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On certain degenerate Whittaker Models for cuspidal representations of $\mathrm{GL}_{k\cdot n}(\mathbb{F}_q)$

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Abstract Let π be an irreducible cuspidal representation of $\mathrm{GL}_{kn}(\mathbb{F}_q)$. Assume that $\pi = \pi_\theta$, corresponds to a regular character θ of $\mathbb{F}^*_{q^{kn}}$. We consider the twisted Jacquet module of π with respect to a non-degenerate character of the unipotent radical corresponding to the partition $(n, n, \ldots, n)$ of kn. We show that, as a $\mathrm{GL}_n(\mathbb{F}_q)$-representation, this Jacquet module is isomorphic to $\pi_\theta\restriction_{\mathbb{F}^*_n} \otimes \mathrm{St}^{\otimes(k-1)}$, where St is the Steinberg representation of $\mathrm{GL}_n(\mathbb{F}_q)$. This generalizes a theorem of D. Prasad, who considered the case $k = 2$. We prove and rely heavily on a formidable identity involving q-hypergeometric series and linear algebra.

1 Introduction

Let $\mathbb{F} := \mathbb{F}_q$ be the finite field of size q. We fix a nontrivial character ψ_0 of $\mathbb{F}$. Denote by $\mathbb{F}_m := \mathbb{F}_{q^m}$ the unique degree m field extension of $\mathbb{F}$. For a positive integer r, we denote the diagonal subgroup of $(\mathrm{GL}_\ell(\mathbb{F}))^r$ by

$$\Delta^r\left(\mathrm{GL}_\ell(\mathbb{F})\right) := \left\{ (g, \ldots, g) \in (\mathrm{GL}_\ell(\mathbb{F}))^r \;\middle|\; g \in \mathrm{GL}_\ell(\mathbb{F}) \right\}.$$

For a partition $\rho = (k_1, k_2, \ldots, k_s)$ of ℓ, denote by P_ρ the corresponding standard parabolic subgroup of $\mathrm{GL}_\ell(\mathbb{F})$. Let M_ρ and N_ρ be the corresponding standard Levi subgroup and unipotent radical.

Fix $k \geq 1$. Let $\rho = (n, n, \ldots, n)$ be the partition of kn consisting of k parts of size n. In this paper we denote $G := \mathrm{GL}_{kn}(\mathbb{F})$, $P := P_\rho$, $M := M_\rho$ and $N := N_\rho$. We have the Levi

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decomposition $P = M \ltimes N$. We write $U \in N$ in the form

$$U = \begin{pmatrix} I_n & X_{1,1} & X_{1,2} & \cdots & X_{1,k-2} & X_{1,k-1} \\ 0 & I_n & X_{2,2} & \cdots & X_{2,k-2} & X_{2,k-1} \\ 0 & 0 & I_n & \cdots & X_{3,k-2} & X_{3,k-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n & X_{k-1,k-1} \\ 0 & 0 & 0 & \cdots & 0 & I_n \end{pmatrix}, \tag{1.1}$$

where the matrices $X_{i,j}$ $(1 \leq i \leq j \leq k-1)$ are elements of $M_n(\mathbb{F})$.

Definition 1.1 A character $\psi : N \to \mathbb{C}^*$ is said to be non-degenerate if it is of the form

$$\psi(U) := \psi_0\left(\operatorname{tr}\left(\sum_{i=1}^{k-1} A_i X_{i,i}\right)\right) = \prod_{i=1}^{k-1} \psi_0\left(\operatorname{tr}\left(A_i X_{i,i}\right)\right),$$

where the matrices A_i are invertible.

Let $\psi : N \to \mathbb{C}^*$ be a non-degenerate character. Let π be an irreducible representation of G, acting on a space V_π. We denote by $V_{\pi_{k,N,\psi}}$ the largest subspace of V_π, on which N operates through ψ, i.e.

$$V_{\pi_{k,N,\psi}} = \{v \in V_\pi \mid \pi(U)v = \psi(U)v, \ \forall U \in N\}.$$

This is the (N, ψ)-isotypic subspace of V_π and it is the image of the canonical projection of V_π on $V_{\pi_{k,N,\psi}}$ given by

$$P_{k,N,\psi}(v) = \frac{1}{|N|} \sum_{U \in N} \overline{\psi}(U)\pi(U)v. \tag{1.2}$$

Since M normalizes N, it acts on the characters of N as follows. If $m \in M$, then for all $U \in N$

$$(m \cdot \psi)(U) = \psi\left(m^{-1}Um\right).$$

We have, for $m \in M$,

$$\pi(m)V_{\pi_{k,N,\psi}} = V_{\pi_{k,N,m\cdot\psi}}.$$

Let us compute the stabilizer of ψ in M. If

$$m = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix},$$

where $B_i \in \mathrm{GL}_n(\mathbb{F})$ for all $1 \leq i \leq k$, then

$$(m \cdot \psi)(U) = \psi_0\left(\operatorname{tr}\left(\sum_{i=1}^{k-1} A_i B_i^{-1} X_{i,i} B_{i+1}\right)\right).$$

Thus, $m \cdot \psi = \psi$ if and only if $B_i = B_{i+1}$ for all $1 \leq i \leq k-1$. In other words,

$$\operatorname{stab}_M \psi = \Delta^k\left(\mathrm{GL}_n(\mathbb{F})\right) \cong \mathrm{GL}_n(\mathbb{F}).$$

Therefore, $V_{\pi_{k,N,\psi}}$ is a $\mathrm{GL}_n(\mathbb{F})$-module. We denote by $\pi_{k,N,\psi}$ the resulting representation of $\mathrm{GL}_n(\mathbb{F})$ on $V_{\pi_{k,N,\psi}}$. It is easy to see that by conjugation with an element in the standard

Levi subgroup, we may simply take all the A_i to be the identity matrix. The corresponding twisted Jacquet modules are isomorphic. In the rest of the paper we assume $A_i = I_n$ and fix

$$\psi(U) := \psi_0\left(\operatorname{tr}\left(\sum_{i=1}^{k-1} X_{i,i}\right)\right).$$

The goal of this paper is to calculate the character of $\pi_{k,N,\psi}$, and to describe it in terms of more familiar representations, for an irreducible, cuspidal representation $\pi = \pi_\theta$ of $\mathrm{GL}_{kn}(\mathbb{F})$, associated to a regular character θ of $\mathbb{F}_{kn}^*$. The paper generalizes Prasad's result for the case $k = 2$ stated below.

Theorem [11, Thm. 1] *Let π be an irreducible cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{F})$ obtained from a character θ of $\mathbb{F}_{2n}^*$. Then*

$$\pi_{2,N,\psi} \cong \operatorname{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_n^*} . \tag{1.3}$$

Prasad proved this theorem by an explicit calculation of the characters of $\pi_{2,N,\psi}$ and of the induced representation $\operatorname{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_n^*}$. At any element of $\mathrm{GL}_n(\mathbb{F})$ the characters are the same. Therefore, the two representations are equivalent.

The methods used in this paper are generalizations of the methods used by the second author in his thesis [7] for the case $k = 3$. From the character calculation, done in Theorem 3 below, we are able to describe in Theorem 4 $\pi_{k,N,\psi}$ in terms of the representations $\operatorname{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_\ell^*}$, where $\ell \mid n$. This reduces immediately to Prasad's result when $k = 2$. Furthermore, we give a compact description of $\pi_{k,N,\psi}$ in terms of the Steinberg representation in the following theorem.

Theorem 1 *Let $k \geq 1$. Let π_θ be an irreducible cuspidal representation of $\mathrm{GL}_{kn}(\mathbb{F})$ obtained from a character θ of $\mathbb{F}_{kn}^*$. Then*

$$\pi_{k,N,\psi} \cong \pi_{\theta\restriction_{\mathbb{F}_n^*}} \otimes \mathrm{St}^{\otimes(k-1)},$$

where $\pi_{\theta\restriction_{\mathbb{F}_n^*}}$ is the irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ obtained from $\theta \restriction_{\mathbb{F}_n^*}$, and $\mathrm{St}^{\otimes(k-1)}$ is the $(k-1)$-fold tensor product of the Steinberg representation of $\mathrm{GL}_n(\mathbb{F})$ with itself.

Note that for $n = 1$, Theorem 1 gives $\pi_{k,N,\psi} \cong \theta \restriction_{\mathbb{F}^*}$, which also follows from Gel'fand–Graev [4] in case of $\mathrm{GL}_k(\mathbb{F})$ (cf. [12, Ch. 8.1]).

We are currently investigating an analogous construction for a non-Archimedean local field.

1.1 Structure of the paper

In Sect. 2 we set the background material from several topics that are needed in the paper: linear algebra, representation theory, q-hypergeometric identities and arithmetic identities.

In Sect. 3 we calculate the dimension of $\pi_{k,N,\psi}$. Green's formula allows us to express the dimension as rather complicated sum. We use q-hypergeometric identities and linear algebra to show that this sum admits the following compact form.

Theorem 2 *Let $k \geq 2$. We have*

$$\dim\left(\pi_{k,N,\psi}\right) = q^{(k-2)\frac{n(n-1)}{2}} \frac{|\mathrm{GL}_n(\mathbb{F})|}{q^n - 1}.$$

In Sect. 4 we compute the character of $\pi_{k,N,\psi}$, denoted by $\Theta_{k,N,\psi}$. Apart from the tools used in Theorem 2 this requires understanding of some conjugacy classes of $\mathrm{GL}_n(\mathbb{F})$. When $d \mid m$, we have an embedding $\mathbb{F}_d^* \hookrightarrow \mathrm{GL}_m(\mathbb{F})$ (see Sect. 2.1). The elements in $\mathrm{GL}_m(\mathbb{F})$ conjugate to an element in the image of this embedding are said to come from $\mathbb{F}_d$.

Theorem 3 *Let $k \geq 2$. Let $g = s \cdot u$ be the Jordan decomposition of an element g in $\mathrm{GL}_n(\mathbb{F})$, where s and u are the semisimple part and unipotent part, respectively.*

(I) If s does not come from $\mathbb{F}_n$, then

$$\Theta_{k,N,\psi}(g) = 0.$$

(II) If the $u \neq I_n$, then

$$\Theta_{k,N,\psi}(g) = 0.$$

(III) Assume that $u = I_n$ and s comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. Let λ be an eigenvalue of s which generates $\mathbb{F}_d$ over $\mathbb{F}$. Then,

$$\Theta_{k,N,\psi}(s) = (-1)^{k(n-d')} q^{(k-2)\frac{n(d'-1)}{2}} \cdot \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i})\right] \cdot \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1},$$

where $d' = n/d$.

In Sect. 5 we obtain from Theorem 3 and Lemma 2.10 an isomorphism of representation relating between $\pi_{k,N,\psi}$ and $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_\ell^*}$ for all $\ell \mid n$. We write $a|b|c$ for $a|b$ and $b|c$. For any ℓ dividing n and any $k \geq 2$, let

$$a_{k;n,\ell}(q) = \frac{q^\ell - 1}{q^n - 1} \sum_{m:\, \ell|m|n} \mu\left(\frac{m}{\ell}\right) (-1)^{k(n-\frac{n}{m})} q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}, \tag{1.4}$$

where μ is the Möbius function.

