$$\sum_{|I|=N} A^N (I \cdot \vec{m})^{\frac{1}{2} I \cdot \vec{m}} |\xi|^{-\frac{1}{2} I \cdot \vec{m} - \frac{1}{2p}} \leq A^N \left[\left(\frac{N}{|\xi|} \right)^{N/2} + \left(\frac{Nm}{|\xi|} \right)^{\frac{Nm}{2}} \right].$$

Recall that N was chosen to minimize $M_{N'}(\frac{|\xi|}{H})^{-N'}$ and $M_{N'} \geq N'!$. Consequently, since $m \geq 1$ and allowing A and H to grow (if need be),

$$\begin{aligned} A^N \left[\left(\frac{N}{|\xi|} \right)^{N/2} + \left(\frac{Nm}{|\xi|} \right)^{\frac{Nm}{2}} \right] &\leq \left\{ A^N \left(\frac{N}{|\xi|} \right)^N \right\}^{1/2} + \left\{ \left(\frac{N}{|\xi|} mA \right)^N \right\}^{m/2} \\ &\leq \left\{ M_N \left(\frac{H}{|\xi|} \right)^N \right\}^{1/2} + \left\{ M_N \left(\frac{H}{|\xi|} \right)^N \right\}^{m/2} \\ &\leq e^{-\frac{1}{2} M(\frac{|\xi|}{H})}. \end{aligned} \tag{6.11}$$

Finally, we investigate the behavior in J. Observe that (using the same B as above, though it we may require it to grow later)

$$\begin{aligned} \sum_{\gamma_1 \subset J} A^{|J|} |\xi|^{|J| - \frac{1}{2}|\gamma_1|} |\gamma_1|^{\frac{|\gamma_1|}{2}} &= C A^{|J|} B^{|J|} |\xi|^{|J|} \sum_{\gamma \subset J} \frac{|\gamma|^{\frac{|\gamma|}{2}}}{B^{|J|} |\xi|^{\frac{|\gamma|}{2}}} \\ &\leq C A^{|J|} B^{|J|} |\xi|^{|J|} \left(\frac{1}{|B|^{|J|}} + \frac{|J|^{|J|/2}}{B^{|J|} |\xi|^{|J|/2}} \right) \\ &\leq C A^{|J|} B^{|J|} \left(\frac{|\xi|^{|J|}}{|B|^{|J|}} + \frac{|J|^{|J|/2} |\xi|^{|J|/2}}{B^{|J|}} \right). \end{aligned} \tag{6.12}$$

Putting together our estimates (6.10)–(6.12), and choosing B sufficiently large (but independent of $|\xi| \geq 1$, A, J, N), there exists $a > 0$ so we can estimate (6.7) by

$$\begin{aligned} &\left\| D_x^J \langle u, \tilde{R}_y^N \{ e^{i(x-y)\cdot\xi} e^{-\langle \xi \rangle (x-y)^2} \alpha(x-y, \xi) \} \rangle \right\|_{L^r(\mathbb{R}^d)} \\ &\leq C_{AB} e^{-\frac{1}{c} M(\frac{|\xi|}{A})} \left(\frac{|\xi|^{|J|}}{|B|^{|J|}} + \frac{|J|^{|J|/2} |\xi|^{|J|/2}}{B^{|J|}} \right) A^{|J|} |\xi|^{|J|} \\ &\leq C_{AB} e^{-\frac{1}{c} M(\frac{|\xi|}{A})} \left(M_{|J|} \frac{|\xi|^{|J|}}{M_{|J|} |B|^{|J|}} + M_{|J|/2} \frac{|J|^{|J|/2} M_{|J|/2} |\xi|^{|J|/2}}{B^{|J|}} \right) A^{|J|} |\xi|^{|J|} \\ &\leq C_{AB} e^{-\frac{1}{2} M(\frac{|\xi|}{H})} e^{M(\frac{|\xi|}{B})} A^{|J|} M_{|J|} \end{aligned}$$

where, as usual, A, C_{AB}, and B only depends on A. By possibly increasing B and allowing B to depend on H, we may use Eq. (B.3) and bound $e^{-\frac{1}{2} M(\frac{|\xi|}{H})} e^{M(\frac{|\xi|}{B})} \leq C e^{-\frac{1}{c} M(\frac{|\xi|}{H})}$ for some fixed $c > 2$ and C depending on C and the constants in (A.9). □

Recall that a partial differential operator is elliptic if $P_m(x, \xi) \neq 0$ if $\xi \neq 0$.