Theorem 4 *Let $k \geq 2$.*

(I) If k is even or n is odd, we have

$$\pi_{k,N,\psi} \cong \bigoplus\nolimits_{\ell|n} a_{k;n,\ell}(q) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_\ell^*}. \tag{1.5}$$

(II) If k is odd and n is even, we have

$$\left(\pi_{k,N,\psi} \oplus \bigoplus_{\ell:\, \ell|n, 2\nmid\frac{n}{\ell}} (-a_{k;n,\ell}(q)) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_\ell^*}\right) \cong \bigoplus_{\ell:\, \ell|n, 2|\frac{n}{\ell}} a_{k;n,\ell}(q) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \restriction_{\mathbb{F}_\ell^*}. \tag{1.6}$$

We note that the coefficients in Theorem 4 are non-negative integers. Indeed, when $k = 2$, it is easily shown (see Lemma 2.10) that $a_{2;n,\ell}(q) = \delta_{\ell,n}$, which gives (1.3). If $k > 2$ we show in Lemma 2.10 that $a_{k;n,\ell}(q)$ is a positive integer, except when k is odd, n is even and $2 \nmid \frac{n}{\ell}$, in which case $-a_{k;n,\ell}(q)$ is a positive integer.

In Sect. 6 we deduce Theorem 1 from Theorem 3.

2 Preliminaries

2.1 Cuspidal representations

We review the irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{F})$ as in Gel'fand [3, Sect. 6] (originally in Green [5]). Irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{F})$, from which all the other irreducible representations of $\mathrm{GL}_m(\mathbb{F})$ are obtained via the process of parabolic induction, are associated to regular characters of $\mathbb{F}_m^*$. A multiplicative character θ of $\mathbb{F}_m^*$ is called *regular* if, under the action of the Galois group of $\mathbb{F}_m$ over $\mathbb{F}$, the orbit of θ consists of m distinct characters of $\mathbb{F}_m^*$.

We denote the irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$ by π_θ and the character of the representation π_θ by Θ_θ.

Given $a \in \mathbb{F}_m$, consider the map $m_a : \mathbb{F}_m \to \mathbb{F}_m$, defined by $m_a(x) = ax$. The map $a \mapsto m_a$ is an injective homomorphism of algebras $\mathbb{F}_m \hookrightarrow \mathrm{End}_{\mathbb{F}}(\mathbb{F}_m)$. This way, every element of $\mathbb{F}_m^*$ gives rise to a well-defined conjugacy class in $\mathrm{GL}_m(\mathbb{F})$. The elements in the conjugacy classes in $\mathrm{GL}_m(\mathbb{F})$, which are so obtained from elements of $\mathbb{F}_m^*$, are said to come from $\mathbb{F}_m^*$.

We summarize the information about the character Θ_θ in the following theorem. We refer to the paper [11, Thm. 2] for the statement of this theorem in this explicit form, which is originally due to Green [5, Thm. 14] (cf. [3,14]).

Theorem 2.1 (Green [5]) *Let Θ_θ be the character of a cuspidal representation π_θ of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$. Let $g = s \cdot u$ be the Jordan decomposition of an element g in $\mathrm{GL}_m(\mathbb{F})$ (s is a semisimple element, u is unipotent and s, u commute). If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $\mathbb{F}_m^*$. Suppose that s comes from $\mathbb{F}_m^*$. Let λ be an eigenvalue of s in $\mathbb{F}_m^*$, and let $t = \dim_{\mathbb{F}_m} \ker(g - \lambda I)$. Then*

$$\Theta_\theta(s \cdot u) = (-1)^{m-1} \left[\sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{t-1}) \tag{2.1}$$

where q^d is the cardinality of the field generated by λ over $\mathbb{F}$, and the summation is over the various distinct Galois conjugates of λ.

Corollary 2.2 *The value $\Theta_\theta(g)$ is determined by the eigenvalue of g and the number of Jordan blocks of g, which, in turn, is determined by $\dim_{\mathbb{F}_m} \ker(g - \lambda I)$.*

2.2 Characters induced from subfields

The following lemma summarizes the information about the character of $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta \restriction_{\mathbb{F}_\ell^*})$, where $\ell \mid n$ and θ is a character of $\mathbb{F}_n^*$.

Lemma 2.3 [7, Lem. 2.4] *Let θ be a character of $\mathbb{F}_n^*$. Suppose that $s \in \mathrm{GL}_n(\mathbb{F})$ comes from $\mathbb{F}_d \subseteq \mathbb{F}_\ell$ ($d \mid \ell$ is minimal). Let λ be an eigenvalue of s in $\mathbb{F}_d^*$. Then, the character $\Theta_{\mathrm{Ind}_\ell}$ of $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta \restriction_{\mathbb{F}_\ell^*})$ at s is given by*

$$\Theta_{\mathrm{Ind}_\ell}(s) = \frac{1}{q^\ell - 1} \sum_{\substack{g \in \mathrm{GL}_n(\mathbb{F}) \\ g^{-1}sg \in \mathbb{F}_\ell^*}} \theta(g^{-1}sg) \tag{2.2}$$

$$= \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^\ell - 1} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right], \tag{2.3}$$

where $d' = n/d$, and the last sum is over the various distinct Galois conjugates of λ. The value of the character $\Theta_{\mathrm{Ind}_\ell}$ at an element of $\mathrm{GL}_n(\mathbb{F})$ which does not come from $\mathbb{F}_\ell$ is zero.

Remark 2.4 Recall that in (2.2) $\mathbb{F}_\ell^*$ is considered a subgroup of $\mathrm{GL}_n(\mathbb{F})$ by the injective map $a \mapsto [m_a]$, where $[m_a]$ is the representing matrix of m_a with respect to a fixed basis of $\mathbb{F}_n$ over $\mathbb{F}$. Note that the choice of basis for $[m_a]$ does not affect the values of $\Theta_{\mathrm{Ind}_\ell}$.

2.3 On some conjugacy classes of $\mathrm{GL}_n(\mathbb{F})$

2.3.1 Analogue of Jordan form

Let $g \in \mathrm{GL}_n(\mathbb{F})$ and $g = s \cdot u$ be its Jordan decomposition. Assume that s comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ ($d \mid n$ is minimal). Let $\lambda \in \mathbb{F}_d^*$ be an eigenvalue of s, which generates the field $\mathbb{F}_d$ over $\mathbb{F}$. Denote by f the characteristic polynomial of λ (of degree d), and by $L_f \in \mathrm{GL}_d(\mathbb{F})$ the companion matrix of f. For $\ell \geq 1$ we denote

$$L_{f,\ell} = \begin{pmatrix} L_f & I_d & & \\ & L_f & & \\ & & \ddots & I_d \\ & & & L_f \end{pmatrix} \in \mathrm{GL}_{\ell \cdot d}(\mathbb{F}).$$

This is an analogue of a Jordan block. As in [3,5], there exists $\rho = (\ell_1, \ldots, \ell_r)$, a partition of $\frac{n}{d}$, $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r$, such that g is conjugate to

$$L_\rho(f) := \begin{pmatrix} L_{f,\ell_1} & & & \\ & L_{f,\ell_2} & & \\ & & \ddots & \\ & & & L_{f,\ell_r} \end{pmatrix},$$

i.e. there exists $R \in \mathrm{GL}_n(\mathbb{F})$ such that

$$R^{-1} g R = L_\rho(f). \tag{2.4}$$

Notice that in case $u = I_n$ (g is semisimple), we have $\rho = (1^{n/d})$ and there exists $R \in \mathrm{GL}_n(\mathbb{F})$ such that $R^{-1}gR$ is a block diagonal matrix with $d' = n/d$ times L_f on the diagonal. Otherwise, $\ell_1 > 1$ and, in particular, there exists $R \in \mathrm{GL}_n(\mathbb{F})$ such that the upper $2d \times 2d$ left corner of $R^{-1}gR$ is

$$\begin{pmatrix} L_f & I_d \\ & L_f \end{pmatrix}.$$

Now, s (and so g) has d different eigenvalues obtained by applying the Frobenius automorphism σ, which generates the Galois group $\mathrm{Gal}(\mathbb{F}_d/\mathbb{F})$, namely

$$\left\{\lambda, \sigma(\lambda), \ldots, \sigma^{d-1}(\lambda)\right\} = \left\{\lambda, \lambda^q, \ldots, \lambda^{q^{d-1}}\right\},$$

all of multiplicity $d' = n/d$ in the characteristic polynomial of s. Let $0 \neq v_0 \in \mathbb{F}_d^d$ satisfy $L_f \cdot v_0 = \lambda v_0$. So $L_f \cdot \sigma^i(v_0) = \lambda^{q^i}\sigma^i(v_0)$, for $0 \leq i \leq d-1$. Hence, $B = \left\{v_0, \sigma(v_0), \ldots, \sigma^{d-1}(v_0)\right\} \subseteq \mathbb{F}_d^d$ is linearly independent over $\mathbb{F}_d$, since its elements are eigenvectors of L_f for different eigenvalues. Let $T \in \mathrm{GL}_d(\mathbb{F}_d)$ be the diagonalizing matrix of L_f obtained by B, i.e.

$$T^{-1}L_f T = D, \tag{2.5}$$

where

$$D := \mathrm{diag}\left(\lambda, \ldots, \lambda^{q^{d-1}}\right).$$

Denote by $\Delta^{d'}(T)$ the block diagonal matrix with d' times T on the diagonal. Explicitly, the columns of $\Delta^{d'}(T)$ are the vectors of the basis

$$C = \{v_0(i,j)\}_{0\leq i\leq d-1}^{0\leq j\leq d'-1}, \tag{2.6}$$

whose $(j\cdot d+i)$-th vector is given by

$$v_0(i,j) = \begin{pmatrix} \underline{0}_{j\cdot d} \\ \sigma^i(v_0) \\ \underline{0}_{n-(j+1)\cdot d}\end{pmatrix} \in \mathbb{F}_d^n,$$

where $0 \leq i \leq d-1$ and $0 \leq j \leq d'-1$. Thus, in case $u = I_n$

$$\Delta^{d'}\left(T^{-1}\right)R^{-1}gR\Delta^{d'}(T) = \begin{pmatrix} D & & \\ & \ddots & \\ & & D\end{pmatrix}.$$

Otherwise

$$\Delta^{d'}\left(T^{-1}\right)R^{-1}gR\Delta^{d'}(T) = \begin{pmatrix} D & I_d & & & \\ & D & & & \\ & & D & * & \\ & & & \ddots & * \\ & & & & D\end{pmatrix},$$

where $*$ means either I_d or 0_d above the diagonal. We denote

$$g_\rho := g_{\rho,R} = \Delta^{d'}\left(T^{-1}\right)R^{-1}gR\Delta^{d'}(T). \tag{2.7}$$

The matrix g_ρ is sometimes referred to as an analogue of the Jordan form of g [3, Sect. 0].