Corollary 6.1 *If P is an elliptic, constant coefficient differential operator and M be a sequence so that $\mathcal{G}^{q,1}(\mathbb{R}^d) \subset \mathcal{E}^{q,M}(\mathbb{R}^d)$, then*

$$WF_{\mathcal{E}^{q,M}}(Pu) = WF_{\mathcal{E}^{q,M}}(u).$$

Appendix A: On the sequence $M = (M_j)$

Definition A.1 Let $M = (M_j)$ be a sequence of positive real numbers satisfying the following properties:
(**Initial conditions**)

$$M_0 = M_1 = 1. \tag{A.1}$$

(**Strong non-quasianalyticity**) There exists a constant $A > 1$ such that for all $p = 1, 2, \ldots,$ we have

$$\sum_{j=p}^{\infty} \frac{M_j}{M_{j+1}} \leq Ap\frac{M_p}{M_{p+1}}. \tag{A.2}$$

(**Strong logarithmic convexity**) For some fixed $A > 0$ and for any r, with $0 \leq r < 1/A$, if we set $P_j = M_j/(j!)^r$, then

$$\text{the sequence } \left(\frac{P_j}{jP_{j-1}}\right) \text{ is increasing.} \tag{A.3}$$

(**Stability under ultradifferential operators**) There are constants $A > 1$ and $H > 1$, independent of n, such that for all $n = 1, 2, 3, \ldots,$ we have

$$M_n \leq AH^n \min_{0\leq j\leq n} M_j M_{n-j}. \tag{A.4}$$

A.1. Some consequences

We refer to the paper [27] for consequences of the conditions listed in Definition A.1. For instance, condition (A.3) implies: (i) the (usual) **logarithmic convexity** condition: For all $j = 1, 2, 3, \ldots$

$$M_j^2 \leq M_{j-1}M_{j+1}; \tag{A.5}$$

(ii) for all $0 \leq j \leq n$,

$$\binom{n}{j} M_j M_{n-j} \leq M_n \tag{A.6}$$

and (iii)

$$\text{the sequence } \left(\frac{M_j}{j!}\right)^{1/j} \text{ is increasing.} \tag{A.7}$$

Condition (A.6) insures that the class $C^M(U)$ is invariant under composition and, in particular, that for all $0 \leq j \leq n$,

$$M_j M_{n-j} \leq M_n. \tag{A.8}$$

The condition (A.4) implies the (usual) ***Stability under differential operators*** condition; i.e., There are constants $A > 1$ and $H > 1$, independent of n and j, such that for all $1 \leq j \leq n$, we have

$$M_n \leq AH^{n-1} M_j M_{n-j}. \tag{A.9}$$

We will often replace AH^{n-1} with C^n.

If the sequence M satisfies conditions (A.1) and (A.3), then it satisfies the following condition: for all $n = 1, 2, 3, \ldots$

$$M_n \geq n! \tag{A.10}$$

Condition (A.10) insures that every analytic function belongs to the class C^M.

A.2. Associated functions

Definition A.2 For each sequence $\left(M_j\right)$ of positive numbers we define its **associated function** $M(t)$ on $(0, \infty)$ by

$$M(t) = \sup_j \log \frac{t^j}{M_j}. \tag{A.11}$$

For the reader who is interested in learning more about associated functions and how each of the conditions which we impose on the sequence can be written in terms of the associated function, we recommend the paper by Komatsu [27]. In particular, it is not difficult to show that if $\left(M_j\right)$ satisfies conditions (A.1) and (A.10), then for all $t > 0$,

$$\log t \leq M(t) \leq t. \tag{A.12}$$

Appendix B: Some estimates

Lemma B.1 (See [25]) *If the sequence $M = (M_j)_{j\in\mathbb{N}}$ satisfies* (A.4) *and* (A.8), *then for each $\theta > 0$ and $k, r, \ell \in \mathbb{N}$ such that $k \geq r \geq 0$ we have*

$$t^r M_{k-r} \leq A \frac{H^{\ell r}}{\theta^r} M_k\, e^{\frac{1}{2^\ell} M(\theta t)}, \quad \textit{for all } t > 0 \tag{B.1}$$

where A and H are given by (A.4).