2.3.2 Conjugating an arbitrary matrix

We use the notation of Sect. 2.3.1. In particular, we have a fixed $g \in \mathrm{GL}_n(\mathbb{F})$ and corresponding R and T as defined in (2.4) and (2.5). Let $A \in M_n(\mathbb{F})$. We study the following conjugation

$$A_\rho := A_{\rho,R} = \Delta^{d'}\left(T^{-1}\right)R^{-1}AR\Delta^{d'}(T) \in M_n(\mathbb{F}_d).$$

Define A_R by $A_R = R^{-1}AR$, and so $A_\rho = \Delta^{d'}\left(T^{-1}\right)A_R\Delta^{d'}(T)$.

Let $B \in M_n(\mathbb{F}_d)$. Let us represent the vectors $B\cdot v_0(0,m)$, for any $0 \leq m \leq d'-1$, as a linear combination of the basis C given in (2.6):

$$B\cdot v_0(0,m) = \sum_{\substack{0\leq i\leq d-1\\0\leq j\leq d'-1}} a_{m,i;j}\cdot v_0(i,j), \qquad a_{m,i;j} \in \mathbb{F}_d.$$

A necessary and sufficient condition for $B \in M_n(\mathbb{F})$ is that for all $0 \leq m \leq d'-1,\ 0 \leq r \leq d-1$,

$$B \cdot v_0(r, m) = \sum_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \sigma^r(a_{m,i;j}) \cdot v_0(i + r \pmod d, j). \tag{2.8}$$

By taking $B = A_R \in M_n(\mathbb{F})$, we get that (2.8) holds for A_R. Therefore, $[A_R]_C = A_\rho$ is a $d' \times d'$ matrix with entries from $M_d(\mathbb{F}_d)$. For $0 \leq m, j \leq d'-1$, the m-th row and j-th column of A_ρ, denoted by $A_{m,j}$, is given by

$$A_{m,j} = \left(\sigma^r\left(a_{m,i-r \pmod d;j}\right)\right)_{0 \leq i,r \leq d-1}, \tag{2.9}$$

i.e. $A_{m,j} \in M_d(\mathbb{F}_d)$ and for $0 \leq i, r \leq d-1$, the i-th row and r-th column of $A_{m,j}$ is $\sigma^r\left(a_{m,i-r \pmod d;j}\right)$. The above discussion can be summarized in the following lemma.

Lemma 2.5 *In the above notations, the map $A \mapsto A_\rho$ induces an $\mathbb{F}$-linear isomorphism $M_n(\mathbb{F}) \to M_{n\times d'}(\mathbb{F}_d) \cong \left[M_{d\times d'}(\mathbb{F}_d)\right]^{d'}$. It is given by*

$$A \mapsto \begin{pmatrix} \left(a_{0,i;j}\right)_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \\ \vdots \\ \left(a_{d'-1,i;j}\right)_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \end{pmatrix},$$

where the $(m \cdot d + i)$-th row and j-th column of the image of A is $a_{m,i;j} \in \mathbb{F}_d$, for $0 \leq m, j \leq d'-1$ and $0 \leq i \leq d-1$.

2.3.3 Trace under conjugation

For $g \in \mathrm{GL}_n(\mathbb{F})$ and $A \in M_n(\mathbb{F})$ we shall be interested in $\mathrm{tr}\left(g^{-1}A\right)$. We use the notation of Sects. 2.3.1 and 2.3.2. By (2.7), we have

$$\mathrm{tr}\left(g^{-1}A\right) = \mathrm{tr}\left(g_\rho^{-1}A_\rho\right).$$

The inverse of an analogue of a Jordan block of order $d \cdot \ell$ is given by

$$\left(\begin{pmatrix} D & I_d & & \\ & \ddots & I_d \\ & & D \end{pmatrix}^{-1}\right)_{i,j} = \begin{cases} (-1)^{j-i} D^{-j+i-1}, & i \leq j \\ 0, & i > j, \end{cases} \tag{2.10}$$

for $0 \leq i, j \leq \ell$, where the LHS of (2.10) denotes the block matrix in the i-th row and j-th column. We have

$$\begin{aligned} \mathrm{tr}\left(g_\rho^{-1}A_\rho\right) &= \textstyle\sum_{m=0}^{d'-1} \mathrm{tr}\left(D^{-1}A_{m,m} + D^{-2}\alpha_m\left(g, D^{-1}, A_\rho\right)\right) \\ &= \mathrm{tr}\left(\textstyle\sum_{m=0}^{d'-1} D^{-1}A_{m,m}\right) + \textstyle\sum_{m=0}^{d'-1} \mathrm{tr}\left(D^{-2}\alpha_m\left(g, D^{-1}, A_\rho\right)\right), \end{aligned} \tag{2.11}$$

where $\alpha_m\left(g, D^{-1}, A_\rho\right)$, for $0 \leq m \leq d'-1$ are determined by the analogous Jordan form of g. Notice, that in case g is semisimple, then $\alpha_m\left(g, D^{-1}, A_\rho\right) = 0$ for all $0 \leq m \leq d'-1$. Otherwise, for $0 \leq m \leq d'-1$, $D^{-2}\alpha_m\left(g, D^{-1}, A_\rho\right)$ equals to a sum of terms of the form $(-1)^\ell D^{-\ell-1}A_{\ell,m}$, where $m < \ell \leq d'-1$.

By (2.9) we have

$$D^{-1}A_{m,m} = \left(\left(\lambda^{-1}\right)^{q^r} \sigma^r\left(a_{m,i-r \pmod d;m}\right)\right)_{1 \leq i,r \leq d-1}.$$

So the first sum in the RHS of (2.11) becomes

$$\sum_{m=0}^{d'-1}\sum_{r=0}^{d-1}\left(\lambda^{-1}\right)^{q^r}\sigma^r\left(a_{m,0;m}\right)=\sum_{r=0}^{d-1}\sigma^r\left(\lambda^{-1}\sum_{m=0}^{d'-1}a_{m,0;m}\right)=\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}\left(\lambda^{-1}\cdot\sum_{m=0}^{d'-1}a_{m,0;m}\right).$$

On the other hand, for each $0\le m\le d'-1$, the term $\mathrm{tr}\left(D^{-2}\alpha_m\left(g,D^{-1},A_\rho\right)\right)$ in (2.11) does not depend on the elements $a_{\ell,0;m}$, where $\ell=m$. Each such term depends only on λ and on $a_{\ell,i,m}$ where $\ell>m$. We summarize the above results in the following lemma.

Lemma 2.6 *In the above notations,*

$$\mathrm{tr}\left(g^{-1}A\right)=\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}\left(\lambda^{-1}\cdot\sum_{m=0}^{d'-1}a_{m,0;m}\right)+\sum_{m=0}^{d'-1}\mathrm{tr}\left(D^{-2}\alpha_m\left(g,D^{-1},A_\rho\right)\right),$$

and each summand $\mathrm{tr}\left(D^{-2}\alpha_m\left(g,D^{-1},A_\rho\right)\right)$ *is independent of* $a_{m,0;m}$ *appearing in the first summand, for all* $0\le m\le d'-1$.

In case $g=s$ is semisimple we have

$$\mathrm{tr}\left(g^{-1}A\right)=\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}\left(\lambda^{-1}\cdot\sum_{m=0}^{d'-1}a_{m,0;m}\right).$$

2.4 q-Hypergeometric identity

In order to calculate the dimension of $\pi_{k,N,\psi}$, we need a combinatorial identity related to ranks of triangular block matrices. We first prove a lemma that is a special case of a q-analogue of the Chu–Vandermonde identity, phrased in a manner that we use in the proof of the combinatorial identity. We recall the definition of the q-Pochhammer symbol:

$$(a;q)_n=\prod_{i=0}^{n-1}(1-aq^i).$$

Lemma 2.7 *Let $R_q(n,m,r)$ be the number of $n\times m$ matrices of rank r over the finite field of size q (n, m may be* 0*, with the convention that the empty matrix has rank* 0*). Let a be an integer greater or equal to $n+m$. Then*

$$\sum_{r\ge 0}R_q(n,m,r)(q;q)_{a-r}=q^{nm}\frac{(q;q)_{a-n}(q;q)_{a-m}}{(q;q)_{a-n-m}}.$$

Proof We start by stating a q-analogue of the Chu–Vandermonde identity [2, Eq. (1.5.2)]:

$$\sum_{r=0}^{i}\frac{(q^{-i};q)_r(b;q)_r}{(c;q)_r(q;q)_r}\left(\frac{cq^i}{b}\right)^r=\frac{(c/b;q)_i}{(c;q)_i},$$

where i is a non-negative integer, and b,c are complex numbers that satisfy $b\ne 0$ and $c\notin\{q^{-1},\dots,q^{-(i-1)}\}$. Choosing $i=n$, $b=q^{-m}$, $c=q^{-a}$, we obtain

$$\sum_{r=0}^{n}\frac{(q^{-n};q)_r(q^{-m};q)_r}{(q^{-a};q)_r(q;q)_r}q^{(n+m-a)r}=\frac{(q^{m-a};q)_n}{(q^{-a};q)_n}. \tag{2.12}$$

We have the following formula for $R_q(n, m, r)$ by Landsberg [9]:

$$R_q(n, m, r) = \frac{(-1)^r (q^{-n}; q)_r (q^{-m}; q)_r q^{(n+m)r - \binom{r}{2}}}{(q; q)_r}.$$

By expressing the r-th summand of (2.12) as

$$\frac{(-1)^r (q^{-n};q)_r (q^{-m};q)_r q^{(n+m)r-\binom{r}{2}}}{(q;q)_r} \cdot \frac{q^{-ar+\binom{r}{2}}}{(-1)^r (q^{-a};q)_r} = R_q(n, m, r) \cdot \frac{q^{-ar+\binom{r}{2}}}{(-1)^r (q^{-a};q)_r},$$

we obtain that

$$\sum_{r=0}^{n} R_q(n, m, r) \frac{q^{-ar+\binom{r}{2}} (-1)^r}{(q^{-a}; q)_r} = \frac{(q^{m-a}; q)_n}{(q^{-a}; q)_n}. \tag{2.13}$$

The proof is concluded by applying to (2.13) the simple identity

$$(q^{-x}; q)_y = (-1)^y q^{\binom{y}{2} - xy} \frac{(q; q)_x}{(q; q)_{x-y}}$$

with $(x, y) \in \{(a, n), (a - m, n), (a, r)\}$. □

We now state our main combinatorial identity needed for computing the dimension. Let k be a positive integer. We define the following family of functions.

$$f_{k,q}\left(a; {}^{n_1,\dots,n_k}_{m_1,\dots,m_k}\right) = \sum_{A} (q; q)_{a - \mathrm{rk} A}\,, \tag{2.14}$$

where $\{n_i\}_{i=1}^k$, $\{m_j\}_{j=1}^k$ are sequences of non-negative integers, a is an integer such that

$$a \geq \max\left\{ \sum_{j=1}^{i} n_j + \sum_{j=i}^{k} m_j \mid 1 \leq i \leq k \right\} \tag{2.15}$$

and the sum is over all matrices $A \in M_{(\sum_{i=1}^k n_i) \times (\sum_{j=1}^k m_j)}(\mathbb{F})$ of the form

$$A = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,k} \\ 0 & Y_{2,2} & \cdots & Y_{2,k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Y_{k,k} \end{pmatrix}, \tag{2.16}$$

where $Y_{i,j} \in M_{n_i \times m_j}(\mathbb{F})$ for all $1 \leq i \leq j \leq k$.

Proposition 2.8 *Let $k \geq 1$. For any sequences of non-negative integers, $\{n_i\}_{i=1}^k$ and $\{m_j\}_{j=1}^k$, and for any integer a satisfying (2.15), we have*

$$f_{k,q}\left(a; {}^{n_1,\dots,n_k}_{m_1,\dots,m_k}\right) = q^{\sum_{1\leq i\leq j\leq k} n_i m_j} \cdot \frac{\prod_{i=0}^{k} (q; q)_{a - \sum_{j=1}^{k-i} n_j - \sum_{j=k-i+1}^{k} m_j}}{\prod_{i=1}^{k} (q; q)_{a - \sum_{j=1}^{k-i+1} n_j - \sum_{j=k-i+1}^{k} m_j}}. \tag{2.17}$$

Proof We use the following notation:

$$I_{r,n,m} = \begin{pmatrix} I_r & 0_{m-r} \\ 0_{n-r} & 0 \end{pmatrix}, \quad (r \leq \min\{n, m\}). \tag{2.18}$$

We prove the proposition by induction on k. Let $k = 1$. Then

$$f_{1,q}\left(a; {}^{n}_{m}\right) = \sum_{A \in M_{n\times m}(\mathbb{F})} (q;q)_{a-\mathrm{rk}A} = \sum_{r\geq 0} R_q(n,m,r)\,(q;q)_{a-r}\,.$$

By Lemma 2.7 we find that

$$f_1\left(a; {}^{n}_{m}\right) = q^{nm}\frac{(q;q)_{a-n}(q;q)_{a-m}}{(q;q)_{a-n-m}},$$

as needed. We now perform the induction step, i.e. assume that (2.17) holds for $k-1$ in place of k, and prove it for k. We split the sum defining $f_{k,q}\left(a; {}^{n_1,\ldots,n_k}_{m_1,\ldots,m_k}\right)$ as follows:

$$f_{k,q}\left(a; {}^{n_1,\ldots,n_k}_{m_1,\ldots,m_k}\right) = \sum_{\substack{Y_{i,i}\in M_{n_i\times m_i}(\mathbb{F})\\ 1\leq i\leq k}} \sum_{\substack{Y_{i,j}\in M_{n_i\times m_j}(\mathbb{F})\\ 1\leq i<j\leq k}} (q;q)_{a-\mathrm{rk}A}\,. \tag{2.19}$$

In the inner sum of (2.19) the ranks of $Y_{i,i}$ are fixed for all $1 \leq i \leq k$, so we set $r_i = \mathrm{rk}(Y_{i,i})$. There exist invertible matrices E_i, C_i such that $Y_{i,i} = E_i I_{r_i,n_i,m_i} C_i$, for all $1 \leq i \leq k$. So, one can write A in the inner sum of (2.19) as $\mathrm{diag}\,(E_1,\ldots,E_k)\cdot\widetilde{A}\cdot\mathrm{diag}\,(C_1,\ldots,C_k)$, where

$$\widetilde{A} = \begin{pmatrix} I_{r_1,n_1,m_1} & \widetilde{Y}_{1,2} & \cdots & \widetilde{Y}_{1,k} \\ 0 & I_{r_2,n_2,m_2} & \cdots & \widetilde{Y}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{r_k,n_k,m_k} \end{pmatrix} \tag{2.20}$$

and $\widetilde{Y}_{i,j} = E_i^{-1} Y_{i,j} C_j^{-1}$ for all $1 \leq i < j \leq k$. Together with the fact that rank is invariant under elementary operations, (2.19) becomes

$$f_{k,q}\left(a; {}^{n_1,\ldots,n_k}_{m_1,\ldots,m_k}\right) = \sum_{\substack{\forall 1\leq i\leq k:\\ r_i\geq 0}} \prod_{i=1}^{k} R_q(n_i,m_i,r_i) \sum_{\widetilde{A}} (q;q)_{a-\mathrm{rk}\widetilde{A}}\,, \tag{2.21}$$

where the inner sum is over matrices $\widetilde{A}$ of the form (2.20). We can use Gaussian elimination operations on $\widetilde{Y}_{i,j}$ for all $1 \leq i < j \leq k$ (which do not affect the rank of $\widetilde{A}$) as follows: the first r_i rows of each $\widetilde{Y}_{i,j}$ are being canceled by the pivot elements in $I_{r_i,n}$ (using elementary row operations) and the first r_j columns of each $\widetilde{Y}_{i,j}$ are being canceled by the pivot elements in $I_{r_j,n}$ (using elementary column operations). Formally, the composition of these elementary operations maps the sequence of matrices $\{\widetilde{Y}_{i,j}\}_{1\leq i<j\leq k}$ $\mathbb{F}$-linearly to a sequence of matrices

$$\left\{\widehat{\widetilde{Y}}_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & Z_{i,j} \end{pmatrix}\right\}_{1\leq i<j\leq k}, \tag{2.22}$$

where $Z_{i,j} \in M_{(n_i-r_i)\times(m_j-r_j)}(\mathbb{F})$. This linear map is a projection by construction. Its kernel is of size $q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^{k} m_\ell + \sum_{t=2}^{k} r_t \sum_{\ell=1}^{t-1}(n_\ell - r_\ell)}$. The dimension of the kernel corresponds to the number of elements which we canceled. Equation (2.21) becomes

$$f_{k,q}\left(a; \begin{smallmatrix} n_1,\ldots,n_k \\ m_1,\ldots,m_k \end{smallmatrix}\right) = \sum_{\substack{\forall 1\le i\le k:\\ r_i\ge 0}} \prod_{i=1}^{k} R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^{k} m_\ell + \sum_{t=2}^{k} r_t \sum_{\ell=1}^{t-1}(n_\ell - r_\ell)} \cdot \sum_{\widehat{\widehat{A}}} (q;q)_{a-\mathrm{rk}\widehat{\widehat{A}}}, \tag{2.23}$$

where the inner sum is over matrices of the form

$$\widehat{\widehat{A}} = \begin{pmatrix} I_{r_1,n_1,m_1} & \widehat{\widehat{Y}}_{1,2} & \cdots & \widehat{\widehat{Y}}_{1,k} \\ 0 & I_{r_2,n_2,m_2} & \cdots & \widehat{\widehat{Y}}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{r_k,n_k,m_k} \end{pmatrix},$$

and $\widehat{\widehat{Y}}_{i,j}$ are as defined in (2.22). Note that $\mathrm{rk}\widehat{\widehat{A}} = \sum_{j=1}^{k} r_j + \mathrm{rk} Z$, where

$$Z = \begin{pmatrix} Z_{1,2} & \cdots & Z_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z_{k-1,k} \end{pmatrix}.$$

Hence, from (2.23) we obtain the following recursive relation:

$$f_{k,q}\left(a; \begin{smallmatrix} n_1,\ldots,n_k \\ m_1,\ldots,m_k \end{smallmatrix}\right) = \sum_{\substack{\forall 1\le i\le k:\\ r_i\ge 0}} \prod_{i=1}^{k} R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^{k} m_\ell + \sum_{t=2}^{k} r_t \sum_{\ell=1}^{t-1}(n_\ell - r_\ell)} \cdot f_{k-1,q}\left(a - \sum_{j=1}^{k} r_j; \begin{smallmatrix} n_1-r_1,\ldots,n_{k-1}-r_{k-1} \\ m_2-r_2,\ldots,m_k-r_k \end{smallmatrix}\right). \tag{2.24}$$

Plugging the induction assumption in (2.24) we get that $f_{k,q}\left(a; \begin{smallmatrix} n_1,\ldots,n_k \\ m_1,\ldots,m_k \end{smallmatrix}\right)$ equals

$$\sum_{\substack{\forall 1\le i\le k:\\ r_i\ge 0}} \prod_{i=1}^{k} R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^{k} m_\ell + \sum_{t=2}^{k} r_t \sum_{\ell=1}^{t-1}(n_\ell - r_\ell)} \cdot q^{\sum_{1\le i\le j\le k-1}(n_i-r_i)\cdot(m_{j+1}-r_{j+1})} \cdot \frac{\prod_{i=0}^{k-1} (q;q)_{a-\sum_{j=1}^{k} r_j - \sum_{j=1}^{k-1-i}(n_j-r_j) - \sum_{j=k-i}^{k-1}(m_{j+1}-r_{j+1})}}{\prod_{i=1}^{k-1} (q;q)_{a-\sum_{j=1}^{k} r_j - \sum_{j=1}^{k-i}(n_j-r_j) - \sum_{j=k-i}^{k-1}(m_{j+1}-r_{j+1})}}. \tag{2.25}$$

Rearranging (2.25), we see that the sum over $r_1, \ldots, r_k$ may be written as a product over k sums, where the i-th sum is over r_i:

$$f_{k,q}\left(a; \begin{smallmatrix} n_1,\ldots,n_k \\ m_1,\ldots,m_k \end{smallmatrix}\right) = \frac{q^{\sum_{1\le i\le j\le k-1} n_i m_{j+1}}}{\prod_{i=1}^{k-1} (q;q)_{a-\sum_{j=1}^{k-i} n_j - \sum_{j=k-i}^{k-1} m_{j+1}}} \cdot \prod_{i=1}^{k} \left(\sum_{r_i\ge 0} R_q(n_i, m_i, r_i)\, (q;q)_{a-r_i-\sum_{j=1}^{i-1} n_j - \sum_{j=i}^{k-1} m_{j+1}} \right). \tag{2.26}$$

Using Lemma 2.7 we substitute each inner sum of (2.26) with

$$q^{n_i \cdot m_i} \frac{(q;q)_{a-\sum_{j=1}^{i} n_j - \sum_{j=i}^{k-1} m_{j+1}} (q;q)_{s-\sum_{j=1}^{i-1} n_j - \sum_{j=i-1}^{k-1} m_{j+1}}}{(q;q)_{a-\sum_{j=1}^{i} n_j - \sum_{j=i-1}^{k-1} m_{j+1}}},$$

and by simplifying we complete the induction step and obtain the desired identity. □

Remark 2.9 Solomon [13] proved a relation between the following two quantities: the number of placements of k non-attacking rooks on a $n \times n$ chessboard, counted with certain weights depending on q, and the number of matrices in $M_{n\times n}(\mathbb{F})$ of rank k. Haglund generalized Solomon's result to any "Ferrers board" [6, Thm. 1], which means that the number of matrices of the form (2.16) over $\mathbb{F}$ of rank k is related to the q-rook polynomial $R_k(B, q)$, where B is a certain Ferrers board associated with (2.16). For the definition of a Ferrers board and $R_k(B, q)$, see the introduction to the paper by Garsia and Remmel [1]. In particular, Proposition 2.8 may be deduced from a result of Garcia and Remmel on q-rook polynomials, see [6, Cor. 2]. Our proof of Proposition 2.8 is direct and so we believe it is more accessible. More importantly, the ideas used in the proof reappear in the proofs of Theorems 2 and 3.