Proof We first note that property (A.4) is equivalent to (see [27, Proposition 3.6])

$$M\left(\frac{t}{H}\right) \leq \frac{1}{2} M(t) + \log \sqrt{A} \tag{B.2}$$

and this implies that for every $\ell \in \mathbb{N}$, the following inequality holds true

$$M\left(\frac{t}{H^\ell}\right) \leq \frac{1}{2^\ell} M(t) + \log \sqrt{A} \sum_{j=0}^{\ell-1} \frac{1}{2^j} \leq \frac{1}{2^\ell} M(t) + 2 \log \sqrt{A}. \tag{B.3}$$

Thus if $A > 0$ and $H > 0$ are given by (A.4), and $\theta > 0$, $k, r, \ell \in \mathbb{N}$ are chosen such that $k \geq r$ then it follows from (A.8), (A.11) and (B.3) respectively that

$$\begin{aligned} t^r M_{k-r} &\leq \frac{H^{r\ell}}{\theta^r} M_k \frac{(\frac{\theta t}{H^\ell})^r}{M_r} \leq \frac{H^{r\ell}}{\theta^r} M_k e^{M((\theta t)/H^\ell)} \\ &\leq \frac{H^{r\ell}}{\theta^r} M_k \exp\{\tfrac{1}{2^\ell} M(\theta t) + 2 \log \sqrt{A}\} \\ &= A \frac{H^{r\ell}}{\theta^r} M_k e^{\frac{1}{2^\ell} M(\theta t)} \end{aligned} \tag{B.4}$$

as we wished to prove. □

Proposition B.2 *Let $k \in \mathbb{N}_0$. Then*

1.

$$\frac{d^{2k}}{dx^{2k}} e^{-ax^2} = e^{-ax^2} \sum_{j=0}^{k} (-1)^{k+j} a^{k+j} x^{2j} b_{2k,j}$$

and

$$\frac{d^{2k+1}}{dx^{2k+1}}e^{-ax^2} = e^{-ax^2}\sum_{j=0}^{k}(-1)^{k+j+1}a^{k+j+1}x^{2j+1}b_{2k+1,j}$$

for some constants $b_{2k,j}, b_{2k+1,j} > 0$.

2. *The constants* $b_{2k,j}, b_{2k+1,j}$ *satisfy the following (recursion) relations.*

   (i) $b_{2k+1,j} = 2b_{2k,j} + 2(j+1)b_{2k,j+1}$;
   (ii) $b_{2k+2,j} = 2b_{2k+1,j-1} + (2j+1)b_{2k+1,j}$

   with the understanding that $b_{2k,j} = 0$ *if* $j \leq -1$ *or* $j \geq \ell+1$ *and* $b_{2k+1,j} = 0$ *if* $j \leq -1$ *or* $j \geq \ell+1$.

3. *The constants* $b_{2k,j}$ *and* $b_{2k+1,j}$ *have the following upper bounds*:

   (i) $b_{2k,k} = 2^{2k}$ *and* $b_{2k+1,k} = 2^{2k+1}$
   (ii) *There exist constants* $A, C > 0$ *so that*

$$b_{2k,j} \leq CA^k k^{k-j} \quad \text{and} \quad b_{2k+1,j} \leq CA^k k^{k-j}.$$

Proof The proofs of 1. and 2. follow easily from induction. The only interesting part is 3., and this will follow from a counting argument. The number $b_{2k+1,j}$ is the coefficient of the term (up to a sign and a power of a) $x^{2j+1}e^{-ax^2}$. Viewing the coefficient $b_{2k+1,j}$ are part of tree, the parents of $b_{2k+1,j}$ are $b_{2k,j}$ and $b_{2k,j+1}$ since

$$\frac{d}{dx}e^{-ax^2}x^{2\ell} = -2ae^{-ax^2}x^{2\ell+1} + 2je^{-ax^2}x^{2\ell-1}.$$