2.5 Arithmetic properties of certain polynomials

For any d dividing n and any $k \geq 2$, let

$$a_{k;n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} x^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} \in \mathbb{Q}(x), \tag{2.27}$$

where $\mu : \mathbb{N} \to \mathbb{C}$ is the Möbius function, defined by $\mu(1) = 1$ and

$$\mu(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some prime } p, \\ (-1)^m & \text{if } n = p_1 p_2 \dots p_m, \text{ where } p_i \text{ are distinct primes.} \end{cases}$$

We recall the following properties of μ [8, Ch. 2].

- The divisor sum $\sum_{d|n} \mu(d)$ is given by

$$\sum_{d|n} \mu(d) = \delta_{1,n}. \tag{2.28}$$

- The Möbius function is multiplicative.

Lemma 2.10 *Let $k \geq 2$. The following hold.*

(I) For any $d \mid n$, $a_{k;n,d}(x)$ is a polynomial in $\mathbb{Z}[x]$. Furthermore, in case $d \notin \{n, \frac{n}{2}\}$, $a_{k;n,d}(x)$ is divisible by $x^d - 1$. In the remaining cases we have

$$a_{k;n,d}(x) = \begin{cases} (-1)^{k(n-1)} & \text{if } d = n, \\ \frac{x^{\frac{(k-2)n}{2}} + (-1)^{k+1}}{x^{\frac{n}{2}} + 1} & \text{if } d = \frac{n}{2}. \end{cases} \tag{2.29}$$

(II) If $k > 2$ we have $\deg\left(a_{k;n,d}\right) = \frac{(n(k-2)-2d)(n-d)}{2d}$, and $a_{k;n,d}$ has leading coefficient $(-1)^{k(n-\frac{n}{d})}$. If $k = 2$, we have $a_{k;n,d} = \delta_{n,d}$.

(III) Assume $k > 2$. For any prime power q, $a_{k;n,d}(q)$ is a non-zero integer. Its sign equals the sign of $(-1)^{k(n-\frac{n}{d})}$, i.e. it is a positive integer unless k is odd, n is even and $2 \nmid \frac{n}{d}$.

Proof We begin by proving the first part of the lemma. If $d \in \{n, \frac{n}{2}\}$, a short calculation reveals that (2.29) holds. From now on we assume that $d \notin \{n, \frac{n}{2}\}$. We shall show that

$$x^n - 1 \mid \sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{k(n-\frac{n}{m})} x^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} \tag{2.30}$$

in $\mathbb{Q}[x]$, which implies that $a_{k;n,d}(x)$ is a polynomial divisible by $x^d - 1$. Gauss's lemma, applied to (2.30), implies that $a_{k;n,d}(x) \in \mathbb{Z}[x]$. We now prove (2.30).

Let z be a root of unity of order dividing n. Assume first that n is odd or that k is even. Then for all $m \mid n$ we have

$$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = (z^n)^{(k-2)\frac{\frac{n}{m}-1}{2}} = 1.$$

Hence, using (2.28),

$$\sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = \sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right) = \sum_{a:\, a|\frac{n}{d}} \mu(a) = \delta_{d,n} = 0. \tag{2.31}$$

Now we assume instead that n is even and k is odd. We are led to consider two cases.

- If $z^{\frac{n}{2}} = -1$ then for all $m \mid n$ we have,

  $$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = (-1)^{\frac{n}{m}-1}.$$

  Hence, using (2.28),

  $$\begin{aligned}\sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} &= -\sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right) \\ &= -\sum_{a|\frac{n}{d}} \mu(a) = -\delta_{d,n} = 0. \end{aligned} \tag{2.32}$$

- If $z^{\frac{n}{2}} = 1$ then for all $m \mid n$ we have,

  $$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = 1.$$

  Hence,

  $$\begin{aligned}&\sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} \\ &= \sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{\frac{n}{m}} = \sum_{a|\frac{n}{d}} \mu(a)(-1)^{\frac{n}{ad}} \\ &= \sum_{\substack{a|\frac{n}{d} \\ 2|\frac{n}{ad}}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2\nmid\frac{n}{ad}}} \mu(a) \\ &= \begin{cases} 0 - \sum_{a|\frac{n}{d}} \mu(a) & \text{if } 2 \nmid \frac{n}{d} \\ \sum_{a|\frac{n}{2d}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2|a}} \mu(2 \cdot \frac{a}{2}) & \text{if } 2 \mid \frac{n}{d}, 4 \nmid \frac{n}{d} \\ \sum_{a|\frac{n}{2d}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2\nmid\frac{n}{ad}}} \mu(4 \cdot \frac{a}{4}) & \text{if } 4 \mid \frac{n}{d} \end{cases}\end{aligned}$$

$$= \begin{cases} -\delta_{d,n} & \text{if } 2 \nmid \frac{n}{d} \\ \delta_{2d,n} - \mu(2)\delta_{2d,n} & \text{if } 2 \mid \frac{n}{d}, 4 \nmid \frac{n}{d} \\ \delta_{2d,n} & \text{if } 4 \mid \frac{n}{d} \end{cases}$$
$$= 0. \tag{2.33}$$

Equations (2.31), (2.32) and (2.33) show that the RHS of (2.30) vanishes on each root of the separable polynomial $x^n - 1$, which establishes (2.30). This concludes the proof of the first part of the lemma.

The second part of the lemma for $k > 2$ follows from the observation that the numerator of $a_{k;n,d}(x)$ has degree $d + (k-2)\frac{n}{2}(\frac{n}{d} - 1)$ (arising from the term corresponding to $m = d$) and leading coefficient equal to $(-1)^{k(n-\frac{n}{d})}$, while the denominator of $a_{k;n,d}(x)$ has degree n and leading coefficient equal to 1.

When $k = 2$, all terms in the sum in (2.27) are constants, and we have

$$a_{2;n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right) = \frac{x^d - 1}{x^n - 1}\delta_{n,d} = \delta_{n,d}.$$

We now turn to the third part of the lemma. Since $a_{k;n,d}(x)$ has integer coefficients, $a_{k;n,d}(q)$ is an integer. We now determine its sign when $k > 2$, and in particular show that it is non-zero.

Since $q^d - 1$, $q^n - 1$, $q^{\frac{n}{2}}$ are positive, we deal with the expression

$$\begin{aligned} \widetilde{a}_{k;n,d}(q) := &\ \tfrac{q^n-1}{q^d-1} q^{(k-2)\frac{n}{2}} \cdot a_{k;n,d}(q) \\ = &\ \textstyle\sum_{m:\, d|m|n} \mu\left(\frac{m}{d}\right)(-1)^{k(n-\frac{n}{m})}(q^{(k-2)\frac{n}{2}})^{\frac{n}{m}} \\ = &\ \textstyle\sum_{a|\frac{n}{d}} \mu(a)(-1)^{k(n-\frac{n}{ad})}(q^{(k-2)\frac{n}{2}})^{\frac{n}{ad}}, \end{aligned}$$

whose sign is the same as the sign of $a_{k;n,d}(q)$. If $d = n$ then

$$(-1)^{k(n-\frac{n}{d})}\widetilde{a}_{k;n,d}(q) = q^{(k-2)\frac{n}{2}} > 0.$$

If $d = \frac{n}{2}$ then

$$(-1)^{k(n-\frac{n}{d})}\widetilde{a}_{k;n,d}(q) = (q^{(k-2)\frac{n}{2}})^2 + (-1)^{k+1} q^{(k-2)\frac{n}{2}} > 0.$$

If $\frac{n}{d} \geq 3$, we set $t = q^{(k-2)\frac{n}{2}}$. Then, $t \geq 2^{\frac{3}{2}} > 2$ and

$$\begin{aligned} (-1)^{k(n-\frac{n}{d})}\widetilde{a}_{k;n,d}(q) &\geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{d}} - \textstyle\sum_{1\leq i\leq\frac{n}{2d}} (q^{(k-2)\frac{n}{2}})^i \geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{d}} - \frac{(q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}}}{1-q^{-(k-2)\frac{n}{2}}} \\ &= (q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} \left((q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} - \frac{1}{1-q^{-(k-2)\frac{n}{2}}} \right) \\ &\geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} \left((q^{(k-2)\frac{n}{2}})^{\frac{3}{2}} - \frac{1}{1-q^{-(k-2)\frac{n}{2}}} \right) \\ &= \frac{(q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}}}{1-q^{-(k-2)\frac{n}{2}}} \left(t^{\frac{1}{2}}(t-1) - 1 \right) > 0. \end{aligned}$$

□

Remark 2.11 The polynomials $a_{k;n,d}(x)$ may be expressed using the necklace polynomials (see Moreau [10]), defined by

$$M_n(x) = \frac{1}{n} \sum_{d|n} \mu(d) x^{\frac{n}{d}}.$$

Indeed,

$$a_{k;n,d}(x) = \frac{x^d - 1}{x^n - 1} \cdot \left(\frac{(-1)^n}{x^{\frac{n}{2}}}\right)^{k-2} \cdot M_{\frac{n}{d}}\left(\left(-x^{\frac{n}{2}}\right)^{k-2}\right).$$

3 Calculation of the dimension of $\pi_{k,N,\psi}$

Here we prove Theorem 2. Recall that Θ_θ is the character of the irreducible cuspidal representation π_θ associated to a regular character θ of $\mathbb{F}_n^*$. Given $U \in N$, we write it in the notation of (1.1). From (1.2),

$$\begin{aligned}\dim\left(\pi_{k,N,\psi}\right) &= \frac{1}{|N|}\sum_{U\in N}\Theta_\theta(U)\overline{\psi}(U) = \frac{1}{q^{\binom{k}{2}n^2}}\sum_{U\in N}\Theta_\theta(U)\,\overline{\psi}(U)\\ &= \frac{1}{q^{\binom{k}{2}n^2}}\sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\1\le i\le k-1}}\ \sum_{\substack{X_{i,j}\in M_n(\mathbb{F})\\1\le i<j\le k-1}}\Theta_\theta(U)\,\overline{\psi}(U).\end{aligned}$$

The character $\psi(U) = \psi\left(X_{1,1},\ldots,X_{k-1,k-1}\right)$ is determined by the traces of $X_{i,i}$, $1 \le i \le k-1$. Hence,

$$\dim\left(\pi_{k,N,\psi}\right) = \frac{1}{q^{\binom{k}{2}n^2}}\sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\1\le i\le k-1}}\overline{\psi}(U)\sum_{\substack{X_{i,j}\in M_n(\mathbb{F})\\1\le i<j\le k-1}}\Theta_\theta(U). \tag{3.1}$$

By Corollary 2.2, the value $\Theta_\theta(U)$ is determined by $\dim_{\mathbb{F}_{kn}}\ker(U-I)$ which is in turn determined by $\mathrm{rank}_{\mathbb{F}_{kn}}(U-I)$. In the inner sum of (3.1) set $r_i = \mathrm{rk}\left(X_{i,i}\right)$ for $1 \le i \le k-1$. We write $I_{r,n} := I_{r,n,n}$ as defined in (2.18). There exist invertible matrices E_i, C_{i+1} such that $X_{i,i} = E_i I_{r_i,n} C_{i+1}$. So, one can write U in the inner sum of (3.1) as I_{kn} plus

$$\mathrm{diag}\left(E_1,\ldots,E_{k-1},I_n\right)\begin{pmatrix}0 & I_{r_1,n} & \cdots & \widetilde{X}_{1,k-2} & \widetilde{X}_{1,k-1}\\ 0 & 0 & \cdots & \widetilde{X}_{2,k-2} & \widetilde{X}_{2,k-1}\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n}\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}\mathrm{diag}\left(I_n,C_2,\ldots,C_k\right),$$

where $\widetilde{X}_{i,j} = E_i^{-1}X_{i,j}C_{j+1}^{-1}$ for all $1 \le i < j \le k-1$.Together with the fact that rank is invariant under elementary operations, we now have

$$\dim\left(\pi_{k,N,\psi}\right) = \frac{1}{q^{\binom{k}{2}n^2}}\sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\1\le i\le k-1}}\overline{\psi}(U) \cdot\sum_{\substack{\widetilde{X}_{i,j}\in M_n(\mathbb{F})\\1\le i<j\le k-1}}\Theta_\theta\left(I_{kn} + \begin{pmatrix}0 & I_{r_1,n} & \cdots & \widetilde{X}_{1,k-2} & \widetilde{X}_{1,k-1}\\ 0 & 0 & \cdots & \widetilde{X}_{2,k-2} & \widetilde{X}_{2,k-1}\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n}\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}\right). \tag{3.2}$$

As in the proof of Proposition 2.8, we can use Gaussian elimination operations on $\widetilde{X}_{i,j}$ for all $1 \le i < j \le k-1$ (which do not affect the rank nor dimension of the kernel of the matrix minus I_{kn}, and the number of Jordan blocks is not affected as well) in such a way that the sequence of matrices $\{\widetilde{X}_{i,j}\}_{1\le i<j\le k-1}$ is mapped $\mathbb{F}$-linearly to a sequence of matrices

$$\left\{\widehat{\widetilde{X}}_{i,j} = \begin{pmatrix}0 & 0\\ 0 & Y_{i,j}\end{pmatrix}\right\}_{1\le i<j\le k-1},$$

where $Y_{i,j} \in M_{(n-r_i)\times(n-r_j)}(\mathbb{F})$. The kernel of this mapping is of size $q^{\sum_{i=1}^{k-2} r_i(k-i-1)n+\sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1}(n-r_j)}$. The dimension of the kernel corresponds to the number of elements which we cancel. Equation (3.2) becomes

$$\begin{aligned} \dim\left(\pi_{k,N,\psi}\right) = \frac{1}{q^{\binom{k}{2}n^2}} & \sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\ 1\le i\le k-1}} \overline{\psi}\,(U)\, q^{\sum_{i=1}^{k-2} r_i(k-i-1)n+\sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1}(n-r_j)} \\ & \cdot \sum_{\substack{Y_{i,j}\in M_{(n-r_i)\times(n-r_j)}(\mathbb{F})\\ 1\le i<j\le k-1}} \Theta_\theta\,(g)\,, \end{aligned} \tag{3.3}$$

where

$$g = I_{kn} + \begin{pmatrix} 0 & I_{r_1,n} & \cdots & \widehat{\widehat{X}}_{1,k-2} & \widehat{\widehat{X}}_{1,k-1} \\ 0 & 0 & \cdots & \widehat{\widehat{X}}_{2,k-2} & \widehat{\widehat{X}}_{2,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Using the character formula (2.1), we can calculate $\Theta_\theta(g)$. In this case $m = kn$, $g = s\cdot u$ where $s = I_{kn}$, so $\lambda = 1$ and

$$t = \dim\ker(g-I) = kn - \mathrm{rk}(g-I) = kn - \sum_{i=1}^{k-1} r_i - \mathrm{rk}A,$$

where

$$A = \begin{pmatrix} Y_{1,2} & \cdots & Y_{1,k-1} \\ \vdots & & \vdots \\ 0 & \cdots & Y_{k-2,k-1} \end{pmatrix}, \quad 1 \le i < j \le k-1. \tag{3.4}$$

So,

$$\begin{aligned} \Theta_\theta\,(g) &= (-1)^{kn-1}(1-q)(1-q^2)\cdots(1-q^{kn-\sum_{i=1}^{k-1} r_i-\mathrm{rk}A-1}) \\ &= (-1)^{kn-1}(q;q)_{kn-\sum_{i=1}^{k-1} r_i-\mathrm{rk}A-1}. \end{aligned}$$

Equation (3.3) can now be written as

$$\begin{aligned} \dim\left(\pi_{k,N,\psi}\right) = \frac{1}{q^{\binom{k}{2}n^2}} & \sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\ 1\le i\le k-1}} \overline{\psi}\,(U)\, q^{\sum_{i=1}^{k-2} r_i(k-i-1)n+\sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1}(n-r_j)} \\ & \cdot(-1)^{kn-1} \sum_{A} (q;q)_{kn-\sum_{i=1}^{k-1} r_i-\mathrm{rk}A-1}, \end{aligned} \tag{3.5}$$

where the inner sum is over all matrices of the form (3.4) and by the definition (2.14) it is equal to

$$f_{k-2,q}\left(kn - \sum_{i=1}^{k-1} r_i - 1;\, {}^{n-r_1,\ldots,n-r_{k-2}}_{n-r_2,\ldots,n-r_{k-1}}\right).$$

By applying Proposition 2.8 we replace the inner sum in (3.5) by

$$q^{\sum\limits_{1\le i\le j\le k-2}(n-r_i)\cdot(n-r_{j+1})} \cdot \frac{\prod_{i=0}^{k-2}(q;q)_{kn-\sum_{j=1}^{k-1} r_j-1-\sum_{j=1}^{k-2-i}(n-r_j)-\sum_{j=k-i-1}^{k-2}(n-r_{j+1})}}{\prod_{i=1}^{k-2}(q;q)_{kn-\sum_{j=1}^{k-1} r_j-1-\sum_{j=1}^{k-i-1}(n-r_j)-\sum_{j=k-i-1}^{k-2}(n-r_{j+1})}},$$

which equals

$$q^{\sum\limits_{1\le i\le j\le k-2}(n-r_i)\cdot(n-r_{j+1})}\cdot\frac{\prod_{i=1}^{k-1}(q;q)_{2n-1-r_i}}{((q;q)_{n-1})^{(k-2)}}.$$

Now (3.5) becomes

$$\dim\left(\pi_{k,N,\psi}\right)=\frac{(-1)^{kn-1}}{((q;q)_{n-1})^{(k-2)}\,q^{(k-1)n^2}}\sum_{\substack{X_{i,i}\in M_n(\mathbb{F})\\1\le i\le k-1}}\prod_{i=1}^{k-1}\overline{\psi_0}\left(\mathrm{tr}\left(X_{i,i}\right)\right)(q;q)_{2n-1-r_i}. \tag{3.6}$$

Changing the order of sum and product in (3.6) we get that

$$\dim\left(\pi_{k,N,\psi}\right)=\frac{(-1)^{kn-1}}{((q;q)_{n-1})^{(k-2)}\,q^{(k-1)n^2}}\prod_{i=1}^{k-1}\sum_{X_{i,i}\in M_n(\mathbb{F})}\overline{\psi_0}\left(\mathrm{tr}\left(X_{i,i}\right)\right)(q;q)_{2n-1-r_i}. \tag{3.7}$$

From Sect. 5 of [11], each inner sum in (3.7) is equal to

$$\sum_{X_{i,i}\in M_n(\mathbb{F})}\overline{\psi_0}\left(\mathrm{tr}\left(X_{i,i}\right)\right)(q;q)_{2n-1-r_i}=(-1)^n\cdot q^{n^2}\cdot q^{\binom{n}{2}}(q;q)_{n-1}. \tag{3.8}$$

Plugging (3.8) in (3.7), we obtain

$$\dim\left(\pi_{k,N,\psi}\right)=q^{(k-1)\binom{n}{2}}(-1)^{n-1}(q;q)_{n-1}=q^{(k-2)\binom{n}{2}}\frac{|\mathrm{GL}_n(\mathbb{F})|}{q^n-1},$$

as needed. □

4 Calculation of the character $\Theta_{k,N,\psi}$

In this section we prove Theorem 3. Namely, we calculate $\Theta_{k,N,\psi}$. From now on we use the following notations:

$$h_{g;U}=\begin{pmatrix} g & X_{1,1} & X_{1,2} & \cdots & X_{1,k-2} & X_{1,k-1}\\ 0 & g & X_{2,2} & \cdots & X_{2,k-2} & X_{2,k-1}\\ 0 & 0 & g & \cdots & X_{3,k-2}, & X_{3,k-1}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & g & X_{k-1,k-1}\\ 0 & 0 & 0 & \cdots & 0 & g \end{pmatrix},$$

where U (and so $X_{i,j}$) is as in (1.1). Note that $h_{I_n,U}=U$. We also define

$$\Delta^r(g)=\mathrm{diag}\,(g,\ldots,g)\in\Delta^r\left(\mathrm{GL}_n(\mathbb{F})\right),\qquad g\in\mathrm{GL}_n(\mathbb{F}).$$

By definition,

$$\begin{aligned}\Theta_{k,N,\psi}(g)&=\mathrm{tr}\left(\pi_{k,N,\psi}(g)\right)=\mathrm{tr}\left(\pi(\Delta^k(g))\restriction_{V_{k,N,\psi}}\right)\\ &=\mathrm{tr}\left(\pi(\Delta^k(g))\circ P_{k,N,\psi}\right).\end{aligned} \tag{4.1}$$

Substituting (1.2) into (4.1) we have

$$\begin{aligned}\Theta_{k,N,\psi}(g) &= \mathrm{tr}\left(\frac{1}{q^{\binom{k}{2}n^2}}\sum_{U\in N}\pi\left[\Delta^k(g)\cdot U\right]\overline{\psi}(U)\right)\\ &= \frac{1}{q^{\binom{k}{2}n^2}}\sum_{U\in N}\mathrm{tr}\left(\pi\left[\Delta^k(g)\cdot U\right]\right)\overline{\psi}(U).\end{aligned} \tag{4.2}$$

Now we perform the change of variables

$$X_{i,j}\mapsto g^{-1}X_{i,j},\qquad 1\le i\le j\le k-1$$

in (4.2) and obtain

$$\Theta_{k,N,\psi}(g) = \frac{1}{q^{\binom{k}{2}n^2}}\sum_{U\in N}\Theta_\theta\left(h_{g;U}\right)\overline{\psi}\left(g^{-1}X_{1,1},\ldots,g^{-1}X_{k-1,k-1}\right). \tag{4.3}$$

In parts Sects. 4.1, 4.2 and 4.3 we prove parts (I), (II) and (III) of Theorem 3, respectively.