We will call $b_{2k,j}$ the left parent of $b_{2k+1,j}$ and $b_{2k,j}$ is the right child of $b_{2k+1,j}$. Similarly, we will call $b_{2k,j+1}$ the right parent of $b_{2k+1,j}$ and $b_{2k+1,j}$ the left child of $b_{2k,j+1}$. To pass from the left parent to the child, the term is multiplied by $-2a$, a doubling of the coefficient and an increase of the power of x. The pass from the right parent to the child, the polynomial term of $e^{-ax^2}x^{2j+2}$ is differentiated in x and consequently the child inherits a $(2j+2)b_{2k,j+1}$ summand. Visually, a tree looks like

$$\begin{array}{ccccccc}
b_{1,0} & & & & & & \\
b_{2,0} & b_{2,1} & & & & & \\
 & b_{3,0} & b_{3,1} & & & & \\
 & b_{4,0} & b_{4,1} & b_{4,2} & & & \\
 & & b_{5,0} & b_{5,1} & b_{5,2} & & \\
 & & b_{6,0} & b_{6,1} & b_{6,2} & b_{6,3} & \\
 & & & b_{7,0} & b_{7,1} & b_{7,2} & b_{7,3} \\
 & & & & \vdots & &
\end{array}.$$

For example, the right child of $b_{3,1}$ is $b_{4,2}$ (down and to the right) and the left child is $b_{4,1}$ (straight down). The key to understand the combinatorics is that in order to have a nonzero right parent and hence a factorially growing term, the power of the polynomial piece must be *bigger* than (in this case) $2j+1$. Observe that if we trace through the tree to get to the $(2k+1)$st row, then it is always the case that for any path

$$2k = \#\text{ right children} + \#\text{ left children}$$

and to arrive at a nonzero term at the $(2k+1)$st row, at least half of the children must be right children. Next, to arrive at $b_{2k+1,0}$, exactly half of the children must be right children and half left children while to arrive at $b_{2k+1,1}$, we need an additional right child and consequently one less left child. As a result, to arrive at $b_{2k+1,j}$, it follows there must be $j+k$ right children and $k-j$ left children in the path. Consequently,

$$2j = \text{\# right children} - \text{\# left children}.$$

The number of left children produce the factorially growing terms, and hence 3.ii. follows as $k-j$ left children mean the factorial contribution to the size of $b_{2k+1,j}$ is k^{k-j}. It follows from this observation that the only way to arrive at $b_{2k,k}$ or $b_{2k+1,k}$ is to follow the path of all right children, hence 3.i. follows. The argument to bound the size of $b_{2k,j}$ is similar. □

Corollary B.3 *There exist constants $C, A > 0$ so that*

1.

$$\left|\frac{d^{2k}}{dx^{2k}}e^{-ax^2}\right| \le Ce^{-ax^2}A^k a^k \sum_{j=0}^{k} a^j x^{2j} k^{k-j}$$

and

$$\left|\frac{d^{2k+1}}{dx^{2k+1}}e^{-ax^2}\right| \le e^{-ax^2}A^k a^{k+1} \sum_{j=0}^{k} a^j x^{2j+1} k^{k-j}$$

for all $k \in \mathbb{N}_0$ and $a > 0$.

2. *If, in addition, $0 \le \ell \le d$, then there exist constants $C_d, A > 0$ so that*

$$\left\|x^\ell \frac{d^k}{dx^k}e^{-ax^2}\right\|_{L^p(\mathbb{R})} \le C_d A^k a^{\frac{k}{2}-\frac{1}{2p}-\frac{\ell}{2}} k^{\frac{k}{2}}.$$