4.1 Character at $g = s\cdot u$ such that the semisimple part s does not come from $\mathbb{F}_n$

Let $g = s\cdot u$. Assume that the semisimple part s does not come from $\mathbb{F}_n$. The semisimple part of $h_{g;U}$ is $\Delta^k(s)$, which also does not come from $\mathbb{F}_n$. By Theorem 2.1, we have $\Theta_\theta\left(h_{g;U}\right) = 0$. Hence, by (4.3) $\Theta_{k,N,\psi}(g) = 0$. □

4.2 Character calculation at a non-semisimple element

Assume that s comes from $\mathbb{F}_d\subseteq\mathbb{F}_n$ and $d\mid n$ is minimal. In addition, $d<n$ since g is not semisimple. Let $\lambda\in\mathbb{F}_d^*$ be an eigenvalue of s which generates the field $\mathbb{F}_d$ over $\mathbb{F}$. We use the notation of Sect. 2.3. Thus, there exist $R\in\mathrm{GL}_n(\mathbb{F})$ and ρ a partition of $d'=n/d$ such that $R^{-1}gR = L_\rho(f)$. There exists $\Delta^{d'}(T)\in\mathrm{GL}_n(\mathbb{F}_d)$ such that

$$g_\rho = \Delta^{d'}\left(T^{-1}\right)R^{-1}gR\Delta^{d'}(T),$$

the analogue of the Jordan form of g. Recall that by Lemma 2.5, the map

$$A\mapsto A_\rho := A_{\rho,R} = \Delta^{d'}\left(T^{-1}\right)R^{-1}AR\Delta^{d'}(T)$$

induces an isomorphism. By the notation of Sect. 2.3.2 we have for each

$$X_{a,b},\qquad \forall 1\le a\le b\le k-1,$$

the corresponding isomorphism of Lemma 2.5

$$X_{a,b}\mapsto (X_{a,b})_\rho = \begin{pmatrix}\left(x^{(a,b)}_{0,i;j}\right)_{\substack{0\le i\le d-1\\0\le j\le d'-1}}\\ \vdots\\ \left(x^{(a,b)}_{d'-1,i;j}\right)_{\substack{0\le i\le d-1\\0\le j\le d'-1}}\end{pmatrix}.$$

Note that

$$\Delta^k\left(\Delta^{d'}\left(T^{-1}\right)\right)\Delta^k\left(R^{-1}\right)h_{g;U}\Delta^k(R)\,\Delta^k\left(\Delta^{d'}(T)\right) = h_{g_\rho;U_\rho}, \tag{4.4}$$

where U_ρ is the element of N with $(X_{a,b})_\rho$ instead of $X_{a,b}$. From (4.4) we obtain

$$\mathrm{rk}\left(h_{g-\lambda I_n;U}\right) = \mathrm{rk}\left(h_{g_\rho-\lambda I_n;U_\rho}\right).$$

We prove that $\mathrm{rk}\left(h_{g-\lambda I_n;U}\right)$ (which by Corollary 2.2 determines the value of $\Theta_\theta\left(h_{g;U}\right)$) is independent of $x_{1,0;1}^{(k-1,k-1)} \in \mathbb{F}_d$. The matrix $h_{g_\rho-\lambda I_n;U_\rho}$ is of the form

$$h_{g_\rho-\lambda I_n;U_\rho} = \begin{pmatrix} g_\rho - \lambda I_n & (X_{1,1})_\rho & \cdots & (X_{1,k-2})_\rho & (X_{1,k-1})_\rho \\ 0 & g_\rho - \lambda I_n & \cdots & (X_{2,k-2})_\rho & (X_{2,k-1})_\rho \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & g_\rho - \lambda I_n & (X_{k-1,k-1})_\rho \\ 0 & 0 & \cdots & 0 & \boxed{g_\rho - \lambda I_n} \end{pmatrix}. \tag{4.5}$$

Consider the boxed block in (4.5). The $2d \times 2d$ upper left block of the boxed matrix $g_\rho - \lambda I_n$ is of the form

$$\begin{pmatrix} 0 & & & \boxed{1} & & & \\ & \lambda^q - \lambda & & & 1 & & \\ & & \ddots & & & 1 & \\ & & & \lambda^{q^{d-1}} - \lambda & & & 1 \\ & & & 0 & & & \\ & & & & \lambda^q - \lambda & & \\ & & & & & \ddots & \\ & & & & & & \lambda^{q^{d-1}} - \lambda \end{pmatrix} \tag{4.6}$$

Let $Z := X_{k-1,k-1}$, $Z_\rho := (X_{k-1,k-1})_\rho$ and $z_{m,i;j} := x_{m,i;j}^{(k-1,k-1)}$. One can eliminate the $(d+1)$-th column in Z_ρ by the boxed 1 from (4.6), i.e. all the elements $\left\{z_{m,i;1}\right\}_{\substack{0 \le i \le d-1 \\ 0 \le m \le d'-1}}$. In particular, $z_{1,0;1} = x_{1,0;1}^{(k-1,k-1)}$ is eliminated. Now, by Lemma 2.6, (4.3) can be written as

$$\begin{aligned} \Theta_{k,N,\psi}(g) = & \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta\left(h_{g;U}\right) \cdot \prod_{i=1}^{k-2} \overline{\psi_0}\left(g^{-1} X_{i,i}\right) \\ & \cdot \overline{\psi}_0 \left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} z_{m,0;m} \right) + \mathrm{tr}\left(D^{-2} \alpha\left(g, D^{-1}, Z_\rho\right)\right) \right). \end{aligned} \tag{4.7}$$

By Lemma 2.5, going over $Z \in M_n(\mathbb{F})$ is equivalent to going over $\left(z_{m,i;j}\right)_{\substack{0 \le i \le d-1 \\ 0 \le j,m \le d'-1}}$, $z_{m,i;j} \in \mathbb{F}_d$. We have just shown that $\Theta_\theta\left(h_{g;U}\right)$ is independent of $z_{1,0;1}$, and by Lemma 2.6 $\mathrm{tr}\left(D^{-2}\alpha\left(g, D^{-1}, Z_\rho\right)\right)$ in (4.7) is also independent of $z_{1,0;1}$. Thus, we may write (4.7) as the following double sum, where the inner sum is over $z_{1,0;1}$ and the outer sum is over the rest of the coordinates of U:

$$\begin{aligned} \Theta_{k,N,\psi}(g) = & \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,j} \in N, (i,j) \neq (k-1,k-1) \\ z_{m,i;j} \in \mathbb{F}_d, (m,i,j) \neq (1,0,1)}} \Theta_\theta\left(h_{g;U}\right) \cdot \prod_{i=1}^{k-2} \overline{\psi_0}\left(g^{-1} X_{i,i}\right) \\ & \cdot \overline{\psi}_0\left(\mathrm{tr}\left(D^{-2}\alpha\left(g, D^{-1}, Z_\rho\right)\right)\right) \cdot \overline{\psi}_0 \left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{\substack{0 \le m \le d'-1 \\ m \neq 1}} z_{m,0;m} \right) \right) \\ & \cdot \sum_{z_{1,0;1} \in \mathbb{F}_d} \overline{\psi_0}\left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}\left(\lambda^{-1} \cdot z_{1,0;1}\right)\right). \end{aligned}$$

Since $\overline{\psi}_0 \circ \mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}$ is a nontrivial character, we have

$$\sum_{z_{1,0;1} \in \mathbb{F}_d} \overline{\psi}_0 \left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot z_{1,0;1} \right) \right) = 0.$$

Thus, $\Theta_{k,N,\psi}(g) = 0$. □

4.3 Character calculation at a semisimple element

Here we use (4.3) to calculate the value of $\Theta_{k,N,\psi}(g)$ for $g = s$ where s is semisimple element which comes from a subfield of $\mathbb{F}_n$ ($u = I_n$). Again, we use the notation of Sect. 2.3. Thus, there exist $R \in \mathrm{GL}_n(\mathbb{F})$, ρ a partition of n/d and $\Delta^{d'}(T) \in \mathrm{GL}_n(\mathbb{F}_d)$ such that

$$s_\rho = \Delta^{d'}\left(T^{-1}\right) R^{-1} s R \Delta^{d'}(T), \tag{4.8}$$

the analogue of the Jordan form of s. We also use the notation of Sect. 2.3.2, and in particular define $(X_{a,b})_\rho$ as in Sect. 4.2.

Let $\lambda \in \mathbb{F}_n^*$ be an eigenvalue of s. If $\lambda \in \mathbb{F}^*$ then $s = \lambda I$, and we have by (4.3)

$$\Theta_{k,N,\psi}(\lambda I) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta \left(h_{\lambda I;U} \right) \overline{\psi} \left(\lambda^{-1} X_{1,1}, \ldots, \lambda^{-1} X_{k-1,k-1} \right).$$

By the change of variables

$$X_{i,j} \mapsto \lambda X_{i,j},$$

we get

$$\Theta_{k,N,\psi}(\lambda I) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta \left(\lambda h_{I;U} \right) \overline{\psi} \left(X_{1,1}, \ldots, X_{k-1,k-1} \right).$$

By Theorem 2.1, we have $\Theta_\theta\left(\lambda \cdot h_{I;U}\right) = \theta(\lambda)\Theta_\theta\left(h_{I;U}\right)$, and so

$$\Theta_{k,N,\psi}(\lambda I) = \theta(\lambda)\Theta_{k,N,\psi}(I) = \theta(\lambda)\dim\left(\pi_{k,N,\psi}\right).$$

By Theorem 2, this proves the case $\lambda \in \mathbb{F}^*$.

If $\lambda \in \mathbb{F}_d^* \subseteq \mathbb{F}_n^*$ is an eigenvalue of s and $1 < d \mid n$ is such that $\mathbb{F}_d$ is generated by λ over $\mathbb{F}$, we have by (4.3)

$$\Theta_{k,N,\psi}(s) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta \left(h_{s;U} \right) \overline{\psi} \left(s^{-1} X_{1,1}, \ldots, s^{-1} X_{k-1,k-1} \right). \tag{4.9}$$

In order to compute $\Theta_\theta(h_{s;U})$, we need to find conditions for $X_{i,j}$, such that $h_{s;U}$ will have a fixed number of Jordan blocks. This is equivalent to saying that $h_{s;U} - \lambda I_{kn}$ will have a given kernel dimension, or a given rank. Rank and trace are invariant under conjugation, so let us denote by h_{s_ρ,U_ρ}, the matrix $h_{s;U}$ conjugated by $\Delta^k(R)\Delta^k\left(\Delta^{d'}(T)\right)$, where R and T are defined by s in (4.8):

$$h_{s_\rho;U_\rho} := \Delta^k\left(\Delta^{d'}\left(T^{-1}\right)\right)\Delta^k\left(R^{-1}\right)h_{s;U}\Delta^k(R)\Delta^k\left(\Delta^{d'}(T)\right).$$

We have a matrix in $\mathrm{GL}_{kn}(\mathbb{F}_d)$ and our goal is to find out how many matrices of the form

$$h_{s_\rho;U_\rho} - \lambda I_{kn} = h_{s_\rho - \lambda I_n;U_\rho},$$

where U varies, have a given rank ℓ.