Proof Part 1. of the corollary follows immediately from Proposition B.2. For the second piece, we estimate that when k is even,

$$\begin{aligned}\left\|x^\ell \frac{d^{2k}}{dx^{2k}}e^{-ax^2}\right\|_{L^p(\mathbb{R})} &\le CA^k a^k \sum_{j=0}^{k} a^j k^{k-j} \left\|x^{\ell+2j}e^{-ax^2}\right\|_{L^p(\mathbb{R})} \\ &= CA^k a^k \sum_{j=0}^{k} a^j k^{k-j} \left(\int_{\mathbb{R}} x^{p(\ell+2j)} e^{-apx^2}\,dx\right)^{1/p} \\ &= CA^k a^k \sum_{j=0}^{k} a^j k^{k-j} \left(\frac{\Gamma(\frac{1}{2}+jp+\frac{\ell p}{2})}{2(ap)^{\frac{1}{2}+jp+\frac{\ell p}{2}}}\right)^{1/p} \\ &= CA^k a^k \sum_{j=0}^{k} a^j k^{k-j} \frac{\Gamma(\frac{1}{2}+jp+\frac{\ell p}{2})^{1/p}}{2(ap)^{\frac{1}{2p}+j+\frac{\ell}{2}}}.\end{aligned}$$

By Stirling's Formula, there exist constants $C_0, A_0 > 0$ (which may grow from line to line) so that

$$\frac{\Gamma(\frac{1}{2}+jp+\frac{\ell p}{2})^{1/p}}{p^{\frac{1}{2p}+j+\frac{\ell}{2}}} \le C_0 \frac{(jp+\frac{\ell p}{2})^{\frac{1}{2p}+j+\frac{\ell}{2}}}{p^{\frac{1}{2p}+j+\frac{\ell}{2}}} \le C_0 A_0^j j^j \le C_0 A_0^j k^j.$$

Similarly,

$$\left\| x^\ell \frac{d^{2k+1}}{dx^{2k+1}} e^{-ax^2} \right\|_{L^p(\mathbb{R})} \le C A^k a^k \sum_{j=0}^{k} a^j k^{k-j} \left\| x^{\ell+2j} e^{-ax^2} \right\|_{L^p(\mathbb{R})}$$
$$= C A^k a^k \sum_{j=0}^{k} a^j k^{k-j} \frac{\Gamma(\frac{1}{2} + p(\frac{1}{2} + j + \frac{\ell p}{2}))^{1/p}}{2(ap)^{\frac{1}{2p} + (\frac{1}{2} + j + \frac{\ell}{2})}} \le C A_0^k k^k.$$

□

Corollary B.4 *Let* $0 \le \ell \le d$, $a > 0$ *and* $|y| \le 1$, *then there exist constants* C_d, $A > 0$ *so that*

$$\left\| (x+iy)^\ell \partial_x^k \{ e^{-a[x+iy]^2} \} \right\|_{L_x^p(\mathbb{R})} \le C_d \, e^{ay^2} A^k \sum_{\substack{k_1+k_2=k \\ 0 \le \ell \le d}} \binom{k}{k_1} a^{\frac{k_1}{2} + k_2 - \frac{1}{2p} - \frac{\ell}{2}} k_1^{\frac{k_1}{2}}. \tag{B.5}$$

Proof First we note that, since $\ell \le d$ and $|y| \le 1$ we have

$$\left|(x+iy)^\ell\right| = \left| \sum_{j=0}^{\ell} \binom{\ell}{j} x^j (iy)^{\ell-j} \right| \le C_d \sum_{\ell=0}^{d} |x|^\ell. \tag{B.6}$$

Also, using Leibniz rule, the derivative of the complex exponential can be written as

$$\partial_x^k \{ e^{-a[x+iy]^2} \} = \sum_{k_1+k_2=k} \binom{k}{k_1} \big(\partial_x^{k_1} \{ e^{-a(x^2-y^2)} \}\big)\big(\partial_x^{k_2} \{ e^{-2aixy} \}\big)$$
$$= e^{ay^2} \sum_{k_1+k_2=k} \binom{k}{k_1} \big(\partial_x^{k_1} \{ e^{-ax^2} \}\big)\big(-2aiy\big)^{k_2} e^{-2aixy}, \tag{B.7}$$

which, recalling that $|y| \le 1$, can be easily estimated by

$$\left| \partial_x^k \{ e^{-a[x+iy]^2} \} \right| \le e^{ay^2} \sum_{k_1+k_2=k} \binom{k}{k_1} \left| \partial_x^{k_1} \{ e^{-ax^2} \} \right| |2a|^{k_2}. \tag{B.8}$$

The proof now is a consequence of (B.6), (B.8) and Corollary B.3. □

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