First, notice that by the invariance of rank under elementary row and column operations on $h_{s_\rho-\lambda I_n;U_\rho}$, we can use the nonzero elements on the diagonal of $s_\rho - \lambda I_n$ to cancel the corresponding elements of $(X_{a,b})_\rho$. These elementary operations map the sequence of matrices $\{(X_{a,b})_\rho\}_{1\le a\le b\le k-1}$ $\mathbb{F}_d$-linearly to the sequence

$$\left\{(\widehat{X}_{a,b})_\rho = \begin{pmatrix} x^{(a,b)}_{0,0;0} & \cdots & x^{(a,b)}_{d'-1,0;0} \\ \vdots & \ddots & \vdots \\ x^{(a,b)}_{0,0;d'-1} & \cdots & x^{(a,b)}_{d'-1,0;d'-1} \end{pmatrix} \in M_{d'}(\mathbb{F}_d)\right\}_{1\le a\le b\le k-1}.$$

The dimension of the kernel of this map is $\binom{k}{2}(n-d')d'$, corresponding to the number of elements we canceled. Hence, the number of matrices $h_{s_\rho-\lambda I_n;U_\rho}$ of rank ℓ is $(q^d)^{\binom{k}{2}(n-d')d'}$ times the number of matrices of the form

$$A := \begin{pmatrix} (\widehat{X}_{1,1})_\rho & \cdots & (\widehat{X}_{1,k-2})_\rho & (\widehat{X}_{1,k-1})_\rho \\ 0 & \cdots & (\widehat{X}_{2,k-2})_\rho & (\widehat{X}_{2,k-1})_\rho \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & (\widehat{X}_{k-1,k-1})_\rho \end{pmatrix} \in M_{(k-1)d'}(\mathbb{F}_d) \tag{4.10}$$

of rank $\ell - k(n-d')$. Using the character formula (2.1), we can calculate $\Theta_\theta(h_{s;U})$. In this case $m = kn$, $g = h_{s;U}$ and

$$t = \dim\ker(h_{s;U} - I) = kn - \mathrm{rk}(h_{s;U} - I) = kn - k(n-d') - \mathrm{rk}A = kd' - \mathrm{rk}A.$$

Thus

$$\begin{aligned}\Theta_\theta\left(h_{s;U}\right) &= (-1)^{kn-1}\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right](1-q^d)(1-(q^d)^2)\cdots(1-(q^d)^{kd'-\mathrm{rk}A-1}) \\ &= (-1)^{kn-1}\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right](q^d;q^d)_{kd'-\mathrm{rk}A-1}. \end{aligned} \tag{4.11}$$

Now, by (4.11) and Lemma 2.6, (4.9) can be written as

$$\begin{aligned}\Theta_{k,N,\psi}(s) = \frac{(-1)^{kn-1}(q^d)^{\binom{k}{2}(n-d')d'}}{q^{\binom{k}{2}n^2}} &\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right]\sum_A (q^d;q^d)_{kd'-\mathrm{rk}A-1} \\ &\cdot\prod_{i=1}^{k-1}\overline{\psi_0}\left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}\left(\lambda^{-1}\cdot\sum_{m=0}^{d'-1}x^{(i,i)}_{m,0;m}\right)\right),\end{aligned} \tag{4.12}$$

where the sum is over matrices A as in (4.10). By the character formula (2.1), the RHS of (4.12) is $(-1)^{k(n-d')}\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right]$ times the RHS of (3.1), when one replaces n with d', q with q^d and ψ_0 with

$$\psi_0' : \mathbb{F}_d \to \mathbb{C}^*, \quad \psi_0'(x) = \psi_0\left(\mathrm{Tr}_{\mathbb{F}_d/\mathbb{F}}(\lambda^{-1}x)\right).$$

Thus, the RHS of (4.12) is equal to $\dim\left(\pi_{k,N,\psi}\right)$ (which is calculated in Theorem 2) after the substitution of n, q, ψ_0 with the relevant values. Hence,

$$\Theta_{k,N,\psi}(s) = (-1)^{k(n-d')}\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right](q^d)^{(k-2)\frac{d'(d'-1)}{2}}\frac{\left|\mathrm{GL}_{d'}(\mathbb{F}_d)\right|}{q^n-1},$$

as desired. □

5 Proof of Theorem 4

Notice first that by part (III) of Lemma 2.10, the coefficients in both (1.5) and (1.6) are positive integers, unless $k = 2$ in which case they may also be zero.

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, both parts of the theorem are equivalent to

$$\forall g \in \mathrm{GL}_n(\mathbb{F}) :\ \Theta_{k;N,\psi}(g) = \sum_{\ell|n} a_{k;n,\ell}(q) \cdot \Theta_{\mathrm{Ind}_\ell}(g), \tag{5.1}$$

where $\Theta_{\mathrm{Ind}_\ell}$ is the character of $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta \restriction_{\mathbb{F}_\ell^*})$. We prove now (5.1) for any $g \in \mathrm{GL}_n(\mathbb{F})$. If g is not semisimple or does not come from $\mathbb{F}_n$ then the LHS of (5.1) is zero by parts (I) and (II) of Theorem 3. The RHS of (5.1) is also zero on such elements by Lemma 2.3.

Let g be a semisimple element, which comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. Let λ be an eigenvalue of s, which generates $\mathbb{F}_d$ over $\mathbb{F}$. For such g, part (III) of Theorem 3 and Lemma 2.3 imply that (5.1) is equivalent to

$$\begin{aligned}(-1)^{k(n-d')}&\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right]q^{(k-2)\frac{n(d'-1)}{2}} \cdot \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^n-1} \\ &= \sum_{\ell:\ d|\ell|n} a_{k;n,\ell}(q)\frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^\ell-1}\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right],\end{aligned} \tag{5.2}$$

where $d' = n/d$. The following identity, which we now prove, establishes (5.2):

$$\frac{(-1)^{k(n-d')}q^{(k-2)\frac{n(d'-1)}{2}}}{q^n-1} = \sum_{\ell:\ d|\ell|n} \frac{a_{k;n,\ell}(q)}{q^\ell-1}. \tag{5.3}$$

Using (1.4), the RHS of (5.3) is

$$\sum_{\ell:\ d|\ell|n}\ \sum_{m:\ \ell|m|n} \frac{\mu\left(\frac{m}{\ell}\right)(-1)^{k(n-\frac{n}{m})}q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n-1}. \tag{5.4}$$

We simplify (5.4) using (2.28) as follows:

$$\begin{aligned}\sum_{\ell:\ d|\ell|n}\ \sum_{m:\ \ell|m|n} &\frac{\mu(\frac{m}{\ell})(-1)^{k(n-\frac{n}{m})}q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n-1} \\ &= \sum_{m:\ d|m|n} \frac{(-1)^{k(n-\frac{n}{m})}q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n-1} \sum_{\ell:\ d|\ell|m} \mu\left(\frac{m}{\ell}\right) \\ &= \sum_{m:\ d|m|n} (-1)^{k(n-\frac{n}{m})}\frac{q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n-1}\delta_{d,m} \\ &= (-1)^{k(n-\frac{n}{d})}\frac{q^{(k-2)\frac{n}{2}(\frac{n}{d}-1)}}{q^n-1},\end{aligned}$$

which is the LHS of (5.3). Hence the proof is complete. □

6 Proof of Theorem 1

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, the theorem is equivalent to

$$\forall g \in \mathrm{GL}_n(\mathbb{F}): \quad \Theta_{k,N,\psi}(g) = \Theta_{\theta\restriction_{\mathbb{F}_n^*}}(g)\cdot(\mathrm{St}(g))^{k-1}, \tag{6.1}$$

where we use the notation St also for the character of the Steinberg representation. We prove now (6.1) for any $g \in \mathrm{GL}_n(\mathbb{F})$.

We first prove (6.1) for $k = 1$. Note that $N = \{I_n\}$ and so

$$V_{\pi_{1,N,\psi}} = \left\{v \in V_{\pi_\theta} \mid \pi(I_n)v = v\right\} = V_{\pi_\theta}.$$

Hence $\pi_{1,N,\psi}(g) = \pi_\theta(g)$ as needed.

Now assume $k \geq 2$. If the semisimple part s of g does not come from $\mathbb{F}_n$, or g is not semisimple, then $\Theta_{k,N,\psi}(g) = 0$ by Theorem 3. From Theorem 2.1, we have $\Theta_{\theta\restriction_{\mathbb{F}_n^*}}(g) = 0$. Hence, (6.1) is proved in that case.

Otherwise, $g = s$ is a semisimple element which comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. We begin by calculating the character value $\mathrm{St}(g)$. For any prime p, let m_p be the p-part of m. By [[12], Thm. 6.5.9],

$$\mathrm{St}(g) = \varepsilon_{\mathrm{GL}_n}\varepsilon_{C(g)^\circ}\left|C(g)^{\mathbb{F}}\right|_{\mathrm{char}(\mathbb{F})},$$

where ε_G is (-1) to the power of the $\mathbb{F}$-rank of G, $C(g)$ is the centralizer of g in $\mathrm{GL}_n(\overline{\mathbb{F}})$, $C(g)^\circ$ is its identity component and $C(g)^{\mathbb{F}}$ is the subgroup of $\mathbb{F}$-rational points in $C(g)$. The $\mathbb{F}$-rank of GL_n is n. Let $\rho = (1, 1, \ldots, 1)$, a partition of $d' = \frac{n}{d}$ and let f be the characteristic polynomial of s. By Sect. 2.3.1, the centralizer $C(g)^{\mathbb{F}}$ is isomorphic to $C(L_{f,\rho})^{\mathbb{F}}$, which in turn is isomorphic to $\mathrm{GL}_{d'}(\mathbb{F}_d)$ (cf. [[5], Lem. 2.4] and the discussion preceding it). Thus, $\varepsilon_{C(g)^\circ} = \varepsilon_{\mathrm{GL}_{d'}} = (-1)^{d'}$ and

$$\left|C(g)^{\mathbb{F}}\right| = q^{\sum_{i=1}^{d'} d(d'-i)}\prod_{k=1}^{d'}\left(q^{dk}-1\right), \quad \left|C(g)^{\mathbb{F}}\right|_{\mathrm{char}(\mathbb{F})} = q^{\frac{n(d'-1)}{2}}.$$

The discussion shows that

$$\mathrm{St}(g) = (-1)^{n-d'}q^{\frac{n(d'-1)}{2}}. \tag{6.2}$$

By Theorem 2.1,

$$\begin{aligned}\Theta_{\theta\restriction_{\mathbb{F}_n^*}}(g) &= (-1)^{n-1}\left[\sum_{\alpha=0}^{d-1}\theta(\lambda^{q^\alpha})\right](1-q^d)(1-(q^d)^2)\cdots(1-(q^d)^{d'-1})\\ &= (-1)^{n-d'}\left[\sum_{\alpha=0}^{d-1}\theta(\lambda^{q^\alpha})\right](q^d-1)(q^{2d}-1)\cdots(q^{n-d}-1)\frac{q^n-1}{q^n-1}\\ &= (-1)^{n-d'}\left[\sum_{\alpha=0}^{d-1}\theta(\lambda^{q^\alpha})\right]\frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{(q^n-1)q^{\frac{n(d'-1)}{2}}},\end{aligned} \tag{6.3}$$

where λ is an eigenvalue of g. By Theorem 3

$$\Theta_{k,N,\psi}(g) = (-1)^{k(n-d')}q^{(k-2)\frac{n(d'-1)}{2}}\cdot\left[\sum_{i=0}^{d-1}\theta(\lambda^{q^i})\right]\cdot\frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^n-1}. \tag{6.4}$$

Multiplying (6.3) by (6.2) raised to the $(k-1)$-th power, we get (6.4) as needed. □

